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Kumaraswamy Alpha Power Inverted Exponential Distribution: Properties and Applications

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Abstract: This article proposes a four parameter class of lifetime model called Kumaraswamy Alpha Power Inverted Exponential (KAPIE) distribution in the family of the alpha power transformation. Various statistical properties of the KAPIE density including quantile and hazard rate functions, skewness, kurtosis, order statistics and entropies are investigated. The parameters of the KAPIE distribution are estimated by a maximum likelihood. The flexibility and behaviour of the estimators were studied through a simulation. The empirical flexibility of the KAPIE distribution was examined by means of real life data. It was observed that the KAPIE distribution can serve as an alternative model to other existing densities in literature for modeling lifetime data.

Key words: Alpha power transformation; Inverted exponential; Kumaraswamy density; Maximum likelihood; Moment generating function
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1. Introduction

Despite many life time distributions in literature, many new flexible distributions are still being developed to suit real life scenarios. Exponential distribution is a continuous memoryless distributions widely used to describe the time interval between events that follows Poisson processes. However, despite its importance in modeling lifetime Poisson data, the exponential distribution has a constant failure rate. However, real life Poisson constant failure rate are rare to see. Thus, the exponential distribution becomes unsuitable to modeling real life scenarios. However, to make up for this deficiency, Keller and Kamath [18] introduced inverted bathtub hazard rate function which has an inverted exponential distribution. Suppose a random variable U has an exponential distribution, then the random variable, say, $Y = \frac{1}{U}$ has inverted exponential (IE) distribution. Hence, the probability density function (pdf) of the random variable is expressed as

$$w(u) = \frac{\lambda}{u^2}\ exp(-\frac{\lambda}{u})\ ; \qquad \lambda > 0, \quad u \geq 0. \tag{1.1}$$

The cumulative distribution function (cdf) that corresponds is given as

$$W(u) = exp(-\frac{\lambda}{u})\ ; \qquad \lambda > 0, \quad u \geq 0; \tag{1.2}$$

where λ is the scale parameter.
The inverted exponential distribution has a wide range of applications in medicine, engineering, computer software and biology where failure rate are modeled [13]. Oguntunde et al. [27] proposed the transmuted inverse exponential distribution. Generalized exponential was proposed in Gupta and Kundu [14]. Oguntunde et al. [28] examined the properties of the exponentiated generalized

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inverted exponential distribution. Eghwerido et al. [8] proposed the extended new generalized exponential distribution. Nadarajah and Okorie [23] proposed a method of alpha power transformation for obtaining the moments of the generalized exponential distribution. Abouammoh and Ashingiti [1] proposed the generalized inverted exponential distribution. The extended generalized exponential distribution was proposed in Olapade [29]. Fatima and Roohi [12] proposed the extended Poisson exponential distribution. Efe-Eyefia et al. [7] proposed the Weibull-alpha power inverted exponential distribution. Eghwerido et al. [9] proposed the Gompertz alpha power inverted exponential distribution. Unal et al. [33] proposed the alpha power inverted exponential distribution. The Harris extended exponential distribution was proposed in Pinho et al. [30]. Hamedani et al. [15] proposed the type 1 general exponential distribution. The generalized odd generalized exponential distribution was proposed in Alizadeh et al. [2]. Extended weighted exponential distribution was proposed in Mahdavi and Jabari [20]. Beta exponential distribution was proposed in Nadarajah and Kotz [22]. Nassar et al. [24] proposed alpha power Weibull distribution.
However, to make the inverted exponential distribution more flexible, Unal et al. [33] adopted the alpha power transformation and defined the cumulative distribution function of the alpha power inverted exponential distribution as

$$W(u) = \begin{cases} \frac{\alpha^{e^{-\frac{\lambda}{u}}}-1}{\alpha-1}, & if \quad \alpha > 0, \alpha \neq 1 \\ \\ e^{-\frac{\lambda}{u}}, & if \quad \alpha = 1 \end{cases} \tag{1.3}$$

and the corresponding probability density function is

$$w(u) = \begin{cases} \frac{\log \alpha}{(\alpha-1)} \frac{\lambda}{u^2} e^{-\frac{\lambda}{u}} \alpha^{e^{-\frac{\lambda}{u}}}, & if \quad \alpha > 0, \alpha \neq 1 \\ \\ \frac{\lambda}{u^2} e^{-\frac{\lambda}{u}}, & if \quad \alpha = 1 \end{cases} \tag{1.4}$$

More so, Eugene et al. [11] made the exponential distribution more flexible by introducing a beta random variable that introduces skewness and a tail weighted into the distribution. This led many authors to introduce alternative distributions [26]. The inadequacy of the beta distribution not being tractable led many researchers suggesting alternative bounded distributions.
Kumaraswamy distribution was named after Ponndi Kumaraswamy in (1980)[19]. The Kumaraswamy distribution provides a better alternative to the beta distribution because of its boundedness, unimodality, decreasing, increasing as well as constant failure rate and its monotonicity property. [17]. Oguntunde et al. [26] proposed the Kumaraswamy inverse exponential distribution.
Cordeiro and de-Castro [4] introduced the Kumaraswamy-G family of distributions with a cdf expressed as

$$W(u) = 1 - \{1 - F(u)^{\psi}\}^{b} \tag{1.5}$$

The corresponding pdf is given as

$$w(u) = \psi b\ f(u)[F(u)]^{\psi-1}\{1 - [F(u)]^{\psi}\}^{b-1} \tag{1.6}$$

where $F(u)$, and $f(u)$ are the baseline cdf and pdf, $b > 0$ and $\psi > 0$ are additional shape parameters. This study is motivated as a result of lack of knowledge of the class of the Kumaraswamy alpha power transformed family of distribution called Kumaraswamy alpha power inverted exponential (KAPIE) distribution with a Kumaraswamy distribution characterization.
In this article, we propose a four-parameter family of distributions called KAPIE distribution, which can be applied to life time data. The statistical properties of the KAPIE family of distributions will be established. In addition, the(MLEs) of the parameters will be obtained in a closed form.

2. KAPIE distribution (KAPIE-D)

The probability density function of the KAPIE-D given as

$$w_{KAPIE}(u) = \frac{\psi b\lambda exp(\frac{-\lambda}{u})\log(\alpha)\alpha^{(e^{\frac{-\lambda}{u}})}}{u^2(\alpha-1)}\left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b-1}\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi-1} \tag{2.1}$$

$$\alpha \in \Re^{+} - \{1\}.$$

The corresponding cdf is expressed as

$$W_{KAPIE}(u) = 1-\left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b} \quad \alpha \in \Re^{+} - \{1\}. \tag{2.2}$$

The plot for KAPIE-D different values of parameters for the probability density function is shown in Figure 1 and Figure 2 is the plot for cumulative density function.

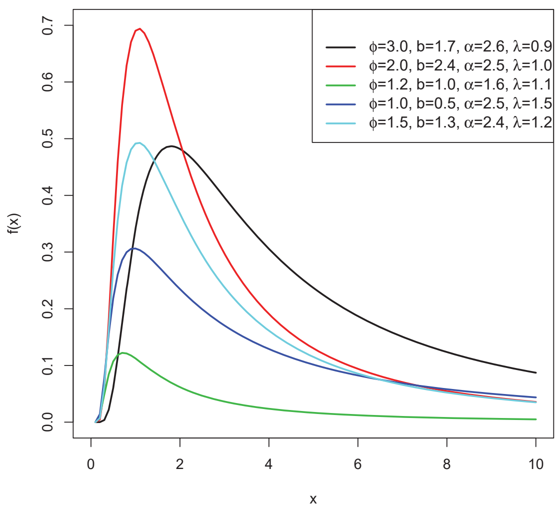

Figure 1. The plots for the KAPIE pdf for different parameter values

Remark 1. The shape of the KAPIE-D pdf in Figure 1 could be inverted bathtub, negatively skewed or positively skewed. In Figure 2, the shape of the cdf of the KAPIE-D shows that it is increasing.

2.1. Survival and hazard rate

The survival function ($S_{KAPIE}(u)$) for the KAPIE-D is given by

$$S_{KAPIE}(u) = 1 - W_{KAPIE}(u) \tag{2.3}$$

Hence,

$$S_{KAPIE}(u) = \left(1-\left(\frac{\alpha^{exp(\frac{-\lambda}{u})}-1}{\alpha-1}\right)^{\psi}\right)^{b} \tag{2.4}$$

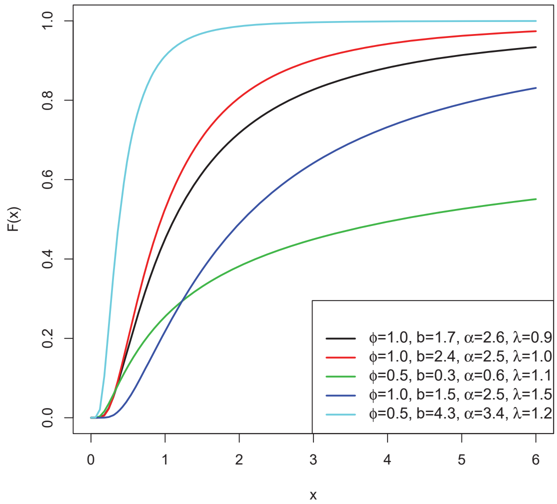

FIGURE 2. The plots for the KAPIE cdf for different parameter values

The hazard rate is also a very important property to consider. It is given by

$$H_{KAPIE}(u) = \frac{w_{KAPIE}}{S_{KAPIE}} \tag{2.5}$$

Substituting Equations (2.1) and (2.4) into Equations (2.5) gives

$$H_{KAPIE}(u) = \frac{\psi b \log(\alpha) \lambda e^{\frac{-\lambda}{u}} \alpha^{(e^{\frac{-\lambda}{u}})}}{u^2(\alpha - 1)} \left(\frac{\alpha^{e^{\frac{-\lambda}{u}}} - 1}{\alpha - 1} \right)^{\psi - 1} \left(1 - \left(\frac{\alpha^{e^{\frac{-\lambda}{u}}} - 1}{\alpha - 1} \right)^{\psi} \right)^{-1} \tag{2.6}$$

Figure 3 shows some examples for the hazard shape of the KAPIE distribution.

REMARK 2. The hazard rate function $H_{KAPIE}(x)$ of the KAPIE-D is right skewed unimodal and decreasing for different parameter value.

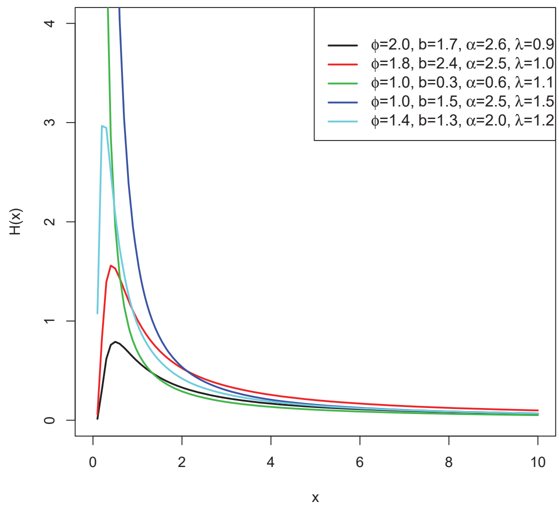

FIGURE 3. The plots for the KAPIE hazard rate function for different parameter values

3. Mixture representation of the KAPIE-D

In this section, we shall derive the algebraic expression of the KAPIE distribution. However, $(x+y)^n = \sum_{n=0}^{m} x^{n-i}y^i$.

Then, let the quantity $(1-(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1})^{\psi})^{b-1}$ in Equation (2.1) be B; where b is a positive integer. Then, by binomial expansion

$$B = \sum_{\rho=0}^{b-1} \binom{b-1}{\rho} (-1)^{\rho} (\alpha-1)^{-\psi\rho} (\alpha^{e^{-\frac{\lambda}{u}}} - 1)^{\psi\rho}$$

This implies that

$$(\alpha-1)^{-\psi}(\alpha^{e^{-\frac{\lambda}{u}}} - 1)^{\psi-1} B = \sum_{\rho=0}^{b-1} \binom{b-1}{\rho} (\alpha-1)^{-\psi(\rho+1)} (\alpha^{e^{-\frac{\lambda}{u}}} - 1)^{\psi(\rho+1)-1} (-1)^{\rho}. \tag{3.1}$$

However, the quantity $(\alpha^{e^{-\frac{\lambda}{u}}} - 1)^{\psi(\rho+1)-1}$ can be expressed as

$$(\alpha^{e^{-\frac{\lambda}{u}}} - 1)^{\psi(\rho+1)-1} = \sum_{j=0}^{\psi(\rho+1)-1} \binom{\psi(\rho+1)-1}{j} (-1)^j \alpha^{e^{-\frac{\lambda}{u}(\psi(\rho+1)-1-j)}}.$$

Then, Equation (3.1) can be expressed as

$$Q_{\rho j} = \sum_{j=0}^{\psi(\rho+1)-1} \sum_{\rho=0}^{b-1} \binom{b-1}{\rho} \binom{\psi(\rho+1)-1}{j} (-1)^{\rho+j} (\alpha-1)^{-\psi(\rho+1)} \alpha^{e^{-\frac{\lambda}{u}(\psi(\rho+1)-j-1)}}. \tag{3.2}$$

Multiplying Equation (3.2) by $\alpha^{e^{-\frac{\lambda}{u}}}$, we have

$$\sum_{j=0}^{\psi(\rho+1)-1} \sum_{\rho=0}^{b-1} \binom{b-1}{\rho} \binom{\psi(\rho+1)-1}{j} (-1)^{\rho+j} (\alpha-1)^{-\psi(\rho+1)} \alpha^{e^{-\frac{\lambda}{u}(\psi(\rho+1)-j)}}. \tag{3.3}$$

More so, the $\alpha^{e^{-\frac{D}{u}}}$ for $D=\lambda(\psi(\rho+1)-j)$ can be expressed as

$$\alpha^{e^{-\frac{D}{u}}}=\sum_{k=0}^{\infty}\frac{(\log\alpha)^k}{k!}e^{-\frac{Dk}{u}}.$$

However, on simplifying Equation (2.1) reduces to

$$w_{KAPIE}(u)=\sum_{j=0}^{\psi(\rho+1)-1}\sum_{\rho=0}^{b-1}\sum_{k=0}^{\infty}\frac{\mu_{\rho jk}}{u^2}exp(-\frac{1}{u}(Dk+\lambda)) \qquad \alpha>0 \quad \alpha\neq 1. \tag{3.4}$$

where

$$\mu_{\rho jk}=\psi b\lambda\binom{\psi(\rho+1)-1}{j}\binom{b-1}{\rho}(-1)^{\rho+j}(\alpha-1)^{-\psi(\rho+1)}\frac{(\log\alpha)^{k+1}}{k!}.$$

The corresponding cdf is given as

$$W_{KAPIE}(u)=1-\sum_{\rho=0}^{b}\sum_{j=0}^{\psi\rho}\sum_{k=0}^{\infty}m_{\rho jk}e^{-\frac{\lambda}{u}k(\psi\rho-j)} \tag{3.5}$$

where

$$m_{\rho jk}=\binom{\psi\rho}{j}\binom{b}{\rho}(-1)^{\rho+j}(\alpha-1)^{-\psi\rho}\frac{(\log\alpha)^k}{k!}$$

4. KAPIE Statistical Properties

This section investigates some basic statistical properties of the KAPIE distribution. These include quantile function, order statistics, moments of the residual and entropies.

4.1. Quantile function

Let U be a random variable such that $X\sim KAPIE(b,\psi,\alpha,\lambda)$. Then, the quantile function of U for $\varrho\in(0,1)$ is given as

$$Q(\varrho)=\inf\{x\in\Re:\varrho\leq W_{KAPIE}(u)\}=W_{KAPIE}^{-1}(u). \tag{4.1}$$

$$u_\varrho=-\lambda\left(\log\left(\left(\log\alpha\right)^{-1}\log\left(\left(1-(1-\varrho)^{\frac{1}{b}}\right)^{\frac{1}{\psi}}(\alpha-1)+1\right)\right)\right)^{-1} \qquad \varrho\in(0,1). \tag{4.2}$$

Setting $\varrho=0.5$ in Equation (4.2), we have the median (M) of the KAPIE random variable X as

$$M=-\lambda\left(\log\left(\left(\log\alpha\right)^{-1}\log\left(\left(1-(0.5)^{\frac{1}{b}}\right)^{\frac{1}{\psi}}(\alpha-1)+1\right)\right)\right)^{-1}. \tag{4.3}$$

However, the 25^{th} and 75^{th} percentile for the random variable X is obtained as

$$Q_1=-\lambda\left(\log\left(\left(\log\alpha\right)^{-1}\log\left(\left(1-(0.75)^{\frac{1}{b}}\right)^{\frac{1}{\psi}}(\alpha-1)+1\right)\right)\right)^{-1}. \tag{4.4}$$

$$Q_3=-\lambda\left(\log\left(\left(\log\alpha\right)^{-1}\log\left(\left(1-(0.25)^{\frac{1}{b}}\right)^{\frac{1}{\psi}}(\alpha-1)+1\right)\right)\right)^{-1}. \tag{4.5}$$

The Bowley's skewness is based on quartiles as follows

$$S_k=\frac{Q(\frac{3}{4})-2Q(\frac{1}{2})+Q(\frac{1}{4})}{Q(\frac{3}{4})-Q(\frac{1}{4})}. \tag{4.6}$$

The Moor's kurtosis is given as

$$K_{KAPIE}=\frac{Q(\frac{7}{8})-Q(\frac{5}{8})-Q(\frac{3}{8})+Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})} \tag{4.7}$$

where $Q(.)$ is the KAPIE quantile function.

4.2. KAPIE entropies

The Rényi entropy $R_\delta(U)$ of the variable U that is KAPIE distributed measures the variation of the uncertainty. Thus, it is given as

$$R_\delta(U) = \frac{1}{1-\delta}\log\int_{-\infty}^{\infty} w^{\delta}_{KAPIE}(u)du \qquad \delta>0, \delta\neq 0. \tag{4.8}$$

However, thus, the Equation (4.8) is expressed as

$$R_\delta(U) = \frac{1}{1-\delta}\log\Big(\sum_{\rho=0}^{b-1}\sum_{j=0}^{\psi(\rho+1)-1}\sum_{k=0}^{\infty}(\frac{\mu_{\rho jk}}{u^2})^{\delta}\int_0^{\infty} e^{-\frac{\lambda\delta}{u}D_{\rho jk}}du\Big)$$

Integrating and simplifying, we have

$$R_\delta(U) = \frac{1}{1-\delta}\log\Big(\sum_{j=0}^{\psi(\rho+1)-1}\sum_{\rho=0}^{b-1}\sum_{k=0}^{\infty}\mu^{*}_{\rho jk}(\delta\lambda D_{\rho jk})^{1-2\delta}\Gamma(2\delta-1)\Big). \tag{4.9}$$

where $\mu^{*}_{\rho jk} = -\mu_{\rho jk}$
The $\delta-$entropy, say $M_\delta(U)$ is expressed as

$$M_\delta(U) = \frac{1}{1-\delta}\log\Big[1-\int_{-\infty}^{\infty} w^{\delta}_{KAPIE}(u)\ du\Big], \qquad \delta>0, \delta\neq 1$$

$$M_\delta(U) = \frac{1}{1-\delta}\log\Big[\Big(\sum_{j=0}^{\psi(\rho+1)-1}\sum_{\rho=0}^{b-1}\sum_{k=0}^{\infty}\mu_{\rho jk}(\delta\lambda D_{\rho jk})^{1-2\delta}\Gamma(2\delta-1)\Big)+1\Big], \qquad \delta>0, \delta\neq 1 \tag{4.10}$$

4.3. Moments of the residual

The β^{th} moment of the residual life, say $d_\beta(t) = E\big[(U-t)^{\beta}|U>t\big]$ for $\beta=1,2,\ldots$ uniquely determines $W_{KAPIE}(u)$ (see Navarro et al. [25]). However, the β^{th} moment of the residual life is given as

$$d_\beta(t) = \frac{1}{1-W_{KAPIE}(t)}\int_t^{\infty}(U-t)^{\beta}\ dW_{KAPIE}(u)$$

$$d_\beta(t) = \frac{1}{1-W_{KAPIE}(t)}\sum_{p=0}^{\beta}\binom{\beta}{p}(-1)^{\beta-p}\int_t^{\beta-p} u^p t^{\beta-p} w_{KAPIE}(t)\ du. \tag{4.11}$$

Integrating and simplifying, we have

$$d_\beta(t) = \frac{1}{W_{KAPIE}(t)-1}\sum_{\rho=0}^{b-1}\sum_{j=0}^{\psi(\rho+1)-1}\sum_{p=0}^{\beta}\sum_{k=0}^{\infty}\binom{\beta}{p}(-1)^{\beta-p}t^{\beta-p}\mu_{\rho jk}$$

$$\times(\delta\lambda D_{\rho jk})^{p-1}\varrho^{1-3p}\gamma(2p,\beta-p) \qquad \varrho>1; \quad p<0 \tag{4.12}$$

where $\gamma(2p,\beta-p)$ is lower incomplete gamma function.

4.4. Order statistics

Let $U_1, U_2, ..., U_\beta$ be a random sample of the KAPIE-D. Let $U_{\rho:n}$ indicates the ρth order statistics for $U_{\rho:\beta}(1 \le \rho \le \beta)$ for KAPIE distribution given by

$$w_{KAPIE_{\rho:\beta}} = \frac{\beta!}{(\rho-1)!(\beta-\rho)!} w_{KAPIE}(u_\rho)[W_{KAPIE}(u_\rho)]^{i-1}[W_{KAPIE}(u_\rho)]^{\beta-\rho} \tag{4.13}$$

$$\begin{aligned} w_{KAPIE_{\rho:\beta}} = {} & \frac{\beta!}{(\rho-1)!(\beta-\rho)!} \frac{\psi b \log(\alpha)\lambda e^{\frac{-\lambda}{u}} \alpha^{(e^{\frac{-\lambda}{u}})}}{u^2(\alpha-1)} \left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi-1} \left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b-1} \\ & \times \left(1-\left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b}\right)^{\rho-1} \left(1-\left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b}\right)^{\beta-\rho} \end{aligned} \tag{4.14}$$

From (4.14), for $\rho = 1$, the pdf of the minimum order statistics of the KAPIE-D is expressed as

$$w_{KAPIE_{1:\beta}} = \frac{\beta!}{(\beta-1)!} w_{KAPIE}(u_1)[W_{KAPIE}(u_1)]^{\beta-1} = \beta w_{KAPIE}(u_1)[W_{KAPIE}(u_1)]^{\beta-1} \tag{4.15}$$

$$\begin{aligned} w_{KAPIE_{1:\beta}} = {} & \frac{\beta!}{(\beta-1)!} \frac{\psi b \log(\alpha)\lambda e^{\frac{-\lambda}{u}} \alpha^{(e^{\frac{-\lambda}{u}})}}{u^2(\alpha-1)} \left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi-1} \left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b-1} \\ & \times \left(1-\left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b}\right)^{\beta-1} \end{aligned} \tag{4.16}$$

also, for $\beta = 1$, the pdf of the maximum order statistics of the KAPIE-D is expressed as

$$w_{KAPIE_{\beta:\beta}} = \frac{\beta!}{(\beta-1)!} w_{KAPIE}(u_\beta)[W_{KAPIE}(u_\beta)]^{\beta-1} \tag{4.17}$$

$$\begin{aligned} w_{KAPIE_{\beta:\beta}} = {} & \frac{\beta!}{(\beta-1)!} \frac{\psi b \log(\alpha)\lambda e^{\frac{-\lambda}{u}} \alpha^{(e^{\frac{-\lambda}{u}})}}{u^2(\alpha-1)} \left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi-1} \left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b-1} \\ & \times \left(1-\left(1-\left(\frac{\alpha^{e^{\frac{-\lambda}{u}}}-1}{\alpha-1}\right)^{\psi}\right)^{b}\right)^{\beta-1} \end{aligned} \tag{4.18}$$

5. Parameter estimation of KAPIE-D

Several approaches have been employed for estimating parameters in literature. The maximum likelihood method is adopted to obtain the parameters of the KAPIE-D in this study.

Let $\mathbf{x} = (u_1, u_2, \ldots, u_\beta)$ be a random sample from the KAPIE model with unknown parameter vector $\theta = (b, \psi, \alpha, \lambda)^T$. Then log-likelihood function, say ν, of the KAPIE is expressed as

$$\begin{aligned}\nu = {} & \beta \log \psi \; b \; \lambda + \beta \log \alpha - \sum_{\rho=1}^{\beta} \frac{\lambda}{u_\rho} - \sum_{\rho=0}^{\beta} u^2 - \beta \log(\alpha - 1) - \beta(\psi - 1)\log(\alpha - 1) \\ & + (\psi - 1)\sum_{\rho=1}^{\beta} \log(\alpha^{e^{-\frac{\lambda}{u_\rho}}} - 1) + (b-1)\sum_{\rho=1}^{\beta} \log(1 - (\frac{\alpha^{e^{-\frac{\lambda}{u_\rho}}} - 1}{\alpha - 1})^{\psi})\end{aligned} \tag{5.1}$$

Let

$$m = (\psi - 1)\sum_{\rho=1}^{\beta} \log(\alpha^{e^{-\frac{\lambda}{u_\rho}}} - 1)$$

and

$$z = (b-1)\sum_{\rho=1}^{\beta} \log(1 - (\frac{\alpha^{e^{-\frac{\lambda}{u_\rho}}} - 1}{\alpha - 1})^{\psi})$$

such that $m'_\psi = \frac{\partial m}{\partial \psi}$; $m'_\lambda = \frac{\partial m}{\partial \lambda}$; $m'_\alpha = \frac{\partial m}{\partial \alpha}$; $z'_\psi = \frac{\partial z}{\partial \psi}$; $z'_\lambda = \frac{\partial z}{\partial \lambda}$; $z'_\alpha = \frac{\partial z}{\partial \alpha}$; $z'_b = \frac{\partial z}{\partial b}$
Then, the partial derivative of ℓ with respect to each parameter and equating to zero is given as

$$\frac{\partial \nu}{\partial \psi} = \frac{\beta}{\psi} - \beta \log(\alpha - 1) + z'_\psi; \tag{5.2}$$

$$\frac{\partial \nu}{\partial b} = \frac{\beta}{b} + z'_b; \tag{5.3}$$

$$\frac{\partial \nu}{\partial \lambda} = \frac{\beta}{\lambda} - \sum_{\rho=1}^{\beta} \frac{1}{u_\rho} + m'_\lambda + z'_\lambda; \tag{5.4}$$

$$\frac{\partial \nu}{\partial \alpha} = \frac{\beta}{\alpha} - \frac{\beta}{\alpha - 1} - \frac{\beta(\psi - 1)}{\alpha - 1} + m'_\alpha + z'_\alpha; \tag{5.5}$$

The parameter estimates of the unknown is obtained by equating the vector to zero. The solution to the vector is obtained analytically using Newton-Raphson algorithm. R, MATLAB, MAPLE Software are used to obtain their estimates.
A simulation is carried out to test the flexibility and efficiency of the KAPIE distribution. The simulation for various values of parameters are shown in Table 1. The simulation of the KAPIE model is performed as follows:

- Data are generated using

$$x_u = -\lambda\Bigg(\log\bigg(\big(\log \alpha\big)^{-1} \log\big((\alpha - 1)\big(1 - (1-u)^{\frac{1}{b}}\big)^{\frac{1}{\psi}} + 1\big)\bigg)\Bigg)^{-1} \qquad 0 < u < 1.$$

- The values of the parameters are set as $\psi = 1.0, 1.5, 2.0, 2.5$ and 3.0; $b = 1.0, 1.5, 2.0, 2.5$ and 3.0; $\alpha = 2.0, 2.5$, and 3.0 and $\lambda = 1.0, 1.5, 2.0, 2.5$ and 3.0
- The sample sizes of $n = 50, 100$ and 150 are taken.
- Each of the sample size is replicated 1000 times.

In this simulation study, we investigate the biases, means squared errors (RMSEs), kurtosis, skewness, median, first and third quatiles.
The bias is calculated by (for $R = \hat{\psi}, \hat{b}, \hat{\alpha}, \hat{\lambda},$)

$$\hat{B}ias_R = \frac{1}{1000}\sum_{\rho=1}^{1000}\Big(\hat{R}_\rho - R\Big).$$

Table 1. A simulation Study of the KAPIE Distribution

Sample size	Parameters	Bias	MSE	Kurtosis	Skewness	Median	25^{th}	75^{th}
50	$\psi=1.00$, $b=1.00$, $\alpha=2.00$, $\lambda=1.00$	0.8747, 0.2797, 4.9919, -0.5442	1.9847, 1.0282, 37.4683, 0.3681	-1.593219	-0.9246245	6.929397	-12.68539	-0.8492979
	$\psi=1.50$, $b=1.00$, $\alpha=2.00$, $\lambda=1.00$	-0.1329, -0.3071, 4.9740, -0.5624	0.6760, 0.2582, 30.6805, 0.3648	-0.9505154	-0.6019234	-2.549937	-1.848672	-0.785338
	$\psi=1.50$, $b=1.50$, $\alpha=2.00$, $\lambda=1.00$	-0.3658, -0.5121, 4.5508, -0.2471	0.2021, 0.4792, 25.9758, 0.6328	-1.160371	-0.7041206	-6.854841	-3.318867	-0.8864735
	$\psi=1.50$, $b=1.50$, $\alpha=2.00$, $\lambda=1.50$	-0.1248, -0.4667, 6.3303, -0.8234	0.2916, 0.7463, 49.1789, 0.8455	-1.160371	-0.7041206	-10.28226	-4.9783	-1.32971
	$\psi=1.50$, $b=1.50$, $\alpha=2.50$, $\lambda=1.50$	1.3389, 2.5130, 16.4693, -0.7842	3.1269, 11.7366, 344.6195, 0.7142	-1.338551	-0.794868	-71.62764	-8.504993	-1.495328
100	$\psi=2.00$, $b=1.00$, $\alpha=2.00$, $\lambda=1.00$	-0.2825, 0.4266, 4.5316, -0.5487	3.7491, 4.7169, 31.9002, 0.4675	-0.7697992	-0.490566	-1.514219	-1.295374	-0.7568395
	$\psi=1.00$, $b=2.00$, $\alpha=2.00$, $\lambda=1.00$	0.8250, -0.4257, 4.9425, -0.5206	1.3707, 1.3038, 41.3284, 0.9442	4.055039	-2.866188	1.276765	1.83493	-1.295374
	$\psi=1.00$, $b=1.00$, $\alpha=2.00$, $\lambda=2.00$	0.6469, 0.2629, 4.9001, -1.5618	0.8926, 0.8531, 30.2054, 2.4698	-1.593219	-0.9246245	13.85879	-25.37078	-1.698596
	$\psi=2.00$, $b=2.00$, $\alpha=2.00$, $\lambda=2.00$	-0.5974, -0.6579, 8.6906, -1.0273	0.4689, 0.5433, 80.1832, 1.1252	-0.9192074	-0.5609955	-5.865997	-4.347139	-1.788027
	$\psi=2.00$, $b=2.00$, $\alpha=2.50$, $\lambda=2.00$	0.3772, 1.4898, 10.9089, -1.5053	1.3376, 11.9132, 155.5866, 2.3040	-1.030194	-0.6251974	-9.258945	-5.967798	-2.01286
150	$\psi=2.50$, $b=2.50$, $\alpha=2.50$, $\lambda=2.50$	-0.1525, 1.6166, 7.0826, -1.6152	0.8977, 5.1402, 52.2378, 2.7973	-0.8505584	-0.5161039	-7.022994	-5.524683	-2.50368
	$\psi=2.50$, $b=2.50$, $\alpha=3.00$, $\lambda=2.50$	-0.3303, 1.4104, 9.2154, -1.6623	1.1916, 4.7327, 90.8108, 3.1617	-0.8883787	-0.5387582	-8.115947	-6.179299	-2.62994
	$\psi=3.00$, $b=2.50$, $\alpha=3.00$, $\lambda=2.50$	1.8888, 1.4702, 16.6342, -2.0169	6.7118, 3.8098, 298.9408, 4.0946	-0.7308612	-0.4427666	-5.295817	-4.52471	-2.481304
	$\psi=3.00$, $b=3.00$, $\alpha=3.00$, $\lambda=2.50$	-0.4463, 1.0699, 9.6314, -1.9316	0.4797, 1.2376, 101.2376, 3.7423	-0.7623993	-0.4600106	-6.042407	-5.051648	-2.601823
	$\psi=3.00$, $b=3.00$, $\alpha=3.00$, $\lambda=3.00$	-0.0346, 1.6108, 10.0964, -2.0497	1.1743, 6.3628, 108.7375, 4.5048	-0.7623993	-0.4600106	-7.250888	-6.061977	-3.122188

Also, the MSE is obtained as

$$\hat{M}SE_R = \frac{1}{1000}\sum_{\rho=1}^{1000}\left(\hat{R}_\rho - R\right)^2,$$

The kurtosis, skewness, median, the first and third quartiles in Table 1 decreases as the parameters increases.

6. Real life application

A gas fiber real life data was applied to the proposed model. This is to examine the performances of the model based on its statistical properties. Some criteria were employed to determine the distribution for best fit: Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Akaike Information Criteria (AIC) and Hannan and Quinn Information Criteria (HQIC). The Cramer-von Mises statistic (W), Anderson Darling (A) statistic, Kolmogorov Smirnov (KS) statistic, and the p value were also provided.

6.1. Glass fiber data

The data set on 1.5 cm glass strengths fibres were collected by workers at the UK National Physical Laboratory. These data were used to compare the performance of the *KAPIE* distribution as used in Smith and Naylor [32], Bourguignon et al. [3], Haq et al. [16], Merovci et al. [21] (2016), Rastogi and Oguntunde [31] and Eghwerido et al. [10]. The observations are as follows:
0.81, 0.84, 0.55, 0.74, 0.77,1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.36, 1.39, 1.42, 1.48, 1.27, 1.28, 1.29, 1.48,1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76,1.84, 2.00, 2.01, 2.24, 1.77, 1.89, 1.81, 1.82, 1.84.
Table 2 is the descriptive statistics of the glass fibers dataset. Table 3 is the measure of comparison for the various distribution under consideration. Figure 4 is the plots of some estimated densities

Table 2. Descriptive statistics for the Glass Fibers dataset to 2 decimal points

Mean	Median	Mode	St.D	IQR	Variance	Skewness	Kurtosis	$25^{th}P.$	$75^{th}P.$
1.51	1.59	1.61	0.32	0.31	0.11	-0.81	0.80	1.38	1.69

of the models under consideration. Figure 5 is the plots of some estimated cdfs. These plots show that the KAPIE distribution produces a better fit when compared with others continuous models.

TABLE 3. Performance rating of the KAPIE distribution with glass fibres dataset

Distribution	Parameter MLEs	AIC	CAIC	BIC	HQIC	W	A
KAPIE	$\hat{\psi} = 1.044277478$ $\hat{b} = 19.303904609$ $\hat{\lambda} = 7.427692354$ $\hat{\alpha} = 0.002053295$	52.71052	53.40017	61.28306	56.08214	0.5063777	2.770684
Kumaraswamy Inverted Exponential	$\hat{\alpha} = 3.023170$ $\hat{\lambda} = 163.215156$ $\hat{\beta} = 2.696144$	53.42339	53.83017	59.85279	55.95211	0.5113895	2.832409
Transmuted Marshall-Olkin Frechét	$\hat{\alpha} = 0.65238$ $\hat{\beta} = 6.874409$ $\hat{\lambda} = 376.26842$ $\hat{\gamma} = 0.149932$	56.5234	57.21306	65.09594	59.89502	2.50127	3.1009
Beta Lomax	$\hat{a} = 18.1737$ $\hat{b} = 26.7645$ $\hat{c} = 10.8769$ $\hat{\alpha} = 0.0329$	56.8068	57.4964	65.3793	60.1784	2.542603	3.1986
Alpha Power Inverted Weibull	$\hat{\alpha} = 61.0991524$ $\hat{\beta} = 0.7750625$ $\hat{\lambda} = 3.8048114$	82.58	82.99	89.01	85.11	0.9853	4.2956
Transmuted Frechét	$\hat{\alpha} = 1.306830$ $\hat{\beta} = 2.7898$ $\hat{\lambda} = 0.129842$	100.125	100.5318	106.5544	102.6537	0.9901	4.2832
Extended Generalized Exponential	$\hat{\alpha} = 144.0791$ $\hat{\beta} = 0.055021$ $\hat{\lambda} = 137.8711$ $\hat{\gamma} = 7.994$	145.3	145.9	153.8	148.6	0.9922	4.2501
Exponential	$\hat{\lambda} = 0.6637$	179.6	181.8	185.9	179.7	0.9969	4.2902
Alpha Power Inverted Exponential	$\hat{\alpha} = 53.5634269$ $\hat{\lambda} = 0.3508747$	196.3253	196.5253	200.6116	198.0111	0.777503	4.238456

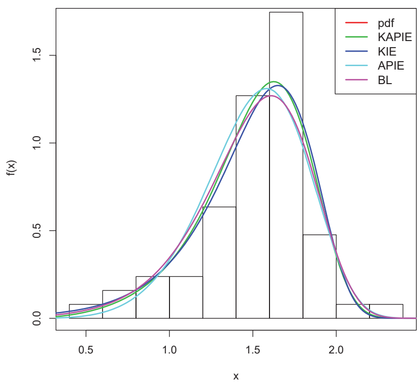

FIGURE 4. The plots of fitted estimated KAPIE density

6.2. Discussion

The proposed model performance is determined by the value of the lowest Akaike Information

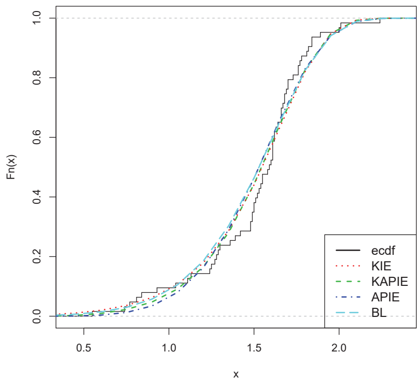

FIGURE 5. The plots of fitted estimated cdf of the KAPIE model

Criteria (AIC). The real life case considered has the KAPIE distribution with AIC value 52.71052. Hence, it is regarded as a better model for the data used compare to some other existing distributions.

7. Conclusion

The KAPIE distribution has been successfully studied. The basic statistical properties of the KAPIE model such as the order statistics distribution, hazard rate function, quantile function, median, odds function have been established. The shape of the KAPIE-D could be inverted bathtub or decreasing (depending on the value of the parameters). An application to the real life data shows that the KAPIE distribution is a strong competitor and can also be used as alternative model in modeling lifetime processes.

Conflicts of interest

Authors declare that there is no conflicts of interest.

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