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# Kumaraswamy Distribution Based on Alpha Power Transformation Methods

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***Authors' contributions***

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

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## Abstract

In this paper, the alpha power Kumaraswamy distribution, new alpha power transformed Kumaraswamy distribution and new extended alpha power transformed Kumaraswamy distribution are presented. Some statistical properties of the three distributions are derived including quantile function, moments and moment generating function, mean residual life and order statistics. Estimation of the unknown parameters based on maximum likelihood estimation are obtained. A simulation study is carried out. Finally, a real data set is applied.

*Keywords:* *Alpha power transformation; new alpha power transformed; new extended alpha power transformed; Kumaraswamy distribution; maximum likelihood estimation.*

## 1 Introduction

The *Kumaraswamy* (Kum) distribution was introduced by Kumaraswamy [1], it is similar to the beta distribution, but much simpler to use, especially in simulation studies; due to the simple closed form of both its *cumulative distribution function* (cdf) and quantile function. This distribution is applicable to many

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natural phenomena whose outcomes have lower and upper bounds, such as the heights of individuals, scores obtained on a test, atmospheric temperatures and hydrological data. Jones [2] studied the basic properties of the Kum distribution, also considered the *maximum likelihood* (ML) estimators and compared the Kum distribution to the Beta distribution. Grag [3] derived the distribution of single order statistics, the joint distribution of two order statistics and the distribution of the product and the quotient of two order statistics when the random variables are independent and identically Kum-distributed. Golizadeh et al. [4] obtained non-Bayesian and Bayesian estimation for the parameters of the Kum distribution. Moreover, non-Bayesian and Bayesian approaches are used to obtain point and interval estimation of the shape parameters, the reliability and the hazard rate functions of the Kum distribution based on generalized order statistics [5].

Considering $X$ is a random variable from Kum distribution, then the *probability density function* (pdf) and cdf are, respectively, given by

$$f(x;\lambda,\beta)=\lambda\beta x^{\lambda-1}(1-x^{\lambda})^{\beta-1}, \qquad 0<x<1,\ \lambda,\beta>0, \tag{1}$$

And

$$F(x;\lambda,\beta)=1-(1-x^{\lambda})^{\beta}\ , \tag{2}$$

where $\lambda$ and $\beta$ are the shape parameters.

In the practice of distribution theory, developing new statistical models is very common in the recent literature. It provides more flexibility to the class of distribution functions, and it can be very useful for data analysis purposes. Several methods of generating new statistical distributions were presented in the literature; for example, Marshall and Olkin [6], Eugene et al. [7], Cordeiro and Castro [8] and Alzaatreh et al. [9]. For more details about methods of generating distributions see, Lee et al. [10] and Jones [11]. Recently, Mahdavi and Kundu [12] proposed a new method for introducing new lifetime distributions named as alpha power transformation (APT). Let $F(x)$ and $f(x)$ be the cdf and pdf of a random variable $X$, as the cdf of APT is given by

$$G_1(x)=\frac{\alpha^{F(x)}-1}{\alpha-1}, \qquad \alpha>0,\alpha\neq1, \tag{3}$$

and the corresponding pdf is

$$g_1(x)=\frac{\ln(\alpha)}{\alpha-1}f(x)\alpha^{F(x)}, \qquad x\in\mathbb{R}, \tag{4}$$

where α is a shape parameter. Mahdavi and Kundu [12] introduced the alpha power exponential (APE) distribution and studied the main properties as well as estimation of the parameters of the proposed distribution. Many authors studied the APT method to re-extend some univariate distributions in particular cases; for example, Nassar et al. [13] introduced the alpha power Weibull distribution. Dey et al. [14] presented the alpha power generalized exponential distribution. Nadarajah and Okorie [15] derived a closed form expression for moment properties of the alpha power generalized exponential distribution. Mead et al. [16] studied the general mathematical properties of the APT family and considered the alpha power exponentiated Weibull distribution. Nassar et al. [17] discussed the estimation of the parameters of the APE distribution using nine methods of estimation. Another technique of APT is used to obtain a new class of lifetime distributions by Elbatal et al. [18]; which is named as *new alpha power transformation* (NAPT), with the following cdf

$$G_2(x)=\frac{F(x)\alpha^{F(x)}}{\alpha}, \qquad \alpha>0,\ \alpha\neq1, \tag{5}$$

and the corresponding pdf

$$g_2(x) = \frac{f(x)\alpha^{F(x)}}{\alpha}[1 + \ln(\alpha)F(x)], \qquad x \in \mathbb{R}. \tag{6}$$

Four special models of the NAPT are presented by Elbatal et al. [18] using Fréchet distribution, Pareto distribution, Lomax distribution and Weibull distribution. Ahmad et al. [19] introduced a new method of APT to construct a new class of lifetime distribution, called *new extended alpha power transformation* (NEAPT) which has the cdf as follows:

$$G_3(x) = \frac{\alpha^{F(x)} - e^{\alpha F(x)}}{\alpha - e^{\alpha}}, \qquad \alpha > 0,\ \alpha \neq e, \tag{7}$$

and the pdf is given by

$$g_3(x) = \frac{f(x)[\ln(\alpha)\alpha^{F(x)} - \alpha e^{\alpha F(x)}]}{\alpha - e^{\alpha}}, \qquad x \in \mathbb{R}. \tag{8}$$

A special sub-case has been considered in details namely, two parameters Weibull distribution based on NEAPT by Ahmad et al [19].

The aim of this paper is to present a new lifetime distribution using the APT, NAPT and NEAPT methods to the Kum distribution and comparing between these methods. The main purpose of the new models is that the additional parameter can give several desirable properties and more flexibility in the form of the density and hazard functions. The rest of the paper is organized as follows: In Section 2, the *alpha power Kumaraswamy* (APKum) distribution is considered, also its properties and estimation of the unknown parameters are studied. The *new alpha power transformed Kumaraswamy* (NAPTKum) distribution is presented in Section 3. In Section 4, the *new extended alpha power transformed Kumaraswamy* (NEAPTKum) distribution is discussed. In Section 5, a simulation study is conducted to evaluate the performance of the different estimators. Finally, in Section 6 a real data set is applied.

# 2 Alpha Power Kumaraswamy Distribution

Let *X* be a random variable which has the Kum distribution, using (2) and (1) in (3) and (4), then the cdf of APKum distribution has the following expression

$$G_1(x;\alpha,\lambda,\beta) = \frac{\alpha^{1-(1-x^{\lambda})^{\beta}} - 1}{\alpha - 1}, \tag{9}$$

and the pdf of APKum distribution is

$$g_1(x;\alpha,\lambda,\beta) = \frac{\ln(\alpha)}{\alpha-1}\lambda\beta x^{\lambda-1}(1-x^{\lambda})^{\beta-1}\alpha^{1-(1-x^{\lambda})^{\beta}}, \qquad 0 < x < 1, \alpha,\lambda,\beta > 0. \tag{10}$$

The *survival function* (sf); $S_1(x)$, and the *hazard rate function* (hrf); $h_1(x)$, are respectively, given by

$$S_1(x) = \frac{\alpha}{\alpha-1}\left(1 - \alpha^{-(1-x^{\lambda})^{\beta}}\right), \qquad \alpha \neq 1, \tag{11}$$

And

$$h_1(x) = \frac{\lambda\beta\ln(\alpha)\,x^{\lambda-1}(1-x^{\lambda})^{\beta-1}}{\alpha^{(1-x^{\lambda})^{\beta}} - 1}, \qquad \alpha \neq 1.$$

The plots for the pdf and hrf of the APKum distribution are displayed in Fig. 1.

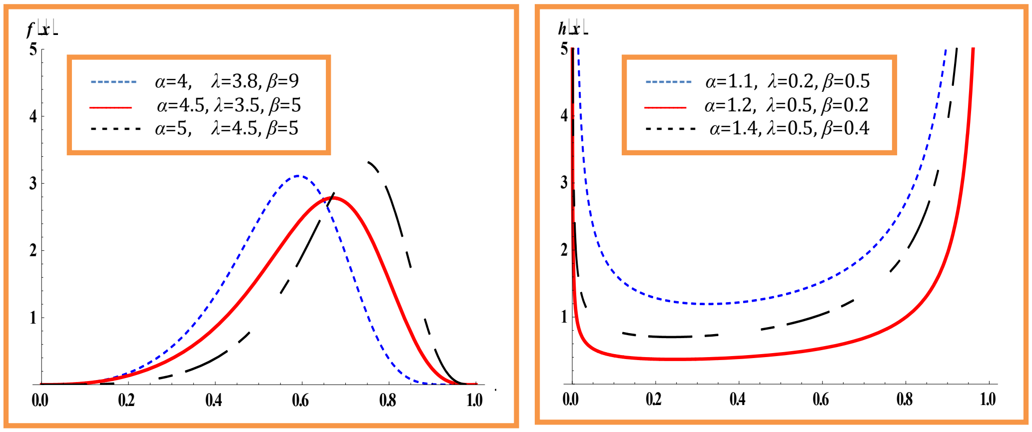


**Fig. 1. Different plots of pdf and hrf of the APKum distribution**

## 2.1 Some statistical properties

In this subsection, some basic properties of the APKum distribution such as quantile function, moments and moment generating function, mean residual life and order statistics are derived.

### 2.1.1 Quantile function

The quantile function of APKum random variable $X$ is given by

$$x_1 = \left[1 - \left[1 - \frac{\ln\left(U(\alpha - 1) + 1\right)}{\ln(\alpha)}\right]^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}}$$

Using the previous equation, a random sample from APKum family can be generated; by using $U$ as uniform random number.

### 2.1.2 Moments and moment generating function

Let $X \sim APKum(x; \alpha, \lambda, \beta)$, then the $r^{\text{th}}$ moment of $X$ can be obtained as follows:

$$\mu'_{r1} = \frac{\alpha\lambda\beta \ln(\alpha)}{\alpha - 1} \int_0^1 x^{\lambda + r - 1}(1 - x^{\lambda})^{\beta - 1} \alpha^{-(1 - x^{\lambda})^{\beta}} dx\,.$$

Using the series representation in the form

$$\alpha^{-\nu} = \sum_{k=0}^{\infty} \frac{(-\nu \ln(\alpha))^k}{k!}\ , \qquad (12)$$

Then

$$\mu'_{r1} = \frac{\alpha\beta \ln(\alpha)}{\alpha - 1} \sum_{k=0}^{\infty} \frac{(-\ln(\alpha))^k}{k!} \mathbf{B}\left(\frac{r}{\lambda} + 1, \beta k + \beta\right), \qquad (13)$$

where, $\mathbf{B}(.,.)$ is the beta function. The moment generating function is

$$M_{x1}(t) = \frac{\alpha\beta\ln(\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\ln(\alpha))^k\, t^j\, \mathbf{B}\left(\frac{j}{\lambda}+1, \beta k+\beta\right)}{k!\,j!}.$$

### 2.1.3 Mean residual life

The mean residual life is the expected additional lifetime that a component has survived after a fixed time point *t*, the mean residual life function is given by

$$m1(t) = \frac{1}{S(t)}\left(E(t) - \int_0^t x\, g(x)dx\right) - t \quad , \tag{14}$$

Where

$$\int_0^t x\, g(x)dx = \frac{\alpha\beta\ln(\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\ln(\alpha))^k}{k!}\mathbf{IB}_t\left(\frac{1}{\lambda}+1, \beta k+\beta\right) \ , \tag{15}$$

where, $\mathbf{IB}_t(.,.)$ is the incomplete beta function. Substituting (11), (13) and (15) in (14), then the mean residual life can be written as

$$m1(t) = \frac{\beta\ln(\alpha)\sum_{k=0}^{\infty}\frac{(-\ln(\alpha))^k}{k!}\left(\mathbf{B}\left(\frac{1}{\lambda}+1, \beta k+\beta\right) - \mathbf{IB}_t\left(\frac{1}{\lambda}+1, \beta k+\beta\right)\right)}{1-\alpha^{-(1-t^\lambda)^\beta}} - t.$$

### 2.1.4 Order statistics

Suppose that $X_1, X_2, \dots, X_n$ is a random sample of size *n* from APKum distribution and let

$X_{1:n} < X_{2:n} < \dots < X_{n:n}$ be the corresponding order statistics. Then, the pdf of the *r*th order statistic is given by

$$g_{1r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[G(x;\alpha,\lambda,\beta)]^{r-1}[1-G(x;\alpha,\lambda,\beta)]^{n-r}g(x;\alpha,\lambda,\beta) \ . \tag{16}$$

Using (9) and (10) in (16), hence the pdf of $X_{r:n}$ can be written as

$$g_{1r:n}(x) = \frac{(-1)^{r-1}\alpha^{n-r}\lambda\beta\ln(\alpha)}{\mathbf{B}(r,n-r+1)(\alpha-1)^n}\left[1-\alpha^{1-(1-x^\lambda)^\beta}\right]^{r-1}\left[1-\alpha^{-(1-x^\lambda)^\beta}\right]^{n-r} \\ \times\ x^{\lambda-1}(1-x^\lambda)^{\beta-1}\alpha^{1-(1-x^\lambda)^\beta} \ .$$

Using the binomial expansion, one obtains

$$g_{1r:n}(x) = \frac{\lambda\beta\ln(\alpha)}{\mathbf{B}(r,n-r+1)(\alpha-1)^n}\sum_{k=0}^{r-1}\sum_{j=0}^{n-r}\binom{r-1}{k}\binom{n-r}{j}\frac{(-1)^{r+k+j-1}}{\alpha^{-(n+k-r+1)}}x^{\lambda-1}(1-x^\lambda)^{\beta-1}\alpha^{-(k+j+1)(1-x^\lambda)^\beta}.$$

## 2.2 Maximum likelihood estimation for APKum distribution parameters

Let $X_1, X_2, \dots, X_n$ be a simple random sample of size *n* from the APKum distribution, then from the pdf in (10) the likelihood function is

$$L_1(\alpha,\lambda,\beta;\underline{x}) = \lambda^n\beta^n(\alpha-1)^{-n}(\ln(\alpha))^n\prod_{i=1}^{n}x_i^{\lambda-1}\prod_{i=1}^{n}(1-x_i^{\lambda})^{\beta-1}\ \alpha^{\sum_{i=1}^{n}\left[1-(1-x_i^{\lambda})^{\beta}\right]}.$$

The log likelihood function is

$$\ell_1 = n\ln(\lambda)+n\ln(\beta)-n\ln(\alpha-1)+n\ln(\ln(\alpha))+(\lambda-1)\sum_{i=1}^{n}\ln(x_i)+(\beta-1)\sum_{i=1}^{n}\ln\left(1-{x_i}^{\lambda}\right)$$
$$+\ln(\alpha)\sum_{i=1}^{n}\left[1-\left(1-{x_i}^{\lambda}\right)^{\beta}\right]. \tag{17}$$

The ML estimators of $\alpha$, $\lambda$ and $\beta$ can be obtained by differentiating $\ell_1$ with respect to $\alpha$, $\lambda$ and $\beta$ and equating the results to zero.

$$\frac{\partial\ell_1}{\partial\alpha}=\frac{-n}{\hat{\alpha}-1}+\frac{n}{\hat{\alpha}\ln(\hat{\alpha})}+\frac{1}{\hat{\alpha}}\sum_{i=1}^{n}\left[1-\left(1-x_i^{\hat{\lambda}}\right)^{\hat{\beta}}\right]=0,$$
$$\frac{\partial\ell_1}{\partial\lambda}=\frac{n}{\hat{\lambda}}+\sum_{i=1}^{n}\ln(x_i)-\left(\hat{\beta}-1\right)\sum_{i=1}^{n}\left[\frac{x_i^{\hat{\lambda}}\ln(x_i)}{1-x_i^{\hat{\lambda}}}\right]+\hat{\beta}\ln(\hat{\alpha})\sum_{i=1}^{n}x_i^{\hat{\lambda}}\ln(x_i)\left(1-x_i^{\hat{\lambda}}\right)^{\hat{\beta}-1}=0\,,$$

And

$$\frac{\partial\ell_1}{\partial\beta}=\frac{n}{\hat{\beta}}+\sum_{i=1}^{n}\ln(1-x_i^{\hat{\lambda}})-\ln(\hat{\alpha})\sum_{i=1}^{n}\left(1-x_i^{\hat{\lambda}}\right)^{\hat{\beta}}\ln\left(1-x_i^{\hat{\lambda}}\right)=0.$$

The system of the non-linear equations cannot be solved explicitly, so the ML estimators of $\alpha$, $\lambda$ and $\beta$ can be obtained by using a numerical technique.

# 3 New Alpha Power Transformed Kumaraswamy Distribution

Assuming that $X$ is a random variable which has the Kum distribution, substituting (2) and (1) in (5) and (6), then the cdf of NAPTKum distribution is

$$G_2(x;\alpha,\lambda,\beta)=\frac{\left[1-(1-x^{\lambda})^{\beta}\right]\alpha^{1-(1-x^{\lambda})^{\beta}}}{\alpha}\,, \tag{18}$$

and the pdf is given by

$$g_2(x;\alpha,\lambda,\beta)=\frac{\lambda\beta}{\alpha}x^{\lambda-1}\left(1-x^{\lambda}\right)^{\beta-1}\alpha^{1-(1-x^{\lambda})^{\beta}}\left[1+\ln(\alpha)\left(1-(1-x^{\lambda})^{\beta}\right)\right],\ \ 0<x<1,\alpha,\lambda,\beta>0. \tag{19}$$

The sf and the hrf are, respectively, given by

$$S_2(x)=\frac{\alpha-\left[1-(1-x^{\lambda})^{\beta}\right]\alpha^{1-(1-x^{\lambda})^{\beta}}}{\alpha},\quad \alpha\neq 1, \tag{20}$$

And

$$h_2(x) = \frac{\lambda\beta x^{\lambda-1}\left(1-x^{\lambda}\right)^{\beta-1}\alpha^{1-\left(1-x^{\lambda}\right)^{\beta}}\left[1+\ln(\alpha)\left(1-\left(1-x^{\lambda}\right)^{\beta}\right)\right]}{\alpha-\left[1-\left(1-x^{\lambda}\right)^{\beta}\right]\alpha^{1-\left(1-x^{\lambda}\right)^{\beta}}}, \quad \alpha \neq 1.$$

The plots for the pdf and hrf of the NAPTKum distribution are presented in Fig. 2 distribution are sketched in Fig. 2.

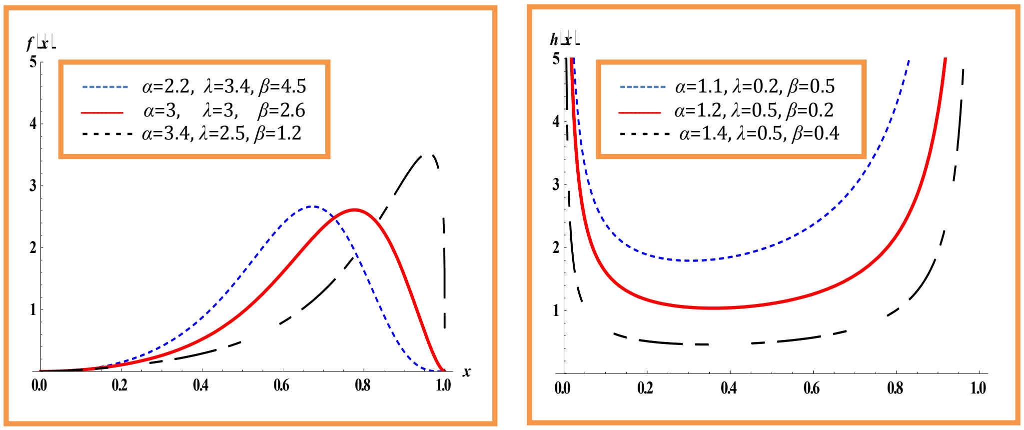


**Fig. 2. Different plots of pdf and hrf of the NAPTKum distribution**

## 3.1 Some statistical properties

Some basic properties of the NAPTKum distribution including the quantile function, moments and moment generating function, mean residual life and order statistics are obtained.

### 3.1.1 Quantile function

The quantile function of NAPTKum random variable $X$ cannot be obtained in closed form, therefore, the quantile can be evaluated numerically from the following equation

$$\ln\left[1-(1-x_2^{\lambda})^{\beta}\right] + \left[1-(1-x_2^{\lambda})^{\beta}\right]\ln(\alpha) - \ln(U\alpha) = 0.$$

### 3.1.2 Moments and moment generating function

Let $X \sim NAPTKum(x;\alpha,\lambda,\beta)$, then the $r^{\text{th}}$ moment of $X$ is obtained as follows:

$$\mu'_{r2} = \lambda\beta\int_0^1 x^{\lambda+r-1}(1-x^{\lambda})^{\beta-1}\alpha^{-(1-x^{\lambda})^{\beta}}dx + \lambda\beta\ln(\alpha)\int_0^1 x^{\lambda+r-1}(1-x^{\lambda})^{\beta-1}\alpha^{-(1-x^{\lambda})^{\beta}}\left[1-(1-x^{\lambda})^{\beta}\right]dx.$$

Using the series representation in (12), then

$$\mu'_{r2} = \beta\sum_{k=0}^{\infty}\frac{(-\ln(\alpha))^k}{k!}\left(\mathbf{B}\left(\frac{r}{\lambda}+1,\beta k+\beta\right)+\ln(\alpha)\ \mathbf{B}\left(\frac{r}{\lambda}+1,\beta k+\beta\right) - \ln(\alpha)\ \mathbf{B}\left(\frac{r}{\lambda}+1,\beta k+2\beta\right)\right), \quad (21)$$

and the moment generating function is

$$M_{x2}(t) = \beta \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-\ln(\alpha))^k \, t^j}{k!\, j!} \left( \mathbf{B}\left(\frac{j}{\lambda} + 1, \beta k + \beta\right) + \ln(\alpha)\ \mathbf{B}\left(\frac{j}{\lambda} + 1, \beta k + \beta\right) - \ln(\alpha)\ \mathbf{B}\left(\frac{j}{\lambda} + 1, \beta k + 2\beta\right)\right).$$

### 3.1.3 Mean residual life

Substituting (20) and (21) in (14), the mean residual life function of NAPTKum distribution is

$$m2(t) = \frac{\beta \sum_{k=0}^{\infty} \frac{(-\ln(\alpha))^k (A-V)}{k!}}{\left[1 - (1 - (1 - t^\lambda)^\beta)\alpha^{-(1-t^\lambda)^\beta}\right]} - t,$$

Where

$$A = \mathbf{B}\left(\frac{1}{\lambda} + 1, \beta k + \beta\right) + \ln(\alpha)\ \mathbf{B}\left(\frac{1}{\lambda} + 1, \beta k + \beta\right) - \ln(\alpha)\ \mathbf{B}\left(\frac{1}{\lambda} + 1, \beta k + 2\beta\right),$$

And

$$V = \mathbf{IB}_t\left(\frac{1}{\lambda} + 1, \beta k + \beta\right) + \ln(\alpha)\ \mathbf{IB}_t\left(\frac{1}{\lambda} + 1, \beta k + \beta\right) - \ln(\alpha)\ \mathbf{IB}_t\left(\frac{1}{\lambda} + 1, \beta k + 2\beta\right).$$

### 3.1.4 Order statistics

Considering a random sample $X_1, X_2, \ldots, X_n$ of size *n* taken from NAPTKum distribution and let

$X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the corresponding order statistics. Using (18) and (19) in (16), then the pdf of the $r^{\text{th}}$ order statistic is given by

$$g_{2r:n}(x) = \frac{\lambda\beta}{\mathbf{B}(r, n-r+1)} \sum_{k=0}^{r-1} \sum_{j=0}^{n-r} \binom{r-1}{k} \binom{n-r}{j} (-1)^{k+j} (1 - x^\lambda)^{\beta k + \beta - 1} \times \alpha^{-(j+r)(1-x^\lambda)^\beta} \left(1 - (1 - x^\lambda)^\beta\right)^j \left[1 + \ln(\alpha)\left(1 - (1 - x^\lambda)^\beta\right)\right].$$

## 3.2 Maximum likelihood estimation for NAPTKum distribution parameters

Considering a random sample $X_1, X_2, \ldots, X_n$ of size *n* from the NAPTKum distribution, then from (19) the likelihood function is

$$L_2(\alpha, \lambda, \beta; \underline{x}) = \lambda^n \beta^n \alpha^{-n} (\ln(\alpha))^n \prod_{i=1}^{n} x_i^{\lambda-1} \prod_{i=1}^{n} \left(1 - x_i^\lambda\right)^{\beta-1} \alpha^{\sum_{i=1}^{n}\left[1-\left(1-x_i^\lambda\right)^\beta\right]} \times \prod_{i=1}^{n} \left[1 + \ln(\alpha)\left(1 - (1 - x_i{}^\lambda)^\beta\right)\right].$$

The log likelihood function is

$$\ell_2 = n\ln(\lambda) + n\ln(\beta) - n\ln(\alpha) + n\ln(\ln(\alpha)) + (\lambda-1)\sum_{i=1}^{n}\ln(x_i) + (\beta-1)\sum_{i=1}^{n}\ln\left(1-x_i{}^{\lambda}\right)$$
$$+\ln(\alpha)\sum_{i=1}^{n}\left[1-\left(1-x_i{}^{\lambda}\right)^{\beta}\right] + \sum_{i=1}^{n}\ln\left[1+\ln(\alpha)\left(1-\left(1-x_i{}^{\lambda}\right)^{\beta}\right)\right]. \quad (22)$$

From (22) the likelihood equations are given by

$$\frac{\partial\ell_2}{\partial\alpha} = \frac{-n}{\hat{\alpha}} + \frac{1}{\hat{\alpha}}\sum_{i=1}^{n}\left[1-(1-x_i^{\hat{\lambda}})^{\hat{\beta}}\right] + \frac{1}{\hat{\alpha}}\sum_{i=1}^{n}\left[\frac{1-(1-x_i{}^{\hat{\lambda}})^{\hat{\beta}}}{1+\ln(\hat{\alpha})\left(1-(1-x_i{}^{\hat{\lambda}})^{\hat{\beta}}\right)}\right] = 0,$$

$$\frac{\partial\ell_2}{\partial\lambda} = \frac{n}{\hat{\lambda}} + \sum_{i=1}^{n}\ln(x_i) - \left(\hat{\beta}-1\right)\sum_{i=1}^{n}\left[\frac{x_i^{\hat{\lambda}}\ln(x_i)}{1-x_i^{\hat{\lambda}}}\right] + \hat{\beta}\ln(\hat{\alpha})\sum_{i=1}^{n}x_i^{\hat{\lambda}}\ln(x_i)(1-x_i{}^{\hat{\lambda}})^{\hat{\beta}-1}$$
$$+\hat{\beta}\ln(\hat{\alpha})\sum_{i=1}^{n}\left[\frac{x_i^{\hat{\lambda}}\ln(x_i)(1-x_i{}^{\hat{\lambda}})^{\hat{\beta}-1}}{1+\ln(\hat{\alpha})\left(1-(1-x_i{}^{\hat{\lambda}})^{\hat{\beta}}\right)}\right] = 0$$

And

$$\frac{\partial\ell_2}{\partial\beta} = \frac{n}{\hat{\beta}} + \sum_{i=1}^{n}\ln(1-x_i^{\hat{\lambda}}) - \ln(\hat{\alpha})\sum_{i=1}^{n}\left(1-x_i^{\hat{\lambda}}\right)^{\hat{\beta}}\ln\left(1-x_i^{\hat{\lambda}}\right) - \ln(\hat{\alpha})\sum_{i=1}^{n}\left[\frac{\ln\left(1-x_i^{\hat{\lambda}}\right)\left(1-x_i{}^{\hat{\lambda}}\right)^{\hat{\beta}}}{1+\ln(\hat{\alpha})\left(1-\left(1-x_i{}^{\hat{\lambda}}\right)^{\hat{\beta}}\right)}\right]$$
$$= 0.$$

The system of the non-linear equations cannot be solved explicitly, so the ML estimators of $\alpha$, $\lambda$ and $\beta$ can be obtained numerically.

# 4 New Extended Alpha Power Transformed Kumaraswamy Distribution

Let $X$ be a random variable with the Kum distribution, substituting (2) and (1) in (7) and (8), then the cdf of NEAPTKum distribution is

$$G_3(x;\alpha,\lambda,\beta) = \frac{\alpha^{1-(1-x^{\lambda})^{\beta}} - e^{\alpha\left(1-(1-x^{\lambda})^{\beta}\right)}}{\alpha - e^{\alpha}}\ , \quad (23)$$

and the pdf is given by

$$g_3(x;\alpha,\lambda,\beta) = \frac{\lambda\beta x^{\lambda-1}}{\alpha-e^{\alpha}}\left(1-x^{\lambda}\right)^{\beta-1}\left[\ln(\alpha)\,\alpha^{1-\left(1-x^{\lambda}\right)^{\beta}} - \alpha e^{\alpha\left(1-(1-x^{\lambda})^{\beta}\right)}\right], 0 < x < 1, \alpha,\lambda,\beta > 0. \quad (24)$$

The sf and the hrf are, respectively, as follows:

$$S_3(x) = \frac{\alpha - e^{\alpha} - \alpha^{\left(1-(1-x^{\lambda})^{\beta}\right)} + e^{\alpha\left(1-(1-x^{\lambda})^{\beta}\right)}}{\alpha - e^{\alpha}}, \quad \alpha \neq 1, \quad (25)$$

And

$$h_3(x) = \frac{\lambda\beta x^{\lambda-1}\left(1-x^{\lambda}\right)^{\beta-1}\left[\ln(\alpha)\,\alpha^{1-\left(1-x^{\lambda}\right)^{\beta}} - \alpha e^{\alpha\left(1-(1-x^{\lambda})^{\beta}\right)}\right]}{\alpha - e^{\alpha} - \alpha^{\left(1-(1-x^{\lambda})^{\beta}\right)} + e^{\alpha\left(1-(1-x^{\lambda})^{\beta}\right)}}, \quad \alpha \neq 1.$$

The plots of pdf and hrf of the NEAPTKum distribution are given in Fig. 3.

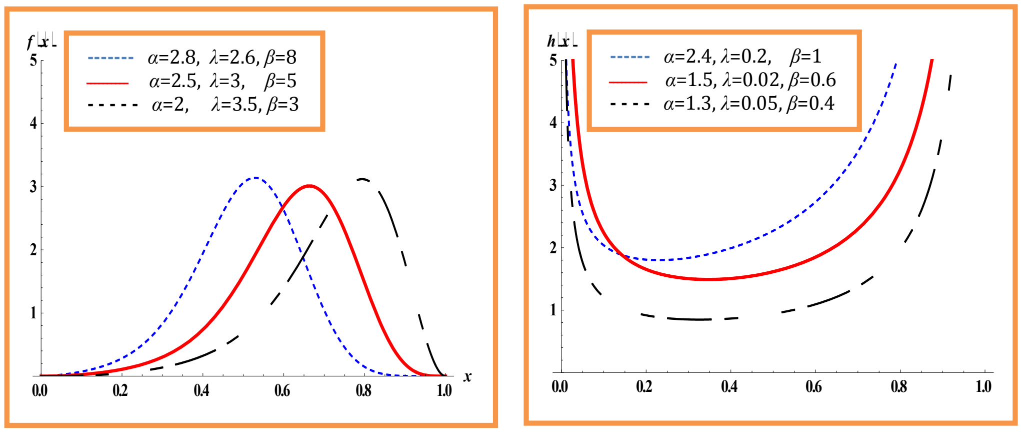


**Fig. 3. Different plots of pdf and hrf of the NEAPTKum distribution**

## 4.1 Some statistical properties

Some general properties of the NEAPTKum distribution including the quantile function, moments and moment generating function, mean residual life and order statistics are obtained.

### 4.1.1 Quantile function

The quantile function of NEAPTKum random variable $X$ cannot be obtained in closed form, therefore, the quantile can be calculated numerically from the following equation

$$\alpha\left(1-(1-x_3^{\lambda})^{\beta}\right)-\ln\left[\alpha^{\left(1-\left(1-x_3^{\lambda}\right)^{\beta}\right)}-U(\alpha-e^{\alpha})\right]=0.$$

### 4.1.2 Moments and moment generating function

Let $X \sim NEAPTKum(x;\alpha,\lambda,\beta)$, then the $r^{\text{th}}$ moment of $X$ is

$$\mu'_{r3}=\frac{\alpha\beta\ln(\alpha)}{\alpha-e^{\alpha}}\sum_{k=0}^{\infty}\frac{(-\ln(\alpha))^k}{k!}\,\mathbf{B}\left(\frac{r}{\lambda}+1,\beta k+\beta\right)-\frac{\alpha e^{\alpha}\beta}{\alpha-e^{\alpha}}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}\mathbf{B}\left(\frac{r}{\lambda}+1,\beta j+\beta\right)\ , \qquad (26)$$

and the moment generating function is given by

$$M_{x3}(t)=\frac{\alpha\beta\ln(\alpha)}{\alpha-e^{\alpha}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\ln(\alpha))^k\,t^j}{k!\,j!}\mathbf{B}\left(\frac{j}{\lambda}+1,\beta k+\beta\right)$$
$$-\frac{\alpha e^{\alpha}\beta}{\alpha-e^{\alpha}}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^i(-\alpha)^m}{i!\,m!}\mathbf{B}\left(\frac{i}{\lambda}+1,\beta m+\beta\right).$$

### 4.1.3 Mean residual life

Substituting (25) and (26) in (14), then the mean residual life function for NEAPTKum distribution is

$$m3(t)=\frac{\alpha\beta\ln(\alpha)\sum_{k=0}^{\infty}\frac{(-\ln(\alpha))^{k}W}{k!}-\alpha e^{\alpha}\beta\sum_{j=0}^{\infty}\frac{(-\alpha)^{j}D}{j!}}{\alpha-e^{\alpha}-\alpha^{1-(1-t^{\lambda})^{\beta}}+e^{\alpha\left(1-(1-t^{\lambda})^{\beta}\right)}}-t,$$

Where

$$W=\mathbf{B}\left(\frac{1}{\lambda}+1,\beta k+\beta\right)-\mathbf{IB}_{t}\left(\frac{1}{\lambda}+1,\beta k+\beta\right),$$

And

$$D=\mathbf{B}\left(\frac{1}{\lambda}+1,\beta j+\beta\right)-\mathbf{IB}_{t}\left(\frac{1}{\lambda}+1,\beta j+\beta\right).$$

**4.1.4 Order statistics**

Let $X_{1:n}<X_{2:n}<\cdots<X_{n:n}$ be the order statistics of a random sample $X_1,X_2,\ldots,X_n$ of size $n$ from NEAPTKum distribution. Substituting (23) and (24) in Equation (16), then the pdf of the $r$th order statistic can be written as

$$g_{3r:n}(x)=\frac{\lambda\beta(\alpha-e^{\alpha})^{-n}}{\mathbf{B}(r,n-r+1)}x^{\lambda-1}\left(1-x^{\lambda}\right)^{\beta-1}\left[\alpha^{1-(1-x^{\lambda})^{\beta}}-e^{\alpha(1-(1-x^{\lambda})^{\beta})}\right]^{r-1}$$
$$\times\left[\ln(\alpha)\,\alpha^{1-(1-x^{\lambda})^{\beta}}-\alpha e^{\alpha(1-(1-x^{\lambda})^{\beta})}\right]\left[\alpha-e^{\alpha}-\alpha^{1-(1-x^{\lambda})^{\beta}}+e^{\alpha(1-(1-x^{\lambda})^{\beta})}\right]^{n-r}.$$

## 4.2 Maximum Likelihood Estimation for NEAPTKum Distribution Parameter

Considering a random sample $X_1,X_2,\ldots,X_n$ of size $n$ from the NEAPTKum distribution, then from (24) the likelihood function is

$$L_3(\alpha,\lambda,\beta;\underline{x})=\lambda^{n}\beta^{n}(\alpha-e^{\alpha})^{-n}\prod_{i=1}^{n}x_i^{\lambda-1}\prod_{i=1}^{n}\left(1-x_i^{\lambda}\right)^{\beta-1}\prod_{i=1}^{n}\left[\ln(\alpha)\,\alpha^{1-(1-x_i^{\lambda})^{\beta}}-\alpha e^{\alpha\left(1-(1-x_i^{\lambda})^{\beta}\right)}\right].$$

The log likelihood function is given by

$$\ell_3=n\ln(\lambda)+n\ln(\beta)-n\ln(\alpha-e^{\alpha})+(\lambda-1)\sum_{i=1}^{n}\ln(x_i)+(\beta-1)\sum_{i=1}^{n}\ln\left(1-x_i{}^{\lambda}\right)$$
$$+\sum_{i=1}^{n}\ln\left[\ln(\alpha)\,\alpha^{1-(1-x_i^{\lambda})^{\beta}}-\alpha e^{\alpha\left(1-(1-x_i^{\lambda})^{\beta}\right)}\right]. \tag{27}$$

From (27), the likelihood equations are given below

$$\frac{\partial\ell_3}{\partial\alpha}=\frac{-n(1-e^{\hat{\alpha}})}{\hat{\alpha}-e^{\hat{\alpha}}}$$
$$+\sum_{i=1}^{n}\left[\frac{\hat{\alpha}^{-(1-x^{\hat{\lambda}})^{\hat{\beta}}}\left[\ln(\hat{\alpha})\left(1-\left(1-x^{\hat{\lambda}}\right)^{\hat{\beta}}\right)+1\right]-e^{\hat{\alpha}\left(1-(1-x_i^{\hat{\lambda}})^{\hat{\beta}}\right)}\left[\hat{\alpha}\left(1-\left(1-x_i{}^{\hat{\lambda}}\right)^{\hat{\beta}}\right)+1\right]}{\ln(\hat{\alpha})\,\hat{\alpha}^{1-(1-x^{\hat{\lambda}})^{\hat{\beta}}}-\hat{\alpha}e^{\hat{\alpha}\left(1-(1-x^{\hat{\lambda}})^{\hat{\beta}}\right)}}\right]=0,$$

$$\frac{\partial \ell_3}{\partial \lambda} = \frac{n}{\hat{\lambda}} + \sum_{i=1}^{n} \ln(x_i) - (\hat{\beta} - 1) \sum_{i=1}^{n} \left[ \frac{x_i^{\hat{\lambda}} \ln(x_i)}{1 - x_i^{\hat{\lambda}}} \right] + \hat{\beta} \sum_{i=1}^{n} \left[ \frac{x_i^{\hat{\lambda}} (1 - x^{\hat{\lambda}})^{\hat{\beta}-1} \ln(x_i) \left[ (\ln \hat{\alpha})^2 \hat{\alpha}^{1-(1-x^{\hat{\lambda}})^{\hat{\beta}}} - \hat{\alpha}^2 e^{\hat{\alpha}\left(1-(1-x_i^{\hat{\lambda}})^{\hat{\beta}}\right)} \right]}{\ln(\hat{\alpha})\, \hat{\alpha}^{1-(1-x^{\hat{\lambda}})^{\hat{\beta}}} - \hat{\alpha} e^{\hat{\alpha}\left(1-(1-x^{\hat{\lambda}})^{\hat{\beta}}\right)}} \right] = 0\,,$$

And

$$\frac{\partial \ell_3}{\partial \beta} = \frac{n}{\hat{\beta}} + \sum_{i=1}^{n} \ln\left(1 - x_i^{\hat{\lambda}}\right) - \sum_{i=1}^{n} \left[ \frac{(1 - x^{\hat{\lambda}})^{\hat{\beta}} \ln\left(1 - x^{\hat{\lambda}}\right) \left[ (\ln \hat{\alpha})^2 \hat{\alpha}^{1-(1-x^{\hat{\lambda}})^{\hat{\beta}}} - \hat{\alpha}^2 e^{\hat{\alpha}\left(1-(1-x_i^{\hat{\lambda}})^{\hat{\beta}}\right)} \right]}{\ln(\hat{\alpha})\, \hat{\alpha}^{1-(1-x^{\hat{\lambda}})^{\hat{\beta}}} - \hat{\alpha} e^{\hat{\alpha}\left(1-(1-x^{\hat{\lambda}})^{\hat{\beta}}\right)}} \right] = 0.$$

The system of the non-linear equations can be solved numerically to obtain the ML estimators of $\alpha$, $\lambda$ and $\beta$.

# 5 Simulation Study

In this section, a simulation study is carried out to illustrate the performance of the ML estimators in terms of the sample size $n$ for APKum distribution, NAPTKum distribution and NEAPTKum distribution. The simulation study is conducted by choosing 1000 simple random samples of sizes 30, 50, 100 and 150 which were generated from APKum, NAPTKum and NEAPTKum distributions. The performance of the estimates for the parameters has been studied in terms of their *relative absolute bias* (RAB) and *estimated risk* (ER) using Mathematica 9. For the APKum distribution, the ML averages, RABs and ERs are displayed in Table 1, for two sets of the population parameter values ($\alpha$=0.7, $\lambda$=0.3, $\beta$=0.5) and ($\alpha$=0.5, $\lambda$=0.5, $\beta$=0.5). For NAPTKum distribution, the ML averages, RABs and ERs are presented in Table 2 for the population parameter values ($\alpha$=1.1, $\lambda$=0.03, $\beta$=0.5) and ($\alpha$=1.2, $\lambda$=0.02, $\beta$=0.4). Also, for the NEAPTKum distribution the results are presented in Table 3 for the population parameter values ($\alpha$=1.2, $\lambda$=0.5, $\beta$=0.4) and ($\alpha$=0.5, $\lambda$=0.5, $\beta$=0.5). From Tables 1, 2 and 3, it can be noted that the RABs and ERs for the estimates of $\alpha$, $\lambda$ and $\beta$ are decreasing when the sample size $n$ is increasing.

**Table 1. ML averages, RABs and ERs from APKum distribution**

| ***n*** | **Parameter** | **($\alpha$=0.7, $\lambda$=0.3, $\beta$=0.5)** | | | **($\alpha$=0.5, $\lambda$=0.5, $\beta$=0.5)** | | |
|---|---|---|---|---|---|---|---|
| | | **Average** | **RAB** | **ER** | **Average** | **RAB** | **ER** |
| | $\alpha$ | 1.1886 | 0.6980 | 0.9378 | 0.9012 | 0.8024 | 0.5544 |
| 30 | $\lambda$ | 0.3048 | 0.0162 | 0.0072 | 0.5062 | 0.0124 | 0.0179 |
| | $\beta$ | 0.5619 | 0.1239 | 0.0275 | 0.5826 | 0.1651 | 0.0358 |
| | $\alpha$ | 1.0897 | 0.5567 | 0.6758 | 0.7933 | 0.5866 | 0.3607 |
| 50 | $\lambda$ | 0.2985 | 0.0049 | 0.0043 | 0.4983 | 0.0034 | 0.0116 |
| | $\beta$ | 0.5447 | 0.0895 | 0.0176 | 0.5555 | 0.1110 | 0.0216 |
| | $\alpha$ | 0.9976 | 0.4252 | 0.4073 | 0.7274 | 0.4548 | 0.2458 |
| 100 | $\lambda$ | 0.2981 | 0.0064 | 0.0024 | 0.4931 | 0.0137 | 0.0058 |
| | $\beta$ | 0.5328 | 0.0656 | 0.0099 | 0.5339 | 0.0677 | 0.0108 |
| | $\alpha$ | 0.9199 | 0.3141 | 0.2888 | 0.7050 | 0.4100 | 0.1915 |
| 150 | $\lambda$ | 0.2979 | 0.0071 | 0.0018 | 0.4913 | 0.0175 | 0.0041 |
| | $\beta$ | 0.5230 | 0.0460 | 0.0063 | 0.5313 | 0.0625 | 0.0083 |

**Table 2. ML averages, RABs and ERs from NAPTKum distribution**

| ***n*** | **Parameter** | **($\alpha$=1.1, $\lambda$=0.03, $\beta$=0.5)** | | | **($\alpha$=1.2, $\lambda$=0.02, $\beta$=0.4)** | | |
|---|---|---|---|---|---|---|---|
| | | **Average** | **RAB** | **ER** | **Average** | **RAB** | **ER** |
| | $\alpha$ | 2.6951 | 1.4501 | 2.56507 | 2.9862 | 1.4886 | 3.3702 |
| 30 | $\lambda$ | 0.0485 | 0.6170 | 0.00045 | 0.0430 | 1.1478 | 0.0006 |
| | $\beta$ | 0.7479 | 0.4958 | 0.09570 | 0.6080 | 0.5199 | 0.0633 |
| | $\alpha$ | 2.6735 | 1.4305 | 2.48394 | 2.9128 | 1.4273 | 3.0547 |
| 50 | $\lambda$ | 0.0473 | 0.5778 | 0.00037 | 0.0414 | 1.0738 | 0.0005 |
| | $\beta$ | 0.7343 | 0.4685 | 0.07284 | 0.5844 | 0.4610 | 0.0455 |
| | $\alpha$ | 2.6612 | 1.4193 | 2.43879 | 2.8255 | 1.3546 | 2.6876 |
| 100 | $\lambda$ | 0.0463 | 0.5432 | 0.00030 | 0.0412 | 1.0586 | 0.0005 |
| | $\beta$ | 0.7163 | 0.4326 | 0.05499 | 0.5773 | 0.4432 | 0.0364 |
| | $\alpha$ | 2.6607 | 1.4188 | 2.43674 | 2.8033 | 1.3361 | 2.5933 |
| 150 | $\lambda$ | 0.0462 | 0.5395 | 0.00028 | 0.0410 | 1.0498 | 0.0005 |
| | $\beta$ | 0.7129 | 0.4258 | 0.05001 | 0.5705 | 0.4265 | 0.0324 |

**Table 3. ML averages, RABs and ERs from NEAPTKum distribution**

| ***n*** | **Parameter** | **($\alpha$=1.2, $\lambda$=0.5, $\beta$=0.4)** | | | **($\alpha$=0.5, $\lambda$=0.5, $\beta$=0.5)** | | |
|---|---|---|---|---|---|---|---|
| | | **Average** | **RAB** | **ER** | **Average** | **RAB** | **ER** |
| | $\alpha$ | 1.5644 | 0.3037 | 0.6193 | 0.7398 | 0.4796 | 0.2524 |
| 30 | $\lambda$ | 0.5168 | 0.0337 | 0.0491 | 0.4917 | 0.0166 | 0.0272 |
| | $\beta$ | 0.4308 | 0.0769 | 0.0092 | 0.5567 | 0.1134 | 0.0267 |
| | $\alpha$ | 1.3945 | 0.1621 | 0.4398 | 0.6955 | 0.3910 | 0.2029 |
| 50 | $\lambda$ | 0.5254 | 0.0509 | 0.0422 | 0.4965 | 0.0071 | 0.0180 |
| | $\beta$ | 0.4132 | 0.0331 | 0.0049 | 0.5387 | 0.0773 | 0.0165 |
| | $\alpha$ | 1.4004 | 0.1670 | 0.3488 | 0.6375 | 0.2750 | 0.1154 |
| 100 | $\lambda$ | 0.4893 | 0.0213 | 0.0212 | 0.4884 | 0.0232 | 0.0105 |
| | $\beta$ | 0.4092 | 0.0229 | 0.0026 | 0.5257 | 0.0513 | 0.0092 |
| | $\alpha$ | 1.3664 | 0.1387 | 0.2510 | 0.6326 | 0.2653 | 0.0960 |
| 150 | $\lambda$ | 0.4895 | 0.0211 | 0.0171 | 0.4815 | 0.0370 | 0.0073 |
| | $\beta$ | 0.4083 | 0.0207 | 0.0019 | 0.5221 | 0.0442 | 0.0063 |

# 6 Application

In this section, a comparison study of the APKum, NAPTKum, NEAPTKum and Kum distributions is presented. The data set represents the monthly water capacity data of the Shasta reservoir in California, USA and was taken for the month of February from 1991 to 2010. The maximum capacity of the reservoir is 4552000 *acre-foot* (AF). The data was transformed to the interval [0, 1] to apply the data set to APKum, NAPTKum, NEAPTKum and Kum distributions by dividing the capacities over the capacity of the reservoir. Actual data and transformed data are given in Table 4, [see Nader et al. [20]]. The Kolmogorov–Smirnov test showed that the data set follows the APKum, NAPTKum, NEAPTKum and Kum distributions, and the *p* values are given, respectively, by 0.3356, 0.08106, 0.1745 and 0.3356. The analytical measures of goodness of fit such as the *Akaike information criterion* (AIC), *consistent Akaike information criterion* (CAIC) and *Bayesian information criterion* (BIC) are considered to compare the proposed distributions, a distribution with smaller values of these analytical measures indicate better fit to the data. Table 5 shows the estimated parameters, standard errors and the analytical results for the proposed distributions. The NEAPTKum distribution provides a better fit than the other distributions regard to AIC, CAIC and BIC.

**Table 4. Monthly capacity for February and proportion of total capacity for Shasta reservoir**

| Year | Capacity | Proportion of total capacity | Year | Capacity | Proportion of total capacity |
|---|---|---|---|---|---|
| 1991 | 1542838 | 0.338936 | 2001 | 3495969 | 0.768007 |
| 1992 | 1966077 | 0.431915 | 2002 | 3839544 | 0.843485 |
| 1993 | 3459209 | 0.759932 | 2003 | 3584283 | 0.787408 |
| 1994 | 3298496 | 0.724626 | 2004 | 3868600 | 0.849868 |
| 1995 | 3448519 | 0.757583 | 2005 | 3168056 | 0.695970 |
| 1996 | 3694201 | 0.811556 | 2006 | 3834224 | 0.842316 |
| 1997 | 3574861 | 0.785339 | 2007 | 3772193 | 0.828689 |
| 1998 | 3567220 | 0.783660 | 2008 | 2641041 | 0.580194 |
| 1999 | 3712733 | 0.815627 | 2009 | 1960458 | 0.430681 |
| 2000 | 3857423 | 0.847413 | 2010 | 3380147 | 0.742563 |

**Table 5. ML estimates, standard errors in parentheses and analytical results of the fitted distributions to the data set**

| Distribution | $\hat{\alpha}$ | $\hat{\lambda}$ | $\hat{\beta}$ | AIC | CAIC | BIC |
|---|---|---|---|---|---|---|
| APKum | 1.8713 | 0.4058 | 0.6608 | 14.862 | 16.362 | 17.849 |
| | (0.2118) | (0.0376) | (0.0470) | | | |
| NAPTKum | 1.1980 | 0.3971 | 0.5780 | 84.722 | 86.222 | 87.709 |
| | (0.0099) | (0.0370) | (0.0374) | | | |
| NEAPTKum | 2.3578 | 0.4384 | 0.8787 | 2.374 | 3.874 | 5.361 |
| | (0.2332) | (0.0406) | (0.0521) | | | |
| Kum | — | 0.3780 | 0.4988 | 16.636 | 17.342 | 18.628 |
| | | (0.0356) | (0.0332) | | | |

# 7 Conclusion

In this paper, three lifetime distributions using the APT, NAPT and NEAPT methods to the Kum distribution are presented. Some statistical properties of the three distributions are obtained. Estimation for the parameters using the ML method is considered and a simulation study is carried out. A real data set is applied and some certain accuracy measures are evaluated. These measures showed that the NEAPTKum distribution provides a better fit to this data than the other distributions.

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# Competing Interests

Authors have declared that no competing interests exist.

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