L^1 STABILITY OF PATTERNS OF NON-INTERACTING LARGE SHOCK WAVES

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Abstract. We consider a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension

$$u_t + f(u)_x = 0,$$

together with Cauchy initial data

$$u(0,x) = \bar{u}(x),$$

that is a small $BV \cap L^1$ perturbation of fixed Riemann data (u_0^-, u_0^+). We a priori assume that the latter problem is solved by M large shocks ($2 \le M \le n$) of different characteristic families, each of them Majda stable and Lax compressive.

We prove that under a suitable Finiteness Condition the problem has a unique solution defined globally in space and time, while a stronger Stability Condition guarantees the existence of a Lipschitz semigroup of solutions.

1. Introduction

In this paper we consider a strictly hyperbolic system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

In the n-dimensional state space $M+1$ ($M \in \{2 \dots n\}$) distinct states $\{u_0^q\}_{q=0}^M$ are fixed, with their corresponding open disjoint neighbourhoods $\{\Omega^q\}_{q=0}^M$ such that:

- $f : \Omega \longrightarrow \mathbf{R}^n$ is smooth and defined on $\Omega = \bigcup_{q=0}^M \Omega^q \subset \mathbf{R}^n$.
- f is strictly hyperbolic in Ω, that is: at every point $u \in \Omega$ the matrix $Df(u)$ has n real and simple eigenvalues $\lambda_1(u) < \ldots < \lambda_n(u)$. Note that consequently one has:
$$|\lambda_k(u) - \lambda_s(v)| \ge c \qquad \forall k \ne s \ \forall q : 0 \dots M \ \forall u, v \in \Omega^q \tag{1.2}$$
with some positive constant c, if only the neighbourhoods Ω^q are sufficiently small.
- Each characteristic field of (1.1) is either linearly degenerate or genuinely nonlinear, that is: with a basis $\{r_k(u)\}_{k=1}^n$ of corresponding right eigenvectors of $Df(u)$, $Df(u)r_k(u) = \lambda_k(u)r_k(u)$, each of the n directional derivatives $r_k \nabla \lambda_k$ vanishes either identically or nowhere.

We assume that the Riemann problem (1.1) with:

$$u(0, \cdot) = \bar{u}, \tag{1.3}$$

$$\bar{u}(x) = \begin{cases} u_0^0 & x < 0 \\ u_0^M & x > 0 \end{cases} \tag{1.4}$$

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has an M-shock solution:

$$u(t,x) = \begin{cases} u_0^0 & x < \Lambda^1 t \\ u_0^q & \Lambda^q t < x < \Lambda^{q+1} t, \quad q : 1 \dots M-1 \\ u_0^M & x > \Lambda^M t, \end{cases} \tag{1.5}$$

in which the 'basic' states u_0^q are joined by M (large) shocks (u_0^{q-1}, u_0^q), $q : 1 \dots M$, travelling with respective speeds Λ^q.

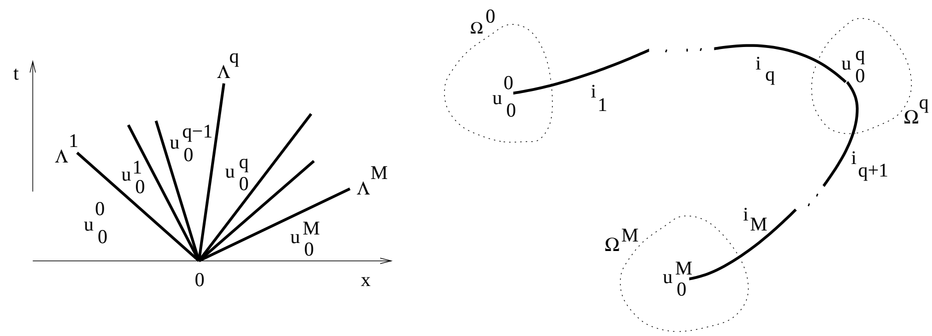

FIGURE 1.1

The goal of this article is to treat the Cauchy problem (1.1) (1.3), if the initial data is a small perturbation of $\bar{u}$ in (1.4).

The following standard conditions on the nature of the large shocks are assumed. For (1.5) to be a distributional solution of (1.1) (1.3) (1.4), we need that for every shock $q : 1 \dots M$ the Rankine-Hugoniot conditions are satisfied:

$$f(u_0^{q-1}) - f(u_0^q) = \Lambda^q (u_0^{q-1} - u_0^q). \tag{1.6}$$

Moreover, the shocks (u_0^{q-1}, u_0^q) are said to belong to the corresponding i_q-characteristic families $(1 \le i_1 < i_2 < \dots < i_M \le n)$ and assumed to be compressive in the sense of Lax [L]:

$$\lambda_{i_q}(u_0^{q-1}) > \Lambda^q > \lambda_{i_q}(u_0^q). \tag{1.7}$$

Note that (1.7) yields in particular that the shocks of characteristic families carrying bigger indices travel with the faster speed: $\Lambda^1 < \dots < \Lambda^M$, as in Figure 1.1.

We require that all large shocks are stable in the sense of Majda [M], that is:

The n vectors

$$r_1(u_0^{q-1}), \dots, r_{i_q - 1}(u_0^{q-1}), u_0^q - u_0^{q-1}, r_{i_q + 1}(u_0^q), \dots, r_n(u_0^q) \tag{1.8}$$

are linearly independent.

for every $q : 1 \dots M$.

The assumptions introduced so far are not sufficient to ensure global-in-time wellposedness of system (1.1) near its multiple shock solution (1.5). We will use the following Finiteness and Stability conditions.

Finiteness Condition

There exist a constant $\theta \in (0,1)$ and positive weights $w_1^q, \dots, w_n^q$ (for every $q : 0 \dots M$) such that the following holds. Consider a small wave of a family $k \leq i_q$, hitting from the right the large initial i_q-shock (u_0^{q-1}, u_0^q), as in Figure 1.2. Then

$$\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| < \theta \tag{1.9}$$

at $\epsilon_1^{in} = \ldots = \epsilon_k^{in} = \ldots = \epsilon_n^{in} = 0$.

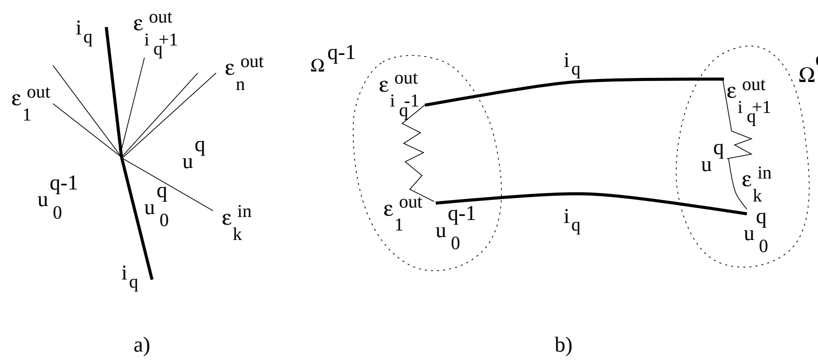

Figure 1.2

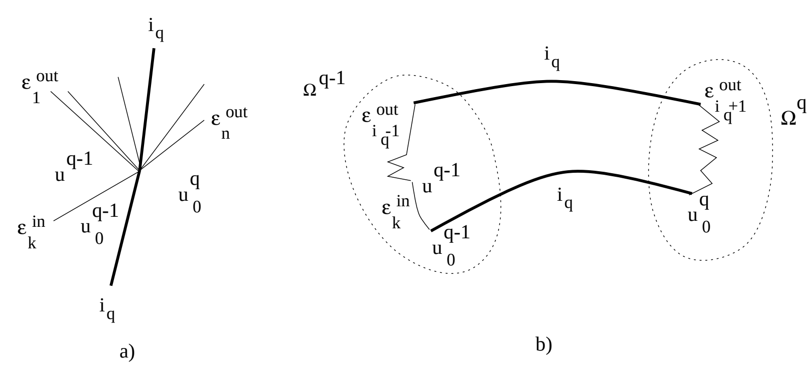

Figure 1.3

Symetrically, in case when a small k-wave with $k \geq i_q$ hits the shock (u_0^{q-1}, u_0^q) from the left (compare Figure 1.3), there holds:

$$\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \epsilon_s^{out} \right| < \theta \tag{1.10}$$

at $\epsilon_1^{in} = \ldots = \epsilon_k^{in} = \ldots = \epsilon_n^{in} = 0$.

Remark. Regarding w_s^q as the weight given to an s-wave located in the region between the $q-1$ and the q-th large shock, conditions (1.9) (1.10) simply say that, every time a small wave hits a large shock, the total weighted strength of the outgoing small waves is smaller than the weighted strength of the incoming wave.

STABILITY CONDITION

In the setting of Figure 1.2:

$$\begin{aligned}\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| \\ + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^q} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| < \theta \end{aligned} \tag{1.11}$$

at $\epsilon_1^{in} = \ldots = \epsilon_k^{in} = \ldots = \epsilon_n^{in} = 0$, while in the setting of Figure 1.3:

$$\begin{aligned}\sum_{s=1}^{i_q-1} \frac{w_s^{q-1}}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| \\ + \sum_{s=i_q+1}^{n} \frac{w_s^q}{w_k^{q-1}} \cdot \left| \frac{\partial}{\partial \epsilon_k^{in}} \left(\frac{\epsilon_s^{out} \cdot (\lambda_s^{out} - \Lambda^q)}{(\lambda_k^{in} - \Lambda^q)} \right) \right| < \theta \end{aligned} \tag{1.12}$$

at $\epsilon_1^{in} = \ldots = \epsilon_k^{in} = \ldots = \epsilon_n^{in} = 0$.

Observe that it is always possible to define the weights $\{w_k^0\}$ and $\{w_k^M\}$ such that (1.9) - (1.12) are satisfied, provided that the suitable weights $\{w_k^q\}$, $q \notin \{0, M\}$ exist.

Another remark is that our Stability Condition (1.11) (1.12) is indeed stronger than the Finiteness Condition (1.9) (1.10); see Theorem 3.1 in [Le1].

Assume that the neighbourhoods Ω^q of the basic states u_0^q have been chosen sufficiently small. We start our analysis recalling a basic fact on the solvability of Riemann problems with initial states in Ω. This issue has been addressed in [Le2]; in our particular setting the conditions (1.6) (1.7) (1.8) guarantee

Proposition 1.1. *Let the Finiteness Condition (1.9) (1.10) hold. With any Riemann data* (u^-, u^+), $u^- \in \Omega^i, u^+ \in \Omega^j, 0 \leq i \leq j \leq M$, *(1.1) has a unique self-similar solution, attaining* $n+1$ *states, consecutively connected by:*

- *weak waves of the corresponding families (if both left and right states of a pair under consideration belong to the same set* Ω^q, $i \leq q \leq j$*),*
- $j - i$ *large shocks, joining the states belonging to different sets* Ω^q*,*

as in Figure 1.4.

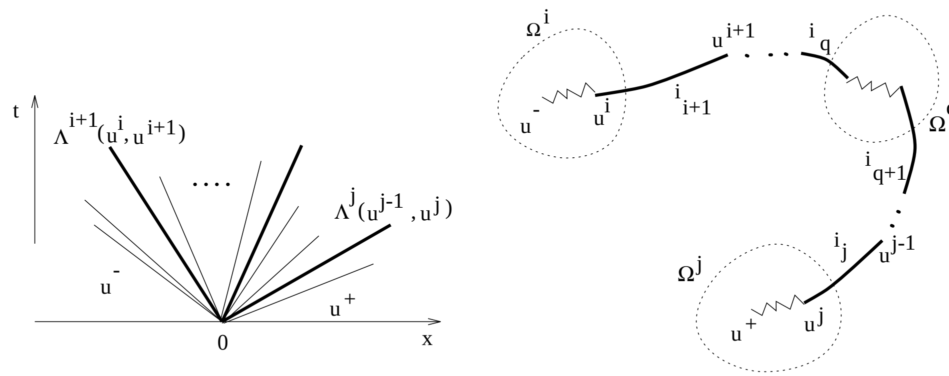

FIGURE 1.4

Now we turn to the main point of this article. Define the domain $\widetilde{\mathcal{D}}_{\delta_0}$ by:

$$\widetilde{\mathcal{D}}_{\delta_0} = \mathrm{cl}\Bigg\{ u : \mathbf{R} \longrightarrow \mathbf{R}^n;\ \text{there exist points } x^1 < x^2 < \ldots < x^M \text{ in } \mathbf{R}$$

$$\text{such that calling } \widetilde{u}(x) = \begin{cases} u_0^0 & x < x^1 \\ u_0^q & x^q < x < x^{q+1},\ q : 1 \ldots M-1 \\ u_0^M & x > x^M \end{cases} \tag{1.13}$$

$$\text{we have: } u - \widetilde{u} \in \mathrm{L}^1(\mathbf{R}, \mathbf{R}^n) \text{ and T.V.}(u - \widetilde{u}) \leq \delta_0 \Bigg\},$$

with the closure taken in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$.

Our main results are the following:

Theorem A *If the Finiteness Condition (1.9) (1.10) is satisfied then there exists* $\delta_0 > 0$ *such that for every* $\bar{u} \in \widetilde{\mathcal{D}}_{\delta_0}$ *(1.1) (1.3) has a solution (defined for all times* $t \geq 0$*).*

Theorem B *If the Stability Condition (1.11) (1.12) is satisfied then there exists* $\delta_0 > 0$, $L > 0$, *a closed domain* $\mathcal{D}_{\delta_0} \subset \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$ *containing* $\widetilde{\mathcal{D}}_{\delta_0}$, *and a continuous semigroup* $S : [0, \infty) \times \mathcal{D}_{\delta_0} \longrightarrow \mathcal{D}_{\delta_0}$ *such that:*

(i) $S(0, \bar{u}) = \bar{u}$,
$S(t+s, \bar{u}) = S(t, S(s, \bar{u})) \quad \forall t, s \geq 0\ \forall \bar{u} \in \mathcal{D}_{\delta_0}$.

(ii) $\| S(t, \bar{u}) - S(s, \bar{w}) \|_{\mathrm{L}^1} \leq L \cdot (|t-s| + \| \bar{u} - \bar{w} \|_{\mathrm{L}^1}) \quad \forall t, s \geq 0\ \forall \bar{u}, \bar{w} \in \mathcal{D}_{\delta_0}$.

(iii) *Each trajectory* $t \mapsto S(t, \bar{u})$ *is a solution of (1.1) (1.3).*

The paper is organised as follows. Towards the proof of Theorem A, in Section 2 we explicitely define the Glimm potentials, measuring the total strength of all small waves in the approximate solutions of (1.1), and the possible amount of interaction between themselves or with the large shocks. Section 3 contains the definition of the Lyapunov functional and the basic L^1 stability estimates for the wave front tracking approximations. Our functional is motivated by the similar one in [BLY]; the difference is that it now contains some extra terms accounting for the interactions and coupling of the small waves against the large shocks. In Section 4 we prove the stated stability estimates, concluding the proof of Theorem B.

We now comment on the relation of this article to other papers. In [Scho], Schochet was the first to introduce a finiteness condition, giving positive answer to question A. This condition is formulated inductively with respect to the number of large shocks M and uses the language of matrix analysis. As shown and accompanied by a more detailed discussion in [Le1], the Schochet finiteness condition and our conditions (1.9) (1.10) are equivalent.

In [BC], Bressan and Colombo consider the general Riemann problem for systems of two equations and assuming the corresponding stability condition, answer question B positively. More recently, the paper [LT] proves Theorems A and B (for systems of $n \geq 2$ equations) in the presence of only two large shocks, of characteristic families i and $j > i$; indeed in the case $M = 2,\ i_1 = i,\ i_2 = j$, the above Finiteness and Stability Conditions reduce to the corresponding conditions of [LT]. Substantial differences between $M = 2$ and $M > 2$ occur in particular in the proof of Theorem B. Namely, the straightforward generalization of the Lyapunov functional introduced in [LT] does not provide a functional decreasing along the wave front tracking solutions, when $M > 2$. On the other hand, our new functional defined in Section 3, reduces when $M = 2$ to a Lyapunov functional that can be seen as a simplification of the one from [LT].

Also, instead of Majda's criterion (1.8), the paper [LT], following [BC], used a differently stated assumption; the forthcoming article [Le1] shows the equivalence of the two conditions.

The Stability Condition (1.11) (1.12), which came up naturally in the investigations leading to this paper, was earlier introduced in [BM] (formulae (3.42) and (3.43)), to guarantee the wellposedness of associated linearized variational systems.

2. Wave front tracking approximations and Glimm's functionals

The purpose of this Section is to establish Theorem A. As, much differently from that of Theorem B, this proof hardly depends on the number of M of large shocks in the reference solution, we are somewhat brief in this Section and refer the reader wishing to see further details to [LT] where Theorem A has been shown for the case $M = 2$.

Given a Cauchy problem (1.1) (1.3), its solution is obtained as a limit (with $\epsilon \to 0$) of piecewise constant ϵ-approximate solutions, given by the wave front tracking algorithm, as described in [LT]. Since the large waves of different families travel with strictly different speeds, they never interact, and thus the analysis in [LT] applies, provided a suitable Glimm's type functional [Gl] [B1] [LT] can be found.

Let $u(t, x)$ be a piecewise constant approximate solution, generated by the wave front tracking algorithm. At a fixed time $t > 0$, the function $u(t, \cdot)$ is piecewise constant, with jumps located at the wave front positions. There are precisely M large jumps, while the others are small, their left and right states belonging to the same set Ω^k.

Definition 2.1. *(Approaching waves)*

(i) *We say that two small (possibly non-physical) fronts α and β, located at points $x_\alpha < x_\beta$ and belonging to the characteristic families $k_\alpha, k_\beta \in \{1 \dots n+1\}$ respectively, approach each other iff the following two conditions hold simultaneously:*
   - *x_α and x_β lay together in one of the $M+1$ intervals (M of them unbounded) into which $\mathbf{R}$ is partitioned by the locations of large shocks. In other words: the states joined by the fronts under consideration both belong to the same set Ω^k.*
   - *Either $k_\alpha < k_\beta$ or else $k_\alpha = k_\beta$ and at least one of the waves is a genuinely nonlinear shock.*

   In this case we write: $(\alpha, \beta) \in \mathcal{A}$.

(ii) *We say that a small wave α of the characteristic family $k_\alpha \in \{1 \dots n+1\}$ located at x_α approaches a large shock of family $k_\beta = i_k$, for some $k : 1 \dots M$, located at a point x_β iff one of the following conditions hold:*
   - *The states u^-, u^+ joined by the small wave under consideration both belong to Ω^{k-1} and $k_\alpha \geq i_k$.*
   - *The states u^-, u^+ belong to Ω^k and $k_\alpha \leq i_k$.*

   We then write: $\alpha \in \mathcal{A}_{i_k}$.

We adopt the following notation. For a small wave of family k and strength ϵ_k, connecting two states u_1 and u_2, we define its weighted strength as

$$b_k = w_k^q \cdot \epsilon_k \qquad \text{if } u_1, u_2 \in \Omega^q. \tag{2.1}$$

The strength of any large wave, connecting two states $u_1 \in \Omega^q$, $u_2 \in \Omega^{q+1}$ is set to be equal to some fixed number $B \leq 1$, bigger than all the possible strengths of the small waves.

Definition 2.2. *For a fixed $t > 0$ we define the following (weighted) total variation and interaction potentials:*

$$\begin{aligned}
V(t) &= \sum \{|b_\alpha|;\ \alpha \text{ - the small waves of all families}\},\\
Q_{\mathcal{A}}(t) &= \sum_{(\alpha,\beta)\in\mathcal{A}} |b_\alpha b_\beta|,\\
Q_{i_k}(t) &= \sum_{\alpha\in\mathcal{A}_{i_k}} |b_\alpha|, \quad k : 1 \dots M,\\
Q(t) &= \kappa Q_{\mathcal{A}}(t) + \sum_{k=1}^{M} Q_{i_k}(t),\\
\gamma(t) &= V(t) + \tilde{\kappa} Q(t) + \sum_{k=1}^{M-1} |q_k(t) - u_0^k|,
\end{aligned}$$

where $\kappa, \tilde{\kappa} > 0$ are constants to be specified later. The vectors $q_k(t)$ are the right states of the i_k-th large shock in $u(t, \cdot)$, respectively.

The following result (analogous to Proposition 3.4. in [LT]) is implied by the assumed Finiteness Condition (1.9) (1.10).

Proposition 2.3. *Assume that the Finiteness Condition holds. Then for some constants $c, \kappa, \tilde{\kappa}, \delta > 0$ the following is satisfied. If $u(0, \cdot)$ is piecewise constant and* *belongs to $\widetilde{\mathcal{D}}_\delta$, then for any $t > 0$ when two wave fronts of families α and β interact we have:*

(i)
$$\Delta Q(t) = Q(t+) - Q(t-) \leq \begin{cases} -c|b_\alpha \cdot b_\beta| & \textit{if both waves are small} \\ -c|b_\alpha| & \textit{if } \alpha \textit{ is a small wave and } \beta \textit{ is a large shock.} \end{cases}$$

(ii) *The same estimate as in (i) above holds for $\Delta\gamma(t) = \gamma(t+) - \gamma(t-)$.*

The above proposition implies the validity of all the main properties acquired by the wave front tracking approximate solutions (Theorem 3.5. in [LT]). Consequently, Theorem A can be proved, as in [LT] [BLY].

3. The Lyapunov functional

This Section serves to define and discuss the properties of the Lyapunov functional Φ [LY1] [LY2] [LY3] [BLY] [LT], measuring the L^1 distance between the time profiles of two arbitrary ϵ-approximate solutions $u, v : [0,\infty) \times \mathbf{R} \longrightarrow \mathbf{R}^n$ constructed by wave front tracking algorithm. The two crucial features of Φ are the following:

$$\frac{1}{C} \cdot \parallel u(t,\cdot) - v(t,\cdot) \parallel_{\mathrm{L}^1} \leq \Phi(u(t,\cdot), v(t,\cdot)) \leq C \cdot \parallel u(t,\cdot) - v(t,\cdot) \parallel_{\mathrm{L}^1}, \tag{3.1}$$

$$\Phi(u(t,\cdot), v(t,\cdot)) \leq \Phi(u(s,\cdot), v(s,\cdot)) + O(1) \cdot \epsilon \cdot (t-s) \quad \forall t > s \geq 0. \tag{3.2}$$

Fix a time $t > 0$ and consider a space point $x \in \mathbf{R}$ which is not a discontinuity point of the functions $u = u(t,\cdot),\ v = v(t,\cdot)$. Let $u(x) \in \Omega^i,\ v(x) \in \Omega^j$, for some $i, j : 0 \dots M$. We define the scalar quantities $\{b_k(x)\}_{k=1}^n$ as the weighted strengths of the corresponding shock waves in the jump $(u(x), v(x))$. More precisely, we consider the Riemann data:

$$(w^-, w^+) = \begin{cases} (u(x), v(x)) & \text{if } i \leq j \\ (v(x), u(x)) & \text{if } i > j. \end{cases} \tag{3.3}$$

By a slight modification of Proposition 1.1 one can see that the Riemann problem (1.1) (3.3) has a unique self-similar solution, whose all small waves are shocks (possibly nonadmissible). The weighted strengths of the waves in this solution are to be called $b_k(x)$. In particular, if for example $u(x), v(x) \in \Omega^0$, then for every $k : 1 \dots n$ we have: $b_k(x) = w_k^0 \cdot \epsilon_k(x)$ where the strengths $\{\epsilon_k(x)\}_{k=1}^n$ are implicitely defined by:

$$v(x) = \mathcal{S}_n(\dots, \mathcal{S}_1(u(x), \epsilon_1(x)), \dots, \epsilon_n(x)).$$

By $\lambda_k(x)$ we denote the corresponding speed of the k-th wave $\epsilon_k(x)$.

We define the functional:

$$\Phi(u, v) := \sum_{k=1}^n \int_{-\infty}^{\infty} |b_k(x)| W_k(x) dx,$$

where the weights W_k are given by:

$$W_k(x) := 1 + \kappa_1 A_k(x) + \kappa_2 [Q(u) + Q(v)]. \tag{3.4}$$

The constants κ_1, κ_2 in (3.4) are to be defined later. Q is our Glimm's interaction potential, introduced in Definition 2.2. The amount $A_k(x)$ of waves in u and v, approaching the wave $\epsilon_k(x)$ is defined in the following way:

$$A_k(x) = \begin{cases} B_k(x) + C_k(x) & \text{if } k\text{-wave } b_k(x) \text{ is small, joining} \\ & \quad\quad \text{the states in } \Omega^s,\ s : 0 \dots M \\ D_k(x) + F_k(x) & \text{if } k\text{-wave } b_k(x) = B \text{ is large,} \\ & \quad\quad k = i_s \text{ for some } s : 1 \dots M \end{cases} \tag{3.5}$$
$$+ \begin{cases} G_k(x) & \text{if } k\text{-field is genuinely nonlinear and } k\text{-wave } b_k(x) \\ & \quad\quad \text{is small, joining the states } \Omega^s,\ s : 0 \dots M \\ 0 & \text{otherwise} \end{cases}$$

The summands in (3.5) are defined:

$$B_k(x) = \left[\sum_{\substack{\alpha\in\mathcal{LS},\ k_\alpha\in\{i_s,i_{s+1}\} \\ x_\alpha<x,\ k_\alpha>k}} + \sum_{\substack{\alpha\in\mathcal{LS},\ k_\alpha\in\{i_s,i_{s+1}\} \\ x_\alpha>x,\ k_\alpha<k}} \right] |\epsilon_\alpha|$$
$$+ \begin{cases} \displaystyle\sum_{\substack{\alpha\in\mathcal{LS} \\ x_\alpha<x,\ k_\alpha=i_s}} |\epsilon_\alpha| & \text{if } k = i_s \\ \displaystyle\sum_{\substack{\alpha\in\mathcal{LS} \\ x_\alpha>x,\ k_\alpha=i_{s+1}}} |\epsilon_\alpha| & \text{if } k = i_{s+1}, \end{cases}$$

$$C_k(x) = \left[\sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha<x,\ k<k_\alpha\leq n, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} + \sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha>x,\ 1\leq k_\alpha<k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} \right] |\epsilon_\alpha|,$$

$$D_k(x) = \left[\sum_{\substack{\alpha\in\mathcal{LS}, \\ x_\alpha>x,\ k_\alpha=i_{s-1}}} + \sum_{\substack{\alpha\in\mathcal{LS}, \\ x_\alpha<x,\ k_\alpha=i_{s+1}}} \right] |\epsilon_\alpha|,$$

$$F_k(x) = \left[\sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha<x,\ k<k_\alpha\leq n, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^{s-1} \text{ or in } \Omega^s}} + \sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha>x,\ 1\leq k_\alpha<k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^{s-1} \text{ or in } \Omega^s}} \right] |\epsilon_\alpha|$$
$$+ \left[\sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha<x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^{s-1}}} + \sum_{\substack{\alpha\in\mathcal{J}\setminus\mathcal{LS},\ x_\alpha>x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} \right] |\epsilon_\alpha|,$$

$$G_k(x) = \begin{cases} \left[\displaystyle\sum_{\substack{\alpha\in\mathcal{J}(u)\setminus\mathcal{LS},\ x_\alpha<x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} + \sum_{\substack{\alpha\in\mathcal{J}(v)\setminus\mathcal{LS},\ x_\alpha>x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} \right] |\epsilon_\alpha| & \text{if } b_k(x) < 0 \\ \left[\displaystyle\sum_{\substack{\alpha\in\mathcal{J}(v)\setminus\mathcal{LS},\ x_\alpha<x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} + \sum_{\substack{\alpha\in\mathcal{J}(u)\setminus\mathcal{LS},\ x_\alpha>x,\ k_\alpha=k, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^s}} \right] |\epsilon_\alpha| & \text{if } b_k(x) > 0. \end{cases}$$

Here ϵ_α stands for the (nonweighted) strength of the wave $\alpha \in \mathcal{J}$, located at point x_α and belonging to the characteristic family k_α. $\mathcal{J} = \mathcal{J}(u) \cup \mathcal{J}(v)$ is the set of all waves in u and v, by $\mathcal{LS}, \mathcal{R}, \mathcal{S}, \mathcal{C}$ we denote respectively: the large shocks, rarefactions, (weak) shocks and non-physical waves in u and v.

We assume the convention that in the above definitions we sum only the terms whose indices lie in their admissible ranges; for example if $s = M$, then obviously there are no large waves with the index i_{s+1} and thus we do not treat the corresponding terms calling the strengths of these waves.

We comment briefly on the formula (3.5). The summands $B_k(x)$ and $D_k(x)$ account for the large waves approaching the k-wave under consideration. However, only these large waves are considered, whose right or left state belongs to the set Ω^s containing at least one of the states joined by the k-wave.

$C_k(x)$ and $G_k(x)$ are the usual summands, identical with the ones in the corresponding definition of $A_k(x)$ in [BLY]. Their presence says that a small k-wave is approached by any wave of a faster family, located to the left and any wave of a slower family, located to the right. Only small physical waves, 'living' in the same set Ω^s as the k-wave, are involved.

The summand $F_k(x)$ contains the strengths of the small physical waves approaching a large k-wave under consideration, according to their locations and speeds. The convention as in the definition of $B_k(x)$ is valid. The presence of the second term in $F_k(x)$ is due to the assumed Lax stability of large shocks.

Let α be a wave in u (or v), located at a point x_α, with speed $\dot{x}_\alpha$. Following [BLY], define:

$$E_{\alpha,k} = |b_k(x_\alpha+)| W_k(x_\alpha+)(\lambda_k(x_\alpha+) - \dot{x}_\alpha) - |b_k(x_\alpha-)| W_k(x_\alpha-)(\lambda_k(x_\alpha-) - \dot{x}_\alpha).$$

The standard argument [BLY] [LT] shows that (3.1) (3.2) are implied by:

$$\sum_{k=1}^{n} E_{\alpha,k} = O(1) \cdot |\epsilon_\alpha| \qquad \forall \alpha \in \mathcal{C} \tag{3.6}$$

$$\sum_{\alpha \in \mathcal{J} \setminus \mathcal{C}} \sum_{k=1}^{n} E_{\alpha,k} = O(1) \cdot \epsilon \tag{3.7}$$

If t is an interaction time of two fronts in u or v then all weights $W_k(x)$ decrease across time t. (3.8)

The statements (3.6) and (3.8) are proved as in [LT], using Definition 2.2. In the remaining part of the article we will focus on (3.7). As usual, if no ambiguity created, we abbreviate the notation and for a particular wave α under consideration write: b_k^+ instead of $b_k(x_\alpha+)$, W_k^- instead of $W_k(x_\alpha-)$, etc.

Keeping in mind a possible 'representative' configuration of wave locations in u and v, as in Figure 3.1 we formulate the following condition:

At least for one wave $\alpha \in \mathcal{LS}$ (of the family i_s) both wave vectors $\{b_k^-\}_{k=1}^n$ and $\{b_k^+\}_{k=1}^n$ contain a large wave of the same family i_k. (3.9)

The proof of (3.7) will be performed according to if (3.9) holds or is violated.

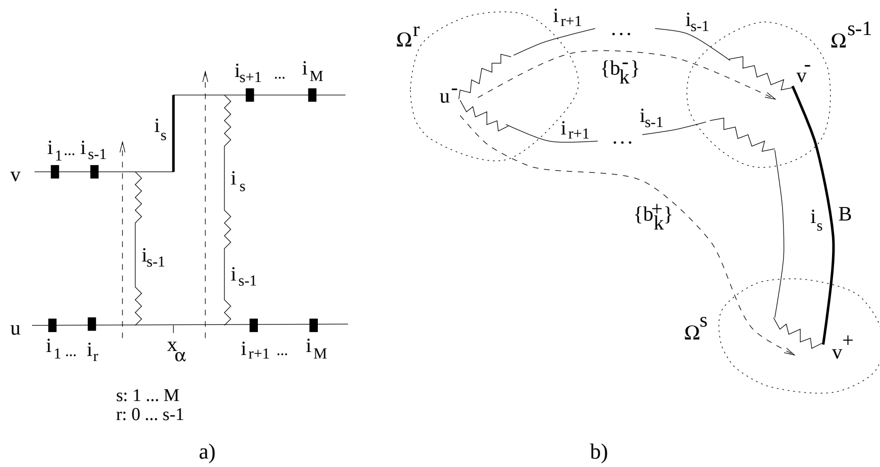

FIGURE 3.1

CASE 1. – (3.9) holds.
Note that one may always take $i_k \in \{i_{s-1}, i_{s+1}\}$ so that, by (3.5):

$$E_{i_k} = B \cdot \left[(W_{i_k}^+ - W_{i_k}^-)(\lambda_{i_k}^\pm - \dot{x}_\alpha) + W_{i_k}^\mp(\lambda_{i_k}^+ - \lambda_{i_k}^-)\right] \le -\kappa_1 B^2 c, \tag{3.10}$$

where $c > 0$ is a small uniform constant, bounded away from zero. The inequality in (3.10) follows from the fact that $\lambda_{i_k}^+ - \lambda_{i_k}^-$ in there is of the order of the sum of all small waves in $\{\epsilon_k^-\}_{k=1}^n$ and $\{\epsilon_k^+\}_{k=1}^n$.

We thus see that E_{i_k} provides a big negative term that eventually overwhelms all the other terms E_k, because:

$$E_k = B \cdot W_k^\pm(\lambda_k^+ - \lambda_k^-) \quad \text{ if } b_k^+ = b_k^- = B \text{ and } k \notin \{i_{s-1}, i_{s+1}\},$$
$$E_k = |b_k^+| \cdot W_k^+(\lambda_k^+ - \dot{x}_\alpha) - |b_k^-| \cdot W_k^-(\lambda_k^- - \dot{x}_\alpha) \quad \text{ if both } b_k^+ \text{ and } b_k^- \text{ are small},$$
$$E_{i_s} = B \cdot W_{i_s}^\pm(\lambda_{i_s}^\pm - \dot{x}_\alpha) - |b_{i_s}^\mp| \cdot W_{i_s}^\mp|\lambda_{i_s}^\mp - \dot{x}_\alpha| \le B \cdot W_{i_s}^\pm(\lambda_{i_s}^\pm - \dot{x}_\alpha).$$

In all the above cases:

$$E_k = O(1) \cdot \left[\sum_{\substack{k=1\\ \alpha_k \notin \mathcal{LS}}}^n |\epsilon_k^-| + \sum_{\substack{k=1\\ \alpha_k \notin \mathcal{LS}}}^n |\epsilon_k^+|\right]. \tag{3.11}$$

Similar analysis works for E_k^β with $\beta \in \mathcal{LS},\ \beta \neq \alpha$. In case $\alpha \in \mathcal{S} \cup \mathcal{R}$, the following estimate will be shown in Section 4:

$$\sum_{k=1}^n E_k = O(1) \cdot |\epsilon_\alpha|. \tag{3.12}$$

Observing that by Proposition 2.3 the quantity

$$\sum_{\alpha \in \mathcal{J} \setminus \mathcal{LS}} |\epsilon_\alpha|$$

is bounded (uniformly in time), one sees that (3.10) - (3.12) imply (3.7) if only κ_1 is big and δ_0 in (1.13) is small enough.

CASE 2. – (3.9) is violated.

The above is possible if and only if no large wave can be found between the locations of any pair of the large shocks of the same family (occuring in u and v). In other words: one of the immediate large successors or predecessors of any large wave in u or v, must be of the same characteristic family as this wave – see Figure 3.2.

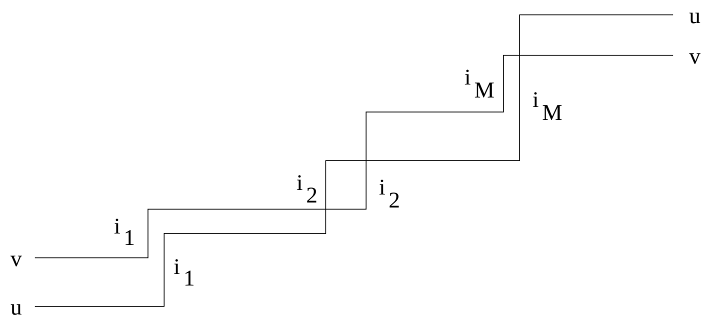

FIGURE 3.2

For a fixed $s : 1 \dots M$, denote by α the wave in v of the family i_s, and by β the large jump in u of the same family. Then, as shown in Section 5:

$$\sum_{k=1}^{n} E_{\alpha,k} + \sum_{k=1}^{n} E_{\beta,k} \leq 0, \tag{3.13}$$

and

$$\sum_{k=1}^{n} E_{\alpha,k} = O(1) \cdot \epsilon \cdot |\epsilon_\alpha| \quad \forall \alpha \in \mathcal{S} \cup \mathcal{R}. \tag{3.14}$$

Certainly (3.13) and (3.14) imply (3.7).

4. PROOFS OF THE STABILITY ESTIMATES

CASE OF LARGE SHOCKS – THE ESTIMATE (3.13)

We assume that the waves location pattern looks as in Figure 4.1, all the other possible configurations can be treated in entirely the same way.

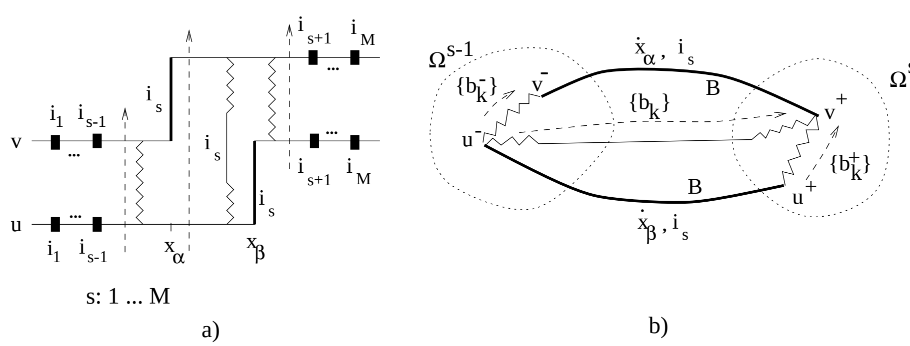

FIGURE 4.1

Using notation of Figure 4.1, we will show that:

$$\begin{aligned}\sum_{k=1}^{n} E_{\alpha,k}+\sum_{k=1}^{n} E_{\beta,k} &= \sum_{k=1}^{n}\left[|b_k|\cdot W_k(\lambda_k-\dot{x}_\alpha)-|b_k^-|\cdot W_k^-(\lambda_k^- -\dot{x}_\alpha)\right]\\ &\quad+\sum_{k=1}^{n}\left[|b_k^+|\cdot W_k^+(\lambda_k^+-\dot{x}_\beta)-|b_k|\cdot W_k(\lambda_k-\dot{x}_\beta)\right]\le 0.\end{aligned}\tag{4.1}$$

First, we estimate $\sum_{k=1}^{n} E_{\alpha,k}$. By Lemma 5.1. in [LT] and definitions (3.5) we get:

$$E_{\alpha,i_s}=B\cdot O(1)\cdot\sum_{k\ge i_s}|b_k^-|-|b_{i_s}^-|\cdot(O(1)+2\kappa_1 B)\cdot|\lambda_{i_s}^- -\dot{x}_\alpha|,\tag{4.2}$$

$$\begin{aligned}\sum_{k\le i_{s-1}} E_{\alpha,k} &= \sum_{k\le i_{s-1}}\Big[|b_k|\cdot(\lambda_k-\dot{x}_\alpha)(W_k-W_k^-)\\ &\qquad\qquad + W_k^-\left(|b_k|(\lambda_k-\dot{x}_\alpha)-|b_k^-|(\lambda_k^- -\dot{x}_\alpha)\right)\Big]\\ &\le \sum_{k\le i_{s-1}}\Big[-|b_k|\cdot|\lambda_k-\dot{x}_\alpha|\kappa_1 B\\ &\qquad\qquad + 2\kappa_1 B\left(|b_k^-||\lambda_k^- -\dot{x}_\alpha|-|b_k||\lambda_k-\dot{x}_\alpha|\right)\Big]\\ &\quad + O(1)\cdot\left[\sum_{k<i_s}|b_k|+\sum_{k\ge i_s}|b_k^-|\right]\\ &= -3\kappa_1 B\cdot\sum_{k\le i_{s-1}}|b_k||\lambda_k-\dot{x}_\alpha|+2\kappa_1 B\cdot\sum_{k\le i_{s-1}}|b_k^-||\lambda_k^- -\dot{x}_\alpha|\\ &\quad + O(1)\cdot\left[\sum_{k<i_s}|b_k|+\sum_{k\ge i_s}|b_k^-|\right],\end{aligned}\tag{4.3}$$

$$\begin{aligned}\sum_{i_{s-1}<k<i_s} E_{\alpha,k} &= \sum_{i_{s-1}<k<i_s}\Big[|b_k|\cdot(\lambda_k-\dot{x}_\alpha)(W_k-W_k^-)\\ &\qquad\qquad + W_k^-\left(|b_k|(\lambda_k-\dot{x}_\alpha)-|b_k^-|(\lambda_k^- -\dot{x}_\alpha)\right)\Big]\\ &\le \sum_{i_{s-1}<k<i_s}\Big[-|b_k|\cdot|\lambda_k-\dot{x}_\alpha|\kappa_1 B\Big]\\ &\quad + O(1)\cdot\left[\sum_{k<i_s}|b_k|+\sum_{k\ge i_s}|b_k^-|\right],\end{aligned}\tag{4.4}$$

$$
\begin{aligned}
\sum_{i_s<k<i_{s+1}} E_{\alpha,k} = & \sum_{i_s<k<i_{s+1}} \Big[|b_k^-|\cdot(\lambda_k^- - \dot{x}_\alpha)(W_k - W_k^-) \\
& \qquad\qquad + W_k\left(|b_k|(\lambda_k - \dot{x}_\alpha) - |b_k^-|(\lambda_k^- - \dot{x}_\alpha)\right)\Big] \\
\leq & \sum_{i_s<k<i_{s+1}} \Big[-|b_k^-|\cdot|\lambda_k^- - \dot{x}_\alpha|\kappa_1 B \\
& \qquad\qquad + \kappa_1 B\left(|b_k||\lambda_k - \dot{x}_\alpha| - |b_k^-||\lambda_k^- - \dot{x}_\alpha|\right)\Big] \\
& + O(1)\cdot\sum_{k\geq i_s}|b_k^-| \\
= & -2\kappa_1 B\cdot\sum_{i_s<k<i_{s+1}}|b_k^-||\lambda_k^- - \dot{x}_\alpha| \\
& + \kappa_1 B\cdot\sum_{i_s<k<i_{s+1}}|b_k||\lambda_k - \dot{x}_\alpha| + O(1)\cdot\sum_{k\geq i_s}|b_k^-|,
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
\sum_{k\geq i_{s+1}} E_{\alpha,k} = & \sum_{k\geq i_{s+1}} \Big[|b_k|\cdot(\lambda_k - \dot{x}_\alpha)(W_k - W_k^-) \\
& \qquad\qquad + W_k^-\left(|b_k|(\lambda_k - \dot{x}_\alpha) - |b_k^-|(\lambda_k^- - \dot{x}_\alpha)\right)\Big] \\
\leq & \sum_{k\geq i_{s+1}} \Big[|b_k|\cdot|\lambda_k - \dot{x}_\alpha|\kappa_1 B \\
& \qquad\qquad + 2\kappa_1 B\left(|b_k||\lambda_k - \dot{x}_\alpha| - |b_k^-||\lambda_k^- - \dot{x}_\alpha|\right)\Big] \\
& + O(1)\cdot\sum_{k\geq i_s}|b_k^-| \\
= & -2\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k^-||\lambda_k^- - \dot{x}_\alpha| + 3\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k||\lambda_k - \dot{x}_\alpha| \\
& + O(1)\cdot\sum_{k\geq i_s}|b_k^-|.
\end{aligned}
\tag{4.6}
$$

As usual, we do not take into account these terms in the above formulae, that contain the 'nonexisting' indices i_{s-1} or i_{s+1}.

Summing the inequalities (4.2) - (4.6) we obtain:

$$
\begin{aligned}
\sum_{k=1}^{n} E_{\alpha,k} \leq & -3\kappa_1 B\cdot\sum_{k\leq i_{s-1}}|b_k||\lambda_k - \dot{x}_\alpha| - \kappa_1 B\cdot\sum_{i_{s-1}<k<i_s}|b_k||\lambda_k - \dot{x}_\alpha| \\
& + \kappa_1 B\cdot\sum_{i_s<k<i_{s+1}}|b_k||\lambda_k - \dot{x}_\alpha| + 3\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k||\lambda_k - \dot{x}_\alpha| \\
& + 2\kappa_1 B\cdot\sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- - \dot{x}_\alpha| - 2\kappa_1 B\cdot\sum_{k\geq i_s}|b_k^-||\lambda_k^- - \dot{x}_\alpha| \\
& + O(1)\cdot\left[\sum_{k<i_s}|b_k| + \sum_{k\geq i_s}|b_k^-|\right].
\end{aligned}
\tag{4.7}
$$

Now we estimate different terms in $\sum_{k=1}^{n} E_{\beta,k}$:

$$
E_{\beta,i_s} = B\cdot O(1)\cdot\sum_{k\leq i_s}|b_k^+| - |b_{i_s}^+|\cdot(O(1) + 2\kappa_1 B)\cdot|\lambda_{i_s}^+ - \dot{x}_\beta|, \tag{4.8}
$$

$$
\begin{aligned}
\sum_{k\le i_{s-1}} E_{\beta,k} &= \sum_{k\le i_{s-1}} \Big[|b_k|\cdot(\lambda_k-\dot{x}_\beta)(W_k-W_k^-) \\
&\qquad\qquad + W_k^+\left(|b_k^+|(\lambda_k^+-\dot{x}_\beta)-|b_k|(\lambda_k-\dot{x}_\beta)\right)\Big] \\
&\le \sum_{k\le i_{s-1}} \Big[|b_k|\cdot|\lambda_k-\dot{x}_\beta|\kappa_1 B \\
&\qquad\qquad + 2\kappa_1 B\left(|b_k||\lambda_k-\dot{x}_\beta|-|b_k^+||\lambda_k^+-\dot{x}_\beta|\right)\Big] \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right] \\
&= 3\kappa_1 B\cdot\sum_{k\le i_{s-1}}|b_k||\lambda_k-\dot{x}_\beta| - 2\kappa_1 B\cdot\sum_{k\le i_{s-1}}|b_k^+||\lambda_k^+-\dot{x}_\beta| \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right],
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\sum_{i_{s-1}<k<i_s} E_{\beta,k} &= \sum_{i_{s-1}<k<i_s} \Big[|b_k^+|\cdot(\lambda_k^+-\dot{x}_\beta)(W_k-W_k^-) \\
&\qquad\qquad + W_k\left(|b_k^+|(\lambda_k^+-\dot{x}_\beta)-|b_k|(\lambda_k-\dot{x}_\beta)\right)\Big] \\
&\le \sum_{i_{s-1}<k<i_s} \Big[-|b_k^+|\cdot|\lambda_k^+-\dot{x}_\alpha|\kappa_1 B\Big] \\
&\qquad\qquad + \kappa_1 B\left(|b_k||\lambda_k-\dot{x}_\beta|-|b_k^+||\lambda_k^+-\dot{x}_\beta|\right)\Big] \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right] \\
&= -2\kappa_1 B\cdot\sum_{i_{s-1}<k<i_s}|b_k^+||\lambda_k^+-\dot{x}_\beta| \\
&\quad + \kappa_1 B\cdot\sum_{i_{s-1}<k<i_s}|b_k||\lambda_k-\dot{x}_\beta| \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right],
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
\sum_{i_s<k<i_{s+1}} E_{\beta,k} &= \sum_{i_s<k<i_{s+1}} \Big[|b_k|\cdot(\lambda_k-\dot{x}_\beta)(W_k-W_k^-) \\
&\qquad\qquad + W_k^+\left(|b_k^+|(\lambda_k^+-\dot{x}_\beta)-|b_k|(\lambda_k-\dot{x}_\beta)\right)\Big] \\
&\le \sum_{i_s<k<i_{s+1}} \Big[-|b_k|\cdot|\lambda_k-\dot{x}_\alpha|\kappa_1 B\Big] \\
&\quad + O(1)\cdot\left[\sum_{k\le i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right],
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
\sum_{k\geq i_{s+1}} E_{\beta,k} = \sum_{k\geq i_{s+1}} & \Big[|b_k|\cdot(\lambda_k-\dot{x}_\beta)(W_k-W_k^-) \\
& \qquad + W_k^+\left(|b_k^+|(\lambda_k^+-\dot{x}_\beta)-|b_k|(\lambda_k-\dot{x}_\beta)\right)\Big] \\
\leq \sum_{k\geq i_{s+1}} & \Big[-|b_k|\cdot|\lambda_k-\dot{x}_\beta|\kappa_1 B \\
& \qquad + 2\kappa_1 B\left(|b_k^+||\lambda_k^+-\dot{x}_\beta|-|b_k||\lambda_k-\dot{x}_\beta|\right)\Big] \\
& + O(1)\cdot\left[\sum_{k\leq i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right] \\
= -3\kappa_1 B\cdot & \sum_{k\geq i_{s+1}}|b_k||\lambda_k-\dot{x}_\beta| + 2\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+-\dot{x}_\beta| \\
& + O(1)\cdot\left[\sum_{k\leq i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right].
\end{aligned}
\tag{4.12}
$$

Thus, in view of (4.8) - (4.12), we get:

$$
\begin{aligned}
\sum_{k=1}^n E_{\beta,k} \leq & 3\kappa_1 B\cdot\sum_{k\leq i_{s-1}}|b_k||\lambda_k-\dot{x}_\beta| + \kappa_1 B\cdot\sum_{i_{s-1}<k<i_s}|b_k||\lambda_k-\dot{x}_\beta| \\
& -\kappa_1 B\cdot\sum_{i_s<k<i_{s+1}}|b_k||\lambda_k-\dot{x}_\beta| - 3\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k||\lambda_k-\dot{x}_\beta| \\
& -2\kappa_1 B\cdot\sum_{k\leq i_s}|b_k^+||\lambda_k^+-\dot{x}_\beta| + 2\kappa_1 B\cdot\sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+-\dot{x}_\beta| \\
& + O(1)\cdot\left[\sum_{k\leq i_s}|b_k^+|+\sum_{k>i_s}|b_k|\right].
\end{aligned}
\tag{4.13}
$$

Summing (4.7) with (4.13) and recalling that:

$$
|\dot{x}_\alpha-\dot{x}_\beta| = O(1)\cdot\left[\sum_{k\geq i_s}|b_k^-|+\sum_{k\leq i_s}|b_k^+|\right],
$$

we receive:

$$
\begin{aligned}
\sum_{k=1}^n & E_{\alpha,k} + \sum_{k=1}^n E_{\beta,k} \\
\leq & 2\kappa_1 B\cdot\Big(\sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- - \dot{x}_\alpha| - \sum_{k\geq i_s}|b_k^-||\lambda_k^- - \dot{x}_\alpha| \\
& \qquad - \sum_{k\leq i_s}|b_k^+||\lambda_k^+-\dot{x}_\beta| + \sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+-\dot{x}_\beta|\Big) \\
& + O(1)\cdot\left[\sum_{k\geq i_s}|b_k^-|+\sum_{k\neq i_s}|b_k|+\sum_{k\leq i_s}|b_k^+|\right].
\end{aligned}
\tag{4.14}
$$

Since

$$\sum_{k\neq i_s} |b_k| = O(1)\cdot\left[\sum_{k\geq i_s}|b_k^-| + \sum_{k\leq i_s}|b_k^+|\right],$$

we see that once we fix $\gamma\in(0,1)$, without loss of generality the following estimate holds:

$$\begin{aligned}\sum_{k=1}^{n} E_{\alpha,k} + \sum_{k=1}^{n} E_{\beta,k}&\\ \leq 2\kappa_1 B\cdot\Big[\sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- - \dot{x}_\beta| + \sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+ - \dot{x}_\beta|&\\ -\gamma\cdot\Big(\sum_{k\geq i_s}|b_k^-||\lambda_k^- - \dot{x}_\beta| + \sum_{k\leq i_s}|b_k^+||\lambda_k^+ - \dot{x}_\beta|\Big)\Big].&\end{aligned}\tag{4.15}$$

Note that if $\gamma>\theta$ then the right hand side of (4.15) is nonpositive by the following two estimates:

$$\begin{aligned}\sum_{k\leq i_{s-1}}|b_k(\lambda_k-\Lambda^s) - b_k^-(\lambda_k^- - \Lambda^s)| + \sum_{k\geq i_{s+1}}|b_k||\lambda_k-\Lambda^s|&\\ \leq \gamma\cdot\sum_{k\geq i_s}|b_k^-||\lambda_k^- - \Lambda^s|,&\end{aligned}\tag{4.16}$$

$$\begin{aligned}\sum_{k\geq i_{s+1}}|b_k(\lambda_k-\Lambda^s) - b_k^+(\lambda_k^+ - \Lambda^s)| + \sum_{k\leq i_{s-1}}|b_k||\lambda_k-\Lambda^s|&\\ \leq \gamma\cdot\sum_{k\leq i_s}|b_k^+||\lambda_k^+ - \Lambda^s|,&\end{aligned}\tag{4.17}$$

that are the consequences of the Stability Condition (1.11) (1.12) and can be proved as in Lemma 5.5. in [LT]. Indeed, summing (4.16) with (4.17), we have:

$$\begin{aligned}\gamma\cdot\Big(\sum_{k\geq i_s}|b_k^-||\lambda_k^- - \Lambda^s| + |b_k^+||\lambda_k^+ - \Lambda^s|\Big)&\\ \geq \Big[\sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- - \Lambda^s| - \sum_{k\leq i_{s-1}}|b_k||\lambda_k-\Lambda^s| + \sum_{k\geq i_{s+1}}|b_k||\lambda_k-\Lambda^s|\Big]&\\ +\Big[\sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+ - \Lambda^s| - \sum_{k\geq i_{s+1}}|b_k||\lambda_k-\Lambda^s| + \sum_{k\leq i_{s-1}}|b_k||\lambda_k-\Lambda^s|\Big]&\\ = \sum_{k\leq i_{s-1}}|b_k^-||\lambda_k^- - \Lambda^s| + \sum_{k\geq i_{s+1}}|b_k^+||\lambda_k^+ - \Lambda^s|\Big],&\end{aligned}$$

that implies (3.13) in view of (4.15).

Case of small physical waves – the estimates (3.12) and (3.14)

Denote by $\dot{y}_\alpha$ the 'real' speed of the α wave under consideration, that is: $\dot{y}_\alpha = \lambda_{k_\alpha}(v^-,v^+)$ in case $\alpha\in\mathcal{S}$ or $\dot{y}_\alpha=\lambda_{k_\alpha}(v^+)$ in case $\alpha\in\mathcal{R}$. For $k:1\ldots n$ let's estimate the difference between E_k and a similar expression where $\dot{y}_\alpha$ replaces $\dot{x}_\alpha$:

$$\begin{aligned} E_k - \big[|b_k^+|W_k^+(\lambda_k^+ - \dot{y}_\alpha) - |b_k^-|W_k^-(\lambda_k^- - \dot{y}_\alpha)\big] \\ = (\dot{y}_\alpha - \dot{x}_\alpha)\big[|b_k^+|W_k^+ - |b_k^-|W_k^-\big] = O(1)\cdot\epsilon\cdot|\epsilon_\alpha|, \end{aligned} \tag{4.18}$$

because $|\dot{y}_\alpha - \dot{x}_\alpha| \leq \epsilon$.

Below we will assume that $\dot{y}_\alpha = \dot{x}_\alpha$ and prove that under this hypothesis (3.12) holds in Case 1, while

$$\sum_{k=1}^{n} E_k \leq 0 \tag{4.19}$$

holds in Case 2. These together with (4.18) will yield, respectively, (3.12) and (3.14).

We assume that α – the wave under consideration is located in Ω^s, for some $s : 0, \ldots, M$. In other words: both states joined by α belong to Ω^s.

Case A. Assume first that both wave vectors $\{b_k^-\}_{k=1}^n$ and $\{b_k^+\}_{k=1}^n$ contain a large wave of the family $i_k \in \{i_s, i_{s+1}\}$. We treat here the case $i_k = i_s$ with wave configuration as in Figure 4.2, the other cases being similar. We have:

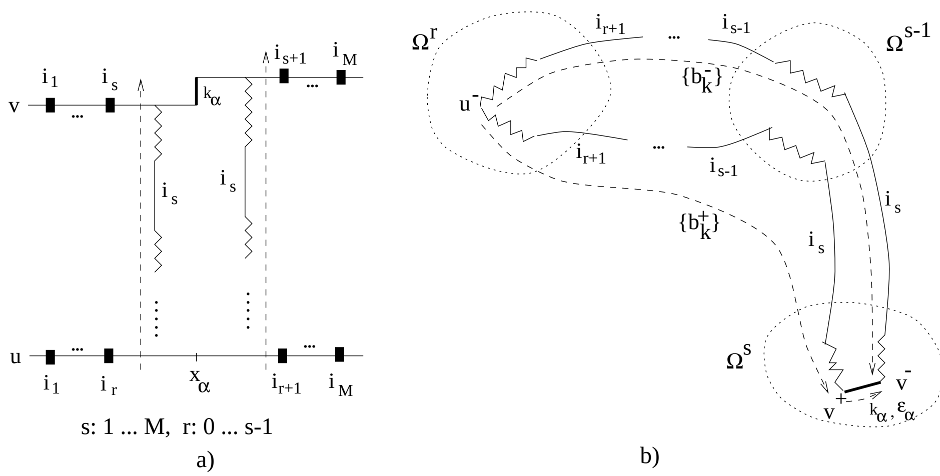

Figure 4.2

$$\begin{aligned} E_{i_s} &= B\cdot\big[(W_{i_s}^+ - W_{i_s}^-)(\lambda_{i_s}^\pm - \dot{x}_\alpha| + W_{i_s}^\mp(\lambda_{i_s}^\pm - \lambda_{i_s}^\mp)\big] \\ &\leq -Bc\kappa_1\cdot|\epsilon_\alpha| + O(1)\cdot B\cdot|\epsilon_\alpha| \end{aligned} \tag{4.20}$$

(the choice of the upper or lower superindices depends on the family number k_α).

For indices k such that b_k^+ and b_k^- are small, as in [LT] we obtain:

$$\begin{aligned} E_k = |b_k^\pm|\cdot(W_k^+ - W_k^-)\cdot(\lambda_k^\pm - \dot{x}_\alpha) \\ + W_k^\mp\big(|b_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |b_k^-|(\lambda_k^- - \dot{x}_\alpha)\big) \\ \leq (O(1) + 4\kappa_1 B)\cdot\big(O(1)\cdot|b_k^+ - b_k^-| + O(1)\cdot|b_k^-||\epsilon_\alpha|\big) + O(1)\cdot|\epsilon_\alpha|. \end{aligned} \tag{4.21}$$

If k wave is large $b_k^+ = b_k^- = B$, but $k \neq i_s$, then $W_k^+ = W_k^-$ and

$$E_k = B \cdot W_k^+(\lambda_k^+ - \lambda_k^-)) = O(1) \cdot (O(1) + 4\kappa_1 B) \cdot |\epsilon_\alpha|. \tag{4.22}$$

Now, summing (4.20) with (4.21) we receive:

$$\begin{aligned} E_{i_s} + \sum_{\substack{k:1\dots n \\ b_k^\pm \neq B}} E_k \leq & -B\kappa_1 c \cdot |\epsilon_\alpha| + O(1) \cdot |\epsilon_\alpha| \\ & + O(1) \cdot 4\kappa_1 B \cdot \sum_{\substack{k:1\dots n \\ b_k^\pm \neq B}} \Big[|b_k^+ - b_k^-| + |b_k^-||\epsilon_\alpha| \Big] \leq 0, \end{aligned} \tag{4.23}$$

if only κ_1 is big enough and all the weights w_k^s small.

Note that in Case 2:

$$\sum_{k=1}^n E_k = E_{i_s} + \sum_{\substack{k:1\dots n \\ b_k^\pm \neq B}} E_k,$$

so (4.19) follows from (4.23).

In Case 1 some terms of the form (4.22) may be added to (4.23), thus we can hope only for the weaker inequality (3.12). Indeed, it follows from (4.20) (4.21) (4.22).

Case B. – Figure 4.3

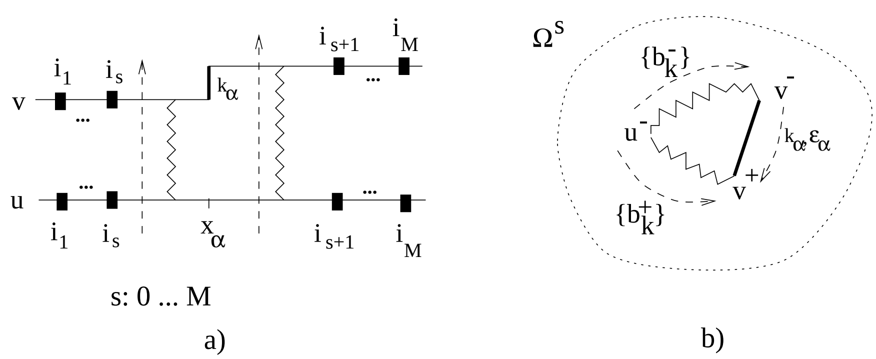

Figure 4.3

This case has been treated in [BLY]. If the constant B is small enough and κ_1 big (with respect to the uniform constants $O(1)$ in all the formulae), we get (4.19) as in [BLY].

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