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L^2-BOUNDEDNESS OF A SINGULAR INTEGRAL OPERATOR

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Abstract

In this paper we study a singular integral operator T with rough kernel. This operator has singularity along sets of the form $\{x = Q(|y|)y'\}$, where $Q(t)$ is a polynomial satisfying $Q(0) = 0$. We prove that T is a bounded operator in the space $L^2(R^n)$, $n \geq 2$, and this bound is independent of the coefficients of $Q(t)$.

We also obtain certain Hardy type inequalities related to this operator.

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $n \geq 2$, with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)|x|^{-n}$ be a homogeneous function of degree $-n$, with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$.

Suppose $b(|x|)$ is an L^∞ function. We consider the distribution $K = p.v.\ b(|x|)\Omega(x)|x|^{-n}$ and study the boundedness of the singular integral operator $T_{Q,b}(f)$ defined by

$$T_{Q,b}(f)(x) = \int_{\mathbb{R}^n} K(y) f(x - Q(|y|)y')\, dy \tag{1.2}$$

where $y' = y/|y| \in S^{n-1}$ and $Q(t) = \sum_{k=1}^{m} b_k t^k$ is a polynomial of degree m.

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For the sake of simplicity, we denote $T_{Q,b} = T_b$ if $Q(t) = t$ and $T_{Q,b} = T$ if $Q(t) = t$ and $b(x) \equiv 1$.

The maximal operator $T_b^*(f)(x)$ now is defined by

$$T_b^*(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} K(y)f(x-y)\,dy \right|. \tag{1.3}$$

The singular integral operator Tf was first studied by Calderón and Zygmund in their pioneering papers [**CZ1**] and [**CZ2**]. In [**CZ2**], Calderón and Zygmund proved that if $\Omega \in L\operatorname{Log}^+ L(S^{n-1})$ satisfies the mean zero condition (1.1) then the operator T with kernel $\Omega(x')|x|^{-n}$ is a bounded operator in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Below let us recall briefly the idea used in Calderón-Zygmund's proof.

Suppose that $\Omega \in L'(S^{n-1})$ is an odd function, then one can easily show that $Tf(x)$ is equal to

$$\begin{aligned} &\int_{\mathbb{R}^n} f(x-y)\Omega(y')|y|^{-n}\,dy \\ &\qquad = \frac{1}{2}\int_{S^{n-1}} \Omega(y') \left\{ \int_{-\infty}^{\infty} f(x-ty')t^{-1}\,dt \right\} d\sigma(y'). \end{aligned} \tag{1.4}$$

By the method of rotation and the well-known L^p boundedness of the Hilbert transform one then obtains the L^p boundedness of T under the weak condition $\Omega \in L^1(S^{n-1})$.

For even kernels, the condition $\Omega \in L^1(S^{n-1})$ is insufficient. It turns out the right condition is $\Omega \in L\operatorname{Log}^+ L(S^{n-1})$ (as far as the size of Ω is concerned). The idea of Calderón-Zygmund is to compose the operator T with the Riesz transform R_j, $1 \le j \le n$, and show that R_jT is a singular integral operator with an appropriate odd kernel. Thus $\|R_jT\psi\|_p \le C_p\|\psi\|_p$ for all test functions $\psi \in \mathcal{L}$. Furthermore, one can obtain

$$\begin{aligned} \|T\psi\|_p &= \left\| \left(\sum_{j=1}^n R_j^2 \right) T\psi \right\|_p \\ &\le \sum_{j=1}^n \|R_j(R_jT\psi)\|_p \le na_p\|R_jT\psi\|_p \le na_pC_p\|\psi\|_p \end{aligned}$$

for all $\psi \in \mathcal{L}$, since $\sum_{j=1}^n R_j^2$ is equal to the identity map.

In [**Fe**], R. Fefferman generalized this Calderón-Zygmund singular integral by replacing the kernel $\Omega(x')|x|^{-n}$ by $b(|x|)\Omega(x')|x|^{-n}$, where b is an arbitrary L^∞ function. This allows the kernel to be rough not only on the sphere, but also in the radial direction. For the singular integral opeator T_b with the kernel $K(x) = b(|x|)\Omega(x')|x|^{-n}$, the formula (1.4) now is

$$T_b f(x) = \int_{S^{n-1}} \Omega(y') \left\{ \int_0^\infty f(x - ty')b(t)t^{-1}\, dt \right\} d\sigma(y'). \tag{1.4'}$$

Clearly, the method by Calderón and Zygmund can no longer be used to estimate the above integral in (1.4') even if Ω is odd, since the integral in the parenthesis can not be reduced to the Hilbert transform for an arbitrary $b(t)$. Thus one needs to find a new approach.

Using a method which is different from Calderón and Zygmund, in [**Fe**] R. Fefferman showed that if Ω satisfies a Lipschitz condition then T_b is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. J. Namazi [**Na**] improved Fefferman's theorem by using the assumption $\Omega \in L^q(S^{n-1})$. The same L^p result was also obtained by L. Chen for the maximal operator T_b^* (see [**Ch**]). In [**Fa**], one of the authors obtained the L^2 boundedness for T_b under the significantly weaker condition $\Omega \in H^1(S^{n-1})$, where $H^1(S^{n-1})$ is the Hardy space on S^{n-1}. The condition $b \in L^\infty$ is also replaced by a weaker conditon

$$R^{-1} \int_0^R |b(\rho)|^q\, d\rho \le A, \text{ for all } R > 0 \text{ and some } q > 1 \tag{1.5}$$

(see also [**St**] or [**DR**]).

The definition of Hardy space will be reviewed in Section 2. But we should mention here that on S^{n-1}, it is well-known that for $q > 1$,

$$L^q \subseteq L \operatorname{Log}^+ L \subseteq H^1(S^{n-1}) \subseteq L^1$$

and all inclusions are proper.

The main purpose of this paper is to study the L^2 boundedness for the more general singular integral operator $T_{Q,b}(f)$ defined in (1.2) as well as the maximal operator $T_b^*(f)$ with $\Omega \in H^1(S^{n-1})$. In a forthcoming paper, we will study the L^p boundedness for another singular integral T_Φ that also takes T_b as a model case.

The following is the main theorem in this paper:

Theorem 1. *Let $T_{Q,b}$ be the singular integral operator defined by (1.2) and T_b^* be the maximal operator defined in (1.3). If $\Omega \in H^1(S^{n-1})$ satisfies (1.1) then both these two operators are bounded in $L^2(\mathbb{R}^n)$.*

More precisely, we have

$$\|T_{Q,b}(f)\|_2 \le C\|b\|_\infty \|\Omega\|_{H^1(S^{n-1})} \|f\|_2; \tag{1.6}$$

$$\|T_b^*(f)\|_2 \le C\|b\|_\infty \|\Omega\|_{H^1(S^{n-1})} \|f\|_2, \tag{1.7}$$

where C is a constant independent of b, Ω, f and the coefficients of Q.

By the proof in Theorem 1, we can further obtain the following result:

Theorem 2 (Hardy-type inequalities).

(i) *Let $Q(t) = \sum_{k=1}^{m} b_k t^k$ be a polynomial in $\mathbb{R}$ and $\Omega \in H^1(S^{n-1})$ satisfy the mean zero property (1.1). Then we have*

$$\int_{\mathbb{R}^n} |x|^{-n} \left| \int_{S^{n-1}} e^{iQ(|x|)\langle x',\xi'\rangle)} \Omega(\xi')\, d\sigma(\xi') \right| dx \le C\|\Omega\|_{H^1(S^{n-1})}, \tag{1.8}$$

where C is a constant independent of Ω and the coefficients of Q.

(ii) *If Ω is a distribution in the Hardy space $H^p(S^{n-1})$, $0 < p < 1$, with property (1.1) then*

$$\int_{\mathbb{R}^n} |x|^{(1-n)(2-p)-1} \left| \int_{S^{n-1}} e^{i\langle x,\xi'\rangle} \Omega(\xi')\, d\sigma(\xi') \right|^p dx \le C_p\|\Omega\|^p_{H^p(S^{n-1})}, \tag{1.9}$$

where C is a constant independent of Ω.

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but independent of the essential variables.

2. Definitions and Lemmas

Recall that the Poisson kernel on S^{n-1} is defined by

$$P_{ry'}(x') = (1 - r^2)/|ry' - x'|^n,$$

where $0 \le r < 1$ and x', $y' \in S^{n-1}$.

For any $f \in \mathcal{L}'(S^{n-1})$, we define the radial maximal function $P^+f(x')$ by

$$P^+f(x') = \sup_{0\le r<1} \left| \int_{S^{n-1}} f(y')P_{rx'}(y')\, d\sigma(y') \right|,$$

where $\mathcal{L}'(S^{n-1})$ is the space of Schwartz distributions on S^{n-1}. The Hardy space $H^p(S^{n-1})$, $0 < p \le 1$, is the linear space of distribution $f \in \mathcal{L}'(S^{n-1})$ with the finite norm $\|f\|_{H^p(S^{n-1})} = \|P^+ f\|_{L^p(S^{n-1})} < \infty$. The space $H^p(S^{n-1})$ was studied in [**Co**] (see also [**CTW**]). In particular, it is known that

$$H^p(S^{n-1}) \supseteq L^1(S^{n-1}) \supseteq H^1(S^{n-1}) \supseteq L\log^+ L(S^{n-1}) \supseteq L^q(S^{n-1})$$

for any $q > 1 > p > 0$.

Another important property of $H^p(S^{n-1})$ is the atomic decomposition, which will be reviewed below.

An *exceptional atom* is an L^∞ function $E(x)$ satisfying $\|E\|_\infty \le 1$. A *regular (p, q) atom* is an L^q $(1 < q \le \infty)$ function $a(\cdot)$ that satisfies

$$\text{supp}(a) \subset \{x' \in S^{n-1},\ |x' - x'_0| < \rho \quad \text{for some } x'_0 \in S^{n-1} \text{ and } \rho > 0\}; \tag{2.1}$$

$$\int_{S^{n-1}} a(\xi')Y(\xi')\,d\sigma(\xi') = 0, \tag{2.2}$$

for any spherical harmonic polynomial with degree $\le N$, where N is any fixed integer larger than $[(n-1)(1/p-1)]$;

$$\|a\|_q \le \rho^{(n-1)(1/q-1/p)}. \tag{2.3}$$

From [**Co**] or [**CTW**], we find that any $\Omega \in H^p(S^{n-1})$ has an atomic decomposition $\Omega = \sum \lambda_j a_j$, where the a_j's are either exceptional atoms or regular (p, q) atoms and $\sum |\lambda_j|^p \le C\|\Omega\|^p_{H^p(S^{n-1})}$. In particular, if $\Omega \in H^p(S^{n-1})$ has the mean zero property (1.1) then all the atoms a_j in the atomic decomposition can be chosen to be regular (p, q) atoms.

In the rest of the paper, for any non-zero $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, we write $\xi/|\xi| = \xi' = (\xi'_1, \dots, \xi'_n) = (\zeta_1, \dots, \zeta_n) = \zeta$. Thus $\zeta \in S^{n-1}$. Also we use ζ_* to denote $(\zeta_2, \dots, \zeta_n)$ and use ξ_* to denote $(\xi_2, \dots, \xi_n)$.

The following lemma is essentially Proposition 2.5 in [**FP**].

Lemma 2.1. *Suppose $n \ge 3$ and $a(\cdot)$ is a $(1, \infty)$ atom on S^{n-1} supported in $S^{n-1} \cap B(\zeta, \rho)$, where $B(\zeta, \rho)$ is the ball with radius ρ and center $\zeta = \xi' \in S^{n-1}$. Let*

$$F_a(s) = (1 - s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}} a(s, (1-s^2)^{1/2}\tilde{y})\,d\sigma(\tilde{y}).$$

Then, up to a constant multiplier independent of $a(\cdot)$, $F_a(s)$ *is a* $(1,\infty)$ *atom on* $\mathbb{R}$. *More precisely, there are* $s_0 \in \mathbb{R}$ *and a constant* C *which is independent of* $a(\cdot)$ *such that*

$$\operatorname{supp}(F_a) \subseteq (s_0 - 2r, s_0 + 2r); \tag{2.4}$$

$$\|F_a\|_\infty \leq C/r; \tag{2.5}$$

$$\int_{\mathbb{R}} F_a(s)\, ds = 0, \tag{2.6}$$

where $r = r(\xi') = |\xi|^{-1}|A_\rho \xi|$ *and* $A_\rho \xi = (\rho^2 \xi_1, \rho\xi_2, \dots, \rho\xi_n)$.

Proof: If $\rho < 1/4$, the proof can be found in [**FP**]. Suppose $\rho \geq 1/4$, then, clearly $\operatorname{supp}(F_a) \subseteq (-1,1)$ and $\|F_a\|_\infty \leq C$. It is also easy to see that F_a satisfies (2.6). ■

Lemma 2.2. *Suppose* $n = 2$ *and that* $a(\cdot)$ *is a* $(1,\infty)$ *atom supported in* $S^1 \cap B(\zeta,\rho)$ *and satisfies (2.1)-(2.3) with* $p = 1$. *Let*

$$F_a(s) = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)\left(a(s,(1-s^2)^{1/2}) + a(s,-(1-s^2)^{1/2})\right).$$

Then, up to a constant multiplier independent of $a(\cdot)$, $F_a(s)$ *is a* $(1,q)$ *atom on* $\mathbb{R}$, *where* q *is any fixed number in the interval* $(1,2)$. *The radius of* $\operatorname{supp}(F_a)$ *is equal to* $r = r(\xi') = |\xi|^{-1}(\rho^4\xi_1^2 + \rho^2\xi_2^2)^{1/2}$.

Proof: By the discussion in Lemma 2.1, without loss of generality, we may assume that $a(\cdot)$ is supported in $S^{n-1} \cap B(\zeta,\rho)$ with a sufficiently small ρ, where $\zeta \in S^1$. Let $\zeta = (\zeta_1, (1-\zeta_1^2)^{1/2}\sigma)$ for $\sigma \in \{\pm 1\}$. If $F_a(s) \neq 0$ then $(s, (1-s)^2)^{1/2}\delta) \in B(\zeta,\rho)$ for some $\delta \in \{\pm 1\}$. Therefore we have

$$(s-\zeta_1)^2 + \{\delta(1-s^2)^{1/2} - \sigma(1-\zeta_1^2)^{1/2}\}^2 < \rho^2.$$

Noting that either $\delta = \sigma$ or $\delta = -\sigma$, we easily see that

$$(s-\zeta_1)^2 + |(1-s^2)^{1/2} - (1-\zeta_1^2)^{1/2}|^2 \leq \rho^2 \tag{2.7}$$

which implies that

$$|s-\zeta_1| \leq \rho; \tag{2.8}$$

$$|(1-s^2)^{1/2} - (1-\zeta_1^2)^{1/2}| \leq \rho; \tag{2.9}$$

and

$$|s-\zeta_1| \le \rho^2 + 2\rho(1-\zeta_1^2)^{1/2}. \tag{2.10}$$

Inequalities (2.8) and (2.9) follow from (2.7) trivially. To see (2.10) we shall consider the following two cases.

Case A: $|\zeta_1| > 3/4$. Then by (2.8) and (2.9) we have

$$|s+\zeta_1| \ge 2|\zeta_1| - |s-\zeta_1| > 1$$

and

$$\begin{aligned}|s-\zeta_1| &\le |s^2-\zeta_1^2| \\ &= |(1-s^2)^{1/2} - (1-\zeta_1^2)^{1/2}| \\ &\quad |2(1-\zeta_1^2)^{1/2} + (1-s^2)^{1/2} - (1-\zeta_1^2)^{1/2}| \\ &\le \rho(\rho + 2(1-\zeta_1^2)^{1/2}) = \rho^2 + 2\rho(1-\zeta_1^2)^{1/2}.\end{aligned}$$

Case B: $|\zeta_1| \le 3/4$. Then $(1-\zeta_1^2)^{1/2} \ge 1/2$. By (2.8) we find

$$|s-\zeta_1| \le \rho < \rho^2 + 2\rho(1-\zeta_1^2)^{1/2},$$

which proves (2.10).

Recalling $\xi' = \zeta$, we easily see that in both Case A and Case B,

$$|s-\zeta_1| \le 2|\xi|^{-1}|A_\rho \xi|.$$

By letting $s_0 = \zeta_1$, $r = r(\xi') = |\xi|^{-1}|A_\rho\xi|$, we see that (2.4) and (2.6) are satisfied.

It remains to show, for $1<q<2$, $\|F_a\|_q \le Cr^{-1+1/q}$. To this end, we first assume that $(1-\zeta_1^2)^{1/2} > 99\rho$. By (2.9) we find

$$1/2(1-\zeta_1^2)^{1/2} \le (1-s^2)^{1/2} \le 2(1-\zeta_1^2)^{1/2}. \tag{2.11}$$

Thus by the definition of F_a we have $\|F_a\|_\infty \le C\rho^{-1}(1-\zeta_1^2)^{-1/2} \le Cr^{-1}$. Now by the support condition (2.4) we have $\|F_a\|_q \le Cr^{-1+1/q}$.

If $(1-\zeta_1^2)^{1/2} = |\zeta_2| \le 99\rho$, then by (2.9) we know $(1-s^2)^{1/2} \le 100\rho$. So by the definition of F_a we have

$$\|F_a\|_q \le C\rho^{-1}\left\{\int_{1-s^2\le 10000\rho^2} |1-s^2|^{-q/2}\,ds\right\}^{1/q}.$$

Noting that we can assume that ρ is sufficiently small so that $10000\rho^2 \leq 1/16$, thus we easily show $\|F_a\|_q \leq C\rho^{2(-1+1/q)} \leq C(\rho^2|\zeta_1|+\rho|\zeta_2|)^{-1+1/q} \leq Cr^{-1+1/q}$. Lemmma 2.2 is proved. ■

Remarks.

1. Let $a(\cdot)$ be a (p,∞) atom with support in a ball of radius ρ and let F_a be defined in Lemma 2.1. Since $\rho^{(1/p-1)(n-1)}a(\cdot)$ is a $(1,\infty)$ atom, by Lemma 2.1 we have

$$\|F_a\|_\infty \leq r^{-1}\rho^{(1-1/p)(n-1)}. \tag{2.12}$$

2. Since any spherical harmonic polynomial is the restriction to S^{n-1} of a polynomial in $\mathbb{R}^n$, from the condition (2.2) for $a(\cdot)$ we easily see

$$\int_{\mathbb{R}} F_a(s)s^k\,ds = 0 \text{ for any integer } k \in [0,N], \tag{2.13}$$

where N is an integer larger than $[(n-1)(1/p-1)]$.

3. Proof of Theorem 1

We first prove (1.6) in Theorem 1. By Fubini's Theorem we easily see that the Fourier transform of $T_{Q,b}(f)$ is equal to $\hat{f}(\xi)\tilde{K}_\Omega(\xi)$, where

$$\tilde{K}_\Omega(\xi) = \int_{\mathbb{R}^n} |y|^{-n}b(|y|)\Omega(y')e^{-iQ(|y|)|\xi|\langle y',\xi'\rangle}\,dy. \tag{3.1}$$

By Plancherel's Theorem, we only need to prove that

$$\|\tilde{K}_\Omega\|_\infty \leq C\|b\|_\infty\,\|\Omega\|_{H^1(S^{n-1})}. \tag{3.2}$$

Since $\Omega \in H^1(S^{n-1})$ satisfies the mean zero property (1.1), we can write $\Omega = \sum \lambda_j a_j$, where $\sum|\lambda_j| \leq C\|\Omega\|_{H^1(S^{n-1})}$ and each a_j is a $(1,\infty)$ atom. Therefore to prove (3.2), it suffices to show

$$\|\tilde{K}_{a_j}\|_\infty \leq C\|b\|_\infty \tag{3.3}$$

for any atom $a_j = a$, where C is independent of the coefficients of Q and the atom $a(\cdot)$. By the method of rotation, we can assume $x' = (1,0,\dots,0)$. Let $y' = (s,y_2,y_3,\dots,y_n)$. Then it is easy to see that

$$\tilde{K}_a(x) = \int_0^\infty b(t)t^{-1}\int_{\mathbb{R}} F_a(s)e^{-iQ(t)|x|s}\,ds\,dt \tag{3.4}$$

where $F_a(s)$ is the function defined in Lemma 2.1 or Lemma 2.2. By Lemma 2.1 and Lemma 2.2, without loss of generality, we may assume that F_a is a $(1,q)$ atom with support in $(-r,r)$ for $1<q<2$. Thus $A(\cdot)=rF_a(r\cdot)$ is a $(1,q)$ atom with support in the interval $(-1,1)$.

For the polynomial

$$Q(t)|x| = \sum_{k=1}^{m} |x| b_k t^k \text{ with } b_m \neq 0,$$

we let

$$\beta_k = (|x| r b_k) \text{ and } \tilde{Q}(t) = -\sum_{k=1}^{m} \beta_k t^k.$$

Then after changing variables we have

$$\tilde{K}_a(x) = \int_0^\infty t^{-1} b(t) \int_{\mathbb{R}} A(s) e^{i\tilde{Q}(t)s}\, ds\, dt. \tag{3.4'}$$

Let $|\beta_\kappa|^{1/\kappa} = \max\{|\beta_k|^{1/k}, k=1,2,\ldots,m\}$ and $\beta = |\beta_\kappa|^{-1/\kappa}$.

Then $\tilde{K}_a(x)$ is bounded by

$$\|b\|_\infty \int_0^\beta \left| \int_{\mathbb{R}} A(s)\{e^{i\tilde{Q}(t)s} - 1\}\, ds \right| t^{-1}\, dt \\ + \|b\|_\infty \int_\beta^\infty t^{-1} \left| \int_{\mathbb{R}} A(s) e^{i\tilde{Q}(t)s}\, ds \right| dt = I_1 + I_2.$$

By the choice of β, it is easy to see that I_1 is bounded by

$$C\|b\|_\infty \sum_{k=1}^{m} |\beta_k| \int_0^\beta t^{-1+k}\, dt \le C\|b\|_\infty$$

where C is independent of β_k's.

To estimate I_2, we let $R_j = [2^j, 2^{j+1})$ for any integer j and let $\Psi \in C^\infty(\mathbb{R})$ satisfy

$$\Psi(t) \equiv 1 \quad \text{for } |t| \le 1 \\ \Psi \equiv 0 \quad \text{for } |t| \ge 2.$$

Define T_j by

$$(T_j f)(t) = \chi_{R_j}(t) \int_{\mathbb{R}} e^{is\tilde{Q}(t)} \Psi(s) f(s)\, ds.$$

From the estimate on page 60 in [**Pa**], we can find an $N > 0$ such that

$$\|T_j\|_{L^2\to L^2} \le C2^{j/2}|\beta_\kappa|^{-1/2N}2^{-j\kappa/2N}.$$

By the trivial estimate $\|T_j\|_{L^1\to L^\infty} \le C$ and interpolation we now have

$$\|T_j\|_{L^p\to L^q} \le C2^{j/q}|\beta_\kappa|^{-1/qN}2^{-j\kappa/qN} \tag{3.5}$$

where $1/p + 1/q = 1$ and $q \ge 2$.

Choosing an integer J such that $2^J \le \beta < 2^{J+1}$, then, we have

$$\begin{aligned} I_2 &\le \|b\|_\infty \int_\beta^\infty t^{-1}\left|\int_{\mathbb{R}} A(s)e^{i\tilde{Q}(t)s}\,ds\right|\,dt \\ &\le C\|b\|_\infty \sum_{j\ge J}\int_{2^j}^{2^{j+1}} t^{-1}|T_j(A)(t)|\,dt \\ &\le C\|b\|_\infty \sum_{j\ge J}\left\{\int_{2^j}^{2^{j+1}} t^{-p}\,dt\right\}^{1/p}\|T_j(A)\|_{L^q}. \end{aligned}$$

Thus by (3.5) we have

$$I_2 \le C\|b\|_\infty \sum_{j\ge J} 2^{-j/q}\,2^{j/q}\|A\|_{L^p}2^{-j\kappa/qN}|\beta_\kappa|^{-1/qN} \le C\|b\|_\infty,$$

because $2^J \ge (1/2)|\beta_\kappa|^{-1/\kappa}$.

The first part in Theorem 1 is proved.

To prove (1.7) in Theorem 1, we notice that

$$\begin{aligned} T_b^* f(x) = \sup_{\varepsilon>0}&\left|\int_{|y|>\varepsilon} b(|y|)|y|^{-n}\Omega(y')f(x-y)\,dy\right| \\ &\le \sum|\lambda_j|\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} b(|y|)|y|^{-n}a_j(y')f(x-y)\,dy\right| \end{aligned}$$

where $\sum|\lambda_j| \le C\|\Omega\|_{H^1(S^{n-1})}$ and all a_j's are $(1,\infty)$ atoms.

So it suffices to show that for any $(1,\infty)$ atom $a(\cdot)$ on S^{n-1}

$$\left\|\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} b(|y|)|y|^{-n}a(y')f(\cdot-y)\,dy\right|\right\|_2 \le C\|b\|_\infty\|f\|_2 \tag{3.6}$$

with a constant C independent of b, f and $a(\cdot)$. Without loss of generality, we may assume $\operatorname{supp} a(\cdot) \subseteq B(\mathbf{1},\rho)\cap S^{n-1}$ where $\mathbf{1} = (1,0,\dots,0)$.

In order to prove (3.6) we need to prove the following:

Proposition 3.1. *Suppose that* $b \in L^\infty$ *and* $a(\cdot)$ *is a* $(1,\infty)$ *atom supported in* $B(\mathbf{1},\rho)\cap S^{n-1}$. *For* $\varepsilon > 0$, *let*

$$T_{b,\varepsilon}f(x) = (a(\cdot)b(|\cdot|)|\cdot|^{-n}\chi_{\{|y|>\varepsilon\}}(\cdot) * f)(x).$$

We have

$$\left\| \sup_{0<s<\infty} \frac{1}{s}\int_0^s |T_{b,\varepsilon}f(\cdot)|\,d\varepsilon \right\|_2 \le C\|b\|_\infty \,\|f\|_2 \tag{3.7}$$

where C *is independent of* b, f *and* $a(\cdot)$.

Proof: The Fourier transform of $T_{b,\varepsilon}f$ is $m_\varepsilon(\xi)\hat{f}(\xi)$, where

$$m_\varepsilon(\xi) = \int_\varepsilon^\infty b(t)t^{-1}\int_{S^{n-1}} a(y')e^{-it\langle\xi,y'\rangle}\,d\sigma(y')\,dt.$$

For each fixed $\xi \neq 0$, we choose a rotation O such that $O(\xi) = |\xi|\,\mathbf{1}$. Thus

$$m_\varepsilon(\xi) = \int_\varepsilon^\infty b(t)t^{-1}\int_{S^{n-1}} a(O^{-1}(y'))e^{-it|\xi|\langle\mathbf{1},y'\rangle}\,d\sigma(y')\,dt.$$

Now $a(O^{-1}(y'))$ is an atom supported in $B(\zeta,\rho)\cap S^{n-1}$ so that

$$m_\varepsilon(\xi) = \int_\varepsilon^\infty b(t)t^{-1}\int_{\mathbb{R}} F_a(s)e^{-it|\xi|s}\,ds\,dt,$$

where F_a is the function as in Lemma 2.1 if $n > 2$ and in Lemma 2.2 if $n = 2$. Without loss of generality, we assume $\mathrm{supp}(F_a) \subseteq (-r,r)$.

For the above $r = r(\xi')$, we take a radial function $\Phi \in C^\infty(\mathbb{R}^n)$ such that its Fourier transform $\hat{\Phi}$ satisfies $\hat{\Phi}(\xi) = 1$ if $|\xi| \le 1$ and $\hat{\Phi}(\xi) = 0$ if $|\xi| > 2$ and define Φ_ε by $\hat{\Phi}_\varepsilon(\xi) = \hat{\Phi}(\varepsilon r(\xi')|\xi|)$. It is easy to see that the maximal function $\Phi^*(f) = \sup_{\varepsilon>0}|\Phi_\varepsilon * f|$ is bounded in $L^p(\mathbb{R}^n)$. Now we define a g-function

$$g(f)(x) = \left\{\int_0^\infty |T_{b,\varepsilon}f(x) - \Phi_\varepsilon * T_bf(x)|^2\varepsilon^{-1}\,d\varepsilon\right\}^{1/2}.$$

Then

$$\frac{1}{s}\int_0^s |T_{b,\varepsilon}f(x)|\,d\varepsilon \le g(f)(x) + \frac{1}{s}\int_0^s |\Phi_\varepsilon * T_bf(x)|\,d\varepsilon.$$

Thus

$$\sup_{\varepsilon>0} \frac{1}{s}\int_0^s |T_{b,\varepsilon} f(x)|\,d\varepsilon \le g(f)(x) + \Phi^*(T_b f)(x).$$

By (1.6) we have

$$\|\Phi^*(T_b f)\|_2 \le C\|T_b f\|_2 \le C\|b\|_\infty\,\|f\|_2.$$

So it remains to show

$$\|g(f)\|_2 \le C\|b\|_\infty\,\|f\|_2. \tag{3.8}$$

By Plancherel's Theorem we know that

$$\|g(f)\|_2^2 = C\int_{\mathbb{R}^n}\int_0^\infty |m_\varepsilon(\xi) - \hat\Phi(\varepsilon r|\xi|)m_0(\xi)|^2\,|\hat f(\xi)|^2\varepsilon^{-1}\,d\varepsilon\,d\xi.$$

So we only need show

$$\int_0^\infty |m_\varepsilon(\xi) - \hat\Phi(\varepsilon r(\xi')|\xi|)m_0(\xi)|^2\varepsilon^{-1}\,d\varepsilon \le C\|b\|_\infty^2 \tag{3.9}$$

where C is a constant independent of b, ξ and r.

By the definition of m_ε and changing of variables we have

$$\begin{aligned}
&\int_0^\infty |m_\varepsilon(\xi) - \hat\Phi(\varepsilon r|\xi|)m_0(\xi)|^2\varepsilon^{-1}\,d\varepsilon\\
&\quad= C\int_0^\infty \Bigg|\int_{\varepsilon|\xi|}^\infty b(t/|\xi|)t^{-1}\int_{S^{n-1}} a(y')e^{-it\langle\xi',y'\rangle}\,d\sigma(y')\,dt\\
&\quad-\hat\Phi(\varepsilon r|\xi|)\int_0^\infty b(t/|\xi|)t^{-1}\int_{S^{n-1}} a(y')e^{-it\langle\xi',y'\rangle}\,d\sigma(y')\,dt\Bigg|^2\varepsilon^{-1}\,d\varepsilon\\
&\quad= C\int_0^\infty \Bigg|\int_{\varepsilon}^\infty b(t/|\xi|)t^{-1}\int_{S^{n-1}} a(y')e^{-it\langle\xi',y'\rangle}\,d\sigma(y')\,dt\\
&\quad-\hat\Phi(r\varepsilon)\int_0^\infty b(t/|\xi|)t^{-1}\int_{S^{n-1}} a(y')e^{-it\langle\xi',y'\rangle}\,d\sigma(y')\,dt\Bigg|^2\varepsilon^{-1}\,d\varepsilon\\
&\quad\le C\int_0^{1/r} \Bigg|\int_0^{\varepsilon} b(t/|\xi|)t^{-1}\int_{S^{n-1}} a(y')e^{-it\langle\xi',y'\rangle}\,d\sigma(y')\,dt\Bigg|^2\varepsilon^{-1}\,d\varepsilon\\
&\quad+ C\|b\|_\infty^2\int_{1/r}^{2/r}\varepsilon^{-1}\,d\varepsilon\left\{\int_0^\infty t^{-1}\left|\int_{S^{n-1}} a(y')e^{-it\langle\xi',y'\rangle}\,d\sigma(y')\right|dt\right\}^2\\
&\quad+ C\int_{2/r}^\infty \Bigg|\int_{\varepsilon}^\infty b(t/|\xi|)t^{-1}\int_{S^{n-1}} a(y')e^{-it\langle\xi',y'\rangle}\,d\sigma(y')\,dt\Bigg|^2\varepsilon^{-1}\,d\varepsilon\\
&\quad= I_1 + I_2 + I_3.
\end{aligned}$$

Using the method of rotation again and the cancellation condition of F_a we have

$$I_1 \le C\|b\|_\infty^2 \int_0^{1/r} \left| \int_0^\varepsilon t^{-1} \left| \int_{\mathbb{R}} F_a(s)\{e^{-its} - 1\}\, ds \right| dt \right|^2 \varepsilon^{-1}\, d\varepsilon$$
$$\le C\|b\|_\infty^2 \int_0^{1/r} r^2 \varepsilon\, d\varepsilon = C\|b\|_\infty^2.$$

$$I_2 \le C\|b\|_\infty^2 \left\{ \int_0^\infty t^{-1} \left| \int_{\mathbb{R}} F_a(s) e^{-its}\, ds \right| dt \right\}^2$$
$$\le C\|b\|_\infty^2 \left\{ \int_0^\infty t^{-1} |\hat{F}_a(t)|\, dt \right\}^2 \le C\|b\|_\infty^2.$$

The last inequality is the classical Hardy inequality (page 128 in [**St**]), since F_a is an atom on $\mathbb{R}$

$$I_3 \le C\|b\|_\infty^2 \int_{2/r}^\infty \left\{ \int_\varepsilon^\infty |t^{-1}\hat{F}_a(t)|\, dt \right\}^2 \varepsilon^{-1}\, d\varepsilon$$
$$\le C\|b\|_\infty^2 \int_{2/r}^\infty \left\{ \int_\varepsilon^\infty t^{-p}\, dt \right\}^{2/p} \varepsilon^{-1}\, d\varepsilon \|\hat{F}_a\|_q^2.$$

Thus by the Hausdorff-Young inequality we have

$$I_3 \le C\|b\|_\infty^2 r^{2-2/p} \|F_a\|_p^2 \le C\|b\|_\infty^2.$$

Clearly the constant C in the above estimates is independent of the essential variables and functions. The proposition is proved.

Now we return to prove (3.6). Let

$$W(t,\xi) = \int_{S^{n-1}} a(y') e^{-it|\xi|\langle \xi', y' \rangle}\, d\sigma(y') b(t).$$

Then

$$m_\varepsilon(\xi) = C \int_\varepsilon^\infty W(t,\xi) t^{-1}\, dt.$$

Thus

$$\begin{aligned} \frac{1}{s}\int_0^s m_\varepsilon(\xi)\, d\varepsilon &= C\frac{1}{s}\int_0^s \int_\varepsilon^\infty W(t,\xi) t^{-1}\, dt\, d\varepsilon \\ &= \frac{C}{s}\left\{ \int_0^s \left(\int_0^t W(t,\xi)\, d\varepsilon \right) t^{-1}\, dt \right. \\ &\quad \left. + \int_0^s \int_s^\infty W(t,\xi) t^{-1}\, dt\, d\varepsilon \right\} \\ &= \frac{C}{s}\int_0^s W(t,\xi)\, dt + C m_s(\xi). \end{aligned}$$

Thus

$$\begin{aligned}&\sup_{\varepsilon>0}|T_{b,\varepsilon}f(x)|\\&\quad\le\sup_{s>0}\frac{1}{s}\int_0^s|T_{b,\varepsilon}f(x)|\,d\varepsilon+C\left\{\int_0^\infty|(W(t,\cdot)\hat f(\cdot))^v(x)|^2t^{-1}\,dt\right\}^{1/2},\end{aligned}$$

where $f^v(x)$ is the Fourier inverse of f. By Proposition (3.1) we only need to prove

$$J=\left\|\left\{\int_0^\infty|(W(t,\cdot)\hat f(\cdot))^v(x)|^2t^{-1}\,dt\right\}^{1/2}\right\|_2\le C\|b\|_\infty\,\|f\|_2.$$

Using Plancherel's Theorem, we have

$$J\le C\left\|\left\{\int_0^\infty|W(t,\cdot)|^2t^{-1}\,dt\right\}^{1/2}\hat f(\cdot)\right\|_2.$$

So it suffices to show

$$R(\xi)=\int_0^\infty|W(t,\xi)|^2t^{-1}\,dt\le C\|b\|_\infty^2 \tag{3.10}$$

with C independent of ξ, $b(\cdot)$ and $a(\cdot)$.

In fact,

$$R(\xi)\le C\|b\|_\infty^2\int_0^\infty\left|\int_{S^{n-1}}a(y')e^{-it\langle\xi',y'\rangle}\,d\sigma(y')\right|^2t^{-1}\,dt.$$

So using the same argument as in estimating (3.4), we have

$$\begin{aligned}R(\xi)&\le C\|b\|_\infty^2\int_0^\infty\left|\int_{\mathbb{R}}F_a(s)e^{-its}\,ds\right|^2t^{-1}\,dt\\&=C\|b\|_\infty^2\left\{\int_{1/r}^\infty t^{-1}|\hat F_a(t)|^2\,dt\right.\\&\qquad\left.+\int_0^{1/r}t^{-1}\left|\int_{\mathbb{R}}F_a(s)\{e^{-its}-1\}\,ds\right|\,dt\right\}\\&\le C\|b\|_\infty^2\{1+r^{1/2}\|\hat F_a\|_4^2\}\\&\le C\|b\|_\infty^2\{1+r^{1/2}\|F_a\|_{4/3}^2\}\le C\|b\|_\infty^2\end{aligned}$$

where C is independent of all the essential variables and functions. The proof of Theorem is complete. ■

Proof of Theorem 2

To prove (1.8), using the atomic decomposition it suffices to show

$$\int_{\mathbb{R}^n} |x|^{-n} \left| \int_{S^{n-1}} e^{iQ(|x|)\langle x',\xi'\rangle} a(\xi')\, d\sigma(\xi') \right| dx \le C \tag{4.1}$$

where C is independent of the atom $a(\cdot)$. By the polar coordinate, we can see that the above integral (4.1) is equal to

$$\int_{S^{n-1}} \int_0^\infty t^{-1} \left| \int_{S^{n-1}} e^{iQ(t)\langle x',\xi'\rangle} a(\xi')\, d\sigma(\xi') \right| dt\, d\sigma(x').$$

So similar to the proof in Theorem 1, by the method of rotation, the last integral is equal to

$$\int_{S^{n-1}} \int_0^\infty t^{-1} \left| \int_{\mathbb{R}} F_a(s) e^{isQ(t)}\, ds \right| dt\, d\sigma(x')$$

where F_a is the function defined in Lemma 2.1 or Lemma 2.2, that depends on $x' \in S^{n-1}$. Now following the estimate of (3.4'), we easily obtain that

$$\int_0^\infty t^{-1} \left| \int_{\mathbb{R}} F_a(s) e^{iQ(t)s}\, ds \right| dt \le C$$

with the constant C independent of x'.

To prove the second part of Theorem 2, for simplicity, we only show the case of $n > 2$. In this case, we do not need to use Lemma 2.2 to estimate the L^q norm of F_a.

By the atomic decomposition of $\Omega \in H^p$, it suffices to show

$$\int_{\mathbb{R}^n} |x|^{-1+(p-2)(n-1)} \left| \int_{S^{n-1}} e^{i\langle x,\xi'\rangle} a(\xi')\, d\sigma(\xi') \right|^p dt \le C, \tag{4.2}$$

where the constant C is independent of any (p,∞) atom $a(\cdot)$.

Using the polar coordinate and the method of rotation, we only need to prove

$$I = \int_0^\infty t^{(n-1)p-n} \left| \int_{\mathbb{R}} F_a(s) e^{its}\, ds \right|^p dt \le C. \tag{4.3}$$

Without loss of generality, we may assume $\mathrm{supp}(F_a) \subseteq (-r,r)$. Now we write

$$I = \left\{ \int_0^{1/r} + \int_{1/r}^\infty \right\} t^{(n-1)p-n} \left| \int_{\mathbb{R}} F_a(s) e^{its}\, ds \right|^p dt = I_1 + I_2.$$

In I_1, choosing an integer $N > (1/p-1)(n-1)$ and using the conditions (2.12) and (2.13), we have

$$\left|\int_{\mathbb{R}} F_a(s)e^{its}\,ds\right| \le C(tr)^N \rho^{(1-1/p)(n-1)}.$$

Noting $r \le C\rho$, so we further have

$$\left|\int_{\mathbb{R}} F_a(s)e^{its}\,ds\right| \le Ct^N r^{N+(1-1/p)(n-1)}.$$

Now it is easy to see that

$$I_1 \le Cr^{p\{N+(n-1)(1-1/p)\}} \int_0^{1/r} t^{-1} t^{p\{N+(1-1/p)(n-1)\}}\,dt \le C.$$

To estimate I_2, by Hölder's inequality,

$$I_2 \le \|\hat{F}_a\|_2^p \left(\int_{1/r}^{\infty} t^{2(n-np+p)/(p-2)}\,dt\right)^{(2-p)/2}.$$

By (2.12) we have

$$\|\hat{F}_a\|_2^p \le C\|F_a\|_2^p \le Cr^{-1/2}\rho^{p(n-1)-(n-1)}.$$

So we easily obtain that $I_2 \le C$ since $r \le C\rho$. The theorem is proved. ■

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