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L^2 harmonic forms and stability of minimal hypersurfaces

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0. Introduction.

The Bernstein conjecture states that any complete minimal graph in E^{m+1} is a hyperplane. This was proved to be true for $m\leqq 7$ by Bernstein [**2**] ($m=2$), De Giorgi [**5**] ($m=3$), Almgren [**1**] ($m=4$) and Simons [**12**] ($m\leqq 7$); and false for $m\geqq 8$ by Bombieri, De Giorgi and Giusti [**3**].

On the other hand, do Carmo and Peng [**6**] and Fischer-Colbrie and Schoen [**8**] showed that complete orientable and stable minimal surfaces in E^3 are planes. Palmer [**10**] studied a topological restriction of a complete minimal hypersurface M in E^{m+1} which implies instability of M. This topological restriction is related to the existence of nonzero L^2 harmonic 1-forms by Dodziuk's result [**7**].

We denote the space of all L^2 harmonic p-forms on a complete orientable Riemannian manifold M by $\mathcal{H}^p(M)$. $\mathcal{H}^p(M)$ consists of p-forms which are closed and coclosed by a theorem of Andreotti and Vesentini. It is well known that $\mathcal{H}^p(E^m)=\{0\}$ for all p; $0\leqq p\leqq m$.

A complete minimal graph in E^{m+1} is minimizing, and any minimizing minimal hypersurface is stable. Therefore, concerning the Bernstein problem we can pose the following problem: *For a complete orientable and stable minimal hypersurface M in E^{m+1}, does $\mathcal{H}^p(M)=\{0\}$ hold for all p; $0\leqq p\leqq m$.*

The case where $m=2$ is trivial by the result of do Carmo and Peng, Fischer-Colbrie and Schoen as stated in the above. A catenoid shows that the assumption of stability in our problem is essential. Here we have the following:

THEOREM A. *Let $M\subset E^{m+1}$ be a complete orientable and stable minimal hypersurface. If $m\leqq 4$, then $\mathcal{H}^p(M)=\{0\}$ holds for all p; $0\leqq p\leqq m$.*

Palmer [**10**] used the norm of an L^2 harmonic 1-form on M to define a variation vector field and proved that a complete orientable minimal hypersurface $M\subset E^{m+1}$ admitting a nontrivial L^2 harmonic 1-form is unstable. So, Theorem A for the case where $m=3$ is due to Palmer [**10**]. To prove Theorem A it suffices to show the following: Let $M\subset E^5$ be a complete, orientable minimal hypersurface;

If M admits a nontrivial L^2 harmonic 2-form, then M is unstable (Theorem 2.1).

In § 3 we give definition of P_2 positivity of sectional curvature, as a condition weaker than positivity of sectional curvature. In § 4 (Theorem 4.1) we prove the following:

THEOREM B. *Let M be a complete noncompact orientable minimal hypersurface of a Riemannian manifold M^*. We assume that the sectional curvature of M^* is P_2 nonnegative. If M is stable, then there are no nontrivial L^2 harmonic 1-forms on M.*

This is a generalization of Palmer's result [**10**] (where $M^*=E^{m+1}$) and Miyaoka's result [**9**] (where M^* is of nonnegative sectional curvature).

For the case where $m=2$, complete stable minimal surfaces of M^* with nonnegative scalar curvature were classified by Fischer-Colbrie and Schoen [**8**], Schoen and Yau [**11**].

The author is very grateful to the referee whose comments on the infiniteness of the volume of M simplified the proof of Theorem 2.1.

1. L^2 harmonic 2-forms.

Let w be an L^2 harmonic 2-form on a complete orientable Riemannian manifold $M=(M, g)$. It is known that w is closed and coclosed. The Riemannian curvature tensor, the Ricci curvature tensor and the Riemannian connection are denoted by $R=(R^i_{jkl})$, $\rho=(R_{jl})$ and ∇. The expression of Δw is given by

$$\Delta w_{ij} = \nabla^r\nabla_r w_{ij} - R^r_i w_{rj} - R^r_j w_{ir} + R^{rs}{}_{ij} w_{rs} = 0.$$

Putting $\|w\|^2 = w_{rs}w^{rs}$ and $\|\nabla w\|^2 = \nabla_r w_{ij}\nabla^r w^{ij}$, we obtain

$$\Delta\|w\|^2 = 4(\rho\,;\,w,\,w) - 2\langle[R]w,\,w\rangle + 2\|\nabla w\|^2, \tag{1.1}$$

where $[R]$ denotes the curvature operator and

$$(\rho\,;\,w,\,w) = R_{rs}w^r{}_i w^{si}, \quad \langle[R]w,\,w\rangle = R^{rs}{}_{ij} w_{rs} w^{ij}.$$

On the other hand, we obtain

$$\begin{aligned}\Delta\|w\|^2 &= 2\|w\|\Delta\|w\| + 2\|\nabla\|w\|\|^2 \\ &= 2\|w\|\Delta\|w\| + 2\|\nabla w\|^2 - 2F(w),\end{aligned} \tag{1.2}$$

where we have put $F(w) = \|\nabla w\|^2 - \|\nabla\|w\|\|^2$. $F(w) \geqq 0$ is Kato's inequality. The equality $F(w)=0$ holds, if and only if

$$2\|w\|^2\nabla_k w_{ij} = \nabla_k\|w\|^2 \cdot w_{ij}, \tag{1.3}$$

i.e., $w/\|w\|$ is parallel, if $\|w\| \neq 0$. By (1.1) and (1.2) we get

(1.4) $$-\|w\|\Delta\|w\| = -2(\rho\,;w,w)+\langle[R]w,w\rangle-F(w).$$

Next we prove the following Lemma for later use in the next section:

LEMMA 1. *Let A, B be $m\times m$ real matrices such that*

(i) *A is symmetric and trace free,*

(ii) *B is skew-symmetric.*

If $2\leqq m\leqq 4$, then $\|A\|^2\|B\|^2+2\mathrm{Tr}(AB)^2+2\mathrm{Tr}\,A^2B^2\geqq 0$.

PROOF. First we diagonalize A to the form $(a_i\delta_{ij})$ by an orthogonal transformation. Let $B=(b_{ij})$. Then we have the following:

$$\|A\|^2\|B\|^2=(\sum a_i^2)(\sum_{i\neq j}b_{ij}^2),$$

$$\mathrm{Tr}(AB)^2=-\sum_{i\neq j}a_ia_jb_{ij}^2,$$

$$\mathrm{Tr}\,A^2B^2=-\sum_{i\neq j}a_i^2b_{ij}^2.$$

Therefore, we obtain

$$\|A\|^2\|B\|^2+2\mathrm{Tr}(AB)^2+2\mathrm{Tr}\,A^2B^2$$
$$=2b_{12}^2[a_3^2+a_4^2+\cdots+a_m^2-2a_1a_2]+2b_{13}^2[\cdots]+\cdots+2b_{m-1m}^2[\cdots].$$

By assumption $\mathrm{Tr}\,A=0$ we have

$$a_3^2+a_4^2+\cdots+a_m^2-2a_1a_2=a_3^2+a_4^2+\cdots+a_m^2+2a_2^2+2a_2a_3+\cdots+2a_2a_m.$$

Thus, for $m=2, 3, 4$ we obtain $-2a_1a_2=2a_2^2\geqq 0$, and

$$a_3^2-2a_1a_2=a_2^2+(a_2+a_3)^2\geqq 0,$$

$$a_3^2+a_4^2-2a_1a_2=(a_2+a_3)^2+(a_2+a_4)^2\geqq 0,$$

etc.. So, the proof is completed.

REMARK. Note that, if $m=5$, for example:

$$a_3^2+a_4^2+a_5^2-2a_1a_2=-3<0$$

holds for $a_1=3$, $a_2=2$, $a_3=a_4=-2$, $a_5=-1$. Thus, considering a matrix B with components ($b_{12}=-b_{21}\neq 0$, otherwise $=0$), we see that the condition $m\leqq 4$ in Lemma 1 is essential.

2. Minimal hypersurfaces of a Riemannian manifold.

Let M be an orientable minimal hypersurface of an $(m+1)$-dimensional orientable Riemannian manifold $M^*=(M^*,\langle\,,\rangle)$. Let n be a unit normal vector field on M and let A denote the shape operator with respect to n. We denote the Riemannian curvature tensor and the Ricci curvature tensor of M^* by R^* and ρ^*.

We assume that M admits a nontrivial L^2 harmonic 2-form w. After Palmer [**10**] we use the following cut off function h. Let p be a point of M. By $B_r(p)$ we denote the geodesic r-ball centered at p (r-neighborhood of p in M). h is a smooth function such that $0 \leqq h \leqq 1$ and

(i) $h=1$ on $B_{r/2}(p)$ and $h=0$ outside $B_r(p)$,

(ii) $\|\nabla h\|^2 \leqq c/r^2$, where c is a constant.

We consider $V=h\|w\|n$ as a variation vector field of M. Then the second variation formula is given by

$$a''(0) = \int_M (\|\nabla^\perp V\|^2 - \rho^*(V, V) - \|A\|^2\|V\|^2)$$

$$= -\int_M h^2(\|w\|\Delta\|w\| + \rho^*(n, n)\|w\|^2 + \|A\|^2\|w\|^2) + \int_M \|\nabla h\|^2\|w\|^2.$$

By (1.4) we obtain

$$a''(0) = -\int_M h^2[2(\rho\,;w, w) - \langle [R]w, w\rangle + F(w) + \rho^*(n, n)\|w\|^2 + \|A\|^2\|w\|^2] + \int_M \|\nabla h\|^2\|w\|^2.$$

Let $\{e_j, e_0=n\,; 1 \leqq j \leqq m\}$ be a local orthonormal frame along M. Then

$$R^i_{jkl} = R^{*i}_{jkl} + A^i_k A_{jl} - A^i_l A_{jk}\,, \tag{2.1}$$

$$R_{jl} = R^*_{jl} - R^{*0}_{j0l} - A^r_j A_{rl}\,. \tag{2.2}$$

Therefore we obtain

$$(\rho\,; w, w) = (\rho^*\,; w, w) - (R^*_0\,; w, w) - \langle Aw, Aw\rangle,$$

$$\langle [R]w, w\rangle = \langle [R^*]w, w\rangle - 2A^{ik}w_{kl}A^{lj}w_{ji}\,,$$

where $(\rho^*\,; w, w) = \rho^*_{jl}w^{jr}w^l{}_r$, $(R^*_0\,; w, w) = R^{*0}_{j0l}w^j{}_r w^{lr}$, and

$$\langle Aw, Aw\rangle = A^i_r w^{rt} A_{is} w^s{}_t\,, \quad \langle [R^*]w, w\rangle = R^*_{ijkl}w^{ij}w^{kl}.$$

Consequently, we get

$$a''(0) = -\int_M h^2[P^*(w) + F(w) + D(w)] + \int_M \|\nabla h\|^2\|w\|^2,$$

where we have put

$$P^*(w) = 2(\rho^*\,; w, w) - 2(R^*_0\,; w, w) - \langle [R^*]w, w\rangle + \rho^*(n, n)\|w\|^2 \tag{2.3}$$

$$D(w) = \|A\|^2\|w\|^2 - 2\langle Aw, Aw\rangle + 2A^{ik}w_{kl}A^{lj}w_{ji}\,. \tag{2.4}$$

Now we assume that M^* is the $(m+1)$-dimensional Euclidean space E^{m+1}. Then we have

$$a''(0) = -\int_M h^2[F(w)+D(w)]+\int_M \|\nabla h\|^2\|w\|^2. \tag{2.5}$$

THEOREM 2.1. *Let $M\subset E^5$ be a complete, orientable minimal hypersurface. If M admits a nontrivial L^2 harmonic 2-form, then M is unstable.*

PROOF. We assume that a complete orientable minimal hypersurface M in E^5 is stable and M admits a nontrivial L^2 harmonic 2-form w. Lemma 1 implies that $D(w)\geqq 0$ holds on M. Then (2.5) implies the following:

$$0\leqq a''(0)\leqq -\int_{B_{r/2}(p)}[F(w)+D(w)]+(c/r^2)\int_M \|w\|^2.$$

Letting $r\to\infty$, we obtain $F(w)=D(w)=0$. The equality $F(w)=0$ shows (1.3). We consider (1.3) on an open set where $w\neq 0$. $\delta w=0$ implies that $w^k{}_j\nabla_k\|w\|^2=0$ holds. $dw=0$ is equivalent to $\nabla_k w_{ij}+\nabla_i w_{jk}+\nabla_j w_{ki}=0$. By (1.3) and the last equality multiplied by w^{ij}, we get $\nabla_k\|w\|^2=0$, and hence $\|w\|$ is constant. This is a contradiction, because the volume of any complete minimal hypersurface of E^{m+1} is infinite (cf. Burago and Zalgaller [**4**], p. 215). q.e.d.

PROOF OF THEOREM A. It suffices to show the case $m=4$. By the assumption that $M\subset E^5$ is stable we see that $\mathcal{H}^1(M)=\{0\}$ holds by Palmer's result [**10**] and $\mathcal{H}^3(M)=\{0\}$ holds by the duality. On the other hand we have $\mathcal{H}^0(M)=\mathcal{H}^4(M)=\{0\}$. Combining these with Theorem 2.1, proof is completed.

REMARK. It is not clear to the author if there is some nice curvature condition of M^* which implies $P^*(w)\geqq 0$. If there is some, then one may obtain some generalized version of Theorem 2.1.

3. P_2 positivity of sectional curvature.

Let $M=(M, g)$ be an m-dimensional Riemannian manifold. Let p be a point of M and let $\{e_1, e_2, \cdots, e_m\}$ be an orthonormal basis of the tangent space T_pM at p. By K_{rs} we denote the sectional curvature; $K(e_r, e_s)$. The scalar curvature of M is denoted by S.

DEFINITION. The sectional curvature of M is said to be P_2 *positive* (P_2 *nonnegative*), if for any point p of M and for any orthonormal pair $\{X, Y\}$ at p

$$\rho(X, X)+\rho(Y, Y)-K(X, Y)>0 \quad (\geqq 0, \text{ resp.}).$$

It is clear that positivity of sectional curvature of M implies P_2 positivity of sectional curvature of M.

If $m=2$, then $\rho(e_1, e_1)+\rho(e_2, e_2)-K_{12}=\rho(e_1, e_1)=S/2$, and P_2 positivity is equivalent to the fact that the scalar curvature is positive.

If $m=3$, then $\rho(e_1, e_1)+\rho(e_2, e_2)-K_{12}=S/2$, etc., and P_2 positivity is equivalent to the fact that the scalar curvature is positive.

If $m\geqq 4$, for i, j $(1\leqq i<j\leqq m)$ we define P_{ij} by

$$P_{ij}=\sum_{r,s\neq i,j,r<s} K_{rs}.$$

Then the scalar curvature S is given by ${}_{m-2}C_2 S=2\sum_{i<j}P_{ij}$, and

$$\rho(e_i, e_i)+\rho(e_j, e_j)-K_{ij}=S/2-P_{ij}.$$

Therefore, P_2 positivity is expressed by $S/2-P_{ij}>0$.

For example, if $m=5$, we have $S=2\sum_{i<j}K_{ij}$, and $P_{12}=K_{34}+K_{35}+K_{45}$. So,

$$S-2P_{12}=2(K_{12}+K_{13}+K_{14}+K_{15}+K_{23}+K_{24}+K_{25}).$$

Here we give a simple example of a Riemannian manifold which is not of positive sectional curvature, but the sectional curvature is P_2 positive. Let (S^m, g_o) be the odd dimensional unit sphere in E^{m+1}, $m=2n+1$. It admits a standard contact structure η. We define a Riemannian metric $g(t)$ by

$$g(t)=t^{-1}g_o+t^{-1}(t^m-1)\eta\otimes\eta,$$

where t is a real number which is determined later. For $t\geqq 1$, the sectional curvature and the Ricci curvature are bounded as follows:

$$t(4-3t^m)\leqq K_{(t)}(X, Y)\leqq t^{m+1},$$
$$(m+1)t-2t^{m+1}\leqq \rho_{(t)}(X, X)\leqq (m-1)t^{m+1}$$

for any orthonormal pair $\{X, Y\}$ with respect to $g(t)$ (cf. S. Tanno [**13**], [**14**], etc.). Let ε be a small positive number, and define t by $-\varepsilon=4-3t^m$. Then for any orthonormal pair $\{X, Y\}$, we obtain

$$\begin{aligned}\rho_{(t)}(X, X)+\rho_{(t)}(Y, Y)-K_{(t)}(X, Y) &\geqq 2(m+1)t-5t^{m+1}\\ &=[2m+2-(20/3)-(5/3)\varepsilon]t.\end{aligned}$$

The above is positive by suitable choice of ε; while some sectional curvature is negative.

4. L^2 harmonic 1-forms.

Let M be a complete orientable minimal hypersurface of an orientable Riemannian manifold $M^*=(M^*, \langle\,,\rangle)$. We assume that M admits a nontrivial L^2 harmonic 1-form u. We consider the variation of M by $V=h\|u\|n$ as Palmer and Miyaoka did. We put

$$W(u) = \|A\|^2\|u\|^2 - \langle Au, Au\rangle \geqq 0,$$

$$E(u) = \|\nabla u\|^2 - \|\nabla\|u\|\|^2 \geqq 0,$$

where $\|u\|^2 = u_i u^i$. Since $\mathrm{Tr}\,A=0$, $W(u)=0$ holds, if and only if A or u vanishes. $E(u)=0$ holds, if and only if $2\|u\|^2\nabla_i u_j = \nabla_i\|u\|^2 \cdot u_j$.

On the other hand, the second variation formula is given by

$$a''(0) = -\int_M h^2[\|u\|\Delta\|u\| + \rho^*(n, n)\|u\|^2 + \|A\|^2\|u\|^2] + \int_M \|\nabla h\|^2\|u\|^2.$$

Corresponding to (1.1) and (1.2), we obtain

$$\Delta\|u\|^2 = 2\rho(u^\#, u^\#) + 2\|\nabla u\|^2,$$

$$\Delta\|u\|^2 = 2\|u\|\Delta\|u\| + 2\|\nabla u\| - 2E(u),$$

where $u^\#$ denotes the vector field dual to u with respect to the Riemannian metric. Hence, using (2.2) we obtain

$$-\|u\|\Delta\|u\| = -\rho(u^\#, u^\#) + \langle R^*(n, u^\#)u^\#, n\rangle + \langle Au, Au\rangle - E(u).$$

Therefore, we obtain

$$a''(0) = -\int_M h^2[Q^*(u) + E(u) + W(u)] + \int_M \|\nabla h\|^2\|u\|^2,$$

where we have put

$$Q^*(u) = \rho^*(u^\#, u^\#) - \langle R^*(n, u^\#)u^\#, n\rangle + \rho^*(n, n)\|u\|^2.$$

THEOREM 4.1. *Let $M \subset M^*$ be a complete orientable and stable minimal hypersurface. We assume that the sectional curvature of M^* is P_2 nonnegative.*

(i) *If M is compact and if M admits a nontrivial harmonic 1-form u, then u is parallel, M is totally geodesic in M^* and $\rho^*(n, n)=0$.*

(ii) *If M is noncompact, then there are no nontrivial L^2 harmonic 1-forms on M.*

PROOF. We assume that M is noncompact and there is a nontrivial L^2 harmonic 1-form u. P_2 nonnegativity of the sectional curvature of M^* shows $Q^*(u) \geqq 0$. So we have

$$0 \leqq a''(0) \leqq -\int_{B_{r/2}(p)} [Q^*(u) + E(u) + W(u)] + (c/r^2)\int_M \|u\|^2.$$

Letting $r \to \infty$, we have $Q^*(u)=E(u)=W(u)=0$. The equality $E(u)=0$ implies $2\|u\|^2\nabla_i u_j = \nabla_i\|u\|^2 u_j$. So $\delta u=0$ implies $u^i\nabla_i\|u\|^2=0$. Furthermore, $du=0$ implies $\|u\|$ is constant and u is parallel. $W(u)=0$ implies that M is totally geodesic in M^*. We check the condition $Q^*(u)=0$. (2.2) means $Q^*(u)=\rho(u^\#, u^\#)+\rho^*(n, n)\|u\|^2=0$. Since u is parallel, $\rho(u^\#, u^\#)=0$, and hence $\rho^*(n, n)=0$. Let

e be an arbitrary unit tangent vector to M. P_2 nonnegativity for an orthonormal pair $\{e, n\}$ implies

$$\rho^*(e, e)+\rho^*(n, n)-K^*(e, n) \geqq 0.$$

By (2.2) and $\rho^*(n, n)=0$, the left hand side of the last inequality is equal to $\rho(e, e)$. Thus the Ricci curvature of M is nonnegative. Because M is complete and noncompact, the volume of M is infinite. This is a contradiction.

Next, if M is compact, it suffices to put $h=1$ in the above discussion.

REMARK. By Dodziuk's result [**7**] the existence of a nontrivial L^2 harmonic 1-form follows from a topological condition that there exists a cycle of codimension one in M which does not disconnect M (cf. Palmer [**10**]).

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