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L^2 HARMONIC FORMS ON ROTATIONALLY SYMMETRIC RIEMANNIAN MANIFOLDS

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ABSTRACT. The paper contains a vanishing theorem for L^2 harmonic forms on complete rotationally symmetric Riemannian manifolds. This theorem requires no assumptions on curvature.

This paper gives necessary and sufficient conditions for existence of L^2 harmonic forms on a special class of Riemannian manifolds. Manifolds of this class were called models by Greene and Wu and played a crucial part in the study of function theory on open manifolds [GW]. Throughout the paper M will denote a model of dimension n, i.e. a C^∞ Riemannian manifold such that:

(1) there exists a point $o \in M$ for which the exponential mapping is a diffeomorphism of T_oM onto M;

(2) every linear isometry φ: $T_oM \to T_oM$ is realized as the differential of an isometry Φ: $M \to M$, i.e., $\Phi(o) = o$ and $\Phi_*(o) = \varphi$.

Clearly, M is complete and can be identified with T_oM via $\exp_o$. In terms of geodesic polar coordinates $(r, \theta) \in (0, \infty) \times S^{n-1} \cong M \setminus \{o\}$ the Riemannian metric ds^2 of M can be written as

$$ds^2 = dr^2 + f(r)^2\, d\theta^2, \tag{3}$$

where $d\theta^2$ denotes the standard metric on S^{n-1} and the function $f(r)$ is C^∞ on $[0, \infty)$ and satisfies

$$f(0) = 0, \quad f'(0) = 1, \quad f(r) > 0 \quad \text{for } r > 0 \tag{4}$$

(cf. [S, pp. 179–183]).

Complete description of the spaces $\mathcal{H}^*(M)$ of L^2 harmonic forms is contained in the following

THEOREM. *Let M be a model of dimension n* $\geqslant 2$. *Then*

$$\text{(i)} \qquad \mathcal{H}^q(M) = \{0\} \quad \textit{for } q \neq 0, n/2, n,$$

$$\text{(ii)} \qquad \mathcal{H}^0(M) \cong \mathcal{H}^n(M) \cong \begin{cases} \{0\} & \textit{if } \int_0^\infty f(r)^{n-1}\, dr = \infty, \\ \mathbf{R} & \textit{if } \int_0^\infty f(r)^{n-1}\, dr < \infty, \end{cases}$$

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$$\text{(iii)} \qquad \mathcal{H}^k(M) = \{0\} \quad \text{if } n = 2k \text{ and } \int_1^\infty \frac{ds}{f(s)} = \infty,$$

$\mathcal{H}^k(M)$ is a Hilbert space of infinite dimension if $n = 2k$ and

$$\int_1^\infty \frac{ds}{f(s)} < \infty.$$

Remark. The integral in (ii) is a multiple of the volume of M. Finiteness of the integral $\int_1^\infty ds/f(s)$ implies that M is conformally equivalent to an open ball in $\mathbf{R}^n$. If $\int_1^\infty ds/f(s) = \infty$ then M is conformal to $\mathbf{R}^n$.

My interest in L^2 harmonic forms is motivated in part by the well known conjecture (cf. [**C**, p. 44]).

Conjecture 1. Let N be a compact Riemannian manifold of dimension $2k$. If the sectional curvature of N is nonpositive the Euler characteristic $\chi(N)$ satisfies $(-1)^k\chi(N) \geqslant 0$.

I. M. Singer suggested that in view of the L^2 index theorem [**A**] an appropriate vanishing theorem for L^2 harmonic forms on the universal covering of N would imply the conjecture. To see what sort of vanishing theorem to expect, I carried out an explicit computation in the case of constant negative curvature. It turned out that the same computation yielded a more general result which is the subject of this paper. The result itself is rather surprising since the curvature of M has no effect on existence of L^2 harmonic forms of degree $q \neq 0, n/2, n$. The vanishing in this range is a consequence of duality between forms of degree q and $n - q$. The general question of existence of nontrivial L^2 harmonic forms on open manifolds is a very difficult one. Nevertheless, I propose hesitantly the following:

Conjecture 2. Let M be a simply connected complete Riemannian manifold of dimension n and of nonpositive sectional curvature. Then there are no nonzero L^2 harmonic forms on M of degree $q \neq n/2$.

Conjecture 2 combined with the L^2 index theorem implies Conjecture 1. Indeed, the L^2 index theorem, applied to the operator $d + \delta$ whose index is equal to the Euler characteristic, states that L^2 harmonic forms on the universal covering $\tilde{N}$ of N can be used to reckon the Euler characteristic of N. More precisely $\chi(N)$ is equal to the alternating sum

$$\sum_1^{2k} (-1)^p \dim_{\pi_1(N)} \mathcal{H}^p(\tilde{N}),$$

where $\dim_{\pi_1(N)} \mathcal{H}^p(\tilde{N})$ is the normalized dimension of $\mathcal{H}^p(\tilde{N})$ with respect to the natural action of $\pi_1(N)$ on $\mathcal{H}^p(\tilde{N})$ (cf. [**A**]). Thus, if $\mathcal{H}^p(\tilde{N}) = \{0\}$ for $p \neq k$,

$$(-1)^k\chi(N) = \dim_{\pi_1(N)} \mathcal{H}^k(\tilde{N}) \geqslant 0.$$

The following example due to E. Calabi shows that one cannot expect to have $\mathcal{H}^q(M) = \{0\}$ for $q \neq 0, n/2, n$ for every manifold M satisfying (1). Let $(M_i, dr_i^2 + f_i(r_i)^2\, d\theta_i^2)$ be a model of dimension n_i, $i = 1,2$. Suppose that n_2 is even,

$$\int_0^\infty f_1(s)^{n_1-1}\,ds < \infty, \qquad \int_1^\infty \frac{ds}{f_2(s)} < \infty.$$

Then, according to the theorem $\mathcal{H}^q(M_1) \neq \{0\}$ for $q = 0, n_1$, $\mathcal{H}^q(M_2) \neq \{0\}$ when $q = n_2/2$. The Fubini theorem and the identity

$$\Delta_{M_1\times M_2} = \Delta_{M_1} \otimes I + I \otimes \Delta_{M_2}$$

imply that $\mathcal{H}^q(M_1 \times M_2) \neq \{0\}$ when $q = n_2/2, n_2/2 + n_1$.

The above construction cannot be used to produce a counterexample to Conjecture 2. In order that $M_1 \times M_2$, when equipped with the product metric, have nonpositive sectional curvature, M_1 and M_2 must have the same property. This would force the integral $\int_0^\infty f_1(s)^{n_1-1}\,ds$ to diverge since the volume of complete, simply connected Riemannian manifold of nonpositive sectional curvature is infinite.

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PROOF OF THEOREM. According to a theorem of Andreotti and Vesentini (cf. [**dR**, Theorem 26]) an L^2 form ω on M is harmonic if and only if it is closed and coclosed. Thus a C^∞ q-form ω is in $\mathcal{H}^q(M)$ if and only if

$$\int_M \omega \wedge *\omega < \infty, \qquad d\omega = 0, \quad d{*\omega} = 0, \tag{5}$$

where $*$ denotes the duality operator between forms of degree q and $n - q$. Since $*\omega \wedge *(*\omega) = \omega \wedge *\omega$ for every form ω, $*$ establishes an isomorphism between $\mathcal{H}^q(M)$ and $\mathcal{H}^{n-q}(M)$. Let dV denote the volume element of the Riemannian metric of M, and let $\langle\ ,\ \rangle$ and $|\ |$ be the pointwise inner product and norm, respectively, of differential forms on M. The global (integrated) inner product and norm are given by

$$(\omega, \eta) = \int_M \omega \wedge *\eta = \int_M \langle \omega, \eta\rangle\, dV,$$

$$\|\omega\|^2 = \int_M \omega \wedge *\omega = \int_M |\omega|^2\, dV,$$

where ω and η are two forms of equal degrees. Corresponding objects on S^{n-1} equipped with the standard metric will have to be considered. They will be denoted by the same symbols as their counterparts on M with a subscript 0. For example, the volume elements dV and dV_0 of M and S^{n-1}, respectively, are related by $dV = f(r)^{n-1}\, dV_0 \wedge dr$.

The case (ii) of the theorem is now trivial. If ω is an L^2 harmonic function $d\omega = 0$ by (5), i.e., ω is constant. Constants are in L^2 if and only if the total volume of M is finite, which gives (ii). To study the remaining cases one writes the conditions (5) in terms of geodesic polar coordinates (r, θ). If ω is a C^∞ q-form on $M \setminus \{o\}$ of degree $q \neq 0, n$, then

$$\omega = a(r, \theta) \wedge dr + b(r, \theta), \tag{6}$$

where $a(r,\theta)$, $b(r,\theta)$ are smooth forms on S^{n-1}, depending on a parameter $r > 0$, of degree $q-1$ and q, respectively. Formally $a = (-1)^{q-1}\,\iota(\partial/\partial r)\omega$, $b = \omega - a \wedge dr$, where $\iota(\partial/\partial r)$ is the interior product with the radial vector field $\partial/\partial r$. Of course, a and b can be also regarded as forms on $M \setminus \{o\}$.

In terms of decomposition (6) $*\omega$ can be computed as follows:

$$*\omega = (-1)^{n-p} f^{n-2q+1} *_0 a + f^{n-2q-1} *_0 b\, dr. \tag{7}$$

To prove this formula one uses the fact that $*$ consists essentially of taking orthogonal complement together with the identity

$$*_{\lambda^2 g} = \lambda^{n-2q} *_g, \tag{8}$$

relating duality operators on q-forms for two conformal metrics g and $\lambda^2 g$.

Using (5), (6) and (7) one concludes that for $\omega \in \mathcal{H}^q(M)$ the following conditions hold:

$$\begin{gathered}
\int_0^\infty \int_{S^{n-1}} \left(f^{n-2q+1}|a|_0^2 + f^{n-2q-1}|b|_0^2\right) dV_0\, dr < \infty,\\
d_0 b = 0, \qquad d_0 *_0 a = 0,\\
d_0 a + (-1)^q \frac{\partial b}{\partial r} = 0,\\
\frac{\partial}{\partial r}\left(f^{n-2q+1} *_0 a\right) + f^{n-2q-1} d_0 *_0 b = 0.
\end{gathered} \tag{9}$$

Moreover the pointwise norm $|\omega|^2$ is bounded near $r = 0$, i.e.,

$$|\omega|^2 = f^{-2(q-1)}|a|_0^2 + f^{-2q}|b|_0^2 \leq C \quad \text{for } r \in (0, 1].$$

Apply $*_0$ to the last equation in (9) and use commutativity

$$\frac{\partial}{\partial r} *_0 = *_0 \frac{\partial}{\partial r}$$

to obtain the following set of conditions satisfied by $\omega = a \wedge dr + b \in \mathcal{H}^q(M)$ on $M \setminus \{o\}$

$$\begin{aligned}
&\text{(a)} && d_0 b = 0,\\
&\text{(b)} && d_0 *_0 a = 0,\\
&\text{(c)} && d_0 a + (-1)^q \frac{\partial b}{\partial r} = 0,\\
&\text{(d)} && \frac{\partial}{\partial r}\left(f^{n-2q+1} a\right) + (-1)^q f^{n-2q-1} \delta_0 b = 0,\\
&\text{(e)} && f^{-2(q-1)}|a|_0^2 + f^{-2q}|b|_0^2 \leq C \quad \text{for } r \in (0, 1],\\
&\text{(f)} && \int_0^\infty \int_{S^{n-1}} \left(f^{n-2q+1}|a|_0^2 + f^{n-2q-1}|b|_0^2\right) dV_0\, dr < \infty,
\end{aligned} \tag{10}$$

where δ_0 is the formal adjoint of d_0 on S^{n-1}. Observe now that if $\omega \in \mathcal{H}^q(M)$ and $b \equiv 0$, then $a \equiv 0$. Indeed, if $b \equiv 0$, then, by (10b) and (10c), $a(r,\theta)$ is a harmonic form on S^{n-1} for every fixed $r > 0$. Since $0 \leq \deg a \leq n-2$,

$a(r, \theta)$ can be nonzero only if $q - 1 = \deg a = 0$, in which case $a(r, \theta)$ is independent of θ. On the other hand, by (10d),

$$\frac{\partial}{\partial r}(f^{n-1}a) = 0,$$

i.e., $a = C_1 f^{-(n-1)}$ which blows up at $r = 0$ contradicting (10e) unless $C_1 = 0$.

Now eliminate $a(r, \theta)$ from the system consisting of equations (10c) and (10d). Thus apply d_0 to (10d) and use commutativity $d_0 \partial/\partial r = (\partial/\partial r) d_0$ to obtain

$$f^{n-2q-1} d_0 \delta_0 b = \frac{\partial}{\partial r}\left(f^{n-2q+1}\frac{\partial b}{\partial r}\right).$$

Take the inner product (over S^{n-1}) of both sides of this equation with b keeping $r > 0$ fixed to see that

$$\left(\frac{\partial}{\partial r}\left(f^{n-2q+1}\frac{\partial b}{\partial r}\right), b\right)_0 = f^{n-2q-1}(\delta_0 b, \delta_0 b)_0 \geqslant 0.$$

Therefore

$$\frac{d}{dr}\left(f^{n-2q+1}\frac{\partial b}{\partial r}, b\right)_0 = \left(\frac{\partial}{\partial r}\left(f^{n-2q+1}\frac{\partial b}{\partial r}\right), b\right)_0 + f^{n-2q+1}\left(\frac{\partial b}{\partial r}, \frac{\partial b}{\partial r}\right)_0 \geqslant 0.$$

By (10e) and (4) $|b|_0^2 = O(r^{2q})$ for small r. Hence

$$\left(f^{n-2q+1}\frac{\partial b}{\partial r}, b\right)_0 = O(r^n).$$

It follows that

$$\frac{d}{dr}(b, b)_0 = 2\left(\frac{\partial b}{\partial r}, b\right)_0 \geqslant 0$$

for all $r > 0$, i.e. $\|b\|_0^2$ is a nondecreasing function of r. Now suppose $b \neq 0$. Since $\|b\|_0^2$ is monotone and

$$\infty \geqslant \|\omega\|^2 \geqslant \|b\|^2 = \int_0^\infty f^{n-2q-1}\|b\|_0^2\, dr,$$

the integral $\int_1^\infty f^{n-2q-1}\, dr$ is finite. Thus for $q \neq 0, n$, $\mathcal{H}^q(M) \neq \{0\}$ implies that $\int_1^\infty f^{n-2q-1}\, dr$ is finite. By duality $\mathcal{H}^q(M) \cong \mathcal{H}^{n-q}(M)$, i.e. if $\mathcal{H}^q(M) \neq \{0\}$, then the two integrals $\int_1^\infty f^{n-2q-1}\, dr$, $\int_1^\infty f^{-n+2q-1}\, dr$ are simultaneously finite. If $n = 2q$ the two integrands are the same. If, on the other hand, $n - 2q \neq 0$ then $(n - 2q - 1)(-n + 2q - 1) = 1 - (n - 2q)^2$. Thus either one of the exponents is equal to zero, or the two exponents have opposite signs. In both cases one of the integrals has to diverge, which proves that $\mathcal{H}^q(M) = \{0\}$ if $q \neq 0, n/2, n$. This still leaves the possibility that, for $n = 2k$, $\mathcal{H}^k(M) \neq \{0\}$ provided $\int_1^\infty f^{-1}\, dr < \infty$. Such is the case and, in fact, $\mathcal{H}^k(M)$ has infinite dimension. The last assertion will follow from the following:

LEMMA. *Let M be a model with the metric $ds^2 = dr^2 + f(r)^2\, d\theta^2$. Define*

$$R(r) = e^{\int_1^r ds/f(s)}.$$

Then the mapping F: $M \setminus \{o\} \to \mathbf{R}^n \setminus \{o\}$ given (in terms of polar geodesic coordinates (r, θ) on M and polar coordinates on $\mathbf{R}^n$) by $F(r, \theta) = (R, \theta)$ extends to a C^1 conformal diffeomorphism of M onto an open ball of (possible infinite) radius equal to $\int_1^\infty ds/f(s)$ centered at the origin. Moreover, F is C^∞ on $M \setminus \{o\}$.

REMARK. The lemma is due to Milnor [M] for $n = 2$, in which case F is C^∞ everywhere. The proof for $n > 2$ is essentially the same and will not be repeated here. If $n > 2$, the restriction of F to every plane through o is C^∞. It is likely that F is C^∞, but the regularity asserted in the lemma is sufficient for the purpose at hand.

To finish the proof of the theorem assume the lemma and suppose $\dim M = 2k$. By (8) the $*$ operator acting on forms of degree k depends only on the conformal structure. Thus all conditions in (5) are conformally invariant. Assume that $\int_1^\infty ds/f(s) < \infty$ and let B be the open ball in $\mathbf{R}^n$ of radius $\int_1^\infty ds/f(s)$. The space of all C^∞ k-forms η on $\mathbf{R}^n$ which satisfy the equations $d\eta = 0$, $d*\eta = 0$ ($*$ induced by the standard flat metric) has infinite dimension (e.g., if $h(y_1, y_2, \ldots, y_{k+1})$ is a nonconstant harmonic function on $\mathbf{R}^{k+1}$, then

$$\eta = d(h(x_k, x_{k+1}, \ldots, x_n)\, dx_1 \ldots dx_{k-1})$$

satisfies the two equations). Restrictions of such forms to B are clearly in L^2. Thus the space $\mathcal{H}$ of k-forms on B satisfying conditions (5) with respect to the flat metric has infinite dimension. By the lemma and the conformal invariance, the space $F^*\mathcal{H}$ consists of forms ω of degree k which are continuous, square integrable on M, C^∞ on $M \setminus \{o\}$ and satisfy $d\omega = d*\omega = 0$ on $M \setminus \{o\}$. Standard regularity theorem shows that every $\omega \in F^*\mathcal{H}$ is in fact C^∞ and harmonic at every point of M. It follows that F^* establishes an isomorphism between $\mathcal{H}$ and $\mathcal{H}^k(M)$ and that $\mathcal{H}^k(M)$ has infinite dimension.

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