PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 4, Number 2/2003, pp.000-000.

L.C.K-MANIFOLDS ON A TANGENT BUNDLE DETERMINED BY A RIEMANN, FINSLER OR LAGRANGE STRUCTURE

Radu MIRON*, Hiroshi ENDO**

* "Al.I.Cuza"University, Faculty of Mathematics, 6600 Iaşi Romania, E-mail : rmiron@uaic.ro
** Utsunomiya University, Yoto 7-1-2, Utsunomiya-shi Japan, E-mail: hsk-endo@cc.utsunomiya-u.ac.jp

We prove that any Riemann structure (*M, g*) determines a local conformal Kähler manifold on the tangent bundle (*TM*, π ,*M*). The Finsler structures and Lagrange structures have the same properties.

Key words : Lck-manifolds, Lagrange spaces

1. INTRODUCTION

As we know ([5], [7]), any Riemann structure R^n = (*M, g*) determines on the tangent bundle *TM* an almost Kähler structure (*TM*, $\overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}$). We prove that the lift:

$$\left[\mathbb{G} = g_{ij}(x)dx^i \otimes dx^j + e^{2\sigma(x)}g_{ij}(x)\delta y^i \otimes \delta y^j\right]$$

of g to *TM* together with the correspondent almost complex structure:

$$\left[\mathbb{F}(x,y) = -e^{2\sigma(x)}\frac{\partial}{\partial y^i}\otimes dx^i + e^{2\sigma(x)}\frac{\delta}{\delta x^i}\otimes \delta y^i\right]$$

generate a local conformal almost Kähler structure (L.c.k)-structure. The same property holds for the Finsler spaces and Lagrange spaces.

2. THE ALMOST KÄHLER MANIFOLD DETERMINED BY A RIEMANN SPACE

Let $R^n = (M, g(x))$ be a Riemann space. The tangent bundle (*TM*, π ,*M*) has $(x^i, y^i), i = 1,...,n = \dim M$ as canonical local coordinates.

On the manifold *TM* there exist two distributions *N* and *V*, such that the following direct sum of linear spaces holds at any point $u \in TM$

$$[T_u TM = N_u \oplus V_u]$$

The horizontal distribution *N* has a local adapted basis $\frac{\delta}{\delta x^i}, (i = 1,...,n)$:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j} \tag{2.1}$$

where:

$$N_i^j = \gamma_{ih}^j(x)y^h \tag{2.2}$$

and $\gamma_{ih}^j(x)$ are the Christoffel symbols of the metric tensor $g_{ij}(x)$ of R^n.

The vertical distribution *V* is integrable, so that it has the adapted basis:

$$\left[\frac{\partial}{\partial y^i}, (i = 1, ..., n)\right]$$

The dual basis of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ is given by $\left(dx^i, \delta y^i\right)$, where:

$$\delta y^i = dy^i + N_i^j(x, y)dx^j \tag{2.2'}$$

The Sasaki lift ([3], [4]) of the tensor metric $g_{ij}(x)$ to the manifold *TM* is defined by:

$$\overset{\circ}{\mathbb{G}}(x, y) = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^i \otimes \delta y^j \tag{2.3}$$

at any point $u = (x, y) \in TM$.

$\overset{\circ}{\mathbb{G}}$ is a Riemanian metric tensor on the manifold *TM*.

The horizontal distribution *N* determines a natural almost complex structure $\overset{\circ}{\mathbb{F}}$ on *TM*:

$$\overset{\circ}{\mathbb{F}}(x, y) = -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^i \tag{2.4}$$

It is a known fact that $\overset{\circ}{\mathbb{F}}$ is a complex structure if and only if the space R^n is locally flat.

Also, the following result is known ([5], [6], [7]).

Theorem 2.1.

1° The pair $\left(\overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}\right)$ *is an almost Hermitian structure on TM determined only by the Riemannian structure* $g(x)$.

2° The almost symplectic structure associated to $\left(\overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}\right)$ *is given by:*

$$\overset{\circ}{\theta} = g_{ij}(x)\delta y^i \wedge dx^j \tag{2.5}$$

The manifold $\left(TM, \overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}\right)$ is an almost Kählerian space, determined only by a Riemannian structure space $g(x)$ i.e. we have:

$$d\overset{\circ}{\theta} = 0 \tag{2.6}$$

Also, we have:

Theorem 2.2. *The manifold* $\left(TM, \overset{\circ}{\mathbb{G}}, \overset{\circ}{\mathbb{F}}\right)$ *is a Kähler manifold if and only if the Riemann space* $R^n = (M, g)$ *is locally flat.*

3. L.C.K-MANIFOLD DETERMINED BY RIEMANNIAN SPACE R^n

Let us consider on the manifold *M* the conformal change:

$$g(x) \to e^{2\sigma(x)} g(x)$$

of the metric tensor *g*, where $\sigma(x)$ is a differentiable function, locally defined. We denote:

$$\overline{g}_{ij}(x) = e^{2\sigma(x)} g(x) \tag{3.1}$$

On the manifold *TM* we locally define the tensor field:

$$\mathbb{G}(x,y) = g_{ij}(x) dx^i \otimes dx^j + \overline{g}_{ij}(x) \delta y^i \otimes \delta y^i \tag{3.2}$$

where $\overline{g}_{ij}$ is from (3.1). The distributions *N* and *V* are orthogonal with respect to $\mathbb{G}$.

Consider, also, the tensor field:

$$\mathbb{F} = -e^{-2\sigma(x)} \frac{\partial}{\partial y^i} \otimes dx^i + e^{2\sigma(x)} \frac{\delta}{\delta x^i} \otimes \delta y^i \tag{3.3}$$

The following properties of $\mathbb{F}$ hold:

Theorem 3.1.

1° $\mathbb{F}$ is locally defined on the manifold TM.

2° $\mathbb{F}$ depends only on the conformal structure defined by the Riemann structure $g(x)$ on the base manifold M.

3° $\mathbb{F}$ is a tensor field of type (1,1).

4° $\mathbb{F}$ is an almost complex structure.

5° $\mathbb{F}$ is a complex structure if and only if:

a. The space $R^n = (M, g)$ is locally flat and

b. The function $\sigma(x)$ is constant.

The proof of this theorem is not difficult. Also, we can prove:

Theorem 3.2.

1° The pair $(\mathbb{G}, \mathbb{F})$ is an almost Hermitian structure locally determined on TM by the conformal structure $\overline{g}(x)$.

2° The almost symplectic structure $\overline{\theta}$ associated to $(\mathbb{G}, \mathbb{F})$ is:

$$\overline{\theta} = e^{2\sigma(x)} \theta$$

3° The following property holds:

$$d\overline{\theta} = 0, \left(\text{modulo} \overline{\theta}\right)$$

4° The manifold $(TM, \mathbb{G}, \mathbb{F})$ is locally conformal with an almost Kähler manifold.

If R^n is locally flat, then $(TM, \mathbb{G}, \mathbb{F})$ is a local conformal with a Kähler manifold.

4. L.C.K-MANIFOLD ON TM DETERMINED BY A FINSLER STRUCTURE

Let $F^n = (M, \mathbb{F}(x,y))$ be a Finsler manifold, having $\mathbb{F}(x,y)$ as fundamental function, and:

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \forall (x,y) \in \widetilde{TM} = TM \setminus \{0\} \tag{4.1}$$

as fundamental tensor field. We denote by *N* the Cartan nonlinear connection, with coefficients:

$$N^i_j(x,y) = \frac{1}{2}\frac{\partial}{\partial y^j}\left(y^j_{rs}(x,y)y^r y^s\right) \tag{4.2}$$

γ^i_{js} being the Christoffel symbols of the fundamental tensor field $g_{ij}(x,y)$ of the space F^n.

The direct decomposition $T_u TM = N_u \oplus V_u$ holds and an adapted basis to N_u and V_u, $\forall u \in \widetilde{TM} = TM \setminus \{0\}$ is given by $\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} (i = 1,...,n)$, where $\frac{\delta}{\delta x^i}$ are defined by (2.1) and (4.2). Its dual basis is $(dx^i, \delta y^i)$, the 1-forms δy^i being expressed in (2.2)', (4.2).

The Sasaki-Matsumoto ([5], [7]) lift of $g_{ij}(x,y)$ to $\widetilde{TM}$ is:

$$\check{\mathbb{G}}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)dy^i \otimes dy^j \tag{4.3}$$

and the almost complex structure $\check{\mathbb{F}}$ determined by the Cartan nonlinear connection is given by (2.4), (4.2).

The following result is known ([5], [7]):

Theorem 4.1.

1° The pair $\left(\check{\mathbb{G}}, \check{\mathbb{F}}\right)$ *is an almost Hermitian structure determined only by the fundamental function* $\mathbb{F}(x,y)$.

2° The almost symplectic structure associated to $\left(\check{\mathbb{G}}, \check{\mathbb{F}}\right)$ *is given by the 2-form:*

$$\check{\theta}(x,y) = g_{ij}(x,y)\delta y^i \wedge dx^j$$

3° The structure $\check{\theta}$ *is symplectic, i.e.,* $d\check{\theta} = 0$.

4° The manifold $\left(\widetilde{TM}, \check{\mathbb{G}}, \check{\mathbb{F}}\right)$ *is almost Kählerian.*

Now, let us consider the conformal transformation:

$$\overline{\mathbb{F}}(x,y) = e^{2\sigma(x)}\mathbb{F}(x,y), \tag{4.4}$$

of the fundamental function $\mathbb{F}(x,y)$, where $\sigma(x)$ is a local differentiable function on the base manifold *M*.

With respect to (4.4) we have the conformal transformation:

$$\overline{g}_{ij}(x,y) = e^{2\sigma(x)}g_{ij}(x,y), \tag{4.5}$$

of the fundamental tensor field. Now, we consider the following lift of $g_{ij}(x,y)$ and $\overline{g}_{ij}(x,y)$:

$$\overline{\mathbb{G}}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + e^{2\sigma(x)} g_{ij}(x,y) dy^i \otimes dy^j \tag{4.6}$$

and the following tensor field of (1,1) type:

$$\overline{\mathbb{F}}(x,y) = -e^{2\sigma(x)} \frac{\partial}{\partial y^i} + e^{2\sigma(x)} \frac{\delta}{\delta x^i} \otimes \delta y^i \tag{4.7}$$

The proof of the following theorem is not hard.

Theorem 4.2.

1° The pair $\left(\overline{\mathbb{G}},\overline{\mathbb{F}}\right)$ is an almost Hermitian structure locally determined on the manifold $\widetilde{TM}$ by the fundamental functions $\mathbb{F}(x,y)$ and $\overline{\mathbb{F}}(x,y) = e^{2\sigma(x)}\mathbb{F}(x,y)$.

2° The associated almost symplectic structure $\overline{\theta}$ is:

$$\overline{\theta}(x,y) = e^{2\sigma(x)} \check{\theta}(x,y)$$

3° The following property holds:

$$d\overline{\theta} = 0, \text{ (modulo } \overline{\theta}\text{)}$$

4° $\left(\widetilde{TM},\overline{\mathbb{G}},\overline{\mathbb{F}}\right)$ *is a (L.c.k)-manifold.*

Finally, we remark that the previous theory can be extended, step by step, to Lagrange spaces $L^n = \left(M, L(x,y)\right)$ ([4], [5]).

REFERENCES

1. ENDO, H, *On Ricci curvatures of almost cosymplectic manifolds*, Analele Stiintifice Ale Univ. "Al.I.Cuza" Iasi Tomul XL, s.I.a, Mat., 1994, f1. 75-83.
2. ENDO, H, *On invariant submanifolds of cosymplectic manifolds*, Analele Stiintifice Ale Univ. "Al.I.Cuza" Iasi Tomul XLVII, s.I.a, Mat., 2001, f1. 143-152.
3. KASHIWADA, T, *Some properties of locally conformal Kähler manifolds*, Hokkaido Math. J. Vol 8, 1979, 191-198.
4. MIRON, R, *Lagrangian theory of Relativity*, Univ. Timisoara, 84, (1985), 53pg.
5. MIRON, R, ANASTASIEI, M, *The Geometry of Lagrange Spaces; Theory and Applications*, Kluwer Acad publ. FTPH No.59, 1994.
6. MIRON, R., MATSUMOTO, K, *The almost Hermitian structures determined by Riemannian structures on tangent bundle*, to appear in AGG-Hadronic Press.2002.
7. MIRON, R, HRIMIUC, D, SHIMADA, H. , SABĂU, V.S: *The Geometry of Hamilton and Lagrange spaces*, Kluwer Acad. Publ. FTPH. no.118, 2001.
8. VAISMAN, I., *On locally conformal almost Kähler manifolds*, Israel J. of Math., Vol 24, Nos. 3-4, 1976, 338-351.

Received February 3, 2003