Exactly as above this implies

$$\eta_j(x) = \varepsilon_1(x)^{a_{j1}}\varepsilon_2(x)^{a_{j2}}\dots\varepsilon_{m-1}(x)^{a_{j,m-1}} + (1+x+\dots+x^{p-1})f_j(x),$$

where $f_j(x)$ is a polynomial with rational integral coefficients and the determinant $|a_{jk}| = \pm 1$. This implies

$$\zeta\frac{\eta_j'(\zeta)}{\eta_j(\zeta)} = \sum_{k=1}^{m-1} a_{jk}\zeta\frac{\varepsilon_k'(\zeta)}{\varepsilon_k(\zeta)} + M_j(\zeta+2\zeta^2+\dots+(p-1)\zeta^{p-1}). \tag{4.1}$$

Now put

$$\zeta\frac{\eta_j'(\zeta)}{\eta_j(\zeta)} = \sum_{s=0}^{p-1} c_{js}'\zeta^{p^s} \qquad (j = 1, 2, \dots, m-1).$$

Then by (4.1) and (2.10) we have

$$c_{js}' \equiv \sum_{k=1}^{m-1} a_{jk}e_{js} + d_j g^s \pmod p \tag{4.2}$$

$$(j = 1, \dots, m-1;\ g = 0, 1, \dots, p-2).$$

Multiplying both sides of (4.2) by $g^{(2n-1)s}$ and summing over s we get

$$C_{jn}' \equiv \sum_{k=1}^{m-1} a_{jk}C_{kn} \pmod p, \tag{4.3}$$

where

$$C_{jn}' = \sum_{s=0}^{p-2} c_{js}'g^{(2n-1)s}.$$

It follows at once from (4.3) that

$$C' = |C_{jn}'| \equiv \pm C \pmod p. \tag{4.4}$$

References

[1] H. Hasse, *Vandiver's congruence for the relative class number of the p-th cyclotomic field*, J. of Mathematical Analysis and Applications 15 (1966), pp. 87-90.

[2] H. S. Vandiver, *On the first factor of the class number of a cyclotomic field*, Bulletin of the American Mathematical Society 25 (1919), pp. 458-461.

[3] — and G. E. Wahlin, *Algebraic numbers II*, Report of the Committee on Algebraic Numbers, Washington 1928.

Reçu par la Rédaction le 10. 4. 1967

ACTA ARITHMETICA
XIV (1968)

L-functions and character sums for quadratic forms (I)

by

H. M. Stark (Ann Arbor, Mich.)

1. Let $Q(x)$ be a positive definite quadratic form in n variables $x = (x_1, x_2, \dots, x_n)$ with integral coefficients, and let χ be a character $(\mathrm{mod}\,k)$. We define

$$L(s, \chi, Q) = \frac{1}{2}\sum_{x\neq 0}\chi(Q(x))Q(x)^{-s}, \tag{1}$$

the series converges to an analytic function if $\mathrm{Re}\,s > n/2$. This generalization of the Epstein zeta function has been, in the case of binary quadratic forms, closely related to class-number problems for the last thirty years. Recently [5], a rapidly convergent expansion of $L(s, \chi, Q)$ at $s = 1$ was derived for a particular positive definite binary quadratic form with the real character $\chi(j) = \left(\frac{k}{j}\right)$, $k = 8$ and 12. On the basis of this expansion it was shown in [5] that the number of classes of binary quadratic forms of discriminant < -163 is greater than one. Still, the functions $L(s, \chi, Q)$ have not been sufficiently studied for their own sake. Even in [5], since only two different L-functions were studied with the corresponding characters having relatively small moduli (8 and 12), arithmetic was sometimes able to take the place of a general theory. In this paper, we introduce a general L-function for positive definite quadratic forms in n variables. Under certain restrictions, $L(s, \chi, Q)$ can be extended to an entire function in the complex s plane which satisfies a functional equation. In this paper we derive that functional equation and the character identity on which it depends. In [6], we will show how an alternate form of our character identity leads, in general, to an expansion of $L(s, \chi, Q)$ at $s = 1$ similar to that in [5], but with the arithmetic eliminated. Much of the difficulty in the following comes from allowing k to be even; but if we wish to apply these results to [5], it is clear that we must put up with the extra difficulty.

2. Notation and statement of theorems. The letters x, y, z and only these letters will be reserved to be n-dimensional vectors of integers,

$$x = (x_1, x_2, \ldots, x_n), \qquad y = (y_1, y_2, \ldots, y_n), \qquad z = (z_1, z_2, \ldots, z_n).$$

An inequality such as $1 \leqslant x \leqslant k$ shall be interpreted as $1 \leqslant x_j \leqslant k$, $j = 1, 2, \ldots, n$ and

$$\sum_{x=1}^{k} \quad \text{shall mean} \quad \sum_{x_1=1}^{k}\sum_{x_2=1}^{k}\cdots\sum_{x_n=1}^{k}.$$

Similarly $x \equiv 0 \pmod k$ shall mean every $x_j \equiv 0 \pmod k$, $j = 1, \ldots, n$; $x \not\equiv 0 \pmod k$ naturally means that at least one of the $x_j \not\equiv 0 \pmod k$. We will write the usual inner product of two vectors as $(x, y) = x_1y_1 + \ldots + x_ny_n$, or if necessary, as $[x, y]$.

Unless otherwise stated, Q shall denote a quadratic form in n variables with integral coefficients (not necessarily positive definite). We shall now define the discriminant of Q. The various definitions of the discriminant differ in whether or not to include a factor of 2^n. Were it not for our desire to include even k in our theorems, it would not make any difference which definition we use. But for even k, our choice is made for us. We write

$$2Q(x) = (xF, x) \tag{2}$$

where F is a symmetric $n \times n$ matrix of integers. We define the discriminant d of Q to be

$$d = |F| \tag{3}$$

where $|F|$ stands for the determinant of F. As is well known, if Q is positive definite then $d > 0$. It will be assumed throughout that $(d, k) = 1$. Define $\bar{Q}(x)$ by

$$\bar{Q}(x) = \tfrac{1}{2}(xdF^{-1}, x). \tag{4}$$

If χ is a character $(\operatorname{mod} k)$, we put

$$\tau(\chi) = \sum_{j=1}^{k} \chi(j) e_k(j) \tag{5}$$

where for convenience we write

$$e_k(j) = e^{2\pi i j/k},$$

or if necessary, $e_k[j]$; in fact we will abuse the notation slightly and use

$$e_k(x, y) \quad \text{when we mean} \quad e_k\big((x, y)\big).$$

We now define three other characters:

$$\chi_1(j) = \left(\frac{k'}{j}\right), \qquad k' = \begin{cases} (-1)^{(k-1)/2}k & \text{if } k \text{ is odd}, \\ -k & \text{if } k \equiv 0 \pmod 4, \\ 4k & \text{if } k \equiv 2 \pmod 4; \end{cases}$$

$$\chi'(j) = \chi(j)\chi_1(j)^n = \begin{cases} \chi(j) & \text{if } n \text{ is even}, \\ \chi(j)\chi_1(j) & \text{if } n \text{ is odd}; \end{cases} \tag{6}$$

$$\chi_8(j) = \left(\frac{8}{j}\right),$$

the Kronecker symbol has been used in the first and third of these. Note that $(\bar{\chi})' = \overline{(\chi')}$, so that there is no confusion in using $\bar{\chi}'$. Note also that if k is odd, $\chi_1(j) = \left(\frac{j}{k}\right)$ (Jacobi symbol). We now put

$$\begin{aligned} \varepsilon &= \begin{cases} 1 & \text{if } k \equiv 1 \pmod 4 \text{ or if } k \text{ is even and } d \equiv 1 \pmod 4, \\ i & \text{if } k \equiv 3 \pmod 4 \text{ or if } k \text{ is even and } d \equiv 3 \pmod 4; \end{cases} \\ \varepsilon_1 &= \begin{cases} \chi_8(k) & \text{if } k \text{ is odd}, \\ \chi_8(k+d) & \text{if } k \text{ is even}. \end{cases} \end{aligned} \tag{7}$$

Finally, we define

$$a = \frac{(\varepsilon\varepsilon_1)^n \chi_1(d)\chi'(-d)\tau(\chi')}{\tau(\bar{\chi})}. \tag{8}$$

By the well known result quoted as Lemma 1 below, if χ and χ' are primitive characters $(\operatorname{mod} k)$, then $|a| = 1$. We can now state our two main results.

THEOREM 1. *Let $(d, k) = 1$ and either n be even or k odd (or both). Suppose also that χ and χ' are primitive characters* $(\operatorname{mod} k)$. *Then*

$$\sum_{x=1}^{k} \chi\big(Q(x)\big) e_k(x, y) = a k^{n/2} \bar{\chi}'\big(\bar{Q}(y)\big). \tag{9}$$

A stronger looking, but equivalent, form of Theorem 1 is

THEOREM 1'. *Under the hypotheses of Theorem 1 and for any r, $0 \leqslant r \leqslant n$, and any z,*

$$\begin{aligned} \sum_{\substack{x_j=1 \\ j\leqslant r}}^{k} \chi\big(Q(x_1, \ldots, x_r, z_{r+1}, \ldots, z_n)\big) e_k\Big(\sum_{j=1}^{r} x_j z_j\Big) \\ = a k^{(2r-n)/2} \sum_{\substack{x_j=1 \\ j>r}}^{k} \bar{\chi}'\big(\bar{Q}(-z_1, \ldots, -z_r, x_{r+1}, \ldots, x_n)\big) e_k\Big(\sum_{j=r+1}^{n} x_j z_j\Big). \end{aligned} \tag{10}$$

THEOREM 2. *Under the hypotheses of Theorem 1 and the added assumption that Q is positive definite, $L(s,\chi,Q)$ and $L(s,\bar{\chi}',\bar{Q})$ can be continued to entire functions of s and*

$$\left(\frac{kd^{1/n}}{2\pi}\right)^{s}\Gamma(s)L(s,\chi,Q)=a\left(\frac{kd^{(n-1)/n}}{2\pi}\right)^{n/2-s}\Gamma\left(\frac{n}{2}-s\right)L\left(\frac{n}{2}-s,\bar{\chi}',\bar{Q}\right). \tag{11}$$

Remarks. 1. The form of (11) is particularly apt since the discriminant of $\bar{Q}$ is d^{n-1}.

2. If k is even, then we must be sure that the coefficients of $\bar{Q}$ are integers and not just halves of integers. This detail is taken care of in Lemma 4.

3. The hypothesis that either n is even or k odd assures that χ' is a character (mod k) whenever χ is. But further, the hypothesis that χ is primitive accomplishes the same goal. This is because there are no primitive characters (mod k) for $k\equiv 2 \pmod 4$. See for example [2], pp. 420-421, problems 10, 11.

4. If n is odd, then it is easily shown that d is even. Thus the hypothesis that either n is even or k is odd follows from the hypothesis that $(d,k)=1$.

5. Let $k=4$, $\chi(j)=\left(\frac{-4}{j}\right)$, $Q(x)=x_1^2+x_2^2$. Then

$$\sum_{x=1}^{4}\chi\big(Q(x)\big)=8\neq \text{constant}\cdot\bar{\chi}'\big(\bar{Q}(0)\big)$$

no matter how $\bar{\chi}'$ and $\bar{Q}$ are defined. This shows that the condition $(d,k)=1$ is necessary and that further, the factor of 2^n in d which enters from (2) cannot be eliminated.

In the course of the proofs we will need the Gaussian sums

$$\begin{aligned} G_Q(a,k)&=\sum_{x=1}^{k}e_k\big(aQ(x)\big),\\ G(a,k)&=\sum_{j=1}^{k}e_k(aj^2),\end{aligned} \tag{12}$$

and the theta function,

$$\vartheta(\gamma T,y,z)=\sum_{x}e^{\pi i\gamma[(x-z)T,(x-z)]+2\pi i(x,y)-\pi i(y,z)}, \tag{13}$$

defined for symmetric matrices T corresponding to positive definite quadratic forms (xT,x) and complex numbers γ with $\operatorname{Im}\gamma>0$. It will also be convenient to introduce a number d' defined by

$$\begin{aligned} dd'&\equiv 1 \pmod k \quad \text{(always)},\\ d'&\equiv 0 \pmod 2 \quad \text{(if } d \text{ is even)}.\end{aligned} \tag{14}$$

The second condition in (14) can always be fulfilled since k must be odd when d is even by our assumption that $(d,k)=1$.

3. Lemmas and known results.

LEMMA 1 ([1], pp. 312, 313). *If χ if a primitive character* (mod k) *then*

$$|\tau(\chi)|=\sqrt{k}\quad \textit{and}\quad \sum_{j=1}^{k}\chi(j)e_k(aj)=\bar{\chi}(a)\tau(\chi)\quad \textit{for all } a.$$

LEMMA 2 ([3], p. 41).

$$\vartheta(\gamma T,y,z)=(-i\gamma)^{-n/2}|T|^{-1/2}\vartheta\left(-\frac{1}{\gamma}T^{-1},z,-y\right).$$

Here $(-i\gamma)^{-n/2}$ is taken to be positive if γ is purely imaginary and is given by analytic continuation for other values of γ in the upper half plane.

LEMMA 3 ([3], pp. 44, 45). *If k is an odd prime and $(a,k)=1$, then*

$$G(a,k)=\varepsilon\chi_1(a)\sqrt{k}.$$

LEMMA 4. *If d is odd then $\bar{Q}$ has integral coefficients. Thus $(k,d)=1$ implies that $d'\bar{Q}(y)$ is an integer for all y.*

Proof. We note that $dF^{-1}=\operatorname{adj}F$ is a matrix of integers. Thus, it suffices to show that the diagonal elements of dF^{-1} are all even. To this end, put $dF^{-1}=(a_{ij})$ and let

$$x=ydF^{-1}\quad\text{where}\quad y=(\delta_{i1},\delta_{i2},\ldots,\delta_{in})=(0,\ldots,0,1,0,\ldots,0).$$

Then $dy=xF$ and since F is symmetric,

$$\begin{aligned} da_{ii}&=d(ydF^{-1},y)=(dyF^{-1},dy)\\ &=(xFF^{-1},xF)=(xF,x)=2Q(x)\equiv 0 \pmod 2.\end{aligned}$$

The second statement follows since either d is odd and $\bar{Q}(y)$ is an integer or d is even and d' has the factor of 2 necessary to cancel out the possible denominator of 2 in $\bar{Q}$.

LEMMA 5. *Let $(d,k)=(a,k)=1$. If we define a' by $aa'\equiv 1 \pmod k$ then*

$$\sum_{x=1}^{k}e_k[aQ(x)+(x,y)]=e_k\big(-a'd'\bar{Q}(y)\big)G_Q(a,k).$$

Proof.

$$\sum_{x=1}^{k} e_k[aQ(x)+(x,y)] = 2^{-n}\sum_{x=1}^{2k} e_k[aQ(x)+aa'dd'(x,y)]$$

$$= 2^{-n}\sum_{x=1}^{2k} e_{2k}[a(xF,x)+2aa'dd'(x,y)]$$

$$= 2^{-n}e_{2k}[-aa'^2d^2d'^2(yF^{-1},y)]\sum_{x=1}^{2k} e_{2k}[(x+a'd'ydF^{-1})aF,(x+a'd'ydF^{-1})]$$

$$= e_k[-(aa')(dd')a'd'\bar{Q}(y)]\cdot 2^{-n}\sum_{z=1}^{2k} e_{2k}(zaF,z) = e_k\big(-a'd'\bar{Q}(y)\big)G_Q(a,k)$$

since $d'\bar{Q}(y)$ is an integer by Lemma 4.

We now turn to the problem of evaluating $G_Q(a,k)$. The proofs are provided for completeness.

LEMMA 6. *Suppose that* $(k,d)=(k,a)=1$. *Then*

(15) $$G_Q(a,k) = k^{n/2}\chi_1(a^n d)[\varepsilon(a)\varepsilon_1(a)]^n$$

where

$$\varepsilon(a) = \begin{cases} 1 & \text{if } k\equiv 1 \pmod 4 \text{ or if } k \text{ is even and } k+a^n d\equiv 1 \pmod 4,\\ i & \text{if } k\equiv 3 \pmod 4 \text{ or if } k \text{ is even and } k+a^n d\equiv 3 \pmod 4;\end{cases}$$

(16) $$\varepsilon_1(a) = \begin{cases} \chi_8(k) & \text{if } k \text{ is odd},\\ \chi_8(k+a^n d) & \text{if } k \text{ is even}.\end{cases}$$

Note that when n is odd, the lemma does not apply for even k since d is always even for odd n. It suffices to prove Lemma 6 for $a=1$ since the discriminant of $aQ(x)$ is $a^n d$. The proof of Lemma 6 with $a=1$ is the subject of Lemmas 7 through 11. The proofs follow closely the method of evaluating the ordinary Gaussian sum $G(a,k)$ as given, for example, in [3], pp. 44-48.

LEMMA 7. *Suppose that* $k=k_1k_2$ *with* $(k_1,k_2)=1$. *Then*

$$G_Q(a,k) = G_Q(ak_1,k_2)G_Q(ak_2,k_1).$$

Proof. Any vector $x \pmod{k_1k_2}$ determines unique vectors $y \pmod{k_1}$ and $z \pmod{k_2}$ such that

$$x = k_1z+k_2y$$

and conversely. Thus

$$G_Q(a,k) = \sum_{y=1}^{k_1}\sum_{z=1}^{k_2} e_{k_1k_2}[aQ(k_1z+k_2y)]$$

$$= \sum_{y=1}^{k_1} e_{k_1k_2}[aQ(k_2y)]\sum_{z=1}^{k_2} e_{k_1k_2}[aQ(k_1z)] = G_Q(ak_2,k_1)G_Q(ak_1,k_2).$$

LEMMA 8. *Lemma 6 is true for* $a=1$ *and* $k=p^r$ *where* p *is an odd prime.*

Clearly $G_Q(1,1)=1$ which agrees with the lemma when $r=0$. We next tackle the case $r=1$. Here we may write Q as a symmetric matrix of integers $(\operatorname{mod} p)$ and

$$|Q| \equiv \frac{d}{2^n} \pmod p.$$

Also ([4], p. 153), Q may be diagonalized $(\operatorname{mod} p)$ by a nonsingular matrix of integers M, in other words MQM^t is a diagonal matrix $(\operatorname{mod} p)$ with diagonal elements $q_1,\ldots,q_n$. We let $|M|=m$ so that $m\not\equiv 0 \pmod p$ and then

$$q_1q_2\ldots q_n \equiv |MQM^t| \equiv \frac{dm^2}{2^n} \not\equiv 0 \pmod p,$$

thus $(q_j,p)=1$ for all j. If we use M^{-1} to denote a matrix of integers such that $MM^{-1}\equiv I \pmod p$, where I is the identity matrix, then

$$G_Q(1,p) = \sum_{x=1}^{p} e_p(xQ,x) = \sum_{x=1}^{p} e_p(xM^{-1}MQ,xM^{-1}M)$$

$$= \sum_{x=1}^{p} e_p[(xM^{-1})(MQM^t),(xM^{-1})] = \sum_{y=1}^{p} e_p(yMQM^t,y)$$

$$= \prod_{j=1}^{n} G(q_j,p) = \prod_{j=1}^{n}[\varepsilon\chi_1(q_j)\sqrt{p}]$$

by Lemma 3. Thus

$$G_Q(1,p) = \varepsilon^n\chi_1(dm^2)\chi_1(2^n)^{-1}p^{n/2} = \varepsilon^n\chi_1(2)^n\chi_1(d)p^{n/2}$$

which is the lemma for $r=1$ since here $\chi_1(2)=\chi_8(p)$.

We now assume that $r\geqslant 2$ and derive a reduction formula. For $1\leqslant x\leqslant p^r$, we may write uniquely

$$x = y+p^{r-1}z, \quad 1\leqslant y\leqslant p^{r-1}, \quad 0\leqslant z\leqslant p-1.$$

Thus

(17) $$G_Q(1,p^r) = \sum_{y=1}^{p^{r-1}}\sum_{z=0}^{p-1} e_{p^r}[Q(y+p^{r-1}z)]$$

$$= p^n\sum_{\substack{y=1\\ y\equiv 0 \pmod p}}^{p^{r-1}} e_{p^r}[Q(y)] + \sum_{\substack{y=1\\ y\not\equiv 0 \pmod p}}^{p^{r-1}}\sum_{z=0}^{p-1} e_{p^r}[Q(y+p^{r-1}z)].$$

In order to evaluate the second sum in (17) we write Q as a symmetric matrix of integers $(\bmod p^r)$, $Q(x) \equiv (xQ, x) (\bmod p^r)$. Then

$$(18)\qquad \sum_{\substack{y=1\\ y\not\equiv 0 (\bmod p)}}^{p^{r-1}} \sum_{z=0}^{p-1} e_{p^r}[Q(y+p^{r-1}z)]$$

$$= \sum_{\substack{y=1\\ y\not\equiv 0 (\bmod p)}}^{p^{r-1}} e_{p^r}(yQ, y) \sum_{z=0}^{p-1} e_p(2yQ, z) = 0.$$

This is because $\sum_{z=0}^{p-1} e_p(2yQ, z) = 0$ unless $2yQ \equiv 0 (\bmod p)$ which happens only when $y \equiv 0 (\bmod p)$ since $2Q$ is invertable $(\bmod p)$. From (17) and (18),

$$(19)\qquad G_Q(1, p^r) = p^n \sum_{\substack{y=1\\ y\equiv 0 (\bmod p)}}^{p^{r-1}} e_{p^r}[Q(y)]$$

$$= p^n \sum_{x=1}^{p^{r-2}} e_{p^r}[Q(px)] = p^n G_Q(1, p^{r-2}).$$

Lemma 8 now follows easily by induction on r.

LEMMA 9. *Lemma 6 is correct for $a = 1$ and odd k.*

Proof. Lemma 8 is the case of k having only one distinct prime factor. It therefore suffices to show that if $k = k_1 k_2$, $(k_1, k_2) = 1$ and the lemma is true for k_1 and k_2, then it is true for k also. We note again that Lemma 6 for $a = 1$ is equivalent to Lemma 6 for all a such that $(a, k) = 1$. Thus by Lemma 7,

$$(20)\qquad G_Q(1, k) = G_Q(k_2, k_1) G_Q(k_1, k_2)$$

$$= k_1^{n/2} \left(\frac{k_2^n d}{k_1}\right) \varepsilon_{k_1}^n \chi_8(k_1)^n \cdot k_2^{n/2} \left(\frac{k_1^n d}{k_2}\right) \varepsilon_{k_2}^n \chi_8(k_2)^n$$

$$= k^{n/2} \left(\frac{d}{k}\right) \chi_8(k)^n \left[\left(\frac{k_2}{k_1}\right)\left(\frac{k_1}{k_2}\right) \varepsilon_{k_1} \varepsilon_{k_2}\right]^n$$

where

$$\varepsilon_{k_j} = \begin{cases} 1 & \text{if} \quad k_j \equiv 1 (\bmod 4) \\ i & \text{if} \quad k_j \equiv 3 (\bmod 4) \end{cases} \qquad j = 1, 2.$$

By the law of quadratic reciprocity,

$$(21)\qquad \left(\frac{k_2}{k_1}\right)\left(\frac{k_1}{k_2}\right) \varepsilon_{k_1} \varepsilon_{k_2} = \varepsilon(1),$$

as can be easily checked in each of the four possible cases. From (20) and (21),

$$G_Q(1, k) = k^{n/2} \left(\frac{d}{k}\right) [\varepsilon(1) \chi_8(k)]^n$$

which is the result of Lemma 6.

We now turn our attention to even k. To this end, it is convenient to first prove a reciprocity formula for $G_Q(1, k)$.

LEMMA 10. *If Q is positive definite (thus $d > 0$) then whether or not $(d, k) = 1$,*

$$(22)\qquad G_Q(1, k) = \frac{k^{n/2}\sqrt{d}}{(2d)^n} e_8(n) \sum_{x=1}^{2d} e_d(-k\bar{Q}(x)).$$

Proof. Let

$$(23)\qquad \tau = \frac{1}{k} + i\lambda, \qquad \lambda > 0.$$

Then

$$\vartheta(\tau F, 0, 0) = \sum_x e^{\pi i \tau (xF, x)}.$$

We replace x by $y + 2kz$, $1 \leqslant y \leqslant 2k$, $-\infty < z < \infty$. Thus

$$(24)\qquad \vartheta(\tau F, 0, 0) = \sum_{y=1}^{2k} \sum_z e^{(2\pi i/2k - \pi\lambda)[(y+2kz)F, (y+2kz)]}$$

$$= \sum_{y=1}^{2k} e_{2k}(yF, y) \sum_z e^{-4k^2\pi\lambda[(z+y/2k)F, (z+y/2k)]}$$

$$= \sum_{y=1}^{2k} e_k(Q(y)) \vartheta\left(4k^2\lambda i F, 0, -\frac{y}{2k}\right)$$

by the definition of ϑ in (13). We now make use of the transformation formula in Lemma 2,

$$(25)\qquad \vartheta\left(4k^2\lambda i F, 0, -\frac{y}{2k}\right) = (4k^2\lambda)^{-n/2} |F|^{-1/2} \vartheta\left(\frac{i}{4k^2\lambda} F^{-1}, -\frac{y}{2k}, 0\right)$$

$$= \frac{1}{2^n k^n \lambda^{n/2} \sqrt{d}} \vartheta\left(\frac{i}{4k^2\lambda} F^{-1}, -\frac{y}{2k}, 0\right).$$

But

$$\lim_{\lambda \to 0^+} \vartheta\left(\frac{i}{4k^2\lambda} F^{-1}, -\frac{y}{2k}, 0\right) = 1$$

because the series converges uniformly for $0<\lambda\leqslant 1$ and thus one can interchange the limit and summation operations. Therefore by (24) and (25),

$$(26)\qquad \lim_{\lambda\to 0^+}\lambda^{n/2}\vartheta(\tau F,0,0)=\frac{1}{2^n k^n\sqrt{d}}\sum_{y=1}^{2k}e_k\big(Q(y)\big)=\frac{1}{k^n\sqrt{d}}G_Q(1,k).$$

Now put

$$(27)\qquad \tau'=-\frac{1}{\tau}=-k+i\lambda',\qquad \lambda'=\frac{k^2\lambda}{1+i\lambda k}.$$

Where it occurs, it will be useful to remember that dF^{-1} is a matrix of integers. By Lemma 2,

$$(28)\qquad \vartheta(\tau F,0,0)=(-i\tau)^{-n/2}|F|^{-1/2}\vartheta(\tau' F^{-1},0,0)=\frac{(-i\tau)^{-n/2}}{\sqrt{d}}\vartheta(\tau' F^{-1},0,0).$$

Further,

$$\vartheta(\tau' F^{-1},0,0)=\sum_x e^{\pi i\tau'(xF^{-1},x)},$$

and so replacing x by $y+2dz$, $1\leqslant y\leqslant 2d$, $-\infty<z<\infty$, we find that

$$(29)\qquad \vartheta(\tau' F^{-1},0,0)=\sum_{y=1}^{2d}\sum_z e^{(-\pi ik-\pi\lambda')[(y+2dz)F^{-1},(y+2dz)]}$$
$$=\sum_{y=1}^{2d}e^{-\pi ik(yF^{-1},y)}\sum_z e^{-4\pi d^2\lambda'[(z+y/2d)F^{-1},(z+y/2d)]}$$
$$=\sum_{y=1}^{2d}e_d\big(-k\bar{Q}(y)\big)\vartheta\left(4d^2\lambda' iF^{-1},0,-\frac{y}{2d}\right).$$

Again by Lemma 2,

$$(30)\qquad \vartheta\left(4d^2\lambda' iF^{-1},0,-\frac{y}{2d}\right)=(4d^2\lambda')^{-n/2}|F^{-1}|^{-1/2}\vartheta\left(\frac{i}{4d^2\lambda'}F,-\frac{y}{2d},0\right).$$

As $\lambda\to 0^+$, $\operatorname{Re}\left(\frac{1}{\lambda'}\right)=\frac{1}{k^2\lambda}\to+\infty$ and thus

$$(31)\qquad \lim_{\lambda\to 0^+}\vartheta\left(\frac{i}{4d^2\lambda'}F,-\frac{y}{2d},0\right)=1$$

since the series for ϑ in (30) converges uniformly for $0<\lambda\leqslant 1$. Since

$$\lim_{\lambda\to 0^+}\left(\frac{\lambda}{\lambda'}\right)^{n/2}=\lim_{\lambda\to 0^+}\left(\frac{1+i\lambda k}{k^2}\right)^{n/2}=k^{-n},$$

we see from (29), (30) and (31) that

$$(32)\qquad \lim_{\lambda\to 0^+}\lambda^{n/2}\vartheta(\tau' F^{-1},0,0)=\frac{\sqrt{d}}{2^n d^n k^n}\sum_{y=1}^{2d}e_d\big(-k\bar{Q}(y)\big).$$

We recall that for γ in the upper half plane, $(-i\gamma)^{-n/2}$ is defined by analytic continuation from the positive value when γ is purely imaginary. Hence

$$\lim_{\lambda\to 0^+}(-i\tau)^{-n/2}=\lim_{\lambda\to 0^+}\left[-i\left(\frac{1}{k}+i\lambda\right)\right]^{-n/2}=\left|\frac{-i}{k}\right|^{-n/2}e^{-in\theta/2},$$

where $\theta=\arg(-i/k)$, $-\frac{1}{2}\pi\leqslant\theta\leqslant\frac{1}{2}\pi$. Thus $\theta=-\frac{1}{2}\pi$ and

$$\lim_{\lambda\to 0^+}(-i\tau)^{-n/2}=k^{n/2}e_8(n).$$

Therefore by (28) and (32)

$$(33)\qquad \lim_{\lambda\to 0^+}\lambda^{n/2}\vartheta(\tau F,0,0)=\frac{e_8(n)}{2^n d^n k^{n/2}}\sum_{y=1}^{2d}e_d\big(-k\bar{Q}(y)\big).$$

Lemma 10 follows (26) and (33).

LEMMA 11. *Lemma 6 is correct for $a=1$ and even k.*

Proof. We first note that it suffices to prove Lemma 11 only for positive definite Q. This is because every term of (15) remains invarient if the coefficients of Q are changed $(\operatorname{mod} 4k)$ and if the diagonal elements of Q are sufficiently increased, Q becomes positive definite. For positive definite Q, we see from Lemma 10 that

$$(34)\qquad G_Q(1,k+d)=\frac{(k+d)^{n/2}\sqrt{d}}{(2d)^n}e_8(n)\sum_{x=1}^{2d}e_d[-(k+d)\bar{Q}(x)]$$
$$=\frac{(k+d)^{n/2}\sqrt{d}}{(2d)^n}e_8(n)\sum_{x=1}^{2d}e_d\big(-k\bar{Q}(x)\big),$$

since for k even, $(d,k)=1$ implies d is odd and thus $\bar{Q}(x)$ is an integer by Lemma 4. Thus from (22), (34) and Lemma 9,

$$(35)\qquad G_Q(1,k)=\frac{k^{n/2}}{(k+d)^{n/2}}G_Q(1,k+d)=k^{n/2}\left(\frac{d}{k+d}\right)[\varepsilon(1)\varepsilon_1(1)]^n.$$

Now when $k \equiv 2 \pmod 4$, $d \not\equiv k+d \pmod 4$ and hence

$$\left(\frac{d}{k+d}\right) = \left(\frac{k+d}{d}\right) = \left(\frac{k}{d}\right) = \left(\frac{4k}{d}\right) = \chi_1(d),$$

while if $k \equiv 0 \pmod 4$, $d \equiv k+d \pmod 4$ and therefore

$$\left(\frac{d}{k+d}\right) = \left(\frac{-1}{d}\right)\left(\frac{k+d}{d}\right) = \left(\frac{-k}{d}\right) = \chi_1(d).$$

If we insert this into (35), we get Lemma 6 for k even and $a = 1$.

4. Proof of Theorems 1 and 1′. We begin by assuming only that $(d, k) = 1$ and that χ is a primitive character $(\operatorname{mod} k)$. This last condition means that if k is even then $k \equiv 0 \pmod 4$ (see remark 3 after the statement of Theorem 2). By Lemmas 1, 5, and 6,

$$(36)\qquad \sum_{x=1}^{k} \chi\big(Q(x)\big) e_k(x, y) = \frac{1}{\tau(\bar\chi)} \sum_{\substack{a=1\\(a,k)=1}}^{k} \bar\chi(a) \sum_{x=1}^{k} e_k[aQ(x)+(x, y)]$$

$$= \frac{1}{\tau(\bar\chi)} \sum_{\substack{a=1\\(a,k)=1}}^{k} \bar\chi(a) e_k\big(-a'd'\bar Q(y)\big) G_Q(a, k)$$

$$= \frac{k^{n/2}\chi_1(d)}{\tau(\bar\chi)} \sum_{\substack{a=1\\(a,k)=1}}^{k} \bar\chi(a)\chi_1(a)^n[\varepsilon(a)\varepsilon_1(a)]^n e_k\big(-a'd'\bar Q(y)\big)$$

$$= \frac{k^{n/2}\chi_1(d)}{\tau(\bar\chi)} \sum_{\substack{a=1\\(a,k)=1}}^{k} \bar\chi'(a)[\varepsilon(a)\varepsilon_1(a)]^n e_k\big(-a'd'\bar Q(y)\big)$$

where a' is defined for $(a, k) = 1$ as $a'a \equiv 1 \pmod k$. We now include the hypothesis that either n is even or k odd (which is derivable from the condition $(d, k) = 1$, see remark 4 after Theorem 2). Then we see that

$$\varepsilon(a) = \varepsilon, \qquad \varepsilon_1(a) = \varepsilon_1$$

and thus (36) becomes

$$(37)\qquad \sum_{x=1}^{k} \chi\big(Q(x)\big) e_k(x, y) = \frac{k^{n/2}\chi_1(d)\varepsilon^n\varepsilon_1^n}{\tau(\bar\chi)} \sum_{\substack{a=1\\(a,k)=1}}^{k} \bar\chi'(a) e_k\big(-a'd'\bar Q(y)\big)$$

$$= \frac{k^{n/2}\chi_1(d)\varepsilon^n\varepsilon_1^n}{\tau(\bar\chi)} \sum_{\substack{b=1\\(b,k)=1}}^{k} \chi'(b) e_k\big(-bd'\bar Q(y)\big).$$

By Lemma 4, $d'\bar Q(y)$ is an integer and thus if we make the final assumption that χ' is also a primitive character $(\operatorname{mod} k)$, then by (37), Lemma 1 and the definition of d' in (14),

$$\sum_{x=1}^{k} \chi\big(Q(x)\big) e_k(x, y) = \frac{k^{n/2}\chi_1(d)\varepsilon^n\varepsilon_1^n\tau(\chi')}{\tau(\bar\chi)} \bar\chi'\big(-d'\bar Q(y)\big)$$

$$= \frac{(\varepsilon\varepsilon_1)^n\chi_1(d)\chi'(-d)\tau(\chi')}{\tau(\bar\chi)} k^{n/2}\bar\chi'\big(\bar Q(y)\big) = ak^{n/2}\bar\chi'\big(\bar Q(y)\big)$$

which is Theorem 1.

It follows from Theorem 1 that for $0 \leqslant r \leqslant n$,

$$\sum_{\substack{y_j=1\\j>r}}^{k} \sum_{x=1}^{k} \chi\big(Q(x)\big) e_k(x, y) e_k\Big(-\sum_{j=r+1}^{n} y_j z_j\Big) = ak^{n/2} \sum_{\substack{y_j=1\\j>r}}^{k} \bar\chi'\big(\bar Q(y)\big) e_k\Big(-\sum_{j=r+1}^{n} y_j z_j\Big)$$

or

$$k^{n-r} \sum_{\substack{x_j=1\\j\leqslant r}}^{k} \chi\big(Q(x_1, \ldots, x_r, z_{r+1}, \ldots, z_n)\big) e_k\Big(\sum_{j=1}^{r} x_j y_j\Big)$$

$$= ak^{n/2} \sum_{\substack{y_j=1\\j>r}}^{k} \bar\chi'\big(\bar Q(y_1, \ldots, y_r, -y_{r+1}, \ldots, -y_n)\big) e_k\Big(\sum_{j=r+1}^{n} y_j z_j\Big)$$

$$= ak^{n/2} \sum_{\substack{x_j=1\\j>r}}^{k} \bar\chi'\big(\bar Q(-y_1, \ldots, -y_r, x_{r+1}, \ldots, x_n)\big) e_k\Big(\sum_{j=r+1}^{n} x_j z_j\Big)$$

which in a slightly different notation is k^{n-r} times both sides of (10). This proves Theorem 1′. An interesting special case of (10) is the case $r = 0$,

$$(38)\qquad \chi\big(Q(z)\big) = ak^{-n/2} \sum_{x=1}^{k} \bar\chi'\big(\bar Q(x)\big) e_k(x, z).$$

5. Proof of Theorem 2.

LEMMA 12. *Under the hypotheses of Theorem 2, we define for $t > 0$,*

$$(39)\qquad \psi(t, \chi, Q) = \tfrac{1}{2} \sum_{z} \chi\big(Q(z)\big) e_k\big(td^{-1/n} iQ(z)\big).$$

Then for $t > 0$,

$$(40)\qquad \psi(t, \chi, Q) = at^{-n/2}\psi\Big(\frac{1}{t}, \bar\chi', \bar Q\Big)$$

and

$$\int_{1}^{\infty} t^{s-1}\psi(t, \chi, Q)\,dt$$

is an entire function of s.

Proof. The series in (39) converges absolutely and uniformly for $t \geqslant t_0 > 0$. By (38) (Theorem 1' with $r = 0$) and Lemma 2,

$$2\psi(t,\chi,Q) = ak^{-n/2}\sum_{x=1}^{k}\bar{\chi}'(\bar{Q}(x))\sum_{z} e_k[td^{-1/n}iQ(z)+(x,z)]$$

$$= ak^{-n/2}\sum_{x=1}^{k}\bar{\chi}'(\bar{Q}(x))\sum_{z} e^{-\pi t d^{-1/n}/k(zF,z)+2\pi i\left(z,\frac{x}{k}\right)}$$

$$= ak^{-n/2}\sum_{x=1}^{k}\bar{\chi}'(\bar{Q}(x))\vartheta\left(tk^{-1}d^{-1/n}iF, \frac{x}{k}, 0\right)$$

$$= ak^{-n/2}[tk^{-1}d^{-1/n}]^{-n/2}|F|^{-1/2}\sum_{x=1}^{k}\bar{\chi}'(\bar{Q}(x))\vartheta\left(\frac{kd^{1/n}}{t}iF^{-1}, 0, -\frac{x}{k}\right)$$

$$= at^{-n/2}\sum_{x=1}^{k}\bar{\chi}'(\bar{Q}(x))\sum_{y}\exp\left\{-\frac{\pi k d^{1/n}}{t}\left[\left(y+\frac{x}{k}\right)F^{-1},\left(y+\frac{x}{k}\right)\right]\right\}$$

$$= at^{-n/2}\sum_{x=1}^{k}\sum_{y}\bar{\chi}'(\bar{Q}(x+ky))\exp\left[\frac{-2\pi}{kt}d^{-(n-1)/n}\bar{Q}(x+ky)\right]$$

$$= at^{-n/2}\sum_{z}\bar{\chi}'(\bar{Q}(z))e_k\left(\frac{1}{t}d^{-(n-1)/n}i\bar{Q}(z)\right).$$

Since the discriminant of $\bar{Q}$ is d^{n-1}, this last equation is equivalent to (40). It is easily seen that for any real σ_0

$$\int_1^\infty t^{\sigma_0-1}|\psi(t,\chi,Q)|\,dt$$

converges and therefore the integral

$$\int_1^\infty t^{s-1}\psi(t,\chi,Q)\,dt$$

converges absolutely and uniformly for $\operatorname{Re}s \leqslant \sigma_0$. Thus this last integral represents an analytic function for $\operatorname{Re}s < \sigma_0$, and in fact it represents an entire function since σ_0 is arbitrary. This concludes the proof of Lemma 12.

From this point on we assume that the hypotheses of Theorem 2 are satisfied. By a familiar formula for the gamma function, for $\operatorname{Re}s > 0$,

$$\left(\frac{kd^{1/n}}{2\pi}\right)^s\Gamma(s)Q(x)^{-s} = \int_0^\infty t^{s-1}e_k\left(td^{-1/n}iQ(x)\right)dt$$

and thus for $\operatorname{Re}s > n/2$,

$$\left(\frac{kd^{1/n}}{2\pi}\right)^s\Gamma(s)L(s,\chi,Q) = \tfrac{1}{2}\sum_{x\neq 0}\chi(Q(x))\int_0^\infty t^{s-1}e_k\left(td^{-1/n}iQ(x)\right)dt \tag{41}$$

$$= \int_0^\infty t^{s-1}\cdot\tfrac{1}{2}\sum_{x\neq 0}\chi(Q(x))e_k\left(td^{-1/n}iQ(x)\right)dt$$

$$= \int_0^\infty t^{s-1}\psi(t,\chi,Q)\,dt.$$

The interchange of the order of integration and summation in (41) is justified by the usual uniform convergence argument. It is not necessary to justify the interchange for all s with $\operatorname{Re}s > n/2$, for example $\operatorname{Re}s > n+2$ is perfectly sufficient, analytic continuation takes care of the rest.

We now use Lemma 12 and replace t by $1/u$ to get

$$\int_0^1 t^{s-1}\psi(t,\chi,Q)\,dt = a\int_0^1 t^{s-n/2-1}\psi\left(\frac{1}{t},\bar{\chi}',\bar{Q}\right)dt = a\int_1^\infty u^{n/2-s-1}\psi(u,\bar{\chi}',\bar{Q})\,du$$

and thus (41) becomes

$$\left(\frac{kd^{1/n}}{2\pi}\right)^s\Gamma(s)L(s,\chi,Q) = a\int_1^\infty u^{n/2-s-1}\psi(u,\bar{\chi}',\bar{Q})\,du+\int_1^\infty t^{s-1}\psi(t,\chi,Q)\,dt. \tag{42}$$

The right hand side of (42) is an entire function by Lemma 12 and this gives the analytic continuation of the left hand side of (42). Now for $\operatorname{Re}(n/2-s) > n/2$, we replace t by $1/u$ in the last integral and get

$$\left(\frac{kd^{1/n}}{2\pi}\right)^s\Gamma(s)L(s,\chi,Q) = a\int_0^\infty u^{n/2-s-1}\psi(u,\bar{\chi}',\bar{Q})\,du$$

$$= a\left(\frac{kd^{(n-1)/n}}{2\pi}\right)^{n/2-s}\Gamma\left(\frac{n}{2}-s\right)L\left(\frac{n}{2}-s,\bar{\chi}',\bar{Q}\right)$$

by the analogue of (41) for $\bar{Q}$. This gives the analytic continuation of the right hand side of the last equation. The coefficient of $L(s,\chi,Q)$ has no zeros and therefore $L(s,\chi,Q)$ is itself an entire function, the same is true of $L(s,\bar{\chi}',\bar{Q})$. This concludes the proof of Theorem 2.

Some concluding remarks might not be amiss. It is easily seen that Theorem 1 and as a result its logical corollary Theorem 2, can be relaxed to include forms Q with rational coefficients, the only proviso being that the denominators of the coefficients are relatively prime to k. In particular Theorems 1 and 2 hold for Q and χ replaced by $\bar{Q}$ and $\bar{\chi}'$. In fact since $\bar{\bar{\chi}}' = \chi$ and the discriminant of $\bar{Q}$ is d^{n-1} which is relatively prime to k if and only if $(d, k) = 1$, under the original hypotheses of Theorems 1 and 2 the conclusions are valid not only as stated but for Q, χ and a replaced by $\bar{Q}$, $\bar{\chi}'$ and $a_{\bar{Q}}$ as well ($a_{\bar{Q}}$ is defined from (7) and (8) except that d is replaced by d^{n-1} and χ is replaced by $\bar{\chi}'$). Since $\bar{\bar{Q}} = d^{n-2}Q$ and the discriminant of $\bar{\bar{Q}}$ is $d^{(n-1)^2}$, Theorem 2 for $\bar{Q}$ yields

$$\left(\frac{kd^{(n-1)/n}}{2\pi}\right)^s \Gamma(s) L(s, \bar{\chi}', \bar{Q}) = a_{\bar{Q}} \left(\frac{kd^{(n-1)^2/n}}{2\pi}\right)^{n/2-s} \Gamma\left(\frac{n}{2}-s\right) L\left(\frac{n}{2}-s, \chi, d^{n-2}Q\right)$$
$$= a_{\bar{Q}}\chi(d^{n-2}) \left(\frac{kd^{1/n}}{2\pi}\right)^{n/2-s} \Gamma\left(\frac{n}{2}-s\right) L\left(\frac{n}{2}-s, \chi, Q\right).$$

If we compare this with Theorem 2 with s replaced by $n/2-s$ we see that

(43) $$\bar{a} = a_{\bar{Q}}\chi(d^{n-2}).$$

There is unfortunately no new information in (43) although it takes considerable algebraic manipulation to prove (43) directly from (7) and (8).

There is always more than one way to derive a functional equation. Theorem 2 can be easily derived from Theorem 1 and the functional equation for the general Epstein zeta function in much the same way that the functional equation of Dirichlet's L-function is derived from Lemma 1 and the functional equation of the Hurwitz zeta function.

References

[1] Raymond Ayoub, *An introduction to the analytic theory of numbers*, Providence 1963.

[2] Z. I. Borevich and I. R. Shafarevich, *Number theory*, New York 1966.

[3] Martin Eichler, *Introduction to the theory of algebraic numbers and functions*, New York 1966.

[4] Nathan Jacobson, *Lectures in abstract algebra*, vol. 2, Princeton 1953.

[5] H. M. Stark, *A complete determination of the complex quadratic fields of class-number one*, Mich. Math. J. 14 (1967), pp. 1-27.

[6] — *L-functions and character sums for quadratic forms (II)*, to be published.

THE UNIVERSITY OF MICHIGAN

Reçu par la Rédaction le 25. 4. 1967

ACTA ARITHMETICA
XIV (1968)

On certain additive functions (II)

by

P. D. T. A. Elliott (Nottingham)

A function $f(n)$ which is defined on the set of positive integers is said to be *additive* if for any coprime integers a, b, we have the relation

$$f(ab) = f(a)+f(b).$$

Let $0 < a_1 < a_2 \ldots$ be a set of integers, and let $A(x)$ denote the number of these not exceeding x. It was shown recently [4], that if $A(x)$ is not too small, then in the usual terminology $f(a_i)$ has a normal value. However, in order to prove this result a weak but inconvenient condition was introduced. It is our present purpose to show that this condition can be removed. More especially we prove the following:

Theorem 1. *For any irreducible polynomial $g(y)$ with integer coefficients, and any integer u, we define $\varrho(u)$ to be the number of residue classes r for which $g(r) = 0 \pmod u$.*

Let $f(n)$ be an additive function assuming only non-negative values, and for any positive value of x let $\mu_x = \max f(p^a)$ taken over the prime powers not exceeding x, and

$$S_x = \sum_{p^a\leqslant x} \varrho(p^a)f(p^a)p^{-a}.$$

Then if $A(x) > x\exp\left(-\varepsilon(x)\mu_x^{-1}S_x\right)$ for some function $\varepsilon(x)$ which tends to zero as $x\to\infty$, whilst $\mu_x = o(S_x)$, we have the asymptotic relations:

(1) $$\sum_{a_i\leqslant x} f^k\big(g(a_i)\big) \sim A(x)S_x^k, \quad k = 1, 2, \ldots$$

Corollary. *$f\big(g(a_i)\big)$ is normally S_{a_i}.*

We first show that the corollary is satisfied.

It is clear from the theorem that we have

$$\sum_{a_i\leqslant x} \big(f(g(a_i)) - S_x\big)^2 = o\big(A(x)S_x^2\big),$$