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$L \log L$ CRITERION FOR A CLASS OF SUPERDIFFUSIONS

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Abstract

In Lyons, Pemantle and Peres (1995), a martingale change of measure method was developed in order to give an alternative proof of the Kesten–Stigum $L \log L$ theorem for single-type branching processes. Later, this method was extended to prove the $L \log L$ theorem for multiple- and general multiple-type branching processes in Biggins and Kyprianou (2004), Kurtz *et al.* (1997), and Lyons (1997). In this paper we extend this method to a class of superdiffusions and establish a Kesten–Stigum $L \log L$ type theorem for superdiffusions. One of our main tools is a spine decomposition of superdiffusions, which is a modification of the one in Englander and Kyprianou (2004).

Keywords: Diffusions; superdiffusions; Poisson point process; Kesten–Stigum theorem; martingale; martingale change of measure

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1. Introduction and main result

Suppose that $\{Z_n,\ n \geq 1\}$ is a Galton–Watson branching process with each particle having probability p_n of giving birth to n children. Let L stand for a random variable with this offspring distribution. Let $m := \sum_{n=1}^{\infty} np_n$ be the mean number of children per particle. Then Z_n/m^n is a nonnegative martingale. Let W be the limit of Z_n/m^n as $n \to \infty$. Kesten and Stigum [8] proved that if $1 < m < \infty$ (that is, in the supercritical case) then W is nondegenerate (i.e. not almost surely zero) if and only if

$$\mathrm{E}(L \log^+ L) = \sum_{n=1}^{\infty} p_n n \log n < \infty.$$

This result is usually referred to as the Kesten–Stigum $L \log L$ theorem. In [1], Asmussen and Hering generalized this result to the case of branching Markov processes under some conditions.

Lyons *et al.* [14] developed a martingale change of measure method in order to give an alternative proof of the Kesten–Stigum $L \log L$ theorem for single-type branching processes.

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Later, this method was extended to prove the $L\log L$ theorem for multiple- and general multiple-type branching processes in [2], [12], and [13].

In this paper we will extend this method to a class of superdiffusions and establish an $L\log L$ criterion for superdiffusions. To state our main result, we need to introduce the setup we are going to work with first.

Let a_{ij}, $i, j = 1, \dots, d$, be bounded functions in $C^1(\mathbb{R}^d)$ such that all their first partial derivatives are bounded. We assume that the matrix (a_{ij}) is symmetric and satisfies

$$0 < a|\upsilon|^2 \le \sum_{i,j} a_{ij}\upsilon_i\upsilon_j \quad \text{for all } x \in \mathbb{R}^d \text{ and } \upsilon \in \mathbb{R}^d$$

for some positive constant a. Let b_i, $i = 1, \dots, d$, be bounded Borel functions on $\mathbb{R}^d$.

We will use $(Y, \Pi_x, x \in \mathbb{R}^d)$ to denote a diffusion process on $\mathbb{R}^d$ corresponding to the operator

$$L = \tfrac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla.$$

In this paper we will always assume that β is a bounded Borel function on $\mathbb{R}^d$ and that D is a bounded domain in $\mathbb{R}^d$. We will use $(Y^D, \Pi_x, x \in D)$ to denote the process obtained by killing Y upon exiting from D, that is,

$$Y_t^D = \begin{cases} Y_t & \text{if } t < \tau, \\ \partial & \text{if } t \ge \tau, \end{cases}$$

where $\tau = \inf\{t \ge 0 \colon Y_t \notin D\}$ is the first exit time of D and ∂ is a cemetery point. Any function f on D is automatically extended to $D \cup \{\partial\}$ by setting $f(\partial) = 0$. For convenience, we use the following convention throughout this paper. For any probability measure P, we also use P to denote the expectation with respect to P. When there is only one probability measure involved, we sometimes also use E to denote the expectation with respect to that measure.

We will use $\{P_t\}_{t\ge0}$ to denote the following Feynman–Kac semigroup:

$$P_t f(x) = \Pi_x\bigg(\exp\bigg\{\int_0^t \beta(Y_s^D)\,\mathrm{d}s\bigg\} f(Y_t^D)\bigg), \qquad x \in D.$$

It is well known that the semigroup $\{P_t\}_{t\ge0}$ is strongly continuous in $L^2(D)$ and, for any $t > 0$, P_t has a bounded, continuous, and strictly positive density $p(t, x, y)$.

Let $\{\widehat{P}_t\}_{t\ge0}$ be the dual semigroup of $\{P_t\}_{t\ge0}$ defined by

$$\widehat{P}_t f(x) = \int_D p(t, y, x) f(y)\,\mathrm{d}y, \qquad x \in D.$$

It is well known that $\{\widehat{P}_t\}_{t\ge0}$ is also strongly continuous on $L^2(D)$.

Let A and $\widehat{A}$ be the generators of the semigroups $\{P_t\}_{t\ge0}$ and $\{\widehat{P}_t\}_{t\ge0}$ on $L^2(D)$, respectively. We can formally write A as $L|_D + \beta$, where $L|_D$ is the restriction of L to D with Dirichlet boundary condition. Let $\sigma(A)$ and $\sigma(\widehat{A})$ respectively denote the spectrum of A and $\widehat{A}$. It follows from Jentzsch's theorem [16, Theorem V.6.6, p. 337] and the strong continuity of $\{P_t\}_{t\ge0}$ and $\{\widehat{P}_t\}_{t\ge0}$ that the common value $\lambda_1 := \sup \mathrm{Re}(\sigma(A)) = \sup \mathrm{Re}(\sigma(\widehat{A}))$ is an eigenvalue of multiplicity 1 for both A and $\widehat{A}$, and that an eigenfunction ϕ of A associated with λ_1 can be chosen to be strictly positive almost everywhere (a.e.) on D and an eigenfunction $\widetilde{\phi}$ of $\widehat{A}$ associated with λ_1 can be chosen to be strictly positive a.e. on D. We assume that ϕ and $\widetilde{\phi}$

are strictly positive a.e. on D. By [9, Proposition 2.3] we know that ϕ and $\widetilde{\phi}$ are bounded and continuous on D, and they are in fact strictly positive everywhere on D. We choose ϕ and $\widetilde{\phi}$ so that $\int_D \phi(x)\widetilde{\phi}(x)\,\mathrm{d}x = 1$.

Throughout this paper, we make the following assumptions.

Assumption 1.1. $\lambda_1 > 0$.

Assumption 1.2. *The semigroups* $\{P_t\}_{t\geq 0}$ *and* $\{\widehat{P}_t\}_{t\geq 0}$ *are intrinsic ultracontractive, that is, for any* $t > 0$*, there exists a constant* $c_t > 0$ *such that*

$$p(t, x, y) \leq c_t \phi(x)\widetilde{\phi}(y) \quad \textit{for all } (x, y) \in D \times D.$$

Assumption 1.2 is a very weak regularity assumption on D. It follows from [9] and [10] that Assumption 1.2 is satisfied when D is a bounded Lipshitz domain. For other, more general, examples of domain D for which Assumption 1.2 is satisfied, we refer the reader to [10] and the references therein.

Let $\mathcal{E}_t = \sigma(Y_s^D,\ s \leq t)$. For any $x \in D$, we define a probability measure Π_x^ϕ by the martingale change of measure:

$$\left.\frac{\mathrm{d}\Pi_x^\phi}{\mathrm{d}\Pi_x}\right|_{\mathcal{E}_t} = \frac{\phi(Y_t^D)}{\phi(x)} \exp\left\{-\int_0^{t\wedge\tau} (\lambda_1 - \beta(Y_s))\,\mathrm{d}s\right\}.$$

The process (Y^D, Π_x^ϕ) is an ergodic Markov process and its transition density is given by

$$p^\phi(t, x, y) = \frac{\exp\{-\lambda_1 t\}}{\phi(x)}\, p(t, x, y)\phi(y).$$

The function $\phi\widetilde{\phi}$ is the unique invariant density for the process (Y^D, Π_x^ϕ).

By our choices for ϕ and $\widetilde{\phi}$, $\int_D \phi(x)\widetilde{\phi}(x)\,\mathrm{d}x = 1$. Thus, it follows from [9, Theorem 2.8] that

$$\left|\frac{\exp\{-\lambda_1 t\} p(t, x, y)}{\phi(x)\widetilde{\phi}(y)} - 1\right| \leq c\mathrm{e}^{-\nu t}, \qquad x \in D,$$

for some positive constants c and ν, which is equivalent to

$$\sup_{x\in D}\left|\frac{p^\phi(t, x, y)}{\phi(y)\widetilde{\phi}(y)} - 1\right| \leq c\mathrm{e}^{-\nu t}.$$

Thus, for any $f \in L_+^\infty(D)$, we have

$$\sup_{x\in D}\left|\int_D p^\phi(t, x, y) f(y)\,\mathrm{d}y - \int_D \phi(y)\widetilde{\phi}(y) f(y)\,\mathrm{d}y\right| \leq c\mathrm{e}^{-\nu t}\int_D \phi(y)\widetilde{\phi}(y) f(y)\,\mathrm{d}y.$$

Consequently, we have

$$\begin{aligned}
&\lim_{t\to\infty}\sup_{x\in D}\sup_{f\in L_+^\infty(D)}\left(\int_D \phi(y)\widetilde{\phi}(y) f(y)\,\mathrm{d}y\right)^{-1} \\
&\quad\times\left|\int_D p^\phi(t, x, y) f(y)\,\mathrm{d}y - \int_D \phi(y)\widetilde{\phi}(y) f(y)\,\mathrm{d}y\right| \\
&\quad= 0,
\end{aligned}$$

which is equivalent to

$$\lim_{t\to\infty} \frac{\int_D p^\phi(t,x,y) f(y)\,\mathrm{d}y}{\int_D \phi(y)\widetilde{\phi}(y) f(y)\,\mathrm{d}y} = 1 \quad \text{uniformly for } f \in L^\infty(D)_+ \text{ and } x \in D. \tag{1.1}$$

For any finite Borel measure μ on D, we define a probability measure $\Pi^\phi_{\phi\mu}$ as follows:

$$\Pi^\phi_{\phi\mu} = \int_D \mu(\mathrm{d}x) \frac{\phi(x)}{\langle \phi, \mu\rangle} \Pi^\phi_x.$$

Note that, for any $A \in \mathcal{E}_t$,

$$\Pi^\phi_{\phi\mu}(A) = \frac{1}{\langle \phi, \mu\rangle} \Pi_\mu\bigg(\phi(Y^D_t) \exp\bigg\{-\int_0^{t\wedge\tau} (\lambda_1 - \beta(Y_s))\,\mathrm{d}s\bigg\} \mathbf{1}_A\bigg).$$

The superdiffusion X we are going to study is a $(Y, \psi(\lambda) - \beta\lambda)$-superprocess, which is a measure-valued Markov process with underlying spatial motion Y, branching rate $\mathrm{d}t$, and branching mechanism $\psi(\lambda) - \beta\lambda$, where

$$\psi(x,\lambda) = \int_0^\infty (\mathrm{e}^{-r\lambda} - 1 + \lambda r) n(x, \mathrm{d}r)$$

for some σ-finite kernel n from $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, that is, $n(x, \mathrm{d}r)$ is a σ-finite measure on $\mathbb{R}_+$ for each fixed x, and $n(\cdot, A)$ is a measurable function for each Borel set $A \subset \mathbb{R}_+$. In this paper we will always assume that $\sup_{x\in D} \int_0^\infty (r \wedge r^2) n(x, \mathrm{d}r) < \infty$. Note that this assumption implies that, for fixed $\lambda > 0$, $\psi(\cdot, \lambda)$ is bounded on D.

Let $(Y, \Pi_{r,x})$ denote a diffusion with generator L, birth time r, and starting point x. For any $\mu \in M_F(D)$, the family of all finite Borel measures on D, we will use $(X, \mathrm{P}_{r,\mu})$ to denote a $(Y, \psi(\lambda) - \beta\lambda)$-superprocess with starting time r such that $\mathrm{P}_{r,\mu}(X_r = \mu) = 1$. We will simply denote $(X, \mathrm{P}_{0,\mu})$ as (X, P_μ). Let $X^{t,D}$ be the exit measure from $[0, t) \times D$, and let $\partial^{t,D}$ be the union of $(0, t) \times \partial D$ and $\{t\} \times D$.

Define $\phi^t \colon [0, t] \times \overline{D} \to [0, \infty)$ for each fixed $t \ge 0$, such that $\phi^t(u, x) = \phi(x)$ for $(u, x) \in [0, t] \times D$ and $\phi^t(u, x) = 0$ for $(u, x) \in [0, t] \times \partial D$. In particular, we extend ϕ to $\overline{D}$ by setting it to be 0 on the boundary. Then

$$\{M_t(\phi) := \exp\{-\lambda_1 t\}\langle \phi^t, X^{t,D}\rangle,\ t \ge 0\} \tag{1.2}$$

is a P_μ-martingale with respect to $\mathcal{F}_t := \sigma(X^{s,D}, s \le t)$ (see Lemma 2.1, below) and $\mathrm{P}_\mu(M_t(\phi)) = \langle \phi, \mu\rangle$, $t \ge 0$. It is easy to check that $\{M_t(\phi), t \ge 0\}$ is a multiplicative functional of $X^{t,D}$.

To state our main result, we first define a new kernel $n^\phi(x, \mathrm{d}r)$ from $(D, \mathcal{B}(D))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that, for any nonnegative measurable function f on $\mathbb{R}_+$,

$$\int_0^\infty f(r) n^\phi(x, \mathrm{d}r) = \int_0^\infty f(r\phi(x)) n(x, \mathrm{d}r), \qquad x \in D.$$

The following theorem is the main result of the paper.

Theorem 1.1. *Suppose that (X_t) is a $(Y, \psi(\lambda) - \beta\lambda)$-superdiffusion starting from time 0 and with initial value μ. Set*

$$l(y) := \int_1^\infty r \log r\, n^\phi(y, \mathrm{d}r).$$

1. *If* $\int_D \widetilde{\phi}(y)l(y)\,dy < \infty$ *then* $M_\infty(\phi)$ *is nondegenerate under* P_μ *for any* $\mu \in M_F(D)$.
2. *If* $\int_D \widetilde{\phi}(y)l(y)\,dy = \infty$ *then* $M_\infty(\phi)$ *is degenerate for any* $\mu \in M_F(D)$.

The proof of this theorem is accomplished by combining the ideas from [14] with the 'spine decomposition' of [5]. The new feature here is that we consider a different branching mechanism. The new branching mechanism considered here is essential. With this branching mechanism, we can establish a strong (that is, almost-sure) version of the spine decomposition, as opposed to the weak (that is, in distribution) version in [5]. The reason is that the branching mechanism we consider here results in *discrete* immigration points, as opposed to the quadratic branching case where immigration is continuous in time.

In the next section we first give a spine decomposition of the superdiffusion X under a martingale change of measure with the help of Poisson point processes. Then, in Section 3 we use this decomposition to give a proof of Theorem 1.1.

2. Decomposition of superdiffusions under the martingale change of measure

Let $\mathcal{F}_t = \sigma(X^{s,D},\ s \le t)$. We define a probability measure $\widetilde{P}_\mu$ by the martingale change of measure:

$$\left.\frac{d\widetilde{P}_\mu}{dP_\mu}\right|_{\mathcal{F}_t} = \frac{1}{\langle \phi, \mu\rangle} M_t(\phi).$$

The purpose of this section is to give a spine decomposition of X under $\widetilde{P}_\mu$.

The most important step in proving Theorem 1.1 is a decomposition of X under $\widetilde{P}_\mu$. We could decompose X under $\widetilde{P}_\mu$ as the sum of two independent measure-valued processes. The first process is a copy of X under P_μ. The second process is, roughly speaking, obtained by taking an 'immortal particle' that moves according to the law of Y under $\Pi^{\phi}_{\phi\mu}$ and spins off pieces of mass that continue to evolve according to the dynamics of X.

To give a rigorous description of this decomposition of X under $\widetilde{P}_\mu$, let us first recall some results on Poisson point processes. Let $(S, \mathcal{S})$ be a measurable space. We will use $\mathcal{M}$ to denote the family of σ-finite counting measures on $(S, \mathcal{S})$ and $\mathcal{B}(\mathcal{M})$ to denote the smallest σ-field on $\mathcal{M}$ with respect to which all $\nu \in \mathcal{M} \mapsto \nu(B) \in \mathbb{Z}^+ \cup \{\infty\},\ B \in \mathcal{S}$, are measurable. For any σ-finite measure $\widehat{N}$ on $\mathcal{S}$, we call an $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$-valued random variable ξ a Poisson random measure with intensity $\widehat{N}$ if

(a) for each $B \in \mathcal{S}$ with $\widehat{N}(B) < \infty$, $\xi(B)$ has a Poisson distribution with parameter $\widehat{N}(B)$;

(b) for $B_1, \ldots, B_n \in \mathcal{S}$ disjoint, the variables $\xi(B_1), \ldots, \xi(B_n)$ are independent.

Suppose that $\widehat{N}$ is a σ-finite measure on $(0,\infty) \times S$, if $e = (e(t),\ t \ge 0)$ is a process taking values in $S \cup \{\Upsilon\}$, where Υ is an isolated additional point and $e(0) = \Upsilon$, such that the random counting measure $\xi = \sum_{t\ge 0} \delta_{(t,e(t))}$ is a Poisson random measure on $(0,\infty) \times S$ with intensity $\widehat{N}$, then e is called a Poisson point process with compensator $\widehat{N}$. If, for every $t > 0$, $\widehat{N}((0,t] \times S) < \infty$ then e can also be expressed as $e = (((\sigma_i, e_i),\ i = 1, \ldots, N_t),\ t \ge 0)$, where $e_i = e(\sigma_i)$ and N_t is a Poisson process with instant intensity $\widehat{N}(dt \times S)$. The following proposition follows easily from [15, Proposition 19.5].

Proposition 2.1. *Suppose that* $e = (e(t),\ t \ge 0)$ *is a Poisson point process with compensator* $\widehat{N}$. *Let* f *be a nonnegative Borel function on* $(S \cup \{\Upsilon\}) \times [0,\infty)$ *with* $f(\Upsilon, t) = 0$ *for* *all* $t > 0$*. If* $\int_{(0,t]}\int_S |1-\mathrm{e}^{-f(x,s)}|\widehat{N}(\mathrm{d}s,\mathrm{d}x) < \infty$ *for all* $t > 0$ *then*

$$\mathrm{E}\biggl(\exp\biggl\{-\sum_{0\le s\le t} f(e(s),s)\biggr\}\biggr) = \exp\biggl\{-\int_0^t\int_S (1-\mathrm{e}^{-f(x,s)})\widehat{N}(\mathrm{d}s,\mathrm{d}x)\biggr\}.$$

Moreover, if $\int_0^\infty\int_S f(s,x)\widehat{N}(\mathrm{d}s,\mathrm{d}x) < \infty$ *then*

$$\mathrm{E}\biggl(\int_0^\infty\int_S f(x,s)N(\mathrm{d}s,\mathrm{d}x)\biggr) = \int_0^\infty\int_S f(x,s)\widehat{N}(\mathrm{d}s,\mathrm{d}x). \tag{2.1}$$

To give a formula for the one-dimensional distribution of the exit measure process under $\widetilde{\mathrm{P}}_\mu$, we recall some results from [4] first.

According to [4], for any nonnegative bounded continuous function $f\colon \partial^{t,D} \to \mathbb{R}$, we have

$$\mathrm{P}_{r,\mu}(\exp\langle -f, X^{t,D}\rangle) = \exp\langle -U^t(f)(r,\cdot),\mu\rangle, \tag{2.2}$$

where $U^t(f)$ denotes the unique nonnegative solution to

$$\begin{gathered} -\frac{\partial U(s,x)}{\partial s} = LU + \beta U(s,x) - \psi(U(s,x)), \qquad x\in D,\ s\in(0,\ t),\\ U = f \quad \text{on } \partial^{t,D}. \end{gathered} \tag{2.3}$$

More precisely, $U^t(f)$ satisfies the following integral equation:

$$\begin{aligned} &U^t(f)(r,x) + \Pi_{r,x}\int_r^{t\wedge\tau_r}[\psi(U^t(f))(s,Y_s) - \beta(Y_s)U^t(f)(s,Y_s)]\,\mathrm{d}s\\ &\quad = \Pi_{r,x} f(t\wedge\tau_r, Y_{t\wedge\tau_r}), \qquad r\le t,\ x\in D, \end{aligned} \tag{2.4}$$

where $\tau_r = \inf\{t \ge r\colon X_t \notin D\}$. Since Y is a time-homogeneous process, we find that $X^{t,D}$ under $\mathrm{P}_{r,\mu}$ has the same distribution as $X^{t-r,D}$ under P_μ. The first moment of $\langle f, X^{t,D}\rangle$ is given by

$$\mathrm{P}_{r,x}\langle f, X^{t,D}\rangle = \Pi_{r,x}\biggl(f(t\wedge\tau_r, Y_{t\wedge\tau_r})\exp\biggl\{\int_r^{t\wedge\tau_r}\beta(Y_s)\,\mathrm{d}s\biggr\}\biggr). \tag{2.5}$$

Lemma 2.1. $\{M_t(\phi),\ t\ge 0\}$ *is a* P_μ*-martingale with respect to* $\mathcal{F}_t$*.*

Proof. It follows from the first moment formula (2.5) that

$$\begin{aligned} \mathrm{P}_{r,x}\langle \phi^t, X^{t,D}\rangle &= \Pi_{r,x}\biggl(\phi(Y_t)\exp\biggl\{\int_r^t\beta(Y_s)\,\mathrm{d}s\biggr\},\ t<\tau_r\biggr)\\ &= \mathrm{P}_{t-r}\,\phi(x) \quad \text{for } r\le t,\ x\in D. \end{aligned}$$

It is obvious that $\mathrm{P}_{r,x}\langle\phi^t, X^{t,D}\rangle = 0$ for $x\in\partial D$. By the special Markov property of X and the invariance of ϕ under $\exp\{-\lambda_1 t\}P_t$,

$$\begin{aligned} \mathrm{P}_\mu(M_t(\phi)\mid\mathcal{F}_s) &= \exp\{-\lambda_1 s\}\,\mathrm{P}_{X^{s,D}}(\exp\{-\lambda_1(t-s)\}\langle\phi^t, X^{t,D}\rangle)\\ &= \exp\{-\lambda_1 s\}\langle \exp\{-\lambda_1(t-s)\}\,\mathrm{P}_{t-s}\,\phi, X^{s,D}|_D\rangle\\ &= \exp\{-\lambda_1 s\}\langle\phi^s, X^{s,D}\rangle\\ &= M_s(\phi) \quad \text{for } s\le t, \end{aligned}$$

where $X^{s,D}|_D$ is the restriction of the measure $X^{s,D}$ on $\{s\}\times D$.

Now we give a formula for the one-dimensional distribution of X under $\widetilde{\mathrm{P}}_\mu$.

Theorem 2.1. *Suppose that μ is a finite measure on D and that $g \in C_b^+(\partial^{t,D})$. Then*

$$\widetilde{\mathrm{P}}_\mu(\exp\langle -g, X^{t,D}\rangle) = \mathrm{P}_\mu(\exp\langle -g, X^{t,D}\rangle) \times \Pi^\phi_{\phi\mu}\left(\exp\left\{-\int_0^{t\wedge\tau} \psi'(Y_s, U^t(g)(s, Y_s))\,\mathrm{d}s\right\}\right), \tag{2.6}$$

where $U^t(g)$ is the unique solution of (2.3) or, equivalently, (2.4) with f replaced by g.

Proof. This theorem can be proved using the same argument as that given in [5] to obtain Theorem 5 therein, with some obvious modifications. We omit the details.

From (2.6) we can see that the superprocess $(X^{t,D}, \widetilde{\mathrm{P}}_\mu)$ can be decomposed into two independent parts in the sense of distributions. The first part is a copy of the original superprocess and the second part is an immigration process. To explain the second part more precisely, we need to introduce another measure-valued process $(\widehat{X}_t)$. Now we construct the measure-valued process $(\widehat{X}_t)$ as follows.

(a) Suppose that $\widetilde{Y} = (\widetilde{Y}_t,\ t \geq 0)$ is defined on some probability space $(\Omega, \mathrm{P}_{\mu,\phi})$ and that $\widetilde{Y} = (\widetilde{Y}_t,\ t \geq 0)$ has the same law as $(Y, \Pi^\phi_{\phi\mu})$. Here $\widetilde{Y}$ serves as the spine or the immortal particle, which visits every part of D for large times since it is an ergodic diffusion.

(b) Suppose that $m = \{m_t,\ t \geq 0\}$ is a point process taking values in $(0, \infty) \cup \{\Upsilon\}$ such that, conditional on $\sigma(\widetilde{Y}_t,\ t \geq 0)$, m is a Poisson point process with intensity $rn(\widetilde{Y}_t, \mathrm{d}r)$. Now $(0, \infty)$ is the 'space of mass' and $m_t = \Upsilon$ simply means that there is no immigration at t. We suppose that $\{m_t,\ t \geq 0\}$ is also defined on $(\Omega, \mathrm{P}_{\mu,\phi})$. Set $\mathcal{D}_m = \{t\colon m_t(\omega) \neq \Upsilon\}$. Note that $\mathcal{D}_m$ is almost surely (a.s.) countable. The process m describes the immigration mechanism: along the path of $\widetilde{Y}$, at the moment $t \in \mathcal{D}_m$, a particle with mass m_t is immigrated into the system at the position $\widetilde{Y}_t$.

(c) Once the particles are in the system, they begin to move and branch according to a $(Y, \psi(\lambda) - \beta\lambda)$-superprocess independently.

We use $(X^\sigma_t,\ t \geq \sigma)$ to denote the measure-valued process generated by the mass immigrated at time σ and position $\widetilde{Y}_\sigma$. Conditional on $\{\widetilde{Y}_t, m_t,\ t \geq 0\}$, $\{X^\sigma,\ \sigma \in \mathcal{D}_m\}$ are independent $(Y, \psi - \beta\lambda)$-superprocesses. The birth time of X^σ is σ and the initial value of X^σ is $m_\sigma \delta_{\widetilde{Y}_\sigma}$. Set

$$\widehat{X}^{t,D} = \sum_{\sigma \in (0,t] \cap \mathcal{D}_m} X^{\sigma,(t,D)},$$

where, for each $\sigma \in \mathcal{D}_m$, $X^{\sigma,(t,D)}$ is the exit measure of the superprocess X^σ from $[0, t) \times D$. The Laplace functional of $\widehat{X}^{t,D}$ is described in the following proposition.

Proposition 2.2. *The Laplace functional of $\widehat{X}^{t,D}$ under $\mathrm{P}_{\mu,\phi}$ is*

$$\Pi^\phi_{\phi\mu}\left(\exp\left\{-\int_0^t \psi'(Y_s, U^t(g)(Y_s, s))\,\mathrm{d}s\right\}\right).$$

Proof. For any $g \in C_b^+(\partial^{t,D})$, using (2.2), we have

$$\begin{aligned}
\mathrm{P}_{\mu,\phi}(\exp\{-\langle g, \widehat{X}^{t,D}\rangle\}) &= \mathrm{P}_{\mu,\phi}\Big(\mathrm{P}_{\mu,\phi}\Big(\exp\Big\{-\sum_{\sigma\in(0,t]\cap\mathcal{D}_m}\langle g, X^{\sigma,(t,D)}\rangle\Big\}\,\Big|\,\widetilde{Y},\sigma,m\Big)\Big)\\
&= \mathrm{P}_{\mu,\phi}\Big(\prod_{\sigma\in(0,t]\cap\mathcal{D}_m}\exp\{-m_\sigma U^t(g)(\widetilde{Y}_\sigma,\sigma)\}\Big)\\
&= \mathrm{P}_{\mu,\phi}\Big(\mathrm{P}_{\mu,\phi}\Big(\exp\Big\{-\sum_{\sigma\in[0,t]\cap\mathcal{D}_m} m_\sigma U^t(g)(\widetilde{Y}_\sigma,\sigma)\Big\}\,\Big|\,\widetilde{Y}\Big)\Big).
\end{aligned}$$

Using Proposition 2.1, we obtain

$$\begin{aligned}
&\mathrm{P}_{\mu,\phi}(\exp\{-\langle g, \widehat{X}^{t,D}\rangle\})\\
&\quad= \Pi^{\phi}_{\phi\mu}\exp\Big\{-\int_0^t\int_0^\infty(1-\exp\{-rU^t(g)(Y_s,s)\})rn(Y_s,\mathrm{d}r)\,\mathrm{d}s\Big\}\\
&\quad= \Pi^{\phi}_{\phi\mu}\Big(\exp\Big\{-\int_0^t\psi'(Y_s,U^t(g)(s,Y_s))\,\mathrm{d}s\Big\}\Big).
\end{aligned}$$

Without loss of generality, we suppose that $(X_t,\ t\ge 0;\mathrm{P}_{\mu,\phi})$ is a superdiffusion defined on $(\Omega,\mathrm{P}_{\mu,\phi})$, equivalent to $(X_t,\ t\ge 0;\mathrm{P}_\mu)$ and independent of $\widehat{X}$. Proposition 2.2 says that we have the following decomposition of $X^{t,D}$ under $\widetilde{\mathrm{P}}_\mu$: for any $t>0$,

$$(X^{t,D},\widetilde{\mathrm{P}}_\mu) = (X^{t,D}+\widehat{X}^{t,D},\mathrm{P}_{\mu,\phi}) \quad \text{in distribution,} \tag{2.7}$$

where $X^{t,D}$ is the exit measure of X from $[0,t)\times D$. Since $(X_t,\ t\ge 0;\widetilde{\mathrm{P}}_\mu)$ is generated from the time-homogeneous Markov process $(X_t,\ t\ge 0;\mathrm{P}_\mu)$ via a nonnegative martingale multiplicative functional, $(X_t,\ t\ge 0;\widetilde{\mathrm{P}}_\mu)$ is also a time-homogeneous Markov process (see [17, Section 62]). From the construction of $(\widehat{X}^{t,D},\ t\ge 0;\mathrm{P}_{\mu,\phi})$ we see that $(\widehat{X}^{t,D},\ t\ge 0;\mathrm{P}_{\mu,\phi})$ is a time-homogeneous Markov process. For a rigorous proof of $(\widehat{X}^{t,D},\ t\ge 0;\mathrm{P}_{\mu,\phi})$ being a time-homogeneous Markov process, we refer the reader to [6]. Although [6] dealt with the representation of the superprocess conditioned to stay alive forever, we can check that the arguments there work in our case. Therefore, (2.7) implies that

$$(X^{t,D},\ t\ge 0;\widetilde{\mathrm{P}}_\mu) = (X^{t,D}+\widehat{X}^{t,D},\ t\ge 0;\mathrm{P}_{\mu,\phi}) \quad \text{in distribution.}$$

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need some preparations. The following elementary result is taken from [3].

Lemma 3.1. ([3, Exercise 1.3.8].) *Let $Y\ge 0$ with $\mathrm{E}(Y)<\infty$, and let $0\le a<\mathrm{E}(Y)$. Then*

$$\mathrm{P}(Y>a)\ge\frac{(\mathrm{E}(Y)-a)^2}{\mathrm{E}(Y^2)}.$$

Proposition 3.1. *Set $h(x)=\mathrm{P}_{\delta_x}(M_\infty(\phi))/\phi(x)$.*

1. *h is nonnegative and invariant for the process (Y^D,Π^ϕ_x).*
2. *Either M_∞ is nondegenerate under P_μ for all $\mu\in M_F(D)$ or M_∞ is degenerate under P_μ for all $\mu\in M_F(D)$.*

Proof. 1. Since $\phi^t(\cdot, u) = \phi(\cdot)$ for each $u \in [0, t]$ and ϕ is identically 0 on ∂D, we have, by the special Markov property of X,

$$
\begin{aligned}
h(x) &= \frac{1}{\phi(x)} \mathrm{P}_{\delta_x}\Big(\lim_{s\to\infty} \langle \exp\{-\lambda_1(t+s)\}\phi^{t+s}, X^{t+s,D}\rangle\Big) \\
&= \frac{\exp\{-\lambda_1 t\}}{\phi(x)} \mathrm{P}_{\delta_x}\Big(\mathrm{P}_{X^{t,D}}\Big(\lim_{s\to\infty} \langle \exp\{-\lambda_1 s\}\phi^{s}, X^{s,D}\rangle\Big)\Big) \\
&= \frac{\exp\{-\lambda_1 t\}}{\phi(x)} \mathrm{P}_{\delta_x}(\mathrm{P}_{X^{t,D}}(M_\infty)) \\
&= \frac{\exp\{-\lambda_1 t\}}{\phi(x)} \mathrm{P}_{\delta_x}(\langle (h\phi)^t, X^{t,D}\rangle) \\
&= \frac{\exp\{-\lambda_1 t\}}{\phi(x)} \Pi_x\left(\exp\left\{\int_0^t \beta(Y_s)\,\mathrm{d}s\right\}(h\phi)(Y_t),\ t<\tau\right) \\
&= \frac{1}{\phi(x)} \Pi_x\left(\exp\left\{\int_0^{t\wedge\tau} (\beta-\lambda_1)(Y_s)\,\mathrm{d}s\right\}(h\phi)(Y_t^D)\right), \qquad x \in D.
\end{aligned}
$$

By the definition of Π_x^ϕ we obtain $h(x) = \Pi_x^\phi(h(Y_t^D))$. So, h is an invariant function of the process (Y^D, Π_x^ϕ). The nonnegativity of h is obvious.

2. Since h is nonnegative and invariant, if there exists an $x_0 \in D$ such that $h(x_0) = 0$, then $h \equiv 0$ on D. Since $\mathrm{P}_\mu(M_\infty(\phi)) = \langle h\phi, \mu\rangle$, we then have $\mathrm{P}_\mu(M_\infty(\phi)) = 0$ for any $\mu \in M_F(D)$. If $h > 0$ on D then $\mathrm{P}_\mu(M_\infty(\phi)) > 0$ for any $\mu \in M_F(D)$.

Using Proposition 3.1, we see that, to prove Theorem 1.1, we only need to consider the case $\mu(\mathrm{d}x) = \widetilde{\phi}(x)\,\mathrm{d}x$. So, in the remaining part of this paper we will always suppose that $\mu(\mathrm{d}x) = \widetilde{\phi}(x)\,\mathrm{d}x$.

Lemma 3.2. *Let $(m_t,\ t \geq 0)$ be the Poisson point process constructed in Section 2. Define*

$$\sigma_0 = 0, \qquad \sigma_i = \inf\{s \in \mathcal{D}_m : s > \sigma_{i-1},\ m_s\phi(\widetilde{Y}_s) > 1\}, \qquad \eta_i = m_{\sigma_i}, \qquad i = 1, 2, \ldots.$$

If $\int_D \widetilde{\phi}(y) l(y)\,\mathrm{d}y < \infty$ then

$$\sum_{s\in\mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s) < \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

If $\int_D \widetilde{\phi}(y) l(y)\,\mathrm{d}y = \infty$ then

$$\limsup_{i\to\infty} \exp\{-\lambda_1 \sigma_i\}\eta_i \phi(\widetilde{Y}_{\sigma_i}) = \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

Proof. Since ϕ is bounded from above, σ_i is strictly increasing with respect to i. We first prove that if $\int_D \widetilde{\phi}(y) l(y)\,\mathrm{d}y < \infty$ then

$$\sum_{s\in\mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s) < \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

For any $\varepsilon > 0$, we write the sum above as

$$\begin{aligned}
&\sum_{s\in\mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s) \\
&\quad = \sum_{s\in\mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s)\, \mathbf{1}_{\{\phi(\widetilde{Y}_s) m_s \le \mathrm{e}^{\varepsilon s}\}} + \sum_{s\in\mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s)\, \mathbf{1}_{\{m_s \phi(\widetilde{Y}_s) > \mathrm{e}^{\varepsilon s}\}} \\
&\quad = \sum_{s\in\mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s)\, \mathbf{1}_{\{\phi(\widetilde{Y}_s) m_s \le \mathrm{e}^{\varepsilon s}\}} + \sum_{i=1}^{\infty} \exp\{-\lambda_1 \sigma_i\} \eta_i \phi(\widetilde{Y}_{\sigma_i})\, \mathbf{1}_{\{\eta_i \phi(\widetilde{Y}_{\sigma_i}) > \exp\{\varepsilon\sigma_i\}\}} \\
&\quad =: I + II. \qquad (3.1)
\end{aligned}$$

By (2.1) we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathrm{P}_{\mu,\phi}(\eta_i \phi(\widetilde{Y}_{\sigma_i}) > \exp\{\varepsilon\sigma_i\}) &= \sum_{i=1}^{\infty} \mathrm{P}_{\mu,\phi}(\mathrm{P}_{\mu,\phi}(\eta_i \phi(\widetilde{Y}_{\sigma_i}) > \exp\{\varepsilon\sigma_i\} \mid \sigma(\widetilde{Y}))) \\
&= \mathrm{P}_{\mu,\phi}\left(\mathrm{P}_{\mu,\phi}\left(\sum_{i=1}^{\infty} \mathbf{1}_{\{\eta_i > \exp\{\varepsilon\sigma_i\}\phi(\widetilde{Y}_{\sigma_i})^{-1}\}} \,\middle|\, \sigma(\widetilde{Y})\right)\right) \\
&= \Pi^{\phi}_{\phi\mu}\left(\int_0^{\infty}\left(\int_{\phi(Y_s)^{-1}\mathrm{e}^{\varepsilon s}}^{\infty} r n(Y_s, \mathrm{d}r)\right) \mathrm{d}s\right).
\end{aligned}$$

Recall that, under $\Pi^{\phi}_{\phi\mu}$, Y starts at the invariant measure $\phi(x)\mu(\mathrm{d}x) = \phi(x)\widetilde{\phi}(x)\,\mathrm{d}x$. So we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathrm{P}_{\mu,\phi}(\eta_i \phi(\widetilde{Y}_{\sigma_i}) > \exp\{\varepsilon\sigma_i\}) &= \int_0^{\infty} \mathrm{d}s \int_D \mathrm{d}y \phi(y)\widetilde{\phi}(y) \int_{\phi(y)^{-1}\mathrm{e}^{\varepsilon s}}^{\infty} r n(y, \mathrm{d}r) \\
&= \int_D \phi(y)\widetilde{\phi}(y)\,\mathrm{d}y \int_{\phi(y)^{-1}}^{\infty} r n(y, \mathrm{d}r) \int_0^{\ln(r\phi(y))/\varepsilon} \mathrm{d}s \\
&= \varepsilon^{-1} \int_D \widetilde{\phi}(y) l(y)\,\mathrm{d}y.
\end{aligned}$$

By the assumption that $\int_D \widetilde{\phi}(y) l(y)\,\mathrm{d}y < \infty$ and the Borel–Cantelli lemma, we obtain

$$\mathrm{P}_{\mu,\phi}(\eta_i \phi(\widetilde{Y}_{\sigma_i}) > \exp\{\varepsilon\sigma_i\} \text{ infinitely often}) = 0 \quad \text{for all } \varepsilon > 0,$$

which implies that

$$II < \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.} \qquad (3.2)$$

Meanwhile, for $\varepsilon < \lambda_1$,

$$\begin{aligned}
\mathrm{P}_{\mu,\phi}\, I &= \mathrm{P}_{\mu,\phi}\left(\sum_{s\in\mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s)\, \mathbf{1}_{\{m_s \le \mathrm{e}^{\varepsilon s}\phi(\widetilde{Y}_s)^{-1}\}}\right) \\
&= \Pi^{\phi}_{\phi\mu} \int_0^{\infty} \mathrm{d}t \exp\{-\lambda_1 t\} \int_0^{\phi(Y_t)^{-1}\mathrm{e}^{\varepsilon t}} \phi(Y_t) r^2 n(Y_t, \mathrm{d}r) \\
&\le \|\phi\|_{\infty} \Pi^{\phi}_{\phi\mu} \int_0^{\infty} \mathrm{d}t \exp\{-\lambda_1 t\} \int_0^1 r^2 n(Y_t, \mathrm{d}r) \\
&\quad + \Pi^{\phi}_{\phi\mu} \int_0^{\infty} \mathrm{d}t \exp\{-(\lambda_1 - \varepsilon)t\} \int_1^{\infty} r n(Y_t, \mathrm{d}r),
\end{aligned}$$

where for the second term of the last inequality we used the fact that $r \le \phi(Y_t)^{-1}\mathrm{e}^{\varepsilon t}$ implies that $r\phi(Y_t) \le \mathrm{e}^{\varepsilon t}$. By the assumption that $\sup_{x\in D}\int_0^\infty (r \wedge r^2)n(x, \mathrm{d}r) < \infty$ we have $\mathrm{P}_{\mu,\phi}\, I < \infty$, which implies that

$$I < \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.} \tag{3.3}$$

Combining (3.1), (3.2), and (3.3), we see that $\sum_{s\in\mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s) < \infty$, $\mathrm{P}_{\mu,\phi}$-a.s.

Next we prove that if $\int_D \widetilde{\phi}(y) l(y)\,\mathrm{d}y = \infty$ then

$$\limsup_{i\to\infty} \exp\{-\lambda_1\sigma_i\}\eta_i\phi(\widetilde{Y}_{\sigma_i}) = \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

It suffices to prove that, for any $K > 0$,

$$\limsup_{i\to\infty} \exp\{-\lambda_1\sigma_i\}\eta_i\phi(\widetilde{Y}_{\sigma_i}) > K \quad \mathrm{P}_{\mu,\phi}\text{-a.s.} \tag{3.4}$$

Set $K_0 := 1 \vee (\max_{x\in D}\phi(x))$. Then, for $K \ge K_0$,

$$K \inf_{x\in D} \phi(x)^{-1} \ge 1.$$

Note that, for any $T \in (0, \infty)$, conditional on $\sigma(\widetilde{Y})$,

$$\sharp\{i : \sigma_i \in (0, T];\, \eta_i > K\phi(\widetilde{Y}_{\sigma_i})^{-1}\exp\{\lambda_1\sigma_i\}\}$$

is a Poisson random variable with parameter $\int_0^T \mathrm{d}t \int_{K\phi(\widetilde{Y}_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(\widetilde{Y}_t, \mathrm{d}r)$ a.s. Since $(\widetilde{Y}, \mathrm{P}_{\mu,\phi})$ has the same distribution as $(Y, \Pi^\phi_{\mu\phi})$, we have

$$\begin{aligned}
&\mathrm{P}_{\mu,\phi} \int_0^T \mathrm{d}t \int_{K\phi(\widetilde{Y}_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(\widetilde{Y}_t, \mathrm{d}r)\\
&\quad = \int_0^T \mathrm{d}t \int_D \mathrm{d}y \phi(y)\widetilde{\phi}(y) \int_{K\phi(y)^{-1}\exp\{\lambda_1 t\}}^\infty r n(y, \mathrm{d}r)\\
&\quad < \infty;
\end{aligned}$$

thus,

$$\int_0^T \mathrm{d}t \int_{K\phi(\widetilde{Y}_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(\widetilde{Y}_t, \mathrm{d}r) < \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

Consequently, we have

$$\sharp\{i : \sigma_i \in (0, T];\, \eta_i > K\phi(\widetilde{Y}_{\sigma_i})^{-1}\exp\{\lambda_1\sigma_i\}\} < \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

So, to prove (3.4), we need to prove that

$$\int_0^\infty \mathrm{d}t \int_{K\phi(\widetilde{Y}_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(\widetilde{Y}_t, \mathrm{d}r) = \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.},$$

which is equivalent to

$$\int_0^\infty \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(Y_t, \mathrm{d}r) = \infty \quad \Pi^\phi_{\phi\mu}\text{-a.s.} \tag{3.5}$$

For this purpose, we first prove that

$$\Pi^{\phi}_{\phi\mu}\bigg(\int_0^\infty \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty rn(Y_t,\mathrm{d}r)\bigg) = \infty. \tag{3.6}$$

Applying Fubini's theorem, we obtain

$$\begin{aligned}
&\Pi^{\phi}_{\phi\mu}\bigg(\int_0^\infty \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty rn(Y_t,\mathrm{d}r)\bigg)\\
&\quad= \int_D \phi(y)\widetilde{\phi}(y)\,\mathrm{d}y \int_0^\infty \mathrm{d}t \int_{K\phi(y)^{-1}\exp\{\lambda_1 t\}}^\infty rn(y,\mathrm{d}r)\\
&\quad= \int_D \phi(y)\widetilde{\phi}(y)\,\mathrm{d}y \int_{K\phi(y)^{-1}}^\infty rn(y,\mathrm{d}r) \int_0^{(1/\lambda_1)\ln(r\phi(y)/K)} \mathrm{d}t\\
&\quad= \frac{1}{\lambda_1}\int_D \phi(y)\widetilde{\phi}(y)\,\mathrm{d}y \int_{K\phi(y)^{-1}}^\infty (\ln[r\phi(y)] - \ln K) rn(y,\mathrm{d}r)\\
&\quad\ge \frac{1}{\lambda_1}\int_D \phi(y)\widetilde{\phi}(y)\,\mathrm{d}y \bigg(\int_{K\phi(y)^{-1}}^\infty r\ln[r\phi(y)]n(y,\mathrm{d}r) - A\bigg)\\
&\quad= \frac{1}{\lambda_1}\int_D \widetilde{\phi}(y)\,\mathrm{d}y \int_K^\infty r\ln r n^{\phi}(y,\mathrm{d}r) - \frac{A}{\lambda_1}\int_D \widetilde{\phi}(y)\phi(y)\,\mathrm{d}y
\end{aligned}$$

for some positive constant A, where in the inequality we used the facts that $K\phi(y)^{-1} > 1$ for any $y \in D$ and $\sup_{y\in D}\int_1^\infty rn(y,\mathrm{d}r) < \infty$. Since

$$\int_D \widetilde{\phi}(y)\,\mathrm{d}y \int_1^\infty r\ln r\, n^{\phi}(y,\mathrm{d}r) = \infty$$

and

$$\int_D \widetilde{\phi}(y)\,\mathrm{d}y \int_1^K r\ln r n^{\phi}(y,\mathrm{d}r) \le K\log K \int_D \widetilde{\phi}(y) n(y,[\|\phi\|_\infty^{-1},\infty))\,\mathrm{d}y < \infty,$$

we obtain

$$\int_D \widetilde{\phi}(y)\,\mathrm{d}y \int_K^\infty r\ln r n^{\phi}(y,\mathrm{d}r) = \infty,$$

and, therefore, (3.6) holds.

By (1.1), there exists a constant $c > 0$ such that, for any $t > c$ and any $f \in L_+^\infty(D)$,

$$\frac{1}{2}\int_D \phi(y)\widetilde{\phi}(y)f(y)\,\mathrm{d}y \le \int_D p^{\phi}(t,x,y)f(y)\,\mathrm{d}y \le 2\int_D \phi(y)\widetilde{\phi}(y)f(y)\,\mathrm{d}y, \qquad x\in D. \tag{3.7}$$

For $T > c$, we define

$$\xi_T = \int_0^T \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty rn(Y_t,\mathrm{d}r), \qquad A_T = \int_c^T \mathrm{d}t \int_D \widetilde{\phi}(y)\,\mathrm{d}y \int_{K\exp\{\lambda_1 t\}}^\infty rn^{\phi}(y,\mathrm{d}r).$$

Our goal is to prove (3.5), which is equivalent to

$$\xi_\infty := \int_0^\infty \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty rn(Y_t,\mathrm{d}r) = \infty \quad \Pi^{\phi}_{\phi\mu}\text{-a.s.}$$

Since $\{\xi_\infty = \infty\}$ is an invariant event, by the ergodic property of Y under $\Pi^\phi_{\phi\mu}$, it is enough to prove that

$$\Pi^\phi_{\phi\mu}(\xi_\infty = \infty) > 0. \tag{3.8}$$

Note that

$$\Pi^\phi_{\phi\mu}\xi_T = \int_0^T \mathrm{d}t \int_D \widetilde{\phi}(y)\,\mathrm{d}y \int_{K\exp\{\lambda_1 t\}}^\infty r n^\phi(y, \mathrm{d}r) \ge A_T \tag{3.9}$$

and

$$\begin{aligned}\lim_{T\to\infty} \Pi^\phi_{\phi\mu}\xi_T &\ge A_\infty \\ &= \int_c^\infty \mathrm{d}t \int_D \widetilde{\phi}(y)\,\mathrm{d}y \int_{K\exp\{\lambda_1 t\}}^\infty r n^\phi(y, \mathrm{d}r) \\ &= \int_D \widetilde{\phi}(y)\,\mathrm{d}y \int_{K\exp\{\lambda_1 c\}}^\infty \left(\frac{1}{\lambda_1}(\log r - \log K) - c\right) r n^\phi(y, \mathrm{d}r) \\ &\ge C \int_D \widetilde{\phi}(y) l(y)\,\mathrm{d}y \\ &= \infty,\end{aligned} \tag{3.10}$$

where C is a positive constant. By Lemma 3.1,

$$\Pi^\phi_{\phi\mu}\left(\xi_T \ge \frac{1}{2}\Pi^\phi_{\phi\mu}\xi_T\right) \ge \frac{(\Pi^\phi_{\phi\mu}\xi_T)^2}{4\Pi^\phi_{\phi\mu}(\xi_T^2)}. \tag{3.11}$$

If we can prove that there exists a constant $\widehat{C} > 0$ such that, for all $T > c$,

$$\frac{(\Pi^\phi_{\phi\mu}\xi_T)^2}{4\Pi^\phi_{\phi\mu}(\xi_T^2)} \ge \widehat{C}, \tag{3.12}$$

then by (3.11) we would obtain

$$\Pi^\phi_{\phi\mu}\big(\xi_T \ge \tfrac{1}{2}\Pi^\phi_{\phi\mu}\xi_T\big) \ge \widehat{C},$$

and, therefore,

$$\Pi^\phi_{\phi\mu}\big(\xi_\infty \ge \tfrac{1}{2}\Pi^\phi_{\phi\mu}\xi_T\big) \ge \Pi^\phi_{\phi\mu}\big(\xi_T \ge \tfrac{1}{2}\Pi^\phi_{\phi\mu}\xi_T\big) \ge \widehat{C} > 0.$$

Since $\lim_{T\to\infty} \Pi^\phi_{\phi\mu}\xi_T = \infty$ (see (3.10)), the above inequality implies (3.8). Now we only need to prove (3.12). For this purpose, we first estimate $\Pi^\phi_{\phi\mu}(\xi_T^2)$:

$$\begin{aligned}\Pi^\phi_{\phi\mu}\xi_T^2 &= \Pi^\phi_{\phi\mu}\int_0^T \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(Y_t, \mathrm{d}r) \int_0^T \mathrm{d}s \int_{K\phi(Y_s)^{-1}\exp\{\lambda_1 s\}}^\infty u n(Y_s, \mathrm{d}u) \\ &= 2\Pi^\phi_{\phi\mu}\int_0^T \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(Y_t, \mathrm{d}r) \int_t^T \mathrm{d}s \int_{K\phi(Y_s)^{-1}\exp\{\lambda_1 s\}}^\infty u n(Y_s, \mathrm{d}u) \\ &= 2\Pi^\phi_{\phi\mu}\int_0^T \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(Y_t, \mathrm{d}r) \int_t^{(t+c)\wedge T} \mathrm{d}s \int_{K\phi(Y_s)^{-1}\exp\{\lambda_1 s\}}^\infty u n(Y_s, \mathrm{d}u) \\ &\quad + 2\Pi^\phi_{\phi\mu}\int_0^T \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^\infty r n(Y_t, \mathrm{d}r) \int_{(t+c)\wedge T}^T \mathrm{d}s \int_{K\phi(Y_s)^{-1}\exp\{\lambda_1 s\}}^\infty u n(Y_s, \mathrm{d}u) \\ &= III + IV,\end{aligned}$$

where

$$III = 2\Pi_{\phi\mu}^{\phi} \int_0^T \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^{\infty} rn(Y_t, \mathrm{d}r) \int_t^{(t+c)\wedge T} \mathrm{d}s \int_{K\phi(Y_s)^{-1}\exp\{\lambda_1 s\}}^{\infty} un(Y_s, \mathrm{d}u)$$

and

$$\begin{aligned} IV &= 2\Pi_{\phi\mu}^{\phi} \int_0^T \mathrm{d}t \int_{K\phi(Y_t)^{-1}\exp\{\lambda_1 t\}}^{\infty} rn(Y_t, \mathrm{d}r) \int_{(t+c)\wedge T}^{T} \mathrm{d}s \int_{K\phi(Y_s)^{-1}\exp\{\lambda_1 s\}}^{\infty} un(Y_s, \mathrm{d}u) \\ &= 2\int_0^T \mathrm{d}t \int_D \phi(y)\widetilde{\phi}(y)\,\mathrm{d}y \int_{K\phi(y)^{-1}\exp\{\lambda_1 t\}}^{\infty} rn(y, \mathrm{d}r) \\ &\quad \times \int_{(t+c)\wedge T}^{T} \mathrm{d}s \int_D p^{\phi}(s-t, y, z)\,\mathrm{d}z \int_{K\phi(z)^{-1}\exp\{\lambda_1 s\}}^{\infty} un(z, \mathrm{d}u). \end{aligned}$$

By our assumption on the kernel n we have $\| \int_1^{\infty} rn(\cdot, \mathrm{d}r)\|_{\infty} < \infty$. Since $K \inf_{x\in B} \phi(x)^{-1} \geq 1$, we have

$$III \leq C_1 \Pi_{\phi\mu}^{\phi} \xi_T$$

for some positive constant C_1 which does not depend on T. Using (3.7) and the definition of n^{ϕ}, we obtain

$$\begin{aligned} &\int_{(t+c)\wedge T}^{T} \mathrm{d}s \int_D p^{\phi}(s-t, y, z)\,\mathrm{d}z \int_{K\phi(z)^{-1}\exp\{\lambda_1 s\}}^{\infty} un(z, \mathrm{d}u) \\ &\quad \leq 2\int_{(t+c)\wedge T}^{T} \mathrm{d}s \int_D \phi(z)\widetilde{\phi}(z)\,\mathrm{d}z \int_{K\phi(z)^{-1}\exp\{\lambda_1 s\}}^{\infty} un(z, \mathrm{d}u) \\ &\quad \leq 2\int_c^T \mathrm{d}s \int_D \widetilde{\phi}(z)\,\mathrm{d}z \int_0^{\infty} (\phi(z)u)\, \mathbf{1}_{\{\phi(z)u > k\exp\{\lambda_1 s\}\}}\, n(z, \mathrm{d}u) \\ &\quad = 2\int_c^T \mathrm{d}s \int_D \widetilde{\phi}(z)\,\mathrm{d}z \int_{k\exp\{\lambda_1 s\}}^{\infty} rn^{\phi}(z, \mathrm{d}r) \\ &\quad = 2A_T. \end{aligned}$$

Then, using (3.9), we have

$$IV \leq 4A_T \Pi_{\phi\mu}^{\phi} \xi_T \leq 4(\Pi_{\phi\mu}^{\phi} \xi_T)^2.$$

Combining the above estimates for III and IV, we find that there exists a $C_2 > 0$ independent of T such that, for $T > c$,

$$\Pi_{\phi\mu}^{\phi}(\xi_T^2) \leq 4(\Pi_{\phi\mu}^{\phi}\xi_T)^2 + C_1 \Pi_{\phi\mu}^{\phi}\xi_T \leq C_2(\Pi_{\phi\mu}^{\phi}\xi_T)^2.$$

Then we have (3.12) with $\widehat{C} = 1/C_2$, and the proof of the theorem is now complete.

Definition 3.1. Suppose that $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, that $\{\mathcal{F}_t,\ t \geq 0\}$ is a filtration on $(\Omega, \mathcal{F})$, and that $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. A real-valued process U_t on $(\Omega, \mathcal{F}, \mathrm{P})$ is called a $\mathrm{P}(\cdot \mid \mathcal{G})$-martingale with respect to $\{\mathcal{F}_t,\ t \geq 0\}$ if

(i) it is adapted to $\{\mathcal{F}_t \vee \mathcal{G},\ t \geq 0\}$;

(ii) for any $t \geq 0$, $\mathrm{E}(|U_t| \mid \mathcal{G}) < \infty$; and

(iii) for any $t > s$,

$$\mathrm{E}(U_t \mid \mathcal{F}_s \vee \mathcal{G}) = U_s \quad \text{a.s.}$$

We say that U_t on $(\Omega, \mathcal{F}, \mathrm{P})$ is a $\mathrm{P}(\cdot \mid \mathcal{G})$-submartingale or a $\mathrm{P}(\cdot \mid \mathcal{G})$-supermartingale with respect to $\{\mathcal{F}_t,\ t \geq 0\}$ if, in addition to (i) and (ii), for any $t > s$,

$$\mathrm{E}(U_t \mid \mathcal{F}_s \vee \mathcal{G}) \geq U_s \quad \text{a.s.}$$

or, respectively,

$$\mathrm{E}(U_t \mid \mathcal{F}_s \vee \mathcal{G}) \leq U_s \quad \text{a.s.}$$

The following result is a folklore. Since we could not find a reference for this result, we provide a proof for completeness.

Lemma 3.3. *Suppose that $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, that $\{\mathcal{F}_t,\ t \geq 0\}$ is a filtration on $(\Omega, \mathcal{F})$, and that $\mathcal{G}$ is a σ-field of $\mathcal{F}$. If U_t is a $\mathrm{P}(\cdot \mid \mathcal{G})$-submartingale with respect to $\{\mathcal{F}_t,\ t \geq 0\}$ satisfying*

$$\sup_{t\geq 0} \mathrm{E}(|U_t| \mid \mathcal{G}) < \infty \quad \text{a.s.}, \tag{3.13}$$

then there exists a finite random variable U_∞ such that U_t converges a.s. to U_∞.

Proof. By Definition 3.1, U_t is a submartingale with respect to $\{\mathcal{F}_t \vee \mathcal{G},\ t \geq 0\}$. Let $\Omega_n = \{\sup_{t\geq 0} \mathrm{E}(|U_t| \mid \mathcal{G}) \leq n\}$. Assumption (3.13) implies that $\mathrm{P}(\Omega_n) \uparrow 1$. Note that, for each fixed n, $\mathbf{1}_{\Omega_n} U_t$ is a submartingale with respect to $\{\mathcal{F}_t \vee \mathcal{G},\ t \geq 0\}$ with

$$\begin{aligned}
\sup_{t\geq 0} \mathrm{E}(\mathbf{1}_{\Omega_n} |U_t|) &= \sup_{t\geq 0} \mathrm{E}(\mathrm{E}(\mathbf{1}_{\Omega_n} |U_t| \mid \mathcal{G})) \\
&= \sup_{t\geq 0} \mathrm{E}(\mathbf{1}_{\Omega_n} \mathrm{E}(|U_t| \mid \mathcal{G})) \\
&\leq \mathrm{E}\Big(\sup_{t\geq 0} \mathrm{E}(|U_t| \mid \mathcal{G}); \Omega_n\Big) \\
&< \infty.
\end{aligned}$$

The martingale convergence theorem says that there exists a finite random variable U_∞ defined on Ω_n such that U_t converges to U_∞ on Ω_n as $t \to \infty$. Therefore, there exists a finite U_∞ on the whole space Ω such that U_t converges to U_∞ a.s.

The next result is basically [3, Theorem 4.3.3].

Lemma 3.4. *Suppose that $(\Omega, \mathcal{F})$ is a measurable space and that $(\mathcal{F}_t)_{t\geq 0}$ is a filtration on $(\Omega, \mathcal{F})$ with $\mathcal{F}_t \uparrow \mathcal{F}$. If* P *and* Q *are two probability measures on $(\Omega, \mathcal{F})$ such that, for some nonnegative* P*-martingale Z_t with respect to $(\mathcal{F}_t)_{t\geq 0}$,*

$$\left.\frac{\mathrm{dQ}}{\mathrm{dP}}\right|_{\mathcal{F}_t} = Z_t.$$

Then the limit $Z_\infty := \limsup_{t\to\infty} Z_t$ exists and is finite a.s. under P*. Furthermore, for any $F \in \mathcal{F}$,*

$$\mathrm{Q}(F) = \int_F Z_\infty \,\mathrm{dP} + \mathrm{Q}(F \cap \{Z_\infty = \infty\}),$$

and, consequently,

$$\mathrm{P}(Z_\infty = 0) = 1 \iff \mathrm{Q}(Z_\infty = \infty) = 1,$$

$$\int_\Omega Z_\infty \,\mathrm{dP} = \int_\Omega Z_0 \,\mathrm{dP} \iff \mathrm{Q}(Z_\infty < \infty) = 1.$$

Proof of Theorem 1.1. We first prove that if $\int_D \widetilde{\phi}(y)l(y)\,\mathrm{d}y < \infty$ then M_∞ is nondegenerate under P_μ. Since $M_t^{-1}(\phi)$ is a positive supermartingale under $\widetilde{\mathrm{P}}_\mu$, $M_t(\phi)$ converges to some nonnegative random variable $M_\infty(\phi) \in (0, \infty]$ under $\widetilde{\mathrm{P}}_\mu$. By Lemma 3.4, we only need to prove that

$$\widetilde{\mathrm{P}}_\mu(M_\infty(\phi) < \infty) = 1. \tag{3.14}$$

By (2.7), $(X^{t,D}, \widetilde{\mathrm{P}}_\mu)$ has the same law as $(X^{t,D} + \hat{X}^{t,D}, \mathrm{P}_{\mu,\phi})$, where $X^{t,D}$ is the first exit measure of the superprocess X from $(0, t) \times D$ and $\hat{X}^{t,D} = \sum_{\sigma \in (0,t] \cap \mathcal{D}_m} X^{\sigma,(t,D)}$. Define

$$W_t(\phi) := \sum_{\sigma \in (0,t] \cap \mathcal{D}_m} \langle \phi^t, X^{\sigma,(t,D)} \rangle \exp\{-\lambda_1 t\}.$$

Then,

$$(M_t(\phi),\ t \geq 0; \widetilde{\mathrm{P}}_\mu) = (M_t(\phi) + W_t(\phi),\ t \geq 0; \mathrm{P}_{\mu,\phi}) \quad \text{in distribution,} \tag{3.15}$$

where $\{M_t(\phi),\ t \geq 0\}$ is copy of the martingale defined in (1.2) and is independent of $W_t(\phi)$. Let $\mathcal{G}$ be the σ-field generated by $\{\widetilde{Y}_t, m_t,\ t \geq 0\}$. Then, conditional on $\mathcal{G}$, $(X_t^\sigma,\ t \geq \sigma; \mathrm{P}_{\mu,\phi})$ has the same distribution as $(X_{t-\sigma},\ t \geq \sigma; \mathrm{P}_{m_\sigma \delta_{\widetilde{Y}_\sigma}})$ and the $(X_t^\sigma,\ t \geq \sigma; \mathrm{P}_{\mu,\phi})$ are independent for $\sigma \in \mathcal{D}_m$. Then we have

$$W_t(\phi) \stackrel{\mathrm{D}}{=} \sum_{\sigma \in (0,t] \cap \mathcal{D}_m} \exp\{-\lambda_1 \sigma\} M_{t-\sigma}^\sigma(\phi), \tag{3.16}$$

where, for each $\sigma \in \mathcal{D}_m$, $M_t^\sigma(\phi)$ is a copy of the martingale defined by (1.2) with $\mu = m_\sigma \delta_{\widetilde{Y}_\sigma}$ and, conditional on $\mathcal{G}$, the $\{M_t^\sigma(\phi),\ \sigma \in \mathcal{D}_m\}$ are independent. Here '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution. To prove (3.14), by (3.15), it suffices to show that

$$\mathrm{P}_{\mu,\phi}\Big(\lim_{t \to \infty} [M_t(\phi) + W_t(\phi)] < \infty\Big) = 1.$$

Since $(M_t(\phi),\ t \geq 0)$ is a nonnegative martingale under the probability $\mathrm{P}_{\mu,\phi}$, it converges $\mathrm{P}_{\mu,\phi}$-a.s. to a finite random variable $M_\infty(\phi)$ as $t \to \infty$. So we only need to prove that

$$\mathrm{P}_{\mu,\phi}\Big(\lim_{t \to \infty} W_t(\phi) < \infty\Big) = 1. \tag{3.17}$$

Define $\mathcal{H}_t := \mathcal{G} \vee \sigma(X^{\sigma,(s-\sigma,B)}; \sigma \in [0, t] \cap \mathcal{D}_m,\ s \in [\sigma, t])$. Then $(W_t(\phi))$ is a $\mathrm{P}_{\mu,\phi}(\cdot \mid \mathcal{G})$-nonnegative submartingale with respect to $(\mathcal{H}_t)$. By (3.16) and Lemma 3.2,

$$\begin{aligned}
\sup_{t \geq 0} \mathrm{P}_{\mu,\phi}(W_t(\phi) \mid \mathcal{G}) &= \sup_{t \geq 0} \sum_{s \in [0,\ t] \cap \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s) \\
&\leq \sum_{s \in \mathcal{D}_m} \exp\{-\lambda_1 s\} m_s \phi(\widetilde{Y}_s) \\
&< \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}
\end{aligned}$$

Then, by Lemma 3.3, $W_t(\phi)$ converges $\mathrm{P}_{\mu,\phi}$-a.s. to $W_\infty(\phi)$ as $t \to \infty$ and $\mathrm{P}_{\mu,\phi}(W_\infty(\phi) < \infty) = 1$; therefore, (3.17) holds.

Now we turn to the proof of the second part of the theorem. Assume that $\int_D \widetilde{\phi}(y)l(y)\,\mathrm{d}y = \infty$. We are going to prove that $M_\infty(\phi) := \lim_{t\to\infty} M_t(\phi)$ is degenerate with respect to P_μ.

By [7, Proposition 2], $1/M_t(\phi)$ is a supermartingale under $\widetilde{\mathrm{P}}_\mu$, and, thus, $1/(M_t(\phi)+W_t(\phi))$ is a nonnegative supermartingale under $\mathrm{P}_{\mu,\phi}$. Recall that $M_t(\phi)$ is a nonnegative martingale under $\mathrm{P}_{\mu,\phi}$. Then the limits $\lim_{t\to\infty} M_t(\phi)$ and $1/\lim_{t\to\infty}(M_t(\phi) + W_t(\phi))$ exist and are finite $\mathrm{P}_{\mu,\phi}$-a.s. Therefore, $\lim_{t\to\infty} W_t(\phi)$ exists in $[0, \infty]$ $\mathrm{P}_{\mu,\phi}$-a.s. Recall the definition of $(\eta_i, \sigma_i,\ i = 1, 2, \ldots)$ in Lemma 3.2, and note that $\lim_{i\to\infty} \sigma_i = \infty$. By Lemma 3.2,

$$\limsup_{t\to\infty} W_t(\phi) \ge \limsup_{i\to\infty} W_{\sigma_i}(\phi) \ge \limsup_{i\to\infty} \exp\{-\lambda_1\sigma_i\}\eta_i\phi(\widetilde{Y}_{\sigma_i}) = \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

So we have

$$\lim_{t\to\infty} W_t(\phi) = \infty \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

By (3.15),

$$\widetilde{\mathrm{P}}_\mu(M_\infty(\phi) = \infty) = 1.$$

It follows from Lemma 3.4 that $\mathrm{P}_\mu(M_\infty = 0) = 1$.

Remark 3.1. The argument of this paper actually works for general superprocesses. Our main result remains valid for any general $(Y, \psi(\lambda) - \beta\lambda)$-superprocess with Y being a reasonable Markov process such that Assumptions 1.1 and 1.2 are satisfied. For examples of discontinuous Markov processes satisfying Assumption 1.2, we refer the reader to [11] and the references therein.

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