MATHEMATICS OF COMPUTATION
Volume 79, Number 271, July 2010, Pages 1647–1679
S 0025-5718(09)02322-9
Article electronically published on December 2, 2009

L^p BERNSTEIN ESTIMATES AND APPROXIMATION BY SPHERICAL BASIS FUNCTIONS

H. N. MHASKAR, F. J. NARCOWICH, J. PRESTIN, AND J. D. WARD

Abstract. The purpose of this paper is to establish L^p error estimates, a Bernstein inequality, and inverse theorems for approximation by a space comprising spherical basis functions located at scattered sites on the unit n-sphere. In particular, the Bernstein inequality estimates L^p Bessel-potential Sobolev norms of functions in this space in terms of the minimal separation and the L^p norm of the function itself. An important step in its proof involves measuring the L^p stability of functions in the approximating space in terms of the ℓ^p norm of the coefficients involved. As an application of the Bernstein inequality, we derive inverse theorems for SBF approximation in the L^P norm. Finally, we give a new characterization of Besov spaces on the n-sphere in terms of spaces of SBFs.

1. Introduction

Various applications in meteorology, cosmology, and geophysics require a modeling of functions based on *scattered data* collected on (or near) a sphere, i.e., when one does not have any control on where the data sites are located [7, 5, 6]. On $\mathbb{S}^n$, the unit sphere in $\mathbb{R}^{n+1}$, $n \geq 1$, a popular method is to construct the required approximation from spaces of spherical basis functions (SBFs), which are kernels located at points in a discrete set $X = \{\xi_j\}_{j=1}^N \in \mathbb{S}^n$, the set of *centers* or *nodes*.

A function $\phi : [-1, 1] \to \mathbb{R}$ is an SBF on $\mathbb{S}^n$ if, in its expansion in ultraspherical polynomials $P_\ell^{(\lambda_n)}$, $\lambda_n = \frac{n-1}{2}$, the Fourier-Legendre coefficients $\{\hat{\phi}(\ell)\}$ of ϕ are all positive; see section 3 for details. These ϕ are to be used as kernels of the form $\phi(x \cdot y)$, $x, y \in \mathbb{S}^n$, $x \cdot y$ being the usual "dot" product. The approximation space here is the span

$$\mathcal{G}_{\phi,X} := \operatorname{span}\{\phi(x \cdot \xi)\}_{\xi \in X}.$$

Following usage common in the neural network community, we will say that a function $g \in \mathcal{G}_{\phi,X}$ is an *SBF network* associated with ϕ. The SBF ϕ is sometimes called an *activation function* or a *neuron*, but we will not use these terms here.

Received by the editor October 15, 2008 and, in revised form, July 8, 2009.

2000 *Mathematics Subject Classification.* Primary 41A17, 41A27, 41A63, 42C15.

Key words and phrases. Sphere, Bernstein estimates, approximation, spherical basis functions.

The research of the first author was supported by grant DMS-0605209 from the National Science Foundation and grant W911NF-04-1-0339 from the U.S. Army Research Office.

The research of the second author was supported by grants DMS-0504353 and DMS-0807033 from the National Science Foundation.

The research of the fourth was supported by grants DMS-0504353 and DMS-0807033 from the National Science Foundation.

©2009 American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Such ϕ may have singular behavior. This is the case for certain thin-plate splines; $(1 - x \cdot y)^{-1/2}$ is an SBF in $\mathbb{S}^n$, $n \geq 2$, for instance. However, when they are continuous, they are positive definite in Schoenberg's sense [30]. In that case the interpolation matrix $[\phi(\xi_i \cdot \xi_j)]$ is positive definite, and it is possible to use SBFs to interpolate data given at the points in X.

The focus of this paper is approximation. To handle noisy data, both least squares and quasi-interpolants have been used for many years. More recently, the issue in many meshless numerical methods for solving PDEs is how well a network approximates a solution to the PDE. Singular SBFs should prove useful in probing for a corresponding singularity in the solutions.

To be effective, though, such methods require knowing the degree of approximation in various spaces, especially the L^p, $1 \leq p \leq \infty$. The L^2 case for SBFs ϕ with $\hat{\phi}(\ell) \sim (\ell+1)^{-\beta}$, $\beta > n/2$ was recently investigated in [23], with nearly optimal rates being attained by interpolatory networks. The known estimates on the degree of approximation in the case of L^p, $p \neq 2$ provided by interpolatory networks are not asymptotically optimal. This has led to the development of other approximation tools [15, 13, 21], involving SBFs or spherical harmonics, in L^p, $1 \leq p \leq \infty$. A central step in obtaining approximation rates in L^2 was establishing a Bernstein estimate, which was then used to get an inverse approximation theorem.

The paper has three main goals. The first is to derive an L^p Bernstein inequality, for $1 \leq p \leq \infty$; namely, $\|g\|_{H^p_\gamma} \leq Cq^{-\gamma}\|g\|_p$, $0 < \gamma < c_\phi$. Here H^p_γ is a Bessel-potential Sobolev space [32, 34]; it measures derivatives of g (cf. section 2.3). The quantity q is half of the minimal separation of points in X; q^{-1} plays the role of a Nyquist frequency.

In establishing the Bernstein inequalities it is also necessary to measure the L^p stability of the basis $\{\phi(x \cdot \xi)\}_{\xi \in X}$ for $\mathcal{G}_{\phi,X}$. We do this by introducing a new quantity, the L^p stability ratio, $\mathsf{r}_{\mathcal{G},\,p}$, which is defined in equation (1.1) below. The stability ratio is similar to the ℓ^p condition number for a matrix.

The second goal is to obtain L^p error estimates, $1 \leq p \leq \infty$, for approximating a function by networks in $\mathcal{G}_{\phi,X}$. To give a sample of these results, we need two other quantities related to the geometry of X: the mesh norm h, which measures how far points in $\mathbb{S}^n$ can be from those in X, and the mesh ratio $\rho := h/q$, which measures the uniformity of the distribution of points in X. These are discussed in section 2.1.

A typical estimate that applies to an SBF that has a Green's function singularity similar to a thin-plate spline restricted to $\mathbb{S}^n$ is $\mathrm{dist}_{H^p_\gamma}(f, \mathcal{G}_{\phi,X}) \leq Ch^{\beta-\gamma}\rho^n\|f\|_{H^p_\beta}$, where β is related to the singularity of the SBF.

We combine these direct (Favard-Jackson) estimates with the Bernstein inequalities to provide new characterizations of Besov spaces $B^r_{\tau,p}$ on $\mathbb{S}^n$, characterizations that use rates of approximation from the $\mathcal{G}_{\phi,X}$. The Bernstein estimates are then used to establish inverse theorems and obtain nearly optimal rates of approximation. Under the restrictions in Theorem 6.14, if $f \in L^p$ satisfies $\mathrm{dist}_{L^p(\mathbb{S}^n)}(f, \mathcal{G}_{\phi,X}) \leq c_f \frac{h_X^\mu}{\log_2^t(h_X^{-1})}$ for any $\tau > t^{-1} > 0$, $0 < r \leq \mu$, and in addition for all X in a certain class, then $f \in B^r_{\tau,p}$.

The third goal is to show that the results obtained here will apply for nearly all of the SBFs of interest. In particular, they apply to various RBFs restricted to the sphere: the thin-plate splines and Wendland functions, whose Fourier-Legendre

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

coefficients have algebraic decay, and also Gaussians and multiquadrics, whose coefficients decay faster than algebraically. SBFs in the latter class are well known to be difficult to treat.

The paper is organized this way. Section 2 reviews various geometric quantities, such as the set of centers, the mesh norm, and so on. It also discusses spherical harmonics and the Bessel-potential Sobolev spaces. Section 3 discusses SBFs, their Fourier-Legendre expansions, and deals in detail with the SBFs mentioned earlier, along with ones corresponding to certain Green's functions that play a significant role in the paper. It is here that we will show that nearly all of the SBFs of interest have the properties necessary for our results to hold. We also mention that we obtain precise asymptotic expressions for the Fourier-Legendre coefficients in the case of Wendland functions.

The strategy for establishing the Bernstein inequality, which will be detailed below, consists of two key components: L^p approximation results for functions in $\mathcal{G}_{\phi,X}$ by means of spherical polynomials, and L^p stability estimates; these are developed in sections 4 and 5, respectively. The approximation results are based on Marcinkiewicz-Zygmund inequalities developed in [17, 16, 21], as well as frame results from [21]. The stability results, which are of interest in their own right, are for all L^p, not just for interpolation with continuous SBFs. To obtain them, we introduce a *stability ratio*, which provides some measure of the extent to which a finite set in L^p is linearly independent.

In section 6, the results of the previous two sections are combined to yield L^p Bernstein inequalities (section 6.1), direct theorems for approximation by networks in $\mathcal{G}_{\phi,X}$ (section 6.2), characterizations of Besov spaces on $\mathbb{S}^n$ (section 6.3), and inverse theorems for L^p functions approximated at given rates by SBF networks (section 6.4).

Strategy. Let g be an SBF network in $\mathcal{G}_{\phi,X} \subset H^p_\gamma(\mathbb{S}^n)$, so that it has the form

$$g(\mathbf{x}) = \sum_{\xi \in X} a_\xi \phi(\mathbf{x} \cdot \xi).$$

One of our main goals is to obtain an L^p Bernstein inequality for such networks, that is, a bound of the form $\|g\|_{H^p_\gamma} \le C q^{-\gamma} \|g\|_p$, where the norms are those appropriate for $\mathbb{S}^n$ and $\gamma > 0$ is bounded above by a constant depending on ϕ and p.

Our strategy involves approximating g by degree L spherical polynomials on $\mathbb{S}^n$, where $L \sim q^{-1}$. Now, for fixed L and any S, there is a Bernstein inequality, $\|S\|_{H^p_\gamma} \le C L^\gamma \|S\|_p$, which can be found in Theorem 4.10. Using it and manipulations involving the triangle inequality, one has that

$$\|g\|_{H^p_\gamma} \le \|S\|_{H^p_\gamma} + \|g - S\|_{H^p_\gamma} \le C L^\gamma \|S\|_p + \|g - S\|_{H^p_\gamma},$$

which holds for given L and any S.

Obtaining an appropriate polynomial S is crucial to the argument. To do that, we will use the frame operators introduced in [21] and discussed in more detail in section 4.3 below. In particular, we need reconstruction operators B_J, with $J \sim \log_2 L$. These rotationally-invariant operators have other very useful approximation properties, which are given in Proposition 4.9. They take L^p spaces and the space of continuous functions boundedly into spherical polynomials having degree $\mathcal{O}(2^J)$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Consequently, with $S = \mathsf{B}_J g$, we have $\|S\|_p \le C\|g\|_p$, and also

$$\|g\|_{H^p_\gamma} \le C2^{\gamma J}\|g\|_p + \|g - \mathsf{B}_J g\|_{H^p_\gamma} = C2^{\gamma J}\|g\|_p + \frac{|a|_p}{\|g\|_p}\cdot\frac{\|g - \mathsf{B}_J g\|_{H^p_\gamma}}{|a|_p}\cdot\|g\|_p,$$

where $|a|_p = \left(\sum_{\xi\in X}|a_\xi|^p\right)^{1/p}$ is the p-norm of $a = \{a_\xi\}_{\xi\in X}$.

The functions $\{\phi((\cdot)\cdot\xi)\}_{\xi\in X}$ are linearly independent and form a basis for $\mathcal{G}$, and so the pairing $a \leftrightarrow g$ is bijective. Since $\mathcal{G}$ has finite dimension $|\mathcal{G}|$, the ratio

$$\mathsf{r}_{\mathcal{G},\,p} := \max_{\mathcal{G}\ni g\neq 0}\frac{|a|_p}{\|g\|_p} \tag{1.1}$$

is finite; it will be called the p-norm *stability ratio* of the network $\mathcal{G} = \mathcal{G}_{\phi,X}$. This ratio is similar to a condition number in interpolation, but for L^p. With it, the inequality directly above becomes

$$\|g\|_{H^p_\gamma} \le \left(C2^{\gamma J} + C'\mathsf{r}_{\mathcal{G},p}\left(\frac{\|(I - \mathsf{B}_J)g\|_{H^p_\gamma}}{|a|_p}\right)\right)\|g\|_p. \tag{1.2}$$

To obtain the desired Bernstein inequality, we require two bounds: the first on $\|(I - \mathsf{B}_J)g\|_{H^p_\gamma}/|a|_p$ and the second on $\mathsf{r}_{\mathcal{G},p}$. The first bound relies only on approximation results; these we cover in section 4. The second is a bound on the stability ratio. This bound requires a more detailed analysis involving both the geometry of X and properties of ϕ. It is carried out in section 5.

An interesting point is that the two bounds make different demands on the properties required for ϕ. This makes the analysis of both bounds subtle. Fortunately, the common demands are satisfied by large classes of SBBs, including restrictions to $\mathbb{S}^n$ of the most common RBFs: the thin-plate splines, Wendland functions, Gaussians, Hardy multiquadrics, and others.

2. Background

2.1. Background and notation for $\mathbb{S}^n$.

Centers and decompositions of $\mathbb{S}^n$. Let X be a finite set of distinct points in $\mathbb{S}^n$; we will call these the *centers*. For X, we define these quantities: *mesh norm*, $h_X = \sup_{y\in\mathbb{S}^n}\inf_{\xi\in X} d(\xi, y)$, where $d(\cdot,\cdot)$ is the geodesic distance between points on the sphere; the *separation radius*, $q_X = \frac{1}{2}\min_{\xi\neq\xi'} d(\xi,\xi')$; and the *mesh ratio*, $\rho_X := h_X/q_X \ge 1$.

For $\rho \ge 1$, define $\mathcal{F}_\rho = \mathcal{F}_\rho(\mathbb{S}^n)$ to be the family of all sets of centers X with $\rho_X \le \rho$. We say that X is *ρ-uniform* if $X \in \mathcal{F}_\rho$. For every $\rho \ge 2$, $\mathcal{F}_\rho(\mathbb{S}^n)$ is not only nonempty, but it contains *nested* sequences of sets of centers for which h_X becomes arbitrarily small; precisely, the result is this:

Proposition 2.1 ([23, Proposition 2.1]). *Let $\rho \ge 2$ and let $\mathcal{F}_\rho$ be the corresponding ρ-uniform family. Then, there exists a sequence of sets $X_k \in \mathcal{F}_\rho$, $k = 0, 1, \ldots$, such that the sequence is nested, $X_k \subset X_{k+1}$, and such that at each step the mesh norms satisfy $\frac{1}{4}h_{X_k} < h_{X_{k+1}} \le \frac{1}{2}h_{X_k}$.*

We will need to consider a decomposition of $\mathbb{S}^n$ into a finite number of nonoverlapping, connected regions R_ξ, each containing an interior point ξ that will serve for function evaluations as well as labeling. For example, if $\mathcal{X}$ is the Voronoi tessellation for a set of centers X, then we may take R_ξ to be the region associated

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

with $\xi \in X$. In any case, we will let X be the set of the ξ's used for labels and $\mathcal{X} = \{R_\xi \subset \mathbb{S}^n \,|\, \xi \in X\}$. In addition, let $\|\mathcal{X}\| = \max_{\xi\in X}\{\mathrm{diam}(R_\xi)\}$.

2.2. **Spherical harmonics.** Let $n \geq 2$. Let $d\mu$ be the standard measure on the n-sphere, and let the spaces $L^p(\mathbb{S}^n)$, $1 \leq p \leq \infty$, have their usual meanings. In addition, let $\Delta_{\mathbb{S}^n}$ denote the Laplace-Beltrami operator on $\mathbb{S}^n$. The eigenvalues of $\Delta_{\mathbb{S}^n}$ are $-\ell(\ell+n-1)$, $\ell = 0, 1, \ldots$. For $n \geq 2$ and ℓ fixed, the dimension of the eigenspace is

$$d_\ell^n = \frac{\ell+\lambda_n}{\lambda_n}\binom{\ell+n-2}{\ell} \overset{\ell\to\infty}{\sim} \frac{\ell^{n-1}}{\lambda_n(n-2)!},\ \lambda_n := \frac{n-1}{2}. \tag{2.1}$$

For $n = 1$, the case of the circle, $d_0^1 = 1$ and $d_\ell^1 = 2$, $\ell \geq 1$.

A spherical harmonic $Y_{\ell,m}$ is an eigenfunction of $\Delta_{\mathbb{S}^n}$ corresponding to the eigenvalue $-\ell(\ell+n-1)$ [19, 31], where $m = 1, \ldots, d_\ell^n$. The set $\{Y_{\ell,m} : \ell = 0, 1, \ldots; m = 1, \ldots, d_\ell^n\}$ is orthonormal in $L^2(\mathbb{S}^n)$. Denote by $\mathcal{H}_\ell$ the span of the spherical harmonics with fixed order ℓ, and let $\Pi_L = \bigoplus_{\ell=0}^{L} \mathcal{H}_\ell$ be the span of all spherical harmonics of order at most L. The orthogonal projection P_ℓ onto $\mathcal{H}_\ell$ is given by

$$\mathsf{P}_\ell f = \sum_{m=1}^{d_\ell^n} \langle f, Y_{\ell,m}\rangle Y_{\ell,m}\,. \tag{2.2}$$

We regard the sphere $\mathbb{S}^n$ as being the unit sphere in $\mathbb{R}^{n+1}$, and we let the quantity $\xi\cdot\eta$ denote the usual "dot" product for $\mathbb{R}^{n+1}$. Using the addition formula for spherical harmonics, when $n \geq 2$, one can write the kernel for this projection as

$$P_\ell(\xi\cdot\eta) = \sum_{m=1}^{d_\ell^n} Y_{\ell,m}(\xi)\overline{Y_{\ell,m}(\eta)} = \frac{\ell+\lambda_n}{\lambda_n\omega_n} P_\ell^{(\lambda_n)}(\xi\cdot\eta),\ \lambda_n := \frac{n-1}{2}, \tag{2.3}$$

where $P_\ell^{(\lambda_n)}(\cdot)$ is the ultraspherical polynomial of order λ_n and degree ℓ. Also, we have that $\|P_\ell^{(\lambda_n)}\|_\infty \leq P_\ell^{(\lambda_n)}(1) = \frac{d_\ell^n \lambda_n}{\ell+\lambda_n}$. We will briefly discuss these polynomials in section 3, in connection with spherical basis functions. For $n = 1$, $\lambda_1 = 0$. In that case, the kernel for P_ℓ has the form

$$P_\ell(\xi\cdot\eta) = \begin{cases} \frac{1}{2\pi}, & \ell = 0,\\ \frac{1}{\pi}T_\ell(\xi\cdot\eta), & \ell \geq 1, \end{cases} \tag{2.4}$$

where $T_\ell(\cdot)$ is the degree-ℓ Chebyshev polynomial of the first kind, which is a limiting case of the ultraspherical polynomials [33, Section 4.7].

We will also need to consider operators of the form $\sum_{\ell=0}^{\infty} c_\ell \mathsf{P}_\ell$. The kernels for the projections P_ℓ then provide us with kernels $\sum_{\ell=0}^{\infty} c_\ell P_\ell(\xi\cdot\eta)$, which may be distributional.

2.3. **Bessel-potential Sobolev spaces.** The spherical harmonic $Y_{\ell,m}$ is an eigenfunction corresponding to the eigenvalue $-\ell(\ell+n-1) = \lambda_n^2 - (\ell+\lambda_n)^2$ for the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ on $\mathbb{S}^n$. It follows that $\ell+\lambda_n$ is an eigenvalue corresponding to the eigenfunctions $Y_{\ell,m}\,, m = 1, \ldots, d_\ell^n$, of the pseudo-differential operator

$$\mathsf{L}_n := \sqrt{\lambda_n^2 - \Delta_{\mathbb{S}^n}} = \sum_{\ell=0}^{\infty} (\ell+\lambda_n)\mathsf{P}_\ell. \tag{2.5}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Let γ be real, $1 \le p \le \infty$ and $n \ge 2$. If f is a distribution on $\mathbb{S}^n$, define the Bessel-potential Sobolev spaces $H^p_\gamma(\mathbb{S}^n)$ [32, 34] to be all f such that

$$\|f\|_{H^p_\gamma} := \Big\| \sum_{\ell=0}^{\infty} (\ell+\lambda_n)^\gamma \mathsf{P}_\ell f \Big\|_{L^p} < \infty, \tag{2.6}$$

where P_ℓ is from (2.2). The notation we use here is that of Triebel [34]. Strichartz [32] defined these spaces on a complete Riemannian manifold, using the equivalent operator $(1-\Delta_{\mathbb{S}^n})^{\gamma/2}$ to do so. One more thing:

Remark 2.2. The space $H^2_\gamma(\mathbb{S}^n)$ is the domain of L_n^γ [32, Theorem 4.4], which implies that $H^2_\gamma(\mathbb{S}^n)$ is norm equivalent to the usual Sobolev space $W_2^\gamma(\mathbb{S}^n)$.

3. Spherical basis functions

For any real $\lambda > 0$, not just $\lambda_n = \frac{n-1}{2}$, the ultraspherical polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} P_\ell^{(\lambda)}(x) P_k^{(\lambda)}(x)(1-x^2)^{\lambda-\frac{1}{2}}\,dx = \frac{2^{1-\lambda}\pi\Gamma(\ell+2\lambda)}{(\ell+\lambda)\Gamma^2(\lambda)\Gamma(\ell+1)}\delta_{k,\ell}. \tag{3.1}$$

For the circle, we have $\lambda_1 = 0$. With $\ell \ge 1$, as $\lambda \to 0$, the ratio $P_\ell^{(\lambda_n)}(\cdot)/\lambda$ converges to $(2/\ell)T_\ell(\cdot)$, the degree-ℓ Chebyshev polynomial of the first kind [33, Section 4.7].

Consider a function ϕ in L^p or C. We will assume that ϕ has the following expansion in the orthogonal set of ultraspherical polynomials:

$$\phi(\underbrace{\xi\cdot\eta}_{\cos\theta}) := \begin{cases} \frac{1}{2\pi}\hat{\phi}(0) + \frac{1}{\pi}\sum_{\ell=1}^{\infty}\hat{\phi}(\ell)\cos\ell\theta, & n=1,\\ \sum_{\ell=0}^{\infty}\hat{\phi}(\ell)\frac{\ell+\lambda_n}{\lambda_n\omega_n}P_\ell^{(\lambda_n)}(\cos\theta), & n\ge 2, \end{cases} \tag{3.2}$$

where $\omega_n := \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ is the volume of $\mathbb{S}^n$.

Functions of this form are called *zonal*. We will assume that the series converges in at least a distributional sense. The coefficients in the expansion are obtained via the orthogonality relations in (3.1). These are given below:

$$\frac{\ell+\lambda_n}{\lambda_n\omega_n}\hat{\phi}(\ell) = \frac{(\ell+\lambda_n)\Gamma^2(\lambda_n)\Gamma(\ell+1)}{2^{1-\lambda_n}\pi\Gamma(\ell+2\lambda_n)}\int_{-1}^{1}\phi(x)P_\ell^{(\lambda_n)}(x)(1-x^2)^{\lambda_n-\frac{1}{2}}\,dx.$$

Using Rodrigues' formula [33, Eqn. (4.7.12)] for $P_\ell^{(\lambda_n)}(x)$ in the equation above and employing the duplication formula and other standard properties of the Gamma function, one can obtain this expression:

$$\hat{\phi}(\ell) = \frac{(-1)^\ell\omega_n\Gamma(\lambda_n+1)}{2^\ell\sqrt{\pi}\Gamma(\ell+\lambda_n+\frac{1}{2})}\int_{-1}^{1}\phi(x)\frac{d^\ell}{dx^\ell}\left\{(1-x^2)^{\ell+\lambda_n-\frac{1}{2}}\right\}dx,$$

which holds for all ℓ, even when $n=1$; i.e., $\lambda_1 = 0$.

Schoenberg [30] defined ϕ to be positive definite if for every set of centers X the matrix $[\phi(\xi_j\cdot\xi_k)]$ is positive semidefinite. He showed that ϕ is positive definite if and only if the coefficients satisfy $\hat{\phi}(\ell) \ge 0$ for all ℓ and $\sum_{\ell=0}^{\infty}\hat{\phi}(\ell)d_\ell < \infty$. If in addition $\hat{\phi}(\ell) > 0$, then $[\phi(\xi_j\cdot\xi_k)]$ is a positive definite matrix and one can use shifts of ϕ to interpolate any function $f \in C(\mathbb{S}^n)$ on X. We will say that ϕ is a *spherical basis function* (SBF) in this case.

One usually makes the assumption that the sum $\sum_{\ell=0}^{\infty}\hat{\phi}(\ell)d_\ell < \infty$, for then ϕ is continuous and $\phi(1) = \|\phi\|_{L^\infty}$. This is essential if we are doing standard

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

interpolation of a function from its values on X. However, we are more interested in approximation than interpolation, and so we will *not* make this assumption here. Indeed, we will say that any distribution ϕ for which $\hat{\phi}(\ell) > 0$ for all ℓ is a spherical basis function. In general, we will be interested in SBFs in L^p.

Zonal functions that satisfy $\hat{\phi}(\ell) > 0$ for $\ell \geq L > 0$ are said to be *conditionally positive definite* SBFs. In the RBF theory on Euclidean space, the difference between strictly positive definite RBFs and conditionally strictly positive definite RBFs is significant. On $\mathbb{S}^n$, this difference is less important: a conditionally positive definite SBF differs from an SBF by a polynomial of degree $L-1$. This does play a role in interpolation, but is much less significant in approximation problems. That being the case, unless there is a genuine need to distinguish between the two, we will refer to both as simply SBFs.

Below we will list Fourier-Legendre expansion coefficients for some of the more significant SBFs. Apart from certain Green's functions that we will do first, these are restrictions of various Euclidean RBFs in $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$. (The restriction of any RBF in $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$ is an SBF [24, Corollary 4.3].) These include Gaussians, multiquadrics, thin-plate splines, and Wendland functions. Such SBFs are RBFs expressed in terms of the Euclidean distance between ξ and η or its square, $\|\xi-\eta\|^2 = 2 - 2\xi\cdot\eta$; with $t = \xi\cdot\eta$, these give rise to functions of $1-t$.

Green's functions. Let $\beta > 0$. The Green's function solution to $\mathsf{L}_n^\beta G_\beta = \delta$ is a kernel with an expansion in spherical harmonics having coefficients $\widehat{G}_\beta(\ell,m) = (\ell+\lambda)^{-\beta}$. Properties of Green's functions are discussed in more detail in Proposition 4.12. We simply remark that the kernel G_β is an SBF that is in $L^1(\mathbb{S}^n)$ for all $\beta > 0$. For us, G_β will play a significant role. The SBFs we consider will generally be of two types: $\phi = G_\beta + G_\beta * \psi$, where ψ is an L^1 zonal function, or ϕ will be in C^∞. The first type includes the thin-plate splines and Wendland functions, and the second, the Gaussians and multiquadrics.

Thin-plate splines. The thin-plate splines are defined in [35, Section 8.3]; their Fourier-Legendre coefficients are found in [22, §4.2]. These are given below:

$$\begin{cases} \phi_s(t) = \begin{cases} (-1)^{\lceil (s)_+ \rceil}(1-t)^s, & s > -\frac{n}{2},\ s \notin \mathbb{N}, \\ (-1)^{s+1}(1-t)^s \log(1-t), & s \in \mathbb{N}, \end{cases} \\ \\ \hat{\phi}_s(\ell) = C_{s,n}\frac{\Gamma(\ell-s)}{\Gamma(\ell+s+n)} \end{cases} \tag{3.3}$$

where the factor $C_{s,n}$ is given by

$$C_{s,n} := 2^{s+n}\pi^{\frac{n}{2}}\Gamma(s+1)\Gamma(s+\frac{n}{2}) \begin{cases} \frac{\sin(\pi s)}{\pi} & s > -\frac{n}{2},\ s \notin \mathbb{N}, \\ 1, & s \in \mathbb{N}. \end{cases}$$

Let $\nu = \ell + \lambda_n$. For large ν, the Fourier-Legendre coefficients $\phi_s(\ell)$ for the thin-plate splines have the asymptotic form

$$\hat{\phi}_s(\ell) = C_{s,n}\nu^{-2s-n}\left(1 + \sum_{j=1}^{p-1} G_j(n,s)\nu^{-j} + R_p(n,s,\nu)\right), \tag{3.4}$$

where $R_p(n,s,\nu) = \mathcal{O}(\nu^{-p})$ and $G_j(n,s)$ are defined in [25, p. 119].

Two remarks. First, we have made use of $G_0(n,s) = 1$ in the expansion from [25, p. 119]. Second, when s is an integer or half-integer, $\hat{\phi}_s(\ell)$ is a rational function

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

of ℓ, and, hence, of ν. In that case, it follows that the series for $\hat{\phi}_s(\ell)$ is actually a convergent power series in ν^{-1}. For other s, the expansion is only asymptotic.

From the structure of the expansions above and the properties of Green's functions listed in Proposition 4.12, we see that any finite linear combination of thin-plate splines,

$$\phi = \sum_{j=1}^{m} A_j \phi_{s_j}, \ -\frac{n}{2} < s_1 < s_2 < \cdots < s_m, \tag{3.5}$$

has the form

$$\phi = A_1(G_{2s+n} + G_{2s+n} * \psi), \ \psi \in L^1. \tag{3.6}$$

Wendland functions. All of the SBFs we have discussed so far are related to RBFs stemming from completely monotonic functions. These RBFs have the property that they are strictly positive definite or conditionally positive definite in $\mathbb{R}^n$ for all n. The corresponding SBFs are also positive definite in $\mathbb{S}^n$, again for all n. These RBFs are *not* compactly supported, however. This can be remedied, but there is a price: we must give up positive definiteness beyond a certain dimension.

Wendland (cf. [35, Section 9.4]) constructed families of RBFs that are compactly supported on $0 \le r \le R$, strictly positive definite in Euclidean spaces of dimension d or less, have smoothness C^{2k}, and, within their supports, are polynomials of degree $\lfloor \frac{d}{2} \rfloor + 3k + 1$. The quantities d, k, and R are parameters and may be adjusted as needed.

Restricting the Wendland functions to $\mathbb{S}^n$ just requires setting $r = \sqrt{2(1-t)}$ and $R = \sqrt{2(1-t_0)}$, where $-1 < t_0 \le t \le 1$. We will denote these functions by $\phi_{d,k}(t)$. The support of $\phi_{d,k}$ on $\mathbb{S}^n$ is then $0 \le \theta \le \cos^{-1}(t_0) < \pi$. From [35, Theorems 9.12 & 9.13], if $t > t_0$, then these functions are polynomials in $\sqrt{1-t}$ that may be put into the form,

$$p_{d,k}(t) = e_1(1-t) + (1-t)^{k+\frac{1}{2}} e_2(1-t),$$

where e_1 and e_2 are polynomials having $\deg e_1 = \lfloor \frac{1}{2}(\lfloor \frac{d}{2} \rfloor + 3k + 1) \rfloor$ and $\deg e_2 = \lfloor \frac{1}{2}(\lfloor \frac{d}{2} \rfloor + k) \rfloor$. Outside of this interval, the $\phi_{d,k}$ are identically 0. Using a power series argument, we have that, near $t \gtrapprox t_0$, $\phi_{d,k}(t) = A(t-t_0)^{\lfloor \frac{d}{2} \rfloor + 2k+1}\big(1 + \mathcal{O}(t-t_0)\big)$, from which it follows that $\phi_{d,k}(t)$ is piecewise $C^{\lfloor \frac{d}{2} \rfloor + 2k+1}$ near t_0. In addition, it follows that $\psi_{d,k}(t) := \phi_{d,k}(t) - p_{d,k}(t)$ is piecewise $C^{\lfloor \frac{d}{2} \rfloor + 2k+1}$ on the whole interval $[-1, 1]$. Putting all of this together, we conclude that

$$\phi_{d,k}(t) = e_1(1-t) + (1-t)^{k+\frac{1}{2}} e_2(1-t) + \psi_{d,k}(t). \tag{3.7}$$

Our aim is to use this decomposition to obtain large ℓ asymptotics for the Fourier-Legendre coefficients $\hat{\phi}_{d,k}(\ell)$ in $\mathbb{S}^n$. This we now do.

Proposition 3.1. *Let $m = \lfloor \frac{d}{2} \rfloor + 2k + 1$. If $\ell > \deg e_1$, then*

$$\hat{\phi}_{d,k}(\ell) = (\ell + \lambda_n)^{-(2k+1+n)} \left(A_0 + \frac{A_1}{\ell + \lambda_n} + \mathcal{O}(\ell + \lambda_n)^{-2} \right) + \frac{\widehat{\mathsf{L}^m \psi_{d,k}}(\ell)}{(\ell + \lambda_n)^m}.$$

Moreover, if we choose $\lfloor \frac{d}{2} \rfloor > n$, then the $\phi_{d,k}$ have the structure

$$\phi_{d,k} = \text{polynomial } + A_0 \left(G_{2k+n} + G_{2k+n} * \tilde{\psi} \right), \ \tilde{\psi} \in L.$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Proof. The polynomial term $e_1(1-t)$ doesn't contribute to coefficients with $\ell > \deg e_1$. The term $(1-t)^{k+\frac{1}{2}}e_2(1-t)$ is a linear combination of thin-plate splines, starting with $s = k + \frac{1}{2}$. Thus it contributes the first term on the right above. By Remark 2.2, the function $\psi_{d,k}$ is in H_m^2, so it can be written as $\psi_{d,k} = \mathsf{L}_n^{-m}\mathsf{L}_n^m\psi_{d,k}$. The second term on the right follows directly from this fact. Finally, the form of the $\hat{\phi}_{d,k}(\ell)$'s leads to the second statement. □

Before leaving the topic, we point out that, when $\lfloor \frac{d}{2} \rfloor > n$, we have determined the precise asymptotics of the Fourier-Legendre coefficients for the Wendland functions. Heretofore only upper and lower bounds were known.

Gaussians. The Fourier-Legendre coefficients for the Gaussians, which are given below, may be found in [36, Ex. 37, p. 383], [15, Example 5.2], and [22, §4.3]:

$$\begin{cases} \gamma_\sigma(t) = e^{-2\sigma(1-t)},\ \sigma > 0, \\ \hat{\gamma}_\sigma(\ell) = 2\pi\left(\frac{2\pi}{\sigma}\right)^{\lambda_n} e^{-\sigma} I_{\lambda_n+\ell}(\sigma), \end{cases} \tag{3.8}$$

where $I_{\lambda_n+\ell}$ is an order $\lambda_n + \ell$ modified Bessel function of the first kind. For all $\ell \geq 0$, the coefficient $\hat{\gamma}_\sigma(\ell)$ satisfies this bound [22, Proposition 4.3]:

$$\frac{2\sigma^\ell e^{-2\sigma}\pi^{\frac{n+1}{2}}}{\Gamma(\ell + \frac{n+1}{2})} \leq \hat{\gamma}_\sigma(\ell) \leq \frac{2\sigma^\ell \pi^{\frac{n+1}{2}}}{\Gamma(\ell + \frac{n+1}{2})}. \tag{3.9}$$

Multiquadrics. The Hardy multiquadrics are treated in [22, §5]. The results are:

$$\begin{cases} \mathrm{mq}_\alpha(t) = -\sqrt{\delta^2 + 2(1-t)},\ \delta > 0, \\ \\ \widehat{\mathrm{mq}}_\delta(\ell) = \dfrac{\pi^{\lambda_n}\Gamma(\ell - 1/2)}{(\alpha^2+2)^{\ell-1/2}\Gamma(\ell + \lambda_n + 1)} \\ \qquad\qquad \times\ {}_2F_1\left(\dfrac{\ell - 1/2}{2}, \dfrac{\ell + 1/2}{2}; \ell + \lambda_n + 1; \dfrac{4}{(\delta^2+2)^2}\right). \end{cases} \tag{3.10}$$

Here, ${}_2F_1$ is the usual hypergeometric function. Expressions for Fourier-Legendre coefficients for generalized multiquadrics may be found in [22, §5]. Again, this time for ℓ sufficiently large, the coefficient $\widehat{\mathrm{mq}}_\delta(\ell)$ satisfies the following bound [22, Proposition 5.1]:

$$C_1 \ell^{-\frac{n}{2}-1}\left(\frac{1}{\delta^2+2}\right)^{\ell-\frac{1}{2}} < \widehat{\mathrm{mq}}_\delta(\ell) < C_2 \ell^{-1-n}\left(\frac{2}{\delta^2+2}\right)^{\ell-\frac{1}{2}}. \tag{3.11}$$

Ultraspherical generating functions. For $n \geq 2$, the ultraspherical polynomials $P_\ell^{(\lambda_n)}$ are frequently defined in terms of the generating function [33, Equation (4.7.23)] below:

$$\begin{cases} u_{\lambda_n,w}(t) = (1 - 2tw + w^2)^{-\lambda_n},\ 1 > w > 0,\ n \geq 2, \\ \hat{u}_{\lambda_n,w}(\ell) = w^\ell. \end{cases} \tag{3.12}$$

When $n = 1$, $\lambda_1 = 0$, the expansion is in terms of the $T_\ell(t)$'s, the Chebyshev polynomials of the first kind. In this case, the generating function is simply the Poisson kernel:

$$\begin{cases} P_w(t) = \frac{1-w^2}{1-2tw+w^2},\ 1 > w > 0, \\ \widehat{P}_w(\ell) = \begin{matrix} 1,\ \ell = 0, \\ 2w^\ell,\ \ell \geq 1. \end{matrix} \end{cases} \tag{3.13}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

4. Approximation

The approximation part of the analysis makes use of kernels and frames, which are related to them. These were studied in [1, 12, 14, 18, 21] and further developed in [26]; we review them here, along with a number of other results important to attaining the goals of this paper. First, we will develop various types of Marcinkiewicz-Zygmund inequalities for the sphere. Although some of these were previously derived [17, 16, 21], those pertinent to both the approximation and the stability analysis are new.

Second, using frames we establish a Bernstein inequality for spherical polynomials. Again, using frames we establish various distance estimates for $\phi \in H^1_\beta$ and we discuss Green's function solutions to $\mathsf{L}_n^\beta G_\beta = \delta$. As we have mentioned earlier, these form a very important class of SBFs. Finally, at the end of the section we will complete the approximation part of the analysis.

4.1. **Kernels.** Let $\kappa(t) \in C^k(\mathbb{R})$, with $k \geq \max\{2, n-1\}$, be even, not identically 0, and satisfy

$$|\kappa^{(r)}(t)| \leq C_\kappa (1+|t|)^{r-\alpha} \text{ for all } t \in \mathbb{R},\ r = 0, \dots, k, \tag{4.1}$$

where $\alpha > n+k$ and $C_\kappa > 0$ are fixed constants. We remark that all compactly supported, C^k functions that are even satisfy (4.1). Functions in the Schwartz class $\mathcal{S}(\mathbb{R})$ that are even satisfy (4.1) for arbitrarily large k and α. Given such a κ, define the family of operators

$$\mathsf{K}_{\varepsilon,n} := \kappa(\varepsilon \mathsf{L}_n) = \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))\mathsf{P}_\ell,\ 0 < \varepsilon \leq 1,$$

along with the associated family of kernels

$$K_{\varepsilon,n}(\underbrace{\xi \cdot \eta}_{\cos\theta}) := \begin{cases} \frac{1}{2\pi}\kappa(0) + \frac{1}{\pi}\sum_{\ell=1}^{\infty} \kappa(\varepsilon\ell)\cos\ell\theta, & n = 1, \\ \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))\frac{\ell+\lambda_n}{\lambda_n\omega_n} P_\ell^{(\lambda_n)}(\cos\theta), & n \geq 2, \end{cases} \tag{4.2}$$

where $\cos\theta = \xi \cdot \eta$ and $0 < \varepsilon \leq 1$.

It is worthwhile noting that $\kappa(t) = e^{-t^2}$ satisfies (4.1) and that the corresponding kernel is essentially the heat kernel for $\mathbb{S}^n$.

We will need several results concerning these kernels and operators. First of all, we require the estimates on the L^p norms for the kernels. Material closely connected to the theorem below appeared in [12, Proposition 4.1].

Theorem 4.1 ([21, Theorem 3.5 & Corollary 3.6])**.** *Let κ satisfy* (4.1)*, with $k \geq \max\{2, n-1\}$. If $0 \leq \theta \leq \pi$, then there is a constant $\beta_{n,k,\kappa} > 0$ such that the kernel $K_{\varepsilon,n}$ satisfies the bound*

$$|K_{\varepsilon,n}(\cos\theta)| \leq \frac{\beta_{n,k,\kappa}}{1+(\frac{\theta}{\varepsilon})^k}\varepsilon^{-n}. \tag{4.3}$$

Moreover, we have that

$$\|K_{\varepsilon,n}\|_p := \|K_{\varepsilon,n}(\cos\theta)\|_{L^p(\mathbb{S}^n)} \leq C_{n,k,\kappa}\varepsilon^{-n/p'}. \tag{4.4}$$

These operators can be applied to functions in $L^p(\mathbb{S}^n)$ or even distributions in $\mathcal{D}'(\mathbb{S}^n)$, provided κ decays fast enough; compact support will certainly work. As the result below shows, all of them are bounded operators taking $L^p(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Theorem 4.2 ([21, Theorem 3.7])**.** *If κ satisfies* (4.1)*, with $k > \max\{2, n\}$, then, for all $1 \le p \le \infty$ and $1 \le q \le \infty$, the operator $\mathsf{K}_{\varepsilon,n} \colon L^p(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$ is bounded and its norm satisfies*

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \le C_{n,k,\kappa}(4\omega_{n-1}\varepsilon^n)^{-(\frac{1}{p}-\frac{1}{q})_+},$$

where $C_{n,k,\kappa}$ is a constant that depends only on n, k, κ, and where $(x)_+ = x$ for $x > 0$ and $(x)_+ = 0$ otherwise.

We point out that more can be said when κ has restrictions on its support. The result below follows from the spherical harmonics of degree $L \sim 1/\varepsilon$ being in the kernel (i.e., null space) of $\mathsf{K}_{\varepsilon,n}$ when $\kappa(t) = 0$ near $t = 0$.

Remark 4.3. If $\kappa(t) = 0$ for $|t| \le 1$, then for any spherical harmonic in Π_{L_ε}, where $L_\varepsilon = \lfloor \varepsilon^{-1} - \lambda_n^{-1} \rfloor \sim \varepsilon^{-1}$ or less, then we have $g_\varepsilon := \mathsf{K}_{\varepsilon,n} g = \mathsf{K}_{\varepsilon,n}(g - P)$, and hence $\|g_\varepsilon\|_q \le \|\mathsf{K}_{\varepsilon,n}\|_{p,q} E_{L_\varepsilon}(g)_p$.

Another important result for κ supported away from $t = 0$ and having fast decay is the one below, which follows directly from Theorems 4.1 and 4.2. To simplify matters, we will assume that κ is also compactly supported.

Corollary 4.4. *Let $k > \max\{2, n\}$. If the support of κ is compact and does* not *include $t = 0$, then, for every fixed γ in $\mathbb{C}$, the function $\tilde{\kappa}(t) := |t|^\gamma \kappa(t)$ is also an even C^k function that satisfies* (4.2)*. Moreover, $\mathsf{L}^\gamma \mathsf{K}_{\varepsilon,n} = \varepsilon^{-\gamma} \tilde{\mathsf{K}}_{\varepsilon,n}$. Finally, for real γ, we have the two bounds below:*

$$\|\mathsf{L}^\gamma \mathsf{K}_{\varepsilon,n}\|_{p,q} \le C_{n,k,\tilde{\kappa}}(4\omega_{n-1})^{-(\frac{1}{p}-\frac{1}{q})_+} \varepsilon^{-\gamma - n(\frac{1}{p}-\frac{1}{q})_+},$$
$$\|\mathsf{L}^\gamma \mathsf{K}_{\varepsilon,n}\delta\|_p \le C_{n,k,\tilde{\kappa}} \varepsilon^{-\gamma - n/p'},$$

where δ is the Dirac distribution and thus $\mathsf{L}^\gamma \mathsf{K}_{\varepsilon,n}\delta$ is the kernel for $\mathsf{L}^\gamma \mathsf{K}_{\varepsilon,n}$.

4.2. **Marcinkiewicz-Zygmund inequalities.** Marcinkiewicz-Zygmund (MZ) inequalities provide equivalences between norms defined through integrals and ones defined through discrete sums. For $\mathbb{S}^n$, these were developed in [17, 16, 21]. We will need to adapt these MZ inequalities to estimate certain sums.

Let $X \subset \mathbb{S}^n$ be the set of centers; also, let $q = q_X$, $h = h_X$, and $\rho = \rho_X := h/q$ be the separation radius, mesh norm, and mesh ratio, respectively. We will need a decomposition of the sphere into a finite number of nonoverlapping regions. The Voronoi tessellation corresponding to X will serve our purpose here, although many other decompositions will work as well.

Let R_ξ be the Voronoi region containing X. Denote the collection of these regions by $\mathcal{X} = \{R_\xi \subset \mathbb{S}^n \,|\, \xi \in X\}$ and its partition norm by $\|\mathcal{X}\| = \max_{\xi \in X}\{\mathrm{diam}(R_\xi)\}$. It is easy to show that the following geometric inequalities hold:

$$h \le \|\mathcal{X}\| \le 2h \text{ and } \min_{\xi \in X} \mu(R_\xi) \ge c_n q^n. \tag{4.5}$$

Here c_n is a constant related to the volume of $\mathbb{S}^n$. We will need these later. For a sequence space version of the results below, see [13, Proposition 4.1].

Proposition 4.5. *Fix $\zeta \in \mathbb{S}^n$ and $k \ge n+2$. Let $K_\varepsilon(\eta) := K_{\varepsilon,n}(\eta \cdot \zeta)$. Then, there is a constant $C = C_{n,\kappa,k}$ for which*

$$\left| \|K_\varepsilon\|_1 - \sum_{\xi \in X} \mu(R_\xi) |K_\varepsilon(\xi)| \right| \le C \begin{cases} \|\mathcal{X}\|/\varepsilon & \|\mathcal{X}\| \le \varepsilon, \\ (\|\mathcal{X}\|/\varepsilon)^n & \|\mathcal{X}\| \ge \varepsilon. \end{cases} \tag{4.6}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Moreover, if $\zeta \in X$*, then*

(4.7)
$$\left| \int_{\mathbb{S}^n - R_\zeta} |K_\varepsilon(\eta)| d\mu(\eta) - \sum_{X \ni \xi \neq \zeta} \mu(R_\xi) |K_\varepsilon(\xi)| \right| \leq C_{n,\kappa,k} \begin{cases} (\|\mathcal{X}\|/\varepsilon)^{-1} & \|\mathcal{X}\| \leq \varepsilon, \\ (\varepsilon/\|\mathcal{X}\|)^{k-n-2} & \|\mathcal{X}\| \geq \varepsilon. \end{cases}$$

Proof. This is a strengthened version of [21, Proposition 4.1]. Since its proof is similar to that result, we will only sketch it here, referring the reader to [21] for the technical details.

The inequalities in both (4.6) and (4.7) involve bounding sums of contributions from each R_ξ having the form

$$D_\xi := \left| \int_{R_\xi} |K_\varepsilon(\eta)| d\mu(\eta) - \mu(R_\xi) |K_\varepsilon(\xi)| \right| \leq \int_{R_\xi} |K_\varepsilon(\eta) - K_\varepsilon(\xi)| d\mu(\eta).$$

Take ζ to be the north pole of the sphere and θ to be the co-latitude. Divide the sphere into $M \sim \pi/\|\mathcal{X}\|$ bands, B_m, in which $(m-1)\pi/M \leq \theta \leq m\pi/M$, $m = 1, \ldots, M$. Each R_ξ can have nontrivial intersection with at most two adjacent bands, because $\operatorname{diam}(R_\xi) \leq \|\mathcal{X}\| \sim \pi/M$. Thus, if $R_\xi \subset B_m \cup B_{m+1}$, then its lowest and highest co-latitudes satisfy $(m-1)\pi/M \leq \theta_\xi^- \leq \theta_\xi^+ \leq (m+1)\pi/M$. As is shown in [21], for $m = 2, \ldots, M-1$, the sum of the D_ξ from all $R_\xi \subset B_m \cup B_{m+1}$ is bounded above by the quantity

$$\sum_{R_\xi \subset B_m \cup B_{m+1}} D_\xi \leq \frac{C_{n,\kappa,k}}{M\varepsilon} \int_{\frac{m-1}{M\varepsilon}\pi}^{\frac{m+1}{M\varepsilon}\pi} \frac{t^n}{1+t^k} dt. \tag{4.8}$$

If $R_\xi \ni \zeta$, then dealing with the corresponding D_ξ can be done by estimating the integral that bounds the contribution from the region R_ξ in the cap $0 \leq \theta \leq 2\pi/M$,

$$D_\xi \leq C'_{n,\kappa,k} (M\varepsilon)^{-n} \int_0^{\frac{2\pi}{M\varepsilon}} \frac{t dt}{1+t^k} \leq \frac{C''_{n,\kappa,k}}{(M\varepsilon)^n} \begin{cases} (M\varepsilon)^{-2} & M\varepsilon \geq 1, \\ 1 & M\varepsilon \leq 1. \end{cases} \tag{4.9}$$

Now, let $M = \lfloor \pi/\|\mathcal{X}\| \rfloor$, precisely. Adding up the D_ξ for all $\xi \in X$ yields the bound in (4.6), which was implicit in the proof of [21, Proposition 4.1].

To get (4.7), we need to adjust M so that all $R_\xi \not\ni \zeta$ are contained in the bands $B_m \cup B_{m+1}$, $m = 2, \ldots, M-1$. This is easy to do. Just take $M = \lfloor (\pi - q)/\|\mathcal{X}\| \rfloor$. Summing the D_ξ bounded in (4.8) and taking care of some double counting yields

$$\begin{aligned} \left| \int_{\mathbb{S}^n - R_\zeta} |K_\varepsilon(\eta)| d\mu(\eta) - \sum_{X \ni \xi \neq \zeta} \mu(R_\xi) |K_\varepsilon(\xi)| \right| &\leq \frac{C_{n,\kappa,k}}{M\varepsilon} \int_{\frac{\pi}{M\varepsilon}}^{\frac{\pi}{\varepsilon}} \frac{t^n}{1+t^k} dt \\ &\leq \frac{C_{n,\kappa,k}}{M\varepsilon} \int_{\frac{\pi}{M\varepsilon}}^{\infty} \frac{t^n}{1+t^k} dt \\ &\leq C_{n,\kappa,k} \begin{cases} (M\varepsilon)^{-1} & M\varepsilon \geq 1, \\ (M\varepsilon)^{k-n-2} & M\varepsilon \leq 1, \end{cases} \end{aligned}$$

from which (4.7) follows easily. □

Let $f \in L^1(\mathbb{S}^n)$ and set $f_\varepsilon := K_{\varepsilon,n} * f$; the function f is *not* assumed to be zonal. We wish to estimate the difference $E_\mathcal{X} := \left| \|f_\varepsilon\|_1 - \sum_{\xi \in X} |f_\varepsilon(\xi)| \mu(R_\xi) \right|$. It

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

is straightforward to show that

$$E_{\mathcal{X}} \leq \sum_{\xi\in X}\int_{R_\xi} |f_\varepsilon(\eta) - f_\varepsilon(\xi)|d\mu(\eta) \leq \sup_{\zeta\in\mathbb{S}^n} F_{\varepsilon,\mathcal{X}}(\zeta)\|f\|_1\,,$$

where $F_{\varepsilon,\mathcal{X}}(\zeta) := \sum_{\xi\in X}\int_{R_\xi}\left|K_{\varepsilon,n}(\eta\cdot\zeta) - K_{\varepsilon,n}(\xi\cdot\zeta)\right|d\mu(\eta)$, which is the quantity estimated in Proposition 4.5. Applying that proposition and Remark 4.3, we obtain the desired estimate below.

Corollary 4.6. *Let κ satisfy* (4.1)*, with $k \geq n+2$, and, for $f \in L^1(\mathbb{S}^n)$, let $f_\varepsilon = K_{\varepsilon,n} * f$. If $\mathcal{X}$ is the decomposition of $\mathbb{S}^n$ described above, $\|\mathcal{X}\| \geq \varepsilon$ and $L_\varepsilon = \lfloor \varepsilon^{-1} - \lambda_n^{-1}\rfloor \sim \varepsilon^{-1}$, then*

(4.10)

$$\left|\|f_\varepsilon\|_1 - \sum_{\xi\in X}|f_\varepsilon(\xi)|\mu(R_\xi)\right| \leq C_{n,\kappa,k}\left(\|\mathcal{X}\|/\varepsilon\right)^n \begin{cases} E_{L_\varepsilon}(f)_1, & \kappa(t)=0,\ |t|\leq 1,\\ \|f\|_1, & \textit{otherwise}.\end{cases}$$

Remark 4.7. If f is zonal, i.e. $f(\xi) = \psi(\xi\cdot\zeta)$, then the right side (4.10) is independent of the variable ζ. Also, the strict inequality $\|\mathcal{X}\| \geq \varepsilon$ isn't absolutely necessary. The results still hold when $\|\mathcal{X}\|$ and ε are comparable.

For the most part, we will use these results to bound the sums $\left|\sum_{\xi\in X} a_\xi f_\varepsilon(\xi)\right|$, under the assumption that $\|\mathcal{X}\| \geq \varepsilon$. Using Corollary 4.6 for that case, we see that

$$\begin{aligned}\left|\sum_{\xi\in X} a_\xi f_\varepsilon(\xi)\right| &\leq \frac{|a|_\infty}{\min_{\xi\in X}\mu(R_\xi)}\sum_{\xi\in X}\mu(R_\xi)|f_\varepsilon(\xi)|\\ &\leq \frac{|a|_\infty}{\min_{\xi\in X}\mu(R_\xi)}\left(\|f_\varepsilon\|_{L^1} + C_{n,\kappa,k}\left(\|\mathcal{X}\|/\varepsilon\right)^n\|f\|_{L^1}\right).\end{aligned}$$

From Theorem 4.2, (4.5), and $h = \rho q$, with $L_\varepsilon \sim \varepsilon^{-1}$ and $\rho q \approx \|\mathcal{X}\| \geq \varepsilon$. we have that

$$\left|\sum_{\xi\in X} a_\xi f_\varepsilon(\xi)\right| \leq C\rho^n\varepsilon^{-n}|a|_\infty \begin{cases} E_{L_\varepsilon}(f)_1, & \text{if } \kappa(t)=0,\ |t|\leq 1,\\ \|f\|_1, & \text{otherwise}.\end{cases} \tag{4.11}$$

If f is a zonal function, then, by Remark 4.7, we may use the $\|\cdot\|_\infty$ norm on the left above.

We want to make the same kind of estimate, but with f being replaced by δ_ζ, the usual Dirac delta function. Thus f_ε is replaced by $K_\varepsilon(\cdot) := K_{\varepsilon,n} * \delta(\cdot) = K_{\varepsilon,n}((\cdot)\cdot\zeta)$. A nearly identical argument to the one used above, coupled with (4.6) for $\|\mathcal{X}\| \geq \varepsilon$ and the bound on $\|K_\varepsilon\|_1$ from Theorem 4.2, results in

$$\left|\sum_{\xi\in X} a_\xi K_\varepsilon((\cdot)\cdot\xi)\right| \leq C\rho^n\varepsilon^{-n}|a|_\infty. \tag{4.12}$$

The constants on the right above hold uniformly, so we thus have

$$\left\|\sum_{\xi\in X} a_\xi K_\varepsilon((\cdot)\cdot\xi)\right\|_\infty \leq C\rho^n\varepsilon^{-n}|a|_\infty. \tag{4.13}$$

The two bounds above are very similar and can be used in combination. They will be needed to complete the approximation part of the analysis. There is another bound, somewhat different from these two, that we will need in section 5:

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Lemma 4.8. *If $\rho q \sim \|\mathcal{X}\| \geq \varepsilon > 0$ and if $k \geq n+2$, then*

$$\max_{\zeta\in X} \sum_{X\ni\xi\neq\zeta} |K_{\varepsilon,n}(\xi\cdot\zeta)| \leq C_{n,\kappa,k} q^{-n}. \tag{4.14}$$

Proof. In equation (4.7), Proposition 4.5, again for $\|\mathcal{X}\| \geq \varepsilon$, an argument similar to the ones used above gives us

$$\sum_{X\ni\xi\neq\zeta} |K_{\varepsilon,n}(\xi\cdot\zeta)| \leq C'_{n,\kappa,k} q^{-n} \int_{\mathbb{S}^n - R_\zeta} |K_{\varepsilon,n}(\eta\cdot\zeta)| d\mu(\eta) + C''_{n,\kappa,k} q^{-n} \left(\frac{\varepsilon}{\|\mathcal{X}\|}\right)^{k-n-2}.$$

Using $\int_{\mathbb{S}^n - R_\zeta} |K_{\varepsilon,n}(\eta\cdot\zeta)| d\mu(\eta) \leq \|K_{\varepsilon,n}\|_1 \leq C_{n,\kappa,k}$, $\frac{\varepsilon}{\|\mathcal{X}\|} \leq 1$, and maximizing over $\zeta \in X$, we obtain (4.14). □

This estimate is more delicate than (4.13), because the term missing from the sum is $K_{\varepsilon,n}(\zeta\cdot\zeta) = K_{\varepsilon,n}(1)$, which turns out to be $\mathcal{O}(\varepsilon^{-n})$. For ε/q small enough, the sum (4.14) will be majorized by $K_{\varepsilon,n}(1)$. This is needed as part of a diagonal dominance argument.

4.3. **Frames.** We now address the question of the frame decomposition mentioned previously. Our approach follows the one in [21]. As mentioned earlier, others are certainly possible. For this, we need a function $a \in C^k(\mathbb{R})$,which we may assume is even, with support in $[-2,-\frac{1}{2}] \cup [\frac{1}{2},2]$, and satisfying $|a(t)|^2 + |a(2t)|^2 \equiv 1$ on $[\frac{1}{2},1]$. Such a function can be easily constructed out of an *orthogonal wavelet mask* m_0 [2, §8.3]. In fact, if $m_0(\xi) \in C^{k+1}$, then $a(t) := m_0(\pi \log_2(|t|))$ on $[-2,-\frac{1}{2}] \cup [\frac{1}{2},2]$, and 0 otherwise, is a C^k function that satisfies the appropriate criteria. Define $b \in C^k(\mathbb{R})$ by

$$b(t) := \begin{cases} 1 & |t| \leq 1, \\ |a(t)|^2 & |t| > 1. \end{cases} \tag{4.15}$$

Using the properties of a we see that $\sum_{j=-\infty}^{J} |a(t/2^j)|^2 = b(t/2^J)$ if $t > 0$. In the sum on the left, only terms with $j \geq \lfloor \log_2(t) \rfloor$ contribute. Terms with $j < \lfloor \log_2(t) \rfloor$ are identically 0.

The quantity $\lfloor \log_2(t) \rfloor$ is obviously important. On $\mathbb{S}^n$, the integer that corresponds to it is this:

$$j_n := \begin{cases} 0 & n = 1, \\ \lfloor \log_2(\lambda_n) \rfloor & n \geq 2. \end{cases} \tag{4.16}$$

The integer j_n helps us in defining our frame operators, which we now do. Let $\mathsf{A}_j := a(2^{-j-j_n}\mathsf{L}_n)$ and $\mathsf{B}_j := b(2^{-j-j_n}\mathsf{L}_n)$. Taking into account the support of a, we have $\mathsf{B}_J = \sum_{j=0}^{J} \mathsf{A}_j \mathsf{A}_j^*$ for $n \geq 2$. For $n = 1$, a projection P_0 onto the constant function enters, and $\mathsf{B}_J = \mathsf{P}_0 + \sum_{j=0}^{J} \mathsf{A}_j \mathsf{A}_j^*$. We will need the following approximation result concerning these operators.

Proposition 4.9 ([21, Proposition 5.1]). *Let $k > \max\{n, 2\}$, and let b be defined by (4.15), with $a \in C^k(\mathbb{R})$. If $f \in L^p(\mathbb{S}^n)$, $1 \leq p \leq \infty$, and if $L > 0$ is an integer such that $2^{-J-j_n} \leq (L+\lambda_n)^{-1}$, then*

$$\|f - \mathsf{B}_J f\|_p \leq C_{b,k,n} E_L(f)_p\,,\ E_L(f)_p := \mathrm{dist}_{L^p}(f, \Pi_L). \tag{4.17}$$

Also, for $1 \leq p < \infty$ or, if $p = \infty$, for $f \in C(\mathbb{S}^n)$, we have $\lim_{J\to\infty} \mathsf{B}_J f = f$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Bernstein/Nikolskii inequalities. There are several inequalities that follow easily using frames. We will give a Nikolskii-type inequality, which is a well-known inequality ([15, Proposition 2.1] and [21, §3.5]). From our point of view, the most important inequality derived here is a Bernstein theorem for spherical polynomials [28, Theorem 2 (English transl.)]. An independent proof is given in [10, Proposition 4.3]. For the convenience of the reader, short proofs for both are given below.

Theorem 4.10. *Let $S \in \Pi_L$. Then, for $1 \le p, q \le \infty$ and for $\gamma > 0$, we have*

$$\text{(Nikolskii)} \quad \|S\|_q \le C_{p,q,n} L^{n(\frac{1}{p}-\frac{1}{q})_+} \|S\|_p, \tag{4.18}$$

$$\text{(Bernstein)} \quad \|S\|_{H^p_\gamma} \le C_{n,\gamma} L^\gamma \|S\|_p. \tag{4.19}$$

Proof. Let $\gamma > 0$ and suppose $L + \lambda_n \le 2^{J+j_n}$. From the definition of B_J, it is easy to see that B_J reproduces Π_L, and so $\mathsf{B}_J S = S$ for all $S \in \Pi_L$. By Theorem 4.2, with $\kappa = b$ and $\varepsilon = 2^{-J-j_n} \sim L^{-1}$, we see that $\|S\|_q \le C_{p,q,n} L^{n(\frac{1}{p}-\frac{1}{q})_+} \|S\|_p,\ S \in \Pi_L$. Dependence of the constants on b and k disappears upon taking the infimum over these two quantities, yielding (4.18).

We now establish the Bernstein inequality. If $S \in \Pi_L$, then so is $\mathsf{L}^\gamma S$, and we have that $\mathsf{B}_J \mathsf{L}_n^\gamma S = \mathsf{L}_n^\gamma S$, provided $L + \lambda_n \le 2^{J+j_n}$. Using the expansion $\mathsf{B}_J = \sum_{j=0}^J \mathsf{A}_j \mathsf{A}_j^*$, we see that

$$\mathsf{L}^\gamma S = \sum_{j=0}^J \mathsf{A}_j \mathsf{A}_j^* \mathsf{L}^\gamma S = \sum_{j=0}^J \mathsf{L}^\gamma \mathsf{A}_j \mathsf{A}_j^* S.$$

Consequently, we have that $\|S\|_{H_p^\gamma} = \|\mathsf{L}^\gamma S\|_p \le \sum_{j=0}^J \|\mathsf{L}^\gamma \mathsf{A}_j \mathsf{A}_j^*\|_{p,p} \|S\|_p$. Applying Corollary 4.4, with $\kappa(t) = |a(t)|^2$ and $\varepsilon = 2^{-j-j_n}$ for each j, then yields:

$$\begin{aligned}\|S\|_{H_p^\gamma} &\le \left(\sum_{j=0}^J 2^{(j+j_n)\gamma}\right) C_{a,n,\gamma} \|S\|_p \\ &\le \frac{2^{(J+j_n+1)\gamma} - 2^{j_n\gamma}}{2^\gamma - 1} C_{a,n,\gamma} \|S\|_p \le L^\gamma C_{a,n,\gamma} \|S\|_p\,,\end{aligned}$$

where again $L \sim 2^{J+j_n}$. In the last inequality of the chain above, we can take the infimum over all a satisfying the requisite conditions. This yields (4.19). □

Distance estimates. Frames can be used to estimate the distance in $L^p(\mathbb{S}^n)$ from the polynomials to a function in a smoother space. If $f \in L^p$, let $E_L(f)_p := \mathrm{dist}_{L^p}(f, \Pi_L)$. Because $\mathsf{B}_J f$ is a spherical polynomial in $\Pi_{2^{J+j_n+1}}$, we have

$$E_L(f)_p \le \|f - \mathsf{B}_J f\|_p,\ L + \lambda_n \le 2^{J+j_n+1}.$$

Because $\mathsf{B}_J f$ converges to f in all L^p, $1 \le p < \infty$ and $p = \infty$ if $f \in C(\mathbb{S}^n)$, we also have that

$$E_L(f)_p \le \|f - \mathsf{B}_J f\|_p \le \sum_{j=J+1}^\infty \|\mathsf{A}_j \mathsf{A}_j^* f\|_p,$$

where the right side above may be infinite. Now, suppose that $f = \mathsf{L}_n^\gamma h$, $h \in H_\beta^q(\mathbb{S}^n)$. In that case, we have $\mathsf{A}_j \mathsf{A}_j^* \mathsf{L}_n^\gamma h = \mathsf{L}_n^{-(\beta-\gamma)} \mathsf{A}_j \mathsf{A}_j^* \mathsf{L}_n^\beta h$. From this and Corollary 4.4, with $p \leftrightarrow q$, we arrive at

$$\|\mathsf{A}_j \mathsf{A}_j^* \mathsf{L}_n^\gamma h\|_p = \|\mathsf{L}_n^{-(\beta-\gamma)} \mathsf{A}_j \mathsf{A}_j^* \mathsf{L}_n^\beta h\|_p \le 2^{-(\beta-\gamma-n(\frac{1}{q}-\frac{1}{p})_+)(j+j_n)} C_{n,k,a} \|h\|_{H_\beta^q}.$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Insert this in the equation above, sum the appropriate geometric series, and take $L \sim 2^{J+j_n}$ to get

$$E_{2^{J+j_n}}(\mathsf{L}_n^\gamma h)_p \le C'_{\beta-\gamma,a,k,n} 2^{-(\beta-\gamma-n(\frac{1}{q}-\frac{1}{p})_+)(J+j_n)} \|h\|_{H^q_\beta},$$

which was essentially obtained by Kamzolov [9]. Now, since the left side above is unchanged if we replace L_n^γ by $\mathsf{L}_n^\gamma - S$, $S \in \Pi_{2^{J+j_n}}$, we can replace $\|h\|_{H^q_\beta}$ by $E_{2^{J+j_n}}(\mathsf{L}_n^\beta h)_q$. Collecting these results yields the proposition below.

Proposition 4.11. *Let $\gamma \ge 0$, and $\beta > \gamma + n(\frac{1}{q} - \frac{1}{p})_+)$, where $1 \le p, q \le \infty$. If $h \in H^q_\beta$, then there is a constant $C = C_{n,\beta,\gamma,a}$ such that*

$$E_{2^{J+j_n}}(\mathsf{L}_n^\gamma h)_p \le \|(I - \mathsf{B}_J)h\|_{H^p_\gamma} \le C_{n,\beta,\gamma,a} 2^{-(\beta-\gamma-n(\frac{1}{q}-\frac{1}{p})_+))(J+j_n)} E_{2^{J+j_n}}(\mathsf{L}_n^\beta h)_q .$$

Green's functions and their properties. Let $\beta > n/p'$. Recall that the Green's function solution to $\mathsf{L}_n^\beta G_\beta = \delta$ is a kernel with an expansion in spherical harmonics having coefficients $\widehat{G}_\beta(\ell, m) = (\ell + \lambda)^{-\beta}$. Properties of Green's functions (pseudo-differential operator kernels, really) on manifolds have been studied extensively (cf. [8]). Our aim here is to use frames to obtain properties and various distance estimates that we need here quickly, and in a self-contained way, for SBFs of the form $\phi_\beta = G_\beta + G_\beta * \psi$, where $\psi \in L^1$. Because the ϕ_β's are not in any of the Bessel-Sobolev spaces H^p_β, they have to be treated separately from the class in Proposition 4.11 above

We begin with Green's functions themselves. Note that $\mathsf{A}_j\mathsf{A}_j^* G_\beta = \mathsf{L}_n^{-\beta}\mathsf{A}_j\mathsf{A}_j^*\delta$. Since $\mathsf{A}_j\mathsf{A}_j^* = |a|^2(2^{-j-j_n}\mathsf{L}_n)$, where both a and, of course, $|a|^2$, have compact support that excludes $t = 0$, we may apply Corollary 4.4, with $\varepsilon_j := 2^{-(j+j_n)}$, to get this:

$$\|\mathsf{A}_j\mathsf{A}_j^* G_\beta\|_p \le C_{n,\beta,a}\varepsilon_j^{\beta-n/p'} = C_{n,\beta,a} 2^{-(\beta-n/p')(j+j_n)}. \tag{4.20}$$

Thus, for $\beta > n/p'$, the terms in $\sum_{j=0}^\infty \mathsf{A}_j\mathsf{A}_j^* G_\beta$ are bounded by a geometric series, and so the Weierstrass M test implies that the series converges in L^p. That is, we have shown that when $\beta > n/p'$ the limit $\lim_{J\to\infty} \mathsf{B}_J G_\beta$ is in L^p. A simple duality argument then shows that the kernel $G_\beta = \lim_{J\to\infty} \mathsf{B}_J G_\beta$ in $L^p(\mathbb{S}^n)$. Summing the geometric series in (4.20) yields $\|G_\beta - \mathsf{B}_J G_\beta\|_p \le C\, 2^{-(\beta-n/p')(J+j_n)}$.

These results also give us error bounds in $H^p_\gamma(\mathbb{S}^n)$. If $\gamma \ge 0$, then $\mathsf{L}^\gamma G_\beta = G_{\beta-\gamma}$ and $\mathsf{L}^\gamma \mathsf{B}_J G_\beta = \mathsf{B}_J G_{\beta-\gamma}$. This and the estimate above imply that if, in addition, $\beta > \gamma + n/p'$, then

$$\|G_\beta - \mathsf{B}_J G_\beta\|_{H^p_\gamma} = \|G_{\beta-\gamma} - \mathsf{B}_J G_{\beta-\gamma}\|_p \le C\, 2^{-(\beta-\gamma-n/p')(J+j_n)}. \tag{4.21}$$

Perturbations of G_β can be dealt with, too. Let ψ be in L^1. By Theorem 4.2, (4.20) and Remark 4.3, we have that, for all $j \ge J$,

$$\|\mathsf{A}_j\mathsf{A}_j^* G_\beta * \psi\|_p \le \|\mathsf{A}_j\mathsf{A}_j^* G_\beta\|_{1,p}\, E_{2^{j+j_n}}(\psi)_1 \le C 2^{-(\beta-\gamma)(j+j_n)} E_{2^{J+j_n}}(\psi)_1 .$$

Summing a geometric series and using (4.21), we arrive at the following bound.

Proposition 4.12. *Let $\gamma \ge 0$, $\beta > \gamma + n/p'$, $\varepsilon_j = 2^{-(j+j_n)}$, and let $\psi \in L^1$ be a zonal function. If $\phi_\beta = G_\beta + G_\beta * \psi$, then $\phi_\beta \in H^p_\gamma$ and there is a constant $C = C_{n,\beta,\gamma,a}$, which depends only on n, β, γ, and the function a, such that*

$$\begin{aligned} E_{2^{J+j_n}}(\mathsf{L}^\gamma \phi_\beta)_p &\le \|(I - \mathsf{B}_J)\phi_\beta\|_{H^p_\gamma} \\ &\le C_{n,\beta,\gamma,a}\left(1 + \varepsilon_J^{n/p'} E_{2^{J+j_n}}(\psi)_1\right)\varepsilon_J^{\beta-\gamma-n/p'} \end{aligned} \tag{4.22}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

4.4. **Approximation analysis.** The task at hand is to estimate the norms $\|(I-\mathsf{B}_J)g\|_{H^p_\gamma}/|a|_p$, where $g\in\mathcal{G}_{X,\phi}$. Our approach will be to carry this out for $p=1$ and $p=\infty$, then use the Riesz-Thorin theorem to obtain the result for all intermediate values of p.

The easier of the two cases is $p=1$. Since $g\in\mathcal{G}_{X,\phi}$, then $g=\sum_{\xi\in X}a_\xi\phi((\cdot)\cdot\xi)$. Again, let $\varepsilon_j=2^{-(j+j_n)}$. From the triangle inequality, the rotational invariance of the norms involved, and Proposition 4.11 and Proposition 4.12 it follows that

$$\begin{aligned}\|(I-\mathsf{B}_J)g\|_{H^1_\gamma}&\le|a|_1\|(I-\mathsf{B}_J)\phi\|_{H^1_\gamma}\\&\le C\varepsilon_J^{\beta-\gamma}|a|_1\begin{cases}E_{2^{J+j_n}}(\mathsf{L}_n^\beta\phi)_1 & \phi\in H^1_\beta\,,\\(1+E_{2^{J+j_n}}(\psi)_1) & \phi=G_\beta+G_\beta*\psi\,.\end{cases}\end{aligned}$$

The $p=\infty$ case requires using frames. Again, we have that

$$\|(I-\mathsf{B}_J)g\|_{H^\infty_\gamma}\le\sum_{j=J+1}^{\infty}\|\mathsf{A}_j\mathsf{A}_j^*\mathsf{L}_n^\gamma g\|_\infty\,,$$

where $\mathsf{A}_j\mathsf{A}_j^*\mathsf{L}_n^\gamma g=\sum_{\xi\in X}a_\xi\mathsf{A}_j\mathsf{A}_j^*\mathsf{L}_n^\gamma\phi((\cdot)\cdot\xi)$. By equation (4.11), with $f=\mathsf{L}_n^\gamma\phi$, $K_{\varepsilon_j,n}$ corresponding to $\kappa(t)=|a(t)|^2$, $h\ge\varepsilon_J\ge\varepsilon_j$, all $j\ge J$, and $L_\varepsilon\sim2^{j+j_n}$, we have

$$\|\mathsf{A}_j\mathsf{A}_j^*\mathsf{L}_n^\gamma g\|_\infty=\Big\|\sum_{\xi\in X}a_\xi f_\varepsilon((\cdot)\cdot\xi)\Big\|_\infty\le C\rho^n\varepsilon_j^{-n}|a|_\infty E_{2^{j+j_n}}(\mathsf{L}_n^\gamma\phi)_1\,.$$

By Proposition 4.11 and Proposition 4.12, with J there replaced by j, $p=\infty$, we have

$$\|\mathsf{A}_j\mathsf{A}_j^*\mathsf{L}_n^\gamma g\|_\infty\le C|a|_\infty\rho^n\varepsilon_j^{\beta-\gamma-n}\begin{cases}E_{2^{j+j_n}}(\mathsf{L}_n^\beta\phi)_1 & \phi\in H^1_\beta\,,\\(1+\varepsilon_j^nE_{2^{j+j_n}}(\psi)_1) & \phi=G_\beta+G_\beta*\psi\,.\end{cases}$$

Since $E_{2^{j+j_n}}(f)_1\le E_{2^{J+j_n}}(f)_1$ when $j\ge J$, in the inequality above we may replace the distances with respect to 2^{j+j_n} with ones with respect to 2^{J+j_n}. Doing so and again summing a geometric series, we obtain

$$\|(I-\mathsf{B}_J)g\|_{H^\infty_\gamma}\le C|a|_\infty\rho^n\varepsilon_J^{\beta-\gamma-n}\begin{cases}E_{2^{J+j_n}}(\mathsf{L}_n^\beta\phi)_1 & \phi\in H^1_\beta\,,\\(1+\varepsilon_J^nE_{2^{J+j_n}}(\psi)_1) & \phi=G_\beta+G_\beta*\psi\,.\end{cases}$$

Applying the Riesz-Thorin theorem in conjunction with the bounds above, we complete the approximation part of the problem:

Theorem 4.13. *Let $\gamma\ge0$, $1\le p\le\infty$, $\beta>\gamma+n/p'$, $\varepsilon_j=2^{-(J+j_n)}$. If $h_X\ge\varepsilon_j$ and if $g\in\mathcal{G}_{X,\phi}$, then*

(4.23)
$$\frac{\|(I-\mathsf{B}_J)g\|_{H^p_\gamma}}{|a|_p}\le C\rho^{n/p'}\varepsilon_J^{\beta-\gamma-n/p'}\begin{cases}E_{2^{J+j_n}}(\mathsf{L}_n^\beta\phi)_1 & \phi\in H^1_\beta\,,\\(1+E_{2^{J+j_n}}(\psi)_1) & \phi=G_\beta+G_\beta*\psi\,.\end{cases}$$

5. Stability

The problem that we address here is estimating the norm $|a|_p$ in terms of the $L^p(\mathbb{S}^n)$ norm of g, where $g(\mathbf{x})=\sum_{\xi\in X}a_\xi\phi(\mathbf{x}\cdot\xi)$ and $\phi\in L^p$ is an SBF. Specifically, we wish to estimate the p-norm stability ratio

$$\mathsf{r}_{\mathcal{G},\,p}:=\max_{\mathcal{G}\ni g\neq0}\frac{|a|_p}{\|g\|_p},$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

which we defined in (1.1). This quantity exists and is finite because the set $\{\phi(\mathbf{x}\cdot\xi)\}_{\xi\in X}$ is a linearly independent, finite set of functions. The quantity $\mathsf{r}_{\mathcal{G},\,p}$ provides a measure of the linear independence of the set, albeit one that scales with the norm of ϕ. Once ϕ is fixed, it depends completely on the geometry of X.

For a continuous SBF ϕ, this is related to the stability of the interpolation matrix for ϕ and X. However, we are only assuming that ϕ is in L^p, and thus evaluating ϕ on X is meaningless. Even so, using a smoothed version of ϕ allows us to connect the two concepts.

5.1. **Stability ratios and interpolation matrices.** Let $\kappa \geq 0$ be in $C^k(\mathbb{R})$, $k \geq n+2$, and let it satisfy (4.1). Of course, since κ is not identically 0, we also have that there is some open interval on which $\kappa > 0$. Consider the corresponding operator $\mathsf{K}_{\varepsilon,n} = \kappa(\varepsilon \mathsf{L}_n)$ and its kernel $K_{\varepsilon,n}$. To smooth $g(\mathbf{x}) = \sum_{\xi\in X} a_\xi \phi(\mathbf{x}\cdot\xi)$, apply $\mathsf{K}_{\varepsilon,n}$ to both sides. Doing this yields

$$g_\varepsilon(\mathbf{x}) = \mathsf{K}_{\varepsilon,n} g(\mathbf{x}) = \sum_{\xi\in X} a_\xi \underbrace{\mathsf{K}_{\varepsilon,n}\phi(\mathbf{x}\cdot\xi)}_{\phi_\varepsilon(\mathbf{x}\cdot\xi)} . \tag{5.1}$$

We want to relate $\mathsf{r}_{\mathcal{G},\,p}$ to quantities in a standard SBF interpolation problem on X involving ϕ_ε. The function ϕ_ε is a spherical harmonic, with nonnegative Fourier-Legendre coefficients, whose degree depends on the support of κ. It is thus a positive definite function on $\mathbb{S}^n$, but not an SBF.

The interpolation matrix corresponding to ϕ_ε is

$$A_\varepsilon = [\phi_\varepsilon(\eta\cdot\xi)]_{\xi,\eta\in X}.$$

Later, as a by-product of our analysis, we will establish the invertibility of A_ε, provided ε satisfies certain conditions. When ε is sufficiently small, one can also establish it by using a result of Ron and Sun [27, Theorem 6.4]: Let $X \subset \mathbb{S}^n$ be fixed and let ψ be a positive definite function, but not necessarily an SBF (i.e., some of coefficients $\hat{\psi}(\ell)$ may vanish). Then, there is an integer $j_{X,n}$ such that the interpolation matrix A_ψ will be positive definite if the set of integers on which $\hat{\psi}(\ell) > 0$ contains at least $j_{X,n}$ consecutive even integers and $j_{X,n}$ consecutive odd integers. With our assumptions on κ, in particular, that κ is not identically 0, it is clear that for sufficiently small ε there are arbitrarily large sets of consecutive integers for which $\hat{\phi}_\varepsilon(\ell) > 0$. Thus A_ε is (strictly) positive definite, and hence invertible, for all such ε.

Our approach will again be to use the Riesz-Thorin theorem. Let $y_\varepsilon := g_\varepsilon|_X$, the restriction of g_ε to X. Using (5.1), we can interpolate g_ε on X:

$$y_\varepsilon = A_\varepsilon a\,,\ A_\varepsilon = [\phi_\varepsilon(\eta\cdot\xi)]_{\xi,\eta\in X}.$$

Solving and taking the ℓ^1 norm, we see that

$$|a|_1 \leq \|A_\varepsilon^{-1}\|_1 |y_\varepsilon|_1\,,\ |y_\varepsilon|_1 = \sum_{\xi\in X} |g_\varepsilon(\xi)|.$$

By our assumptions on κ and by (4.11), we have that $|y_\varepsilon|_1 \leq C_{n,\kappa,k}\rho^n\varepsilon^{-n}\|g\|_{L^1}$. Consequently, for $\phi \in L^1$ we have that

$$\mathsf{r}_{\mathcal{G},\,1} \leq C_{\kappa,n,k}\rho^n\varepsilon^{-n}\|A_\varepsilon^{-1}\|_1\,.$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Similarly, working with $p = \infty$ we obtain

$$|a|_\infty \leq \|A_\varepsilon^{-1}\|_\infty |y_\varepsilon|_\infty\,,\ |y_\varepsilon|_\infty = \max_{\xi\in X}\{|g_\varepsilon(\xi)|\} \leq \|g\|_\infty.$$

Recall that A_ε^{-1} is a selfadjoint matrix, and that for such matrices the $p = 1$ and $p = \infty$ norms are equal: $\|A_\varepsilon^{-1}\|_\infty = \|A_\varepsilon^{-1}\|_1$. Hence, for $\phi \in C$ $(p = \infty)$, we obtain

$$\mathsf{r}_{\mathcal{G},\,\infty} \leq \|A_\varepsilon^{-1}\|_1\,.$$

Applying the Riesz-Thorin theorem to these bounds yields the following:

Proposition 5.1. *Let $\varepsilon \leq \|\mathcal{X}\|$ and let $\phi \in L^p$. Then,*

$$\mathsf{r}_{\mathcal{G},\,p} \leq C_{\kappa,n,k}^{1/p} \rho^{n/p} \varepsilon^{-n/p} \|A_\varepsilon^{-1}\|_1\,.$$

5.2. **ℓ^1 stability estimates for interpolation matrices.** The estimates we need next are for $\|A_\varepsilon^{-1}\|_1$, and the approach we take to get them will depend on ϕ and the behavior of the $\hat{\phi}(\ell)$'s. We will first deal with the Green's function case, in which $\hat{\phi}(\ell)$ decays algebraically. After that, we will deal with the case in which ϕ is C^∞, and $\hat{\phi}(\ell)$ has very fast decay.

5.2.1. *SBFs that are perturbations of Green's functions.* A straightforward way to estimate the 1-norm of the inverse of a matrix is to use diagonal dominance techniques if the matrix is amenable to them. To that end, split an $n \times n$ matrix A into its diagonal D and off-diagonal F, so $A = D + F$. We then have the following standard norm estimate, whose proof we omit.

Lemma 5.2. *If D is invertible and $\|D^{-1}F\|_1 < 1$, then A is invertible and $\|A^{-1}\|_1 < \|D^{-1}\|_1(1 - \|D^{-1}F\|_1)^{-1}$.*

We can apply this to A_ε. The diagonal part is $D = \phi_\varepsilon(1)I$, and so $\|D^{-1}\|_1 = \phi_\varepsilon(1)^{-1}$ and $\|D^{-1}F\|_1 = \phi_\varepsilon(1)^{-1}\|F\|_1$. Since the 1-norm of a matrix is the maximum of the 1-norms of its columns, our condition becomes

$$\phi_\varepsilon(1)^{-1}\|F\|_1 = \phi_\varepsilon(1)^{-1} \max_{\eta\in X} \sum_{X\ni\xi\neq\eta} |\phi_\varepsilon(\eta\cdot\xi)| < 1. \tag{5.2}$$

We now want to deal with a special ϕ_ε, which is not necessarily generated by an SBF ϕ. Let ψ be a zonal function in L^1, so that

$$\psi(\xi\cdot\eta) = \sum_{\ell=0}^{\infty} \hat{\psi}(\ell) \frac{\ell+\lambda_n}{\lambda_n\omega_n} P_\ell^{(\lambda_n)}(\xi\cdot\eta).$$

We will assume that $1+\hat{\psi}(\ell) > 0$ for all $\ell \geq 0$ and that κ has support in $|t| \in [1,\infty)$. Take $\phi_\varepsilon = K_{\varepsilon,n} + K_{\varepsilon,n} * \psi$, where $K_{\varepsilon,n}$ is the kernel for the operator $\kappa(\varepsilon\mathsf{L})$. In addition, define $\psi_\varepsilon = K_{\varepsilon,n} * \psi$. Since $\hat{\phi}_\varepsilon(\ell) = \kappa(\varepsilon(\ell+\lambda_n))(1+\hat{\psi}(\ell)) \geq 0$, we see that ϕ_ε is a positive definite spherical function, but not an SBF. Using (4.14) yields

$$\begin{aligned}\sum_{X\ni\xi\neq\eta} |\phi_\varepsilon(\eta\cdot\xi)| &\leq \sum_{X\ni\xi\neq\eta} |K_{\varepsilon,n}(\eta\cdot\xi)| + \sum_{X\ni\xi\neq\eta} |\psi_\varepsilon(\eta\cdot\xi)| \\ &\leq C_{n,\kappa,k} q^{-n} + \sum_{\xi\in X} |\psi_\varepsilon(\eta\cdot\xi)|.\end{aligned}$$

Thus, from this and equation (4.11), with $\kappa(t) = 0$, $|t| \leq 1$, we have shown that

$$\sum_{X\ni\xi\neq\eta} |\phi_\varepsilon(\eta\cdot\xi)| \leq C_{n,\kappa,k}(q^{-n} + \rho^n\varepsilon^{-n} E_{L_\varepsilon}(\psi)_1),\ L_\varepsilon = \lfloor 1/\varepsilon - \lambda_n \rfloor. \tag{5.3}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Thus we have bounded the sum involved in the diagonal dominance condition (5.2). Next, we will deal with $\phi_\varepsilon(1)$. We have the following chain of inequalities:

$$\begin{aligned}\phi_\varepsilon(1) &= K_{\varepsilon,n}(1) + K_{\varepsilon,n} * \psi(1)\\ &= \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))(1+\hat{\psi}(\ell))d_\ell^n\\ &\geq c_0 \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))d_\ell^n = c_0 K_{\varepsilon,n}(1),\end{aligned}$$

where $c_0 = \min_{\ell\geq 0}(1+\psi(\ell)) > 0$. (This is true because $\psi \in L^1$ implies that $\hat{\psi}(\ell) \to 0$ as $\ell \to \infty$.) Furthermore, it is easy to see that

$$K_{\varepsilon,n}(1) = \sum_{\ell=0}^{\infty} \kappa(\varepsilon(\ell+\lambda_n))d_\ell^n \sim \varepsilon^{-n} \underbrace{\int_1^\infty \kappa(t)t^{n-1}dt}_{>0}.$$

Thus, $\phi_\varepsilon(1) \geq C''_{n,\kappa,k}\varepsilon^{-n}$. From this and (5.3), we arrive at the bound below:

$$\|D^{-1}F\|_1 \leq C_{n,\kappa,k}\left((\varepsilon/q)^n + \rho^n E_{L_\varepsilon}(\psi)_1\right), \ L_\varepsilon = \lfloor 1/\varepsilon - \lambda_n \rfloor. \tag{5.4}$$

By choosing $\varepsilon \leq q$ sufficiently small, we can make $C_{n,\kappa,k}\rho^n E_{L_\varepsilon}(\psi)_1$ less than 1/4, since $E_{L_\varepsilon}(\psi)_1 \to 0$ as $L_\varepsilon \to \infty$. At this point, the choice of ε depends only on ψ and the mesh ratio ρ. If necessary, we may then choose ε smaller still in order to force the first term on the right to be less than 1/4. With this choice of ε, which depends on ρ, n, κ and k, we obtain $\|D^{-1}F\|_1 < 1/2$. By Lemma 5.2, we get the bound on $\|A_\varepsilon^{-1}\|_1$ below.

Proposition 5.3. *Suppose that κ has support in $|t| \in [1,\infty)$. Let $\phi_\varepsilon = K_{\varepsilon,n} + K_{\varepsilon,n} * \psi$, where $\psi \in L^1$ is a zonal function satisfying $1+\psi(\ell) > 0$ for $\ell \geq 0$. Then there are constants c and C, which depend on ψ, on ρ, n, κ and k, such that whenever $\varepsilon \leq cq$, we have $\|A_\varepsilon^{-1}\|_1 \leq C\varepsilon^n$.*

The proof above required conditions on the support of κ in order to deal with the perturbation generated by ψ. If ψ is 0, then there is no need for such restrictions. Also, the term involving ρ is gone, and it is no longer involved in determining c and C. We collect these observations below.

Remark 5.4. If $\psi = 0$, then Proposition 5.3 holds without restriction on the support of κ, and neither c nor C depends on ρ.

We now take an SBF ϕ of the form $\phi = G_\beta + G_\beta * \psi$, where G_β is the Green's function for L^β and $\psi \in L^1$. Our aim is to establish a bound on the stability ratio for such a ϕ.

Theorem 5.5. *Consider the SBF $\phi = G_\beta + G_\beta * \psi$, where G_β is the Green's function for L^β and $\psi \in L^1$. Let X be a set of centers with separation radius q and mesh ratio ρ. Let $\mathcal{G} = \mathcal{G}_{\phi,X}$ be the corresponding SBF network. Then there is a constant $C = C(n,\phi,\beta)$ such that the stability ratio of $\mathcal{G}$ satisfies*

$$\mathsf{r}_{\mathcal{G},\,p} \leq C\rho^{n/p}q^{n/p'-\beta}. \tag{5.5}$$

Proof. Since we are assuming that ϕ is an SBF, the coefficients of the L^1 function ψ must satisfy $1+\hat{\psi}(\ell) > 0$ for all $\ell \geq 0$. Assume κ satisfies (4.1) and has support in $|t| \in [1,\infty)$. The corresponding ϕ_ε is just $\phi_\varepsilon = \mathsf{K}_{\varepsilon,n}\phi = \mathsf{K}_{\varepsilon,n}(G_\beta + G_\beta * \psi)$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

By Corollary 4.4, we have that $\mathsf{K}_{\varepsilon,n}G_\beta = \varepsilon^\beta \widetilde{\mathsf{K}}_{\varepsilon,n} = \tilde{\kappa}(\varepsilon \mathsf{L})$, where $\tilde{\kappa}(t) = |t|^{-\beta}\kappa(t)$ satisfies (4.1). From this, we have that $\phi_\varepsilon = \varepsilon^\beta \tilde{\phi}_\varepsilon$. If we let $\tilde{A}_\varepsilon$ be the interpolation matrix for $\tilde{\phi}_\varepsilon$, we see that $A_\varepsilon = \varepsilon^\beta \tilde{A}_\varepsilon$. The function $\tilde{\phi}_\varepsilon$ satisfies the conditions on the corresponding function in Proposition 5.3. Thus, by choosing $\varepsilon \le cq$, we have

$$\|A_\varepsilon^{-1}\|_1 = \varepsilon^{-\beta}\|\tilde{A}_\varepsilon^{-1}\|_1 \le C\varepsilon^{n-\beta}.$$

From Proposition 5.1, we obtain

$$\mathsf{r}_{\mathcal{G},\,p} \le C_{\kappa,n,k}^{1/p}\rho^{n/p}\varepsilon^{-n/p}\|A_\varepsilon^{-1}\|_1 \le C'\rho^{n/p}\varepsilon^{n/p'-\beta}.$$

Choosing ε as large as possible, namely $\varepsilon = cq$, we have

$$\mathsf{r}_{\mathcal{G},\,p} \le C\rho^{n/p}q^{n/p'-\beta},$$

where the constant $C = C(n,\kappa,k,\phi,p,\beta)$. By taking the infimum over all κ, p and k, we reduce the dependency of C to $C = C(n,\phi,\beta)$. This completes the proof. $\square$

5.2.2. *Infinitely differentiable SBFs.* Let ϕ be an infinitely differentiable SBF. The fast decay of the Fourier-Legendre coefficient $\hat{\phi}(\ell)$ requires a different approach to bounding $\mathsf{r}_\mathcal{G}$ than the one used to obtain Theorem 5.5. As before, we let A_ε be the $N \times N$ interpolation matrix for $\phi_\varepsilon = \mathsf{K}_{\varepsilon,n}\phi$. In addition, we will let A be the corresponding matrix for ϕ. By standard matrix estimates, the norm $\|A_\varepsilon^{-1}\|_1$ satisfies

$$\|A_\varepsilon^{-1}\|_1 \le N^{1/2}\|A_\varepsilon^{-1}\|_2.$$

Since A_ε is a positive definite selfadjoint matrix, the norm $\|A_\varepsilon^{-1}\|_2$ is equal to the reciprocal of $\lambda_{\min}(A_\varepsilon)$, the smallest eigenvalue of A_ε; that is, $\|A_\varepsilon^{-1}\|_2 = 1/\lambda_{\min}(A_\varepsilon)$. We will begin by estimating this eigenvalue. In preparation for this, we define the quantity

$$\hat{\phi}_{\min}(L) := \min_{0\le\ell\le L}\hat{\phi}(\ell) > 0, \tag{5.6}$$

where the strict positivity follows from ϕ being an SBF.

Proposition 5.6. *Let $\kappa \ge 0$ be in $C^k(\mathbb{R})$, $k \ge n+2$, and let it satisfy* (4.1). *In addition, suppose that* $\operatorname{supp}(\kappa) \subseteq [-2,2]$ *and that $\kappa \le 1$. Then, there are constants $c = c_{n,\kappa,k} > 0$ and $C = C_{n,\kappa,k} > 0$ such that for all $\varepsilon \le cq$,*

$$\lambda_{\min}(A) \ge \lambda_{\min}(A_\varepsilon) \ge C\hat{\phi}_{\min}(L_{\varepsilon/2})\varepsilon^{-n},\ L_{\varepsilon/2} := \lfloor 2/\varepsilon - \lambda_n \rfloor.$$

Proof. Using the Rayleigh-Ritz principle, we thus have

$$\|A_\varepsilon^{-1}\|_2^{-1} = \lambda_{\min}(A_\varepsilon) = \min_{a\in\mathbb{C}^N} a^*A_\varepsilon a,$$

where $A_\varepsilon = [\phi_\varepsilon(\eta\cdot\xi)]_{\xi,\eta\in X}$. Because ϕ_ε is a (positive definite) zonal function, we can use its expansion in spherical harmonics to represent $\lambda_{\min}(A_\varepsilon)$ via

$$\lambda_{\min}(A_\varepsilon) = \min_{a\in\mathbb{C}^N}\left(\sum_{\ell=0}^{\infty}\sum_{m=1}^{d_\ell}\kappa((\ell+\lambda_n)\varepsilon)\hat{\phi}(\ell)\left|\sum_{\xi\in X}Y_{\ell,m}(\xi)a_\xi\right|^2\right). \tag{5.7}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Since the support of κ is $[-2,2]$, the sum above cuts off at $L_{\varepsilon/2} := \lfloor 2/\varepsilon - \lambda_n \rfloor$. Consequently, we can bound below $\lambda_{\min}(A_\varepsilon)$ this way:

$$\lambda_{\min}(A_\varepsilon) \geq \hat{\phi}_{\min}(L_{\varepsilon/2}) \underbrace{\min_{a\in\mathbb{C}^N}\left(\sum_{\ell=0}^{L_{\varepsilon/2}}\sum_{m=1}^{d_\ell}\kappa((\ell+\lambda_n)\varepsilon)\left|\sum_{\xi\in X}Y_{\ell,m}(\xi)a_\xi\right|^2\right)}_{\lambda_{\min}([K_{\varepsilon,n}(\xi\cdot\eta)])},$$

Note that $\lambda_{\min}([K_{\varepsilon,n}(\xi\cdot\eta)]) = \|\,[K_{\varepsilon,n}(\xi\cdot\eta)]^{-1}\|_2^{-1} \leq \|\,[K_{\varepsilon,n}(\xi\cdot\eta)]^{-1}\|_1^{-1}$, because $\|B\|_2 \leq \|B\|_1$ for all selfadjoint B. The existence of c and C and their dependencies, along with $\|\,[K_{\varepsilon,n}(\xi\cdot\eta)]^{-1}\|_1 \leq C\varepsilon^n$ for $\varepsilon \leq cq$, follow from Proposition 5.3 and Remark 5.4. Finally, applying the Rayleigh-Ritz principle, (5.7), and $0 \leq \kappa \leq 1$, we have that $\lambda_{\min}(A) \geq \lambda_{\min}(A_\varepsilon)$. This finishes the proof. □

There are two immediate consequences that follow from Proposition 5.6. The first is a bound on the stability ratio in this case.

Theorem 5.7. *Consider the SBF ϕ, where ϕ is assumed to be infinitely differentiable, and let X be a set of centers with separation radius q and mesh ratio ρ. Let $\mathcal{G} = \mathcal{G}_{\phi,X}$ be the corresponding SBF network. Then there are positive constants $C = C_{n,\kappa,k}$ and $c = c_{n,\kappa,k}$ such that the stability ratio of $\mathcal{G}$ satisfies*

$$\mathsf{r}_{\mathcal{G},\,p} \leq C\rho^{n/p}\frac{q^{n(1/p'-1/2)}}{\hat{\phi}_{\min}(L_{cq/2})}, \text{ where } L_{cq/2} = \lfloor 2/(cq) - \lambda_n \rfloor.$$

Proof. Since $\|A_\varepsilon^{-1}\|_1 \leq N^{1/2}\|A_\varepsilon^{-1}\|_2$, Proposition 5.6 implies that for $\varepsilon \leq cq$,

$$\|A_\varepsilon^{-1}\|_1 \leq C_{n,\kappa,k}\frac{N^{1/2}\varepsilon^n}{\hat{\phi}_{\min}(L_{\varepsilon/2})}.$$

By Proposition 5.1, we then have that

$$\mathsf{r}_{\mathcal{G},\,p} \leq C_{\kappa,n,k,p}\frac{N^{1/2}\rho^{n/p}\varepsilon^{n/p'}}{\hat{\phi}_{\min}(L_{\varepsilon/2})}.$$

Noting that $N \sim q^{-n}$ and choosing $\varepsilon = cq$, which is as large as possible, we obtain the desired inequality. □

The second consequence is a new stability estimate for interpolation via a C^∞ SBF ϕ. Again, let A be the interpolation matrix for ϕ on the set X. By Proposition 5.6, $\|A^{-1}\|_2 = \lambda_{\min}(A)^{-1} \leq C\varepsilon^n/\hat{\phi}_{\min}(L_{\varepsilon/2})$. Taking $\varepsilon = cq$, we obtain a new bound on the norm of A^{-1}:

$$\|A^{-1}\|_2 \leq C\frac{q^n}{\hat{\phi}_{\min}(L_{cq/2})}. \tag{5.8}$$

6. Bernstein inequalities and inverse theorems

In this section, we will discuss both direct and inverse theorems for approximation by SBFs. For an overview of these notions, see [3].

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

6.1. **Bernstein inequalities.** Bernstein inequalities are a primary tool in obtaining inverse theorems. In the introduction, we gave a strategy for obtaining Bernstein theorems. We have completed the preparation required to state and prove them. Our first result is for SBFs that are perturbations of Green's functions.

Theorem 6.1. *Consider the SBF $\phi = G_\beta + G_\beta * \psi$, where G_β is the Green's function for L^β and $\psi \in L^1$. Let X be a set of centers with separation radius q and mesh ratio ρ, and let $\mathcal{G} = \mathcal{G}_{\phi,X}$ be the corresponding SBF network. If $1 \le p \le \infty$, $0 < \gamma < \beta - n/p'$ and $g \in \mathcal{G}$, then*

$$\|g\|_{H^p_\gamma} \le C q^{-\gamma} \|g\|_p. \tag{6.1}$$

Proof. Recall that $\|g\|_{H^p_\gamma} \le \|B_J g\|_{H^p_\gamma} + \|(I - \mathsf{B}_J)g\|_{H^p_\gamma}$, where B_J is the frame reconstruction operator defined in section 4.3. Of course, from (4.17), this operator is bounded independently of J. From the polynomial version of the Bernstein inequality in (4.19), we have that $\|\mathsf{B}_J g\|_{H^p_\gamma} \le C 2^{\gamma J} \|\mathsf{B}_J g\|_p \le C 2^{\gamma J} \|g\|_p$, which implies (1.2). Inserting the approximation estimate (4.23) and the stability-ratio estimate (5.5) into (1.2) yields

$$\begin{aligned} \|g\|_{H^p_\gamma} &\le \left(C 2^{\gamma J} + C' 2^{-(\beta-\gamma-n/p')J} q^{n/p'-\beta} (1 + E_{2^{J+j_n}}(\psi)_1) \right) \|g\|_p \\ &\le q^{-\gamma} \left(C(2^J q)^\gamma + C'(2^{-J} q)^{(\beta-\gamma-n/p')} (1 + \|\psi\|_1) \right) \|g\|_p. \end{aligned}$$

The integer J is still a free parameter. Choose it to be $J = \lfloor -\log_2(q) \rfloor$. The Bernstein inequality (6.1) then follows on noting that $q \le \pi$, $\beta - \gamma - n/p' > 0$, and $\|\psi\|_1$ is finite and fixed. □

Up to a point, an SBF $\phi \in C^\infty$ is handled in the same way as one related to a Green's function. In particular, using the argument above, coupled with the approximation estimate (4.23), with $\beta = \gamma + n$, and the stability estimate in Theorem 5.7, we obtain

$$\|g\|_{H^p_\gamma} \le C L^\gamma \left(1 + C' \rho^n (qL)^{n(1/p'-1/2)} \frac{L^{-(\beta-\frac{n}{2})} E_L(\mathsf{L}_n^\beta \phi)_1}{\hat{\phi}_{\min}(L_{cq/2})} \right) \|g\|_p, \; L = 2^{J+j_n}, \tag{6.2}$$

where $L_{cq/2} = \lfloor 2/cq - \lambda_n \rfloor$. Because $\phi \in C^\infty$, it is in H^p_β for all β. The inequality thus holds for all $\beta > \gamma + n/p'$. The object here is to find a constant $L = \alpha q^{-1}$, where α is independent of q, such that the ratio on the right above is bounded. The other terms will be controlled easily in that case. To obtain a simple, applicable condition, we need the following lemma.

Lemma 6.2. *Let $0 < \mu(\ell) \le \sigma(\ell)$ be eventually decreasing sequences. Assume that for every $\alpha > 0$ there is an integer $m_1 = m_1(\alpha, \sigma) \ge 0$ such that $\ell^\alpha \sigma(\ell) \le \sigma(2^{-m_1}\ell)$. If in addition for all ℓ sufficiently large there is an integer $m_2(\alpha, \mu, \sigma) \ge 0$ such that $\sigma(2^{m_2}\ell) \le C_{\mu,\sigma}\mu(\ell)$, then with $m = m_1 + m_2$,*

$$\frac{1}{\mu(L)} \sum_{\ell=2^m L}^{\infty} \ell^\alpha \sigma(\ell) \le C_{\mu,\sigma} 2^{-m} L^{-1}.$$

Proof. Let $m_1 = m_1(\alpha + 2, \phi)$. Then

$$\sum_{\ell=L}^{\infty} \ell^\alpha \sigma(\ell) \le \sum_{\ell=L}^{\infty} \ell^{-2} \ell^{\alpha+2} \sigma(\ell) \le \sigma(2^{-m_1} L) \sum_{\ell=L}^{\infty} \ell^{-2} \le \frac{\sigma(2^{-m_1} L)}{L}.$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Replace L by $2^m L$ in the inequality above, so that the sum on the left above is bounded by $(2^m L)^{-1}\sigma(2^{m_2}L) \le C_{\mu,\sigma}2^{-m}L^{-1}\mu(L)$. Dividing by $\mu(L)$ yields the desired inequality. □

Lemma 6.3. *If there are two sequences $\mu(\ell)$ and $\sigma(\ell)$ that satisfy the conditions of Lemma* 6.2 *and in addition satisfy $\mu(\ell) \le \hat{\phi}(\ell) \le \sigma(\ell)$, then there is an integer $m = m(\beta, \phi, n)$ such that for all L sufficiently large,*

$$\frac{E_{2^m L}(\mathsf{L}_n^\beta \phi)_1}{\hat{\phi}_{\min}(L)} \le C_{\beta,\phi,n}2^{-m}L^{-1}. \tag{6.3}$$

Proof. Because ϕ is a C^∞ SBF, the error $E_L(\mathsf{L}_n^\beta \phi)_1$ satisfies

$$E_L(\mathsf{L}_n^\beta \phi)_1 \le \omega_n E_L(\mathsf{L}_n^\beta \phi)_\infty \le \sum_{\ell=L}^{\infty} \frac{(\ell+\lambda_n)^\beta P_\ell^{(\lambda_n)}(1)}{\lambda_n}\hat{\phi}(\ell) \le \frac{2^{\beta+n}}{\Gamma(n)}\sum_{\ell=L}^{\infty} \ell^{\beta+n-1}\hat{\phi}(\ell),$$

where we have estimated factors independent of ϕ to get the term on the right. Applying Lemma 6.2 then completes the proof. □

Putting all these results together leads to this theorem.

Theorem 6.4. *Let ϕ be a C^∞ SBF. If there are two sequences $\mu(\ell)$ and $\sigma(\ell)$ that satisfy the conditions of Lemma* 6.2 *and in addition satisfy $\mu(\ell) \le \hat{\phi}(\ell) \le \sigma(\ell)$, then for every $\gamma > 0$ Bernstein's inequality,*

$$\|g\|_{H^p_\gamma} \le C_{\phi,\gamma,p}q^{-\gamma}\|g\|_p,$$

holds for all $g \in \mathcal{G}_{\phi,X}$, $1 \le p \le \infty$. In particular, it holds for the Gaussians, multiquadrics, ultraspherical generating functions, and the Poisson kernel.

Proof. To get the inequality itself, use Lemma 6.4 with $\beta = \gamma + n > \gamma + n/p'$. The statement concerning the list of functions may be established by checking that for each of them the upper and lower bounds given in section 3 satisfy the conditions on $\mu(\ell)$ and $\sigma(\ell)$. □

6.2. **Direct theorems.** In [15, §4], we used a linear process to estimate the distance $\mathrm{dist}_{L^p}(f, \mathcal{G}_{\phi,X})$, given that ϕ is a continuous SBF and $f \in L^p$. In several important cases, including the Gaussian, the process produced a near-best approximant. We will use a similar process here for an SFB of the form $\phi_\beta = G_\beta + G_\beta * \psi$, $\psi \in L^1$, again obtaining the corresponding distance estimates. Such SBFs are at least in L^1, but they might not be continuous. Our approach also makes use of recently developed positive-weight quadrature formulas for $\mathbb{S}^n$, introduced in [17] and further developed in [21]. We remark that a version of Theorem 6.8, with the conditions on ϕ given in terms of sequence spaces involving the $\hat{\phi}(\ell)$'s, was established in [13, Theorem 3.1].

The general framework is this. Let ϕ be an SBF, so that the Fourier-Legendre coefficients $\hat{\phi}(\ell)$ are positive for all ℓ. Define ϕ^{-1} to be the formal expansion

$$\phi^{-1} \sim \sum_{\ell=0}^{\infty} \frac{\ell+\lambda_n}{\lambda_n \omega_n}\hat{\phi}(\ell)^{-1}P_\ell^{(\lambda_n)}.$$

This expansion will converge in a distributional sense if the $\hat{\phi}(\ell)^{-1}$ grow polynomially fast. Otherwise, i.e. for faster growth, the expansion is purely formal. Since we are using it in connection with polynomials of finite degree, this is not a problem.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

For every spherical polynomial $S \in \Pi_L$, we can use ϕ^{-1} to define an inverse for the convolution operator $S \to \phi * S \in \Pi_L$, namely, the expression $\phi^{-1} * S$, which is defined by the expansion

$$\phi^{-1} * S = \sum_{\ell=0}^{L} \sum_{m=1}^{d_\ell^n} \frac{\ell + \lambda_n}{\lambda_n \omega_n} \frac{\hat{S}(\ell, m)}{\hat{\phi}(\ell)} Y_{\ell,m},$$

which is just the convolution of S with the polynomial $\sum_{\ell=0}^{L} \frac{\ell+\lambda_n}{\lambda_n\omega_n} \hat{\phi}(\ell)^{-1} P_\ell^{(\lambda_n)}$.

Suppose that S is a spherical polynomial for which $\deg S + \lambda_n \le 2^{J+j_n}$. By Theorem 4.10, we have that $\mathsf{B}_J S = S$. In addition, $S = \phi * \phi^{-1} * S$. Combining these two then yields

$$S(x) = \mathsf{B}_J \phi * \phi^{-1} * S = \int_{\mathbb{S}^n} (\mathsf{B}_J \phi)(x \cdot \eta)(\phi^{-1} * S)(\eta) d\mu(\eta).$$

The kernel $\mathsf{B}_J \phi(x \cdot \eta)$ is a zonal polynomial with degree less than 2^{J+j_n+1}. In addition, $\phi^{-1} * S$ is a spherical polynomial of degree 2^{J+j_n-1}. Thus, the integrand above is a polynomial of degree less than $2^{J+j_n+1} + 2^{J+j_n-1} < 2^{J+j_n+2}$.

We will discretize this integral by applying the quadrature formula in [21, §4.2]. Let X be a set of centers, with q, h, ρ, and $\mathcal{X}$ being the separation radius, mesh norm, mesh ratio, and Voronoi (or similar) decomposition, respectively. Take $L > 0$ to be an integer. There are positive weights c_ξ, $\xi \in X$ and a constant $s_n > 0$ (cf. [21, §4.1]) such that

$$\int_{\mathbb{S}^n} f(\eta) d\mu(\eta) \doteq \sum_{\xi \in X} c_\xi f(\xi) \tag{6.4}$$

holds exactly for polynomials in Π_L, provided that $h \le \frac{1}{4} s_n^{-1} (L + \lambda_n)^{-1}$. The weights behave like $c_\xi = \mathcal{O}(h^n)$, where the constants hidden by "big" $\mathcal{O}$ are dependent only on the dimension n. Applying the quadrature formula to the integral representing S yields

$$S(x) = \sum_{\xi \in X} c_\xi (\mathsf{B}_J \phi)(x \cdot \xi)(\phi^{-1} * S)(\xi).$$

Of course we are assuming that $h \sim 2^{-J}$. Let $\mathsf{Q} : \Pi_L \to \mathcal{G}_{\phi,X}$ be given via

$$\mathsf{Q}_{\mathcal{G}} S(x) := \sum_{\xi \in X} c_\xi \phi(x \cdot \xi)(\phi^{-1} * S)(\xi),$$

and let $g = \mathsf{Q}_{\mathcal{G}} S$, where Q is used because of the operator's relationship with quadrature. The difference between g and S is thus

$$g - S = \sum_{\xi \in X} c_\xi (I - \mathsf{B}_J)\phi((\cdot) \cdot \xi)(\phi^{-1} * S)(\xi) = (I - \mathsf{B}_J) g.$$

We now want to estimate the H_γ^p norm of the difference $g - S = (I - \mathsf{B}_J) g$ in terms of $\|\phi^{-1} * S\|_p$. It is important to note that the norm $\|\phi^{-1} * S\|_p$ depends on the degree of S and on ϕ. We will deal with it later.

The easiest way to estimate $\|g - S\|_{H_\gamma^p}$ is to employ Theorem 4.13, where the norm ratios $\|(I - \mathsf{B}_J) g\|_{H_\gamma^p} / |a|_p$ have been estimated. Thus, the task to be accomplished is to relate $|a|_p$ to $\|\phi^{-1} * S\|_p$. To do this, we will again use the Riesz-Thorin theorem.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

First of all, we have that a_ξ, which is the coefficient of $\phi((\cdot)\cdot\xi)$ in g, is given by $a_\xi = c_\xi(\phi^{-1} * S)(\xi)$. Thus, $|a|_\infty = \max_{\xi\in X} c_\xi|(\phi^{-1} * S)(\xi)|$. Since $c_\xi = \mathcal{O}(h^n)$, the bound $|a|_\infty \le Ch^n\|\phi^{-1} * S\|_\infty$ holds.

The $p = 1$ case requires more work. Now, $|a|_1 = \sum_{\xi\in X} c_\xi|(\phi^{-1} * S)(\xi)|$. Since $c_\xi = \mathcal{O}(h^n) \le C_n\rho^n q^n \le C''_n\rho^n \min_{\xi\in X}\mu(R_\xi) \le C''_n\rho^n\mu(R_\xi)$, we have

$$|a|_1 \le C''_n\rho^n\left(\sum_{\xi\in X}\mu(R_\xi)|(\phi^{-1} * S)(\xi)|\right) \le \frac{5C''_n\rho^n}{4}\|\phi^{-1} * S\|_1 .$$

The right-hand side above follows on applying the polynomial version of the Marcinkiewicz-Zygmund inequality from [21, Theorem 4.2], with $\delta = 1/4$, to bound the sum in the middle by $(5/4)\|\phi^{-1} * S\|_1$. The Riesz-Thorin theorem then implies that

$$|a|_p \le C_{n,p}\rho^{n/p}h^{n/p'}\|\phi^{-1} * S\|_p.$$

Combining this with the estimate (4.23), where $h \sim \varepsilon_J = 2^{-(J+j_n)}$ and noting that $g - S = (\mathsf{Q}_{\mathcal{G}} - I)S$, we obtain the following result.

Lemma 6.5. *Let $\gamma \ge 0$, $1 \le p \le \infty$, $\beta > \gamma + n/p'$, $h \sim 2^{-(J+j_n)}$. If S is a spherical polynomial of degree 2^{J+j_n-1} or less , then*

$$\|(\mathsf{Q}_{\mathcal{G}} - I)S\|_{H^p_\gamma} \le C_{n,p}\rho^n h^{\beta-\gamma}\|\phi^{-1} * S\|_p \begin{cases} E_{2^{J+j_n}}(\mathsf{L}^\beta_n\phi)_1 & \phi\in H^1_\beta\,, \\ (1 + E_{2^{J+j_n}}(\psi)_1) & \phi = G_\beta + G_\beta * \psi\,. \end{cases}$$

The ϕ_β case. We will now focus on the ϕ_β's. Our immediate concern is estimating $\|\phi_\beta^{-1} * S\|_p$.

Lemma 6.6. *Let $1 \le p \le \infty$, $\beta > 0$, $\psi \in L^1$, and $S \in \Pi_L$. If $\phi_\beta = G_\beta + G_\beta * \psi$, then there is a constant $C = C_{n,p,\psi}$, which is independent of β, L, and S, such that the following holds:*

$$\|\phi_\beta^{-1} * S\|_p \le C_{n,p,\psi}\|S\|_{H^p_\beta}. \tag{6.5}$$

Proof. Note that $\phi_\beta^{-1} * S = (\mathsf{L}^\beta_n\phi_\beta)^{-1} * \mathsf{L}^\beta_n S$. The kernel G_β is a Green's function for L^β_n, and so $\mathsf{L}^\beta_n\phi_\beta = \delta + \delta * \psi = \delta + \psi$, which is to be regarded as a distributional kernel. Finding $(\mathsf{L}^\beta_n\phi_\beta)^{-1}\mathsf{L}^\beta_n S$ requires solving $\mathsf{L}^\beta_n\phi_\beta * T = T + \psi * T = \mathsf{L}^\beta_n S$ for T in Π_L, which can be done directly, coefficient by coefficient. The solution T is of course unique.

There is another way to look at this equation, in an L^p setting. Suppose that we want to solve $Hf := f + \psi * f = h$ in L^p, for $1 \le p < \infty$ and in C (for $p = \infty$). The operator norm for $f \to \psi * f$ is $\|\psi\|_1$. By Theorem 4.9, we have that $\|\psi - \mathsf{B}_J\psi\|_1 \to 0$ as $J \to \infty$. It follows that the convolution operator with kernel ψ is the norm limit of finite rank operators with convolution kernels, $\mathsf{B}_J\psi$. The operator $\psi*$ is therefore compact on all L^p and C; hence, $Hf = f + \psi * f$ has closed range on these spaces. Moreover, a simple coefficient argument shows that $\ker(H) = \{0\}$. The Fredholm Alternative [4, §VII.11] then implies that $\ker(H^*) = \{0\}$, so H^{-1} exists and is bounded on all L^p and C. Since $\phi_\beta^{-1} * S = H^{-1}\mathsf{L}^\beta_n S$, we have that

$$\|\phi_\beta^{-1} * S\|_p \le \|H^{-1}\|_p\|S\|_{H^p_\beta}. \tag{6.6}$$

We emphasize that $\|H^{-1}\|_p$ is *independent* of β, L, and S. It depends only on p, n, and ψ. Consequently, $C_{n,p,\psi} = \|H^{-1}\|_p$, and (6.5) holds. $\square$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

These lemmas lead to the following two direct theorems, the first for $S \in \Pi_L$ and the second for $f \in H^p_\gamma$.

Theorem 6.7. *Let $1 \le p \le \infty$, $\gamma \ge 0$, and $\beta > \gamma + n/p'$. If S is a spherical polynomial of degree 2^{J+j_n-1} or less and if $h = \rho q \sim 2^{-J-j_n}$, then we have for $\phi = \phi_\beta$,*

$$\operatorname{dist}_{H^p_\gamma}(S, \mathcal{G}_{\phi_\beta,X}) \le C_{n,\beta,\gamma,p,\psi}\rho^n h^{\beta-\gamma}\|S\|_{H^p_\beta}\,. \tag{6.7}$$

Proof. The two lemmas, when applied to ϕ_β, yield

$$\|(\mathsf{Q}_\mathcal{G} - I)S\|_{H^p_\gamma} \le C_{n,\beta,\gamma,p,\psi}\rho^n h^{\beta-\gamma}\|S\|_{H^p_\beta}. \tag{6.8}$$

The result follows on observing that $\operatorname{dist}_{H^p_\gamma}(S, \mathcal{G}_{\phi_\beta,X}) \le \|(\mathsf{Q}_\mathcal{G} - I)S\|_{H^p_\gamma}$. Note that the dependence of C on the particular frame operator disappears on minimizing the constants involved over all functions a. □

Theorem 6.8. *Let $1 \le p \le \infty$, $\gamma \ge 0$, and $\beta > \gamma + n/p'$. If $f \in H^p_\beta$, then for $\phi_\beta = G_\beta + G_\beta * \psi$, $\psi \in L^1$,*

$$\operatorname{dist}_{H^p_\gamma}(f, \mathcal{G}_{\phi,X}) \le C_{\beta,\gamma,n,p,\psi}h^{\beta-\gamma}\rho^n\|f\|_{H^p_\beta}\,.$$

Proof. Let $2^{-J-j_n} \sim h$ and choose S to be the polynomial $S = \mathsf{B}_J f$; note that $\mathsf{Q}_\mathcal{G} S \in \mathcal{G}_{\phi_\beta,X}$. From these choices and (6.8), it follows that

$$\begin{aligned}\|f - \mathsf{Q}_\mathcal{G} S\|_{H^p_\gamma} &\le \|f - \mathsf{B}_J f\|_{H^p_\gamma} + \|(\mathsf{Q}_\mathcal{G} - I)S\|_{H^p_\gamma}\\ &\le \|f - \mathsf{B}_J f\|_{H^p_\gamma} + h^{\beta-\gamma}\rho^n C_{\beta,n,p}\|\mathsf{B}_J f\|_{H^p_\beta}.\end{aligned}$$

By Proposition 4.11, with $p = q$, we have

$$\|f - \mathsf{B}_J f\|_{H^p_\gamma} \le C_{\beta,\gamma,n,a}2^{-(\beta-\gamma)(J+j_n)}E_{2^{J+j_n}}(\mathsf{L}^\beta_n f)_p \le C_{\beta,\gamma,n,a}h^{\beta-\gamma}\|f\|_{H^p_\beta}.$$

From Proposition 4.9, we easily see that $\|\mathsf{B}_J f\|_{H^p_\beta} \le C_{\beta,\gamma,n,a}\|f\|_{H^p_\beta}$. Combining all of these inequalities establishes that

$$\|f - \mathsf{Q}_\mathcal{G} S\|_{H^p_\gamma} \le C_{\beta,\gamma,n,a,\psi}\rho^n h^{\beta-\gamma}\|f\|_{H^p_\beta}. \tag{6.9}$$

Since $\operatorname{dist}_{H^p_\gamma}(f, \mathcal{G}_{\phi,X}) \le \|f - \mathsf{Q}_\mathcal{G} S\|_{H^p_\gamma}$, and since the distance itself doesn't depend on the particular frame function, minimizing over the a yields the result, with the constant independent of a. □

The C^∞ case. The case in which the SBF ϕ is C^∞ was in large part done in [15]. However, some adjustments need to be made because the estimates in that paper did not involve H^p_γ. One difference is in estimating the norm $\|\phi^{-1} * S\|_p$.

Lemma 6.9. *Let $1 \le p \le \infty$, $\delta \ge 0$, $L > 0$ an integer, and $S \in \Pi_L$. If $\phi \in H^P_\delta$ is an SBF, then there is a constant $C = C_n$, depending only on n, such that the following holds:*

$$\|\phi^{-1} * S\|_p \le C_n \frac{L^{n\left|\frac{1}{2}-\frac{1}{p}\right|}}{\widehat{\mathsf{L}^\delta_n\phi}_{\min}(L)}\|S\|_{H^p_\delta}, \tag{6.10}$$

where $\widehat{\mathsf{L}^\delta_n\phi}_{\min}(L) = \min_{0\le\ell\le L}(\ell+\lambda_n)^\delta\hat{\phi}(\ell)$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Proof. We begin by estimating $\|\phi^{-1} * S\|_p$. The case in which $\phi \in H^p_\delta$ was essentially done in the proof of [15, Theorem 4.1]; the result, which makes use of the Nikolskii inequality (4.18), is the following. If $S \in \Pi_L$, then the Nikolskii inequality implies that

$$\|\phi^{-1} * S\|_p = \|(\mathsf{L}^\delta_n \phi)^{-1} * \mathsf{L}^\delta_n S\|_p \le C_n L^{n(\frac{1}{2}-\frac{1}{p})_+} \|(\mathsf{L}^\delta_n \phi)^{-1} * \mathsf{L}^\delta_n S\|_2 \,.$$

At this point, we simply use the 2-norm estimate done in [15, Theorem 4.1] and a second application of (4.18) to get

$$\|(\mathsf{L}^\delta_n \phi)^{-1} * \mathsf{L}^\delta_n S\|_2 \le (\widehat{\mathsf{L}^\delta_n \phi}_{\min}(L))^{-1} \|\mathsf{L}^\delta_n S\|_2 \le C_n L^{n(\frac{1}{p}-\frac{1}{2})_+} (\widehat{\mathsf{L}^\delta_n \phi}_{\min}(L))^{-1} \|\mathsf{L}^\delta_n S\|_p \,.$$

Putting the two inequalities together completes the proof. $\square$

Let $\phi \in C^\infty$. We can now estimate the H^p_γ distance of $S \in \Pi_L$ to $\mathcal{G}_{\phi,X}$, in terms of $\|S\|_{H^p_\delta}$, where $\delta > \gamma + n/p'$. In Lemma 6.5, let $\beta = \delta + n/2$. Apply Lemma 6.9, noting that $L \le 2^{J+j_n-1} \le h^{-1}$ implies that $L^{n\left|\frac{1}{2}-\frac{1}{p}\right|} \le L^{n/2} \le h^{-n/2}$ to get the following:

$$\operatorname{dist}_{H^p_\gamma}(S, \mathcal{G}_{\phi,X}) \le \|(\mathsf{Q}_{\mathcal{G}} - I)S\|_{H^p_\gamma} \le C_{n,p} \rho^n h^{\delta-\gamma} \frac{E_{2^{J+j_n}}(\mathsf{L}^{\delta+n/2}_n \phi)_1}{\widehat{\mathsf{L}^\delta_n \phi}_{\min}(L)} \|S\|_{H^p_\delta} \,. \tag{6.11}$$

Theorem 6.10. *Let $1 \le p \le \infty$, $\gamma \ge 0$, $\delta > \gamma + n/p'$, and $\phi \in C^\infty$. If there is an integer $m = m(\delta, \phi) > 0$ such that*

$$\sup_{\ell > 0} \frac{E_{2^m \ell}(\mathsf{L}^{\delta+n/2}_n \phi)_1}{\widehat{\mathsf{L}^\delta_n \phi}_{\min}(\ell)} \le C_{m,n,\delta,\phi} \tag{6.12}$$

holds, and if $S \in \Pi_L$, with $L \le 2^{J+j_n-1-m}$ and $h \sim 2^{-J-j_n}$, then

$$\operatorname{dist}_{H^p_\gamma}(S, \mathcal{G}_{\phi,X}) \le C_{m,n,p,\delta,\gamma} h^{\delta-\gamma} \rho^n \|S\|_{H^p_\delta} \,. \tag{6.13}$$

In addition, for $f \in H^p_\gamma$, we have that

$$\operatorname{dist}_{H^p_\gamma}(f, \mathcal{G}_{\phi,X}) \le C_{m,n,p,\gamma,\delta,\phi} h^{\delta-\gamma} \rho^n \|f\|_{H^p_\delta}. \tag{6.14}$$

Finally, these estimates hold for Gaussians, multiquadrics, ultraspherical generating functions and Poisson kernels.

Proof. If (6.12) holds, then, since $\widehat{\mathsf{L}^\delta_n \phi}_{\min}(L) \ge \widehat{\mathsf{L}^\delta_n \phi}_{\min}(2^{J+j_n-m})$, it follows that

$$\frac{E_{2^{J+j_n}}(\mathsf{L}^{\delta+n/2}_n \phi)_1}{\widehat{\mathsf{L}^\delta_n \phi}_{\min}(L)} \le \frac{E_{2^{J+j_n}}(\mathsf{L}^{\delta+n/2}_n \phi)_1}{\widehat{\mathsf{L}^\delta_n \phi}_{\min}(2^{J+j_n-m})} \le C_{m,n,\delta,\phi},$$

and (6.13) follows from this and (6.11). One can establish the H^p_γ distance estimate (6.14) using a proof virtually identical to that for Theorem 6.8. Essentially the same argument used in section 6.1 can be used here to show that Gaussians, multiquadrics, etc. satisfy (6.12), and so the estimates hold for them, too. $\square$

6.3. **Besov spaces.** In this section, we review the definitions and basic facts regarding Besov spaces on $\mathbb{S}^n$. These spaces, which will interpolate between $L^p(\mathbb{S}^n)$ and H^p_γ, are defined in [34]. Other, equivalent definitions of Besov spaces on $\mathbb{S}^n$ are given in [20]. Below, we will make use of a general construction found in [3, Chapter 6] to characterize these spaces in terms of spaces of SBF networks, $\mathcal{G}_{\phi,X}$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

There are two ingredients. First, we need to introduce certain sequence spaces. If $r > 0$ and $0 < \tau \leq \infty$, we define for a sequence $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ of real numbers,

$$\|\mathbf{a}\|_{\tau,r} := \begin{cases} \left\{ \sum_{n=0}^{\infty} 2^{nr\tau} |a_n|^{\tau} \right\}^{1/\tau}, & \text{if } 0 < \tau < \infty, \\ \sup_{n \geq 0} 2^{nr} |a_n|, & \text{if } \tau = \infty. \end{cases} \tag{6.15}$$

The space of sequences $\mathbf{a}$ for which $\|\mathbf{a}\|_{\tau,r} < \infty$ will be denoted by $\mathsf{b}_{\tau,r}$.

The other ingredient in the definition of Besov spaces is a K–functional [3, Chapter 6]. For $\delta, \gamma > 0$, $1 \leq p \leq \infty$ and $f \in L^p$, the K–functional for L^p and H^p_γ is given by

$$\mathcal{K}_\gamma(p, f, \delta) := \inf_{g \in H^p_\gamma} \{ \|f - g\|_p + \delta^\gamma (\|g\|_p + \|g\|_{H^p_\gamma}) \}. \tag{6.16}$$

If $r > 0$, $0 < \tau \leq \infty$, $r < \gamma$, we define the class of all $f \in L^p$ for which

$$\|f\|_{r,\gamma,\tau,p} := \|f\|_p + \|\{\mathcal{K}_\gamma(p, f, 2^{-n})\}_{n=0}^{\infty}\|_{\tau,r} < \infty \tag{6.17}$$

to be the Besov space $B^r_{\tau,p}$. As we shall see, other than the requirement $r < \gamma$, the γ dependence will disappear from the characterization of the space, so it isn't necessary to keep it in designating the space.

An important problem in approximation theory is to characterize Besov spaces using degrees of approximation of functions. We recall the results [3, Theorems 7.5.1 and 7.9.1] as they apply in the context of the present paper.

Proposition 6.11. *Let $1 \leq p \leq \infty$, $\gamma > 0$, and let $\{V_j\}_{j=0}^{\infty}$, with $V_0 = \{0\}$, be a nested sequence of finite-dimensional linear subspaces of L^p, $p < \infty$ or C, $p = \infty$ Suppose that for $j = 1, 2, \ldots$, one has both the Favard (Jackson) estimate*

$$\operatorname{dist}_{L^p}(f, V_j) \leq C\, 2^{-j\gamma} (\|f\|_p + \|f\|_{H^p_\gamma}), \tag{6.18}$$

for all $f \in H^p_\gamma$, and the Bernstein inequality

$$\|g\|_{H^p_\gamma} \leq C 2^{j\gamma} \|g\|_p, \qquad g \in V_j. \tag{6.19}$$

Then for $0 < r < \gamma$, $0 < \tau \leq \infty$, $f \in B^r_{\tau,p}$ if and only if $\{\operatorname{dist}_{L^p}(f, V_j)\}_{j=0}^{\infty} \in \mathsf{b}_{\tau,r}$.

Proof. This is just [3, Theorem 7.5.1], with the sequence of spaces satisfying all requirements listed in [3, (5.2), p. 216], except possibly density. This requirement is in fact satisfied if the Favard inequality (6.18) is satisfied. To see this, note that H^p_γ contains all of the spherical polynomials, which form a dense set in L^p, $1 \leq p < \infty$ and in C. The Favard inequality (6.18) then implies that the $\bigcup_j V_j$ is dense in H^p_γ and therefore in L^p, $1 \leq p < \infty$, or in C. □

In the important case when $V_j = \Pi_{2^j}$, Proposition 4.10 gives the Bernstein estimate, while Proposition 4.11 provides the Favard estimate. In addition, since the criterion that $\{\operatorname{dist}_{L^p}(f, \Pi_{2^J})\}_{n=0}^{\infty} \in \mathsf{b}_{\tau,r}$ does *not* depend upon γ, it follows that the Besov spaces $B^r_{\tau,p}$ are independent of the different choices of $\gamma > r$ in their definition. This is why we don't need to include the parameter γ to index these spaces.

Remark 6.12. The polynomial characterization of $B^r_{\tau,p}$ is precisely the one given in [20, Proposition 5.3], so that the "needlet" definition [20, Definition 5.1] is equivalent to the one above. (See also [18].) The needlet definition is itself known to be equivalent (cf. [20]) to that given in [34]. It follows that all three are equivalent.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Using the proposition above, one can also characterize Besov spaces using a variety of spherical basis functions. To do this, we must first have an appropriate nested sequence of sets of centers. By Proposition 2.1, we can find a nested sequence $\{X_j\}_{j=0}^{\infty} \in \mathcal{F}_\rho$, $\rho \geq 2$, each X_j having mesh norm $h_j := h_{X_j}$ satisfying $\frac{1}{4}h_j < h_{j+1} \leq \frac{1}{2}h_j \leq \frac{1}{2^j}h_0$. If $\phi \in L^p$ is an SBF, then define the V_j's to be

$$V_j := \mathcal{G}_{\phi,X_j},\ j = 1, 2, \ldots, \text{ and } V_0 = \{0\}. \tag{6.20}$$

These spaces have finite dimension equal to the cardinality of X_j and, by virtue of the X_j's being nested, are themselves nested. At issue then are the Favard and Bernstein inequalities. Since any ϕ that satisfies both will provide us with a Besov space via Proposition 6.11, we have the following result.

Corollary 6.13. *Let $1 \leq p \leq \infty$, $\phi_\beta = G_\beta + G_\beta * \psi$, where $\psi \in L^1$ and $0 < \beta$. Fix $0 < \gamma < \beta - n/p'$ and suppose that $V_j = \mathcal{G}_{\phi_\beta,X_j}$, with X_j as in* (6.20)*. For all $0 < r < \gamma$ and all $0 < \tau \leq \infty$, we have that $f \in B^r_{\tau,p}$ if and only if $\{\mathrm{dist}_{L^p}(f, V_j)\}_{j=0}^{\infty} \in \mathsf{b}_{\tau,r}$. The same conclusion holds true, with any $\gamma > 0$, for all ϕ that simultaneously satisfy* (6.3) *and* (6.12)*, including the Gaussians, multiquadrics, etc.*

Proof. When $V_j = \mathcal{G}_{\phi_\beta,X_j}$, the result follows immediately from the Bernstein inequality in Theorem 6.1 and the Favard inequality in Theorem 6.8. If ϕ satisfies both (6.3) and (6.12), then it also satisfies both the Bernstein inequality in Theorem 6.4 and the Favard inequality in Theorem 6.10. As before, with the same set of V_j's, the same conclusion holds. □

6.4. **Inverse theorems.** Inverse theorems give indications of rates of approximation being best, or nearly best, possible. We now establish inverse theorems for the approximation rates in the previous section and in [15]. These involve Bessel-potential Sobolev spaces, and in addition Besov spaces.

Theorem 6.14. *Let $1 \leq p \leq \infty$ and let ϕ be as in Theorem* 6.1 *or Proposition* 6.4*. If for $f \in L^p$, $1 \leq p < \infty$, or $f \in C(\mathbb{S}^n)$, $p = \infty$, there are constants $0 < \mu \leq \gamma$, $t \in \mathbb{R}$, and $c_f > 0$ such that*

$$\mathrm{dist}_{L^p(\mathbb{S}^n)}(f, \mathcal{G}_{\phi,X}) \leq c_f \frac{h_X^\mu}{\log_2^t(h_X^{-1})} \tag{6.21}$$

holds for all $X \in \mathcal{F}_\rho$, then, for every $0 \leq \nu < \mu$, $f \in H^p_\nu(\mathbb{S}^n)$. If (6.21) *holds for $\nu = \mu$ and some $t > 1$, then $f \in H^p_\mu(\mathbb{S}^n)$. Moreover, if in addition ϕ satisfies the conditions in Corollary* 6.13*, then for any $\tau > t^{-1} > 0$ and $0 < r \leq \mu$, the function f is in the Besov space $B^r_{\tau,p}$.*

Proof. Let the V_j's be as in (6.20), and set $f_j := \mathrm{argmin}\big(\mathrm{dist}_{L^p(\mathbb{S}^n)}(f, V_j)\big)$, which always exists because V_j is finite dimensional. Since the V_j's are nested, we have that $f_j \in V_k$ for all $k \geq j$. We want to show that f_j is a Cauchy sequence in H^p_ν. From the Bernstein estimate in Theorem 6.1 (or Proposition 6.4) and the inequality $h_{j+1}/q_{j+1} \leq \rho$, we have

$$\|f_{j+1} - f_j\|_{H^p_\nu} \leq C\rho^\nu h_{j+1}^{-\nu}\|f_{j+1} - f_j\|_p \leq C\rho^\nu h_{j+1}^{-\nu}\big(\|f_{j+1} - f\|_p + \|f - f_j\|_p\big).$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

By (6.21), we also have

$$\begin{aligned}\|f_{j+1}-f_j\|_{H^p_\nu} &\le Cc_f\rho^\nu h_{j+1}^{-\nu}(h_{j+1}^\mu \log_2^{-t}(h_{j+1})+h_j^\mu \log_2^{-t}(h_j))\\ &\le Cc_f\rho^\nu h_0 2^{-(\mu-\nu)(j+1)}\left((h_0+j+1)^{-t}+2^\mu(h_0+j)^{-t}\right)\\ &\le C' c_f 2^{-(\mu-\nu)j} j^{-t},\end{aligned}$$

where C' is independent of j. Take $k>j$. Using the previous inequality and a standard telescoping-series argument, we arrive at the following:

$$\|f_j-f_k\|_{H^p_\nu}\le C''(\sum_{m=j}^{k} 2^{-(\mu-\nu)m}m^{-t}).$$

Letting $j,k\to\infty$, we see that $\|f_j-f_k\|_{H^p_\nu}\to 0$ when $\mu>\nu$ and $\tau\in\mathbb{R}$ or when $\mu=\nu$ and $t>1$. Thus, f_j is a Cauchy sequence in H^p_ν and is therefore convergent to $\tilde f\in H^p_\nu$. Moreover, by (6.21) with $X=X_j$, we see that $f_j\to f$ in L^p, so $\tilde f=f$ almost everywhere. Hence, we have $f\in H^p_\nu$. The statement concerning Besov spaces follows from two things: the observation that $a_j:=\operatorname{dist}_{L^p}(f,V_j)\le c_f 2^{-\mu j}j^{-t}$, so $\|\mathbf{a}\|_{\tau,r}<\infty$ whenever $0<r\le\mu$ and $\tau t>1$, and Corollary 6.13. □

For the case $\nu=\mu$, $0<t\le 1$, the inverse theorem fails for Bessel-potential Sovolev spaces, but still remains valid for Besov spaces with $\tau>t^{-1}$.

7. Concluding remarks

There are connections between this paper and [15, 13]. In these papers, quasi-interpolatory SBF networks were obtained yielding near-best approximants for functions in Sobolev classes. The associated quasi-interpolation operators were constructed in the Fourier domain. The paper [15] focused on sequences corresponding to the c^∞ case treated within this paper. The paper [10] dealt with sequences connected to the "perturbations of Green's functions" case. For example, let ψ be a perturbation of a Green's function as described in this paper. If the Fourier coefficients of ψ satisfy the "difference condition" as stated in [13], then it is in L^1. The examples given in section 3 satisfy both kinds of conditions.

In [15, 13], the quasi-interpolatory SBF networks were shown to give the best results in the sense of n-widths. In this paper, using the frame approach, we have shown that the quasi-interpolatory networks are also optimal for approximation of individual functions. Also note that in [13], Marcinkiewicz-Zygmund measures generalizing the measure that associates $\mu_q(R_\xi)$ with each ξ were introduced. These measures were used to derive [13, Proposition 4.1 & (4.15)], which have overlap with the current Proposition 4.4, Lemma 4.7 and estimate (5.4).

In [10], the quasi-interpolation polynomial operators were further utilized to show that, in the presence of certain singularities, they exhibited better approximation properties than traditional methods. Also [10, Propositon 4.3] is related to Proposition 4.10 given here. Finally there is material closely connected to Theorem 4.1 appearing in [6] and [12, Proposition 4.1]. Another version of the operator B_J was introduced in [14]: $\sigma_J(f)=\sum_{l=0}^{2^J}h(l/2^J)P_l(f)$, where $h:[0,\infty)\to[0,\infty)$ is a function in C^k, equal to 1 on [0, 1/2] and 0 on $[1,\infty)$. An early form of Theorem 4.1 was Theorem 3.4 of [14]. Frames, based on the $\sigma_J(f)$ operator can be constructed as in [11, 18] using $h(t)-h(2t)$ in place of κ used in the construction given here.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Finally we mention that the idea of using minimal separation for converse theorems and Bernstein inequalities goes back to [29]; see also [11]. Also, for the neural network community, we note that the number of neurons is not used as a measure of complexity, but rather the minimal separation of the nodes.

References

[1] Dai, F. Characterizations of function spaces on the sphere using frames. *Trans. Amer. Math. Soc. 359*, 2 (2007), 567–589 (electronic). MR2255186 (2008a:42023)

[2] Daubechies, I. *Ten Lectures on Wavelets*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992. MR1162107 (93e:42045)

[3] DeVore, R. A., and Lorentz, G. G. *Constructive approximation*, vol. 303 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1993. MR1261635 (95f:41001)

[4] Dunford, N., and Schwartz, J. T. *Linear Operators, Part I.* Wiley-Interscience, New York, 1958. MR0117523 (22:8302)

[5] Freeden, W., Gervens, T., and Schreiner, M. *Constructive Approximation on the Sphere*. Clarendon Press, Oxford, 1998. MR1694466 (2000e:41001)

[6] Freeden, W., and Michel, V. *Multiscale potential theory, with applications to geoscience*. Birkhäuser, Boston, 2004. MR2090018 (2006g:31001)

[7] Freeden, W., Schreiner, M., and Franke, R. A survey on spherical spline approximation. *Surv. Math. Ind. 7* (1997), 29–85. MR1470453 (98m:65019)

[8] Hörmander, L. *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. Springer, Berlin, 1987. MR2304165 (2007k:35006)

[9] Kamzolov, A. I. The best approximation of classes of functions $W_p^\alpha(s^n)$ by polynomials in spherical harmonics. *(Russian) Mat. Zametki 32* (1982), 285–293. (English) 32 (1983), 622–626. MR677597 (83m:41020)

[10] Le Gia, Q. T., and Mhaskar, H. N. Polynomial operators and local approximation of solutions of pseudo-differential equations on the sphere. *Numer. Math. 103* (2006), 299–322. MR2222812 (2006k:65331)

[11] Mhaskar, H. N. A Markov-Bernstein inequality for Gaussian networks. In *Trends and applications in constructive approximation*, vol. 151 of *Internat. Ser. Numer. Math.* Birkhäuser, Basel, 2005, pp. 165–180. MR2147576 (2006d:41014)

[12] Mhaskar, H. N. On the representation of smooth functions on the sphere using finitely many bits. *Appl. Comput. Harmon. Anal. 18*, 3 (2005), 215–233. MR2131286 (2006a:42048)

[13] Mhaskar, H. N. Weighted quadrature formulas and approximation by zonal function networks on the sphere. *J. Complexity 22* (2006), 348–370. MR2229898 (2007b:65031)

[14] Mhaskar, H. N., Narcowich, F. J., Prestin, J., and Ward, J. D. Polynomial frames on the sphere. *Adv. Comput. Math. 13* (2000), 387–404. MR1826335 (2002a:42021)

[15] Mhaskar, H. N., Narcowich, F. J., and Ward, J. D. Approximation properties of zonal function networks using scattered data on the sphere. *Adv. Comput. Math. 11* (1999), 121–137. MR1731693 (2001h:41035)

[16] Mhaskar, H. N., Narcowich, F. J., and Ward, J. D. Corrigendum to spherical Marcinkiewicz-Zygmund inequalities and positive quadrature. *Math. Comp. 71* (2001), 453–454.

[17] Mhaskar, H. N., Narcowich, F. J., and Ward, J. D. Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature. *Math. Comp. 70* (2001), 1113–1130. (Corrigendum: *Math. Comp. 71* (2001), 453–454). MR1710640 (2002a:41032)

[18] Mhaskar, H. N., and Prestin, J. Polynomial frames: A fast tour. In *Approximation theory XI: Gatlinburg 2004*, Mod. Methods Math. Nashboro Press, Brentwood, TN, 2005, pp. 287–318. MR2126687 (2005k:42087)

[19] Müller, C. *Spherical Harmonics*. Springer, Berlin, 1966. MR0199449 (33:7593)

[20] Narcowich, F., Petrushev, P., and Ward, J. Decomposition of Besov and Triebel-Lizorkin spaces on the sphere. *J. Funct. Anal. 238*, 2 (2006), 530–564. MR2253732 (2007i:46033)

[21] Narcowich, F. J., Petrushev, P., and Ward, J. D. Localized tight frames on spheres. *SIAM J. Math. Anal. 38* (2006), 574–594. MR2237162 (2007k:42084)

[22] Narcowich, F. J., Sun, X., and Ward, J. D. Approximation power of RBFs and their associated SBFs: A connection. *Adv. Comput. Math. 27* (2007), 107–124. MR2317924

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

[23] Narcowich, F. J., Sun, X., Ward, J. D., and Wendland, H. Direct and inverse Sobolev error estimates for scattered data interpolation via spherical basis functions. *Found. Comput. Math. 7* (2007), 369–390. MR2335250 (2008i:41023)

[24] Narcowich, F. J., and Ward, J. D. Scattered-data interpolation on spheres: Error estimates and locally supported basis functions. *SIAM J. Math. Anal. 33* (2002), 1393–1410. MR1920637 (2003j:41021)

[25] Olver, F. W. J. *Asymptotics and special functions*. Academic Press, New York, 1974. MR0435697 (55:8655)

[26] Petrushev, P., and Xu, Y. Localized polynomial frames on the ball. *Constr. Approx. 27*, 2 (2008), 121–148. MR2336420 (2008h:42076)

[27] Ron, A., and Sun, X. Strictly positive definite functions on spheres in Euclidean spaces. *Math. Comp. 65* (1996), 1513–1530. MR1370856 (97a:41032)

[28] Rustamov, K. P. On the equivalence of the K-functional and the modulus of smoothness of functions on a sphere. *Mat. Zametki 52*, 3 (1992), 123–129, 160. English transl. in *Math. Notes*, (1993) 965–970. MR1194136 (94f:41011)

[29] Schaback, R., and Wendland, H. Inverse and saturation theorems for radial basis function interpolation. *Math. Comp. 71* (2002), 669–681. MR1885620 (2003a:41018)

[30] Schoenberg, I. J. Positive definite functions on spheres. *Duke Math. J. 9* (1942), 96–108. MR0005922 (3:232c)

[31] Stein, E. M., and Weiss, G. *Fourier Analysis in Euclidean Spaces*. Princeton University Press, Princeton, New Jersey, 1971. MR0304972 (46:4102)

[32] Strichartz, R. S. Analysis of the Laplacian on the complete Riemannian manifold. *J. Funct. Anal. 52* (1983), 48–79. MR705991 (84m:58138)

[33] Szegö, G. *Orthogonal Polynomials*. Fourth edition, Amer. Math. Soc., Providence, RI, 1975. MR0372517 (51:8724)

[34] Triebel, H. Spaces of Besov-Hardy-Sobolev type on complete Riemannian manifolds. *Ark. Mat. 24*, 2 (1986), 299–337. MR884191 (88d:46057)

[35] Wendland, H. *Scattered Data Approximation*. Cambridge University Press, Cambridge, UK, 2005. MR2131724 (2006i:41002)

[36] Whittaker, E. T., and Watson, G. N. *A Course of Modern Analysis*. Cambridge University Press, Cambridge, UK, 1965. MR1424469 (97k:01072)

(H. N. Mhaskar) Department of Mathematics, California State University, Los Angeles, California 90032

(F. J. Narcowich) Department of Mathematics, Texas A&M University, College Station, Texas 77843

(J. Prestin) Institute of Mathematics, University of Lübeck, Wallstrasse 40, 23560, Lübeck, Germany

(J. D. Ward) Department of Mathematics, Texas A&M University, College Station, Texas 77843

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use