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L^p BOUNDEDNESS OF CARLESON TYPE MAXIMAL OPERATORS WITH NONSMOOTH KERNELS

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Abstract. In this paper, the authors give the L^p boundedness of a class of the Carleson type maximal operators with rough kernel, which improves some known results.

1. Introduction. For $f \in L^2([-\pi,\pi])$ and $x \in [-\pi,\pi]$, the Carleson operator $\mathcal{C}^*$ is defined by

$$\mathcal{C}^* f(x) = \sup_{\lambda\in\boldsymbol{R}} \left| \int_{-\pi}^{\pi} \frac{e^{-i\lambda t} f(t)}{x-t} dt \right|. \tag{1.1}$$

In 1966, using the weak type (2,2) of $\mathcal{C}^*$, Carleson [1] proved his celebrated theorem on almost everywhere convergence of Fourier series of L^2 functions on $[-\pi,\pi]$. Following that, Hunt [8] modified Carleson's proof and extended Carleson's theorem to L^p functions on $[-\pi,\pi]$ for $1 < p < \infty$.

In 1970, Sjölin [11] studied several variables analogue of the Carleson operator $\mathcal{C}^*$. Suppose that K is an appropriate Calderón-Zygmund kernel in $\boldsymbol{R}^n$, then the Carleson type maximal operator $\mathcal{S}^*$ on $\boldsymbol{R}^n$ is defined by

$$\mathcal{S}^*(f)(x) = \sup_{\lambda\in\boldsymbol{R}^n} \left| \int_{\boldsymbol{R}^n} e^{-i\lambda\cdot y} K(x-y) f(y) dy \right|, \tag{1.2}$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in \boldsymbol{R}^n$.

THEOREM A (Sjölin, [11]). *Let K satisfy the following conditions*:
(a) $K(tx) = t^{-n}K(x)$, *for* $t > 0$;
(b) $\int_{S^{n-1}} K(x')d\sigma(x') = 0$;
(c) $K \in C^{n+1}(\boldsymbol{R}^n \setminus \{0\})$.
Then $\|\mathcal{S}^*(f)\|_{L^p} \le C_p \|f\|_{L^p}$ *for* $1 < p < \infty$.

In 2001, Stein and Wainger [13] considered to extend Theorem A to a broader context. More precisely, the authors of [13] replaced the linear phase $\lambda \cdot y$ in the definition of $\mathcal{S}^*$ by more general polynomial phase with a fixed degree. Let $P_\lambda(x) = \sum_{2\le|\alpha|\le d} \lambda_\alpha x^\alpha$ be a

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polynomial in $\boldsymbol{R}^n$ with real coefficients $\lambda := (\lambda_\alpha)_{2\le|\alpha|\le d}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{Z}_+^n$ and $|\alpha| = \sum_{j=1}^n \alpha_j$. Define

$$T_\lambda(f)(x) = \int_{\boldsymbol{R}^n} e^{iP_\lambda(y)} K(y) f(x-y)dy\,.$$

Then the Carleson type maximal operator $\mathcal{T}^*$ is defined by

$$\mathcal{T}^* f(x) = \sup_\lambda |T_\lambda(f)(x)|\,, \tag{1.3}$$

where the supremum is taken over all the real coefficients λ of P_λ. Stein and Wainger proved the following result:

THEOREM B (Stein-Wainger, [13]). *Suppose that* $P_\lambda(x) = \sum_{2\le|\alpha|\le d} \lambda_\alpha x^\alpha$ *and* K *satisfies the following conditions*:

(a) K *is a tempered distribution and agrees with a* C^1 *function* $K(x)$ *for* $x \neq 0$;
(b) $\widehat{K} \in L^\infty$;
(c) $|\partial_x^\gamma K(x)| \le A|x|^{-n-|\gamma|}$ *for* $0 \le |\gamma| \le 1$.

Then the Carleson type maximal operator $\mathcal{T}^*$ *defined in* (1.3) *is bounded on* L^p *for* $1 < p < \infty$.

In 2000, Prestini and Sjölin [9] gave the weighted analogue of Theorem A. Recently, we gave also a weighted variant of Theorem B under weaker conditions [4].

In this paper, we will study the L^p boundedness of the Carleson type maximal operators with rough kernels. Before giving our result, let us recall some definitions. Suppose that Ω is a measurable function on $\boldsymbol{R}^n \setminus \{0\}$ and satisfying the following conditions:

$$\Omega(tx) = \Omega(x) \quad \text{for any } x \in \boldsymbol{R}^n \setminus \{0\} \quad \text{and} \quad t > 0\,; \tag{1.4}$$

$$\Omega \in L^1(S^{n-1})\,, \tag{1.5}$$

where S^{n-1} denotes the unit sphere in $\boldsymbol{R}^n$ $(n \ge 2)$ with area measure $d\sigma$;

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0\,. \tag{1.6}$$

Let $Q_\lambda(r) = \sum_{2\le k\le d} \lambda_k r^k$ be a real-valued polynomial on $\boldsymbol{R}$ and $\lambda = (\lambda_2, \dots, \lambda_d) \in \boldsymbol{R}^{d-1}$. With the notations above, the Carleson type maximal operator $\mathcal{T}^*$ associated to polynomial Q is defined by

$$\mathcal{T}^*(f)(x) = \sup_\lambda |T_\lambda(f)(x)|\,, \tag{1.7}$$

where

$$T_\lambda(f)(x) = \int_{\boldsymbol{R}^n} e^{iQ_\lambda(|y|)} K(y) f(x-y)dy \tag{1.8}$$

and Ω satisfies (1.4) through (1.6). Our main result is following:

THEOREM 1.1. *Let $\mathcal{T}^*$ be given as in* (1.7). *If $\Omega \in H^1(S^{n-1})$, the Hardy space on S^{n-1} (see Section 2 for the definition of $H^1(S^{n-1})$), then for $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$\|\mathcal{T}^*(f)\|_{L^p} \le C\|f\|_{L^p}\,. \tag{1.9}$$

Now we want to give two remarks on our main theorem.

REMARK 1. There are the following containing relationship among the function spaces on S^{n-1}:

$$C^1(S^{n-1}) \subsetneq L^\infty(S^{n-1}) \subsetneq L^q(S^{n-1})\,(1 < q < \infty) \subsetneq H^1(S^{n-1}) \subsetneq L^1(S^{n-1}).$$

Hence, in the sense of removing the smoothness assumption on the kernel function K, Theorem 1.1 improves Theorem B.

REMARK 2. We should point out that the study of a singular integral with oscillating factor $e^{iQ_\lambda(|y|)}$ has an important motivation. In fact, the operator T_λ defined in (1.8) is a generalization of the stronger singular convolution operator, which was first studied by C. Fefferman in [6].

The proof of Theorem 1.1 is based on an idea of linearizing maximal operators and Stein-Wainger's TT^* method presented in [13]. However, because the kernel of our objective operator lacks smoothness on the unit sphere, we need some new ideas to overcome the roughness of the kernel. Namely we use Calderón-Zygmund's rotation method.

2. Notations and Lemmas.

Let us begin with recalling the definition of the Hardy space $H^1(S^{n-1})$.

$$\begin{aligned} &H^1(S^{n-1}) \\ &= \left\{ \Omega \in L^1(S^{n-1})\,;\ \|\Omega\|_{H^1(S^{n-1})} = \left\| \sup_{0<r<1} \left| \int_{S^{n-1}} \Omega(y')P_{r(\cdot)}(y')d\sigma(y') \right| \right\|_{L^1(S^{n-1})} < \infty \right\}, \end{aligned}$$

where $P_{rx'}(y')$ denotes the Possion kernel on S^{n-1} defined by

$$P_{rx'}(y') = \frac{1-r^2}{|rx'-y'|^n}\,, \quad 0 \le r < 1 \quad \text{and} \quad x', y' \in S^{n-1}\,.$$

See [2], [5] or [7] for the properties of $H^1(S^{n-1})$.

In the proof of Theorem 1.1, we will apply the 1-dimensional variant of Stein-Wainger's results. For a real polynomial $P(t) = \sum_{1\le k\le d} \lambda_k t^k$ on $\boldsymbol{R}$ with real coefficients $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_d)$, we denote

$$|\lambda| = \sum_{1\le k\le d} |\lambda_k|\,. \tag{2.1}$$

LEMMA 2.1 ([13, Proposition 2.1]). *Assume that φ is a C^1 function defined in the unit interval $I = (-1, 1)$, V is any subinterval of I and $P(t) = \sum_{1\le k\le d} \lambda_k t^k$ is a polynomial on*

R *of degree d. Then*

$$\left|\int_V e^{iP(t)}\varphi(t)dt\right| \le C|\lambda|^{-1/d}\sup_{t\in I}(|\varphi(t)|+|\varphi'(t)|)\,.$$

The constant C depends on the degree d, but not on P, φ or V.

LEMMA 2.2 ([13, Proposition 2.2]). *With the same notation as above in Lemma* 2.1,

$$|\{t\in I\,;\,|P(t)|\le\varepsilon\}|\le C_d\,\varepsilon^{1/d}|\lambda|^{-1/d}\quad \textit{for any}\ \ \varepsilon>0\,.$$

The constant C_d does not depend on the coefficients of P, but on the degree d.

We also need the following L^p boundedness for a variant of the Hardy-Littlewood maximal operator.

LEMMA 2.3 ([13, Proposition 3.1]). *Let $I_2=(-2,2)$, E is the measurable subset of I_2 and χ_E denotes the characteristic function of E. For $\varepsilon>0$, the maximal operator $\mathcal{M}_\varepsilon$ is defined by*

$$\mathcal{M}_\varepsilon(f)(t)=\sup_{\substack{a>0\\|E|\le\varepsilon}}|f|*(\chi_E)_a(t)\,, \tag{2.2}$$

where $(\chi_E)_a(t)=a^{-1}\chi_E(t/a)$ for $a>0$, and the supremum is taken over all subsets E in I_2 of measure less than ε. Then for $1<p<\infty$, there exists a constant $c>0$, independent of ε, such that

$$\|\mathcal{M}_\varepsilon(f)\|_{L^p(\boldsymbol{R})}\le C\varepsilon^{1-1/p}\|f\|_{L^p(\boldsymbol{R})}\,. \tag{2.3}$$

3. The proof of main result. We now turn to the proof of the main result in this paper. It is obvious that

$$\mathcal{T}^*(f)(x)=\sup_\lambda|T_\lambda(f)(x)|\le\sup_{\lambda\ne\mathbf{0}}|T_\lambda(f)(x)|+|T_\Omega(f)(x)|\,, \tag{3.1}$$

where T_Ω denotes the singular integral operator, which is defined by

$$T_\Omega(f)(x)=\text{p.v.}\int_{\boldsymbol{R}^n}\frac{\Omega(y)}{|y|^n}f(x-y)dy\,.$$

Since $\Omega\in H^1(S^{n-1})$, by the L^p boundedness of T_Ω (see [3] and [10]), we may assume that the first supremum in (3.1) is taken over all the nonzero vectors $\lambda=(\lambda_2,\dots,\lambda_d)$.

Let $\psi\in C_0^\infty(\boldsymbol{R}_+)$ be a nonnegative function such that $\mathrm{supp}(\psi)\subseteq\{1/4<t<1\}$ and

$$\sum_{j=-\infty}^{\infty}\psi_j(t)=1\quad\text{for}\ \ t>0\,,$$

where $\psi_j(t)=\psi(2^{-j}t)$. Denote $K(y)=\Omega(y)|y|^{-n}$ and decompose the kernel K by

$$K(y)=\sum_{j=-\infty}^{\infty}K_j(y)\,,$$

where $K_j(y) = \psi_j(|y|)K(y)$. For $\lambda \in \boldsymbol{R}^{d-1} \setminus \{\boldsymbol{0}\}$, let $j_0 \in \boldsymbol{Z}$ such that $2^{j_0} \le 1/N(\lambda) < 2^{j_0+1}$, where $N(\lambda)$ is given by

$$N(\lambda) = \sum_{2\le k\le d} |\lambda_k|^{1/k} .$$

Thus, we may write

$$T_\lambda f(x) = T_\lambda^- f(x) + T_\lambda^+ f(x) , \tag{3.2}$$

where

(3.3)

$$T_\lambda^- f(x) = \sum_{j\le j_0} \int_{\boldsymbol{R}^n} e^{iQ_\lambda(|y|)} K_j(y) f(x-y)dy \quad \text{and} \quad T_\lambda^+ f(x) = T_\lambda f(x) - T_\lambda^- f(x) .$$

We first give the estimate of $\| \sup_\lambda |T_\lambda^-(f)| \|_{L^p}$. Note that $\sum_{j\le j_0} K_j(y) = K(y)$ for $|y| \le 2^{j_0-1}$ and $\psi(|y|) \in C_0^\infty(\boldsymbol{R}^n)$. Thus

$$\begin{aligned} |T_\lambda^-(f)(x)| &\le \left| \int_{|y|\le 2^{j_0-1}} e^{iQ_\lambda(|y|)} K(y) f(x-y)dy \right| \\ &\quad + \int_{2^{j_0-1}\le |y|\le 2^{j_0}} \frac{|\Omega(y)|}{|y|^n} |f(x-y)|dy \\ &=: I + II . \end{aligned} \tag{3.4}$$

It is easy to see that

$$II \le CM_\Omega f(x) ,$$

where $C = C(n)$ and M_Ω is the maximal operator with homogeneous kernel defined by

$$M_\Omega f(x) = \sup_{t>0} \frac{1}{t^n} \int_{|y|\le t} |\Omega(y)||f(x-y)|dy .$$

Now we consider the term I. Note that

$$|e^{iQ_\lambda(|y|)} - 1| \le C \sum_{2\le k\le d} |\lambda_k||y|^k \le C \sum_{2\le k\le d} N(\lambda)^k |y|^k \le CN(\lambda)|y| ,$$

since $|\lambda_k| \le N(\lambda)^k$ and $N(\lambda)|y| < 1$ for $|y| \le 2^{j_0-1}$. Then, the term I can be dominated by

$$\begin{aligned} &\left| \int_{|y|\le 2^{j_0-1}} \frac{\Omega(y)}{|y|^n} f(x-y)dy \right| + \left| \int_{|y|\le 2^{j_0-1}} (e^{iQ_\lambda(|y|)} - 1) \frac{\Omega(y)}{|y|^n} f(x-y)dy \right| \\ &\quad \le \left| \int_{\boldsymbol{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)dy \right| + \sup_{\varepsilon>0} \left| \int_{|y|\ge\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy \right| \\ &\quad + CN(\lambda) \int_{|y|\le \frac{1}{2N(\lambda)}} \frac{|\Omega(y)|}{|y|^{n-1}} |f(x-y)|dy \\ &\quad \le |T_\Omega(f)(x)| + T_\Omega^*(f)(x) + CM_\Omega(f)(x) , \end{aligned}$$

where the constant C is independent on λ and T^*_Ω denotes the truncated singular integral maximal operator with homogeneous kernel, which is defined by

$$T^*_\Omega(f)(x) = \sup_{\varepsilon>0}\left|\int_{|y|\geq\varepsilon}\frac{\Omega(y)}{|y|^n}f(x-y)dy\right|.$$

Hence,

$$|T^-_\lambda(f)(x)| \leq |T_\Omega(f)(x)| + T^*_\Omega(f)(x) + CM_\Omega(f)(x)\,. \tag{3.5}$$

Thus, by the L^p boundedness of T_Ω, T^*_Ω and M_Ω (see [3], [5] or [7]), we have

$$\left\|\sup_\lambda |T^-_\lambda(f)|\right\|_{L^p} \leq C\|f\|_{L^p}\,, \tag{3.6}$$

where the constant C is independent of λ.

Following that, we will estimate $\|\sup_\lambda |T^+_\lambda(f)|\|_{L^p}$. For $\delta > 0$ and $\lambda = (\lambda_2, \dots, \lambda_d)$, we denote

$$\delta\circ\lambda = \sum_{2\leq k\leq d}\delta^k\lambda_k\,.$$

Noticing that j_0 depends on λ and $N(2^j\circ\lambda) = 2^jN(\lambda)$, we have

$$\begin{aligned}
\sup_\lambda |T^+_\lambda f(x)| &= \sup_\lambda\left|\sum_{j>j_0}N(2^j\circ\lambda)^{-\delta_0}N(2^j\circ\lambda)^{\delta_0}T^j_\lambda f(x)\right| \\
&\leq \sup_\lambda\left(\sup_{j>j_0}|N(2^j\circ\lambda)^{\delta_0}T^j_\lambda f(x)|\right)\sum_{j=j_0+1}^{\infty}N(2^j\circ\lambda)^{-\delta_0} \\
&\leq C\sup_\lambda\sup_{2^j>1/N(\lambda)}|N(2^j\circ\lambda)^{\delta_0}T^j_\lambda f(x)| \\
&= C\sup_j\sup_{N(2^j\circ\lambda)>1}|N(2^j\circ\lambda)^{\delta_0}T^j_\lambda f(x)|\,,
\end{aligned} \tag{3.7}$$

where δ_0 is a positive number which will be chosen later. It is trivial that, for $j\in\mathbf{Z}$,

$$Q_\lambda(|y|) = \sum_{2\leq k\leq d}\lambda_k|y|^k = \sum_{2\leq k\leq d}2^{jk}\lambda_k|2^{-j}y|^k = Q_{2^j\circ\lambda}(|2^{-j}y|)$$

and

$$\begin{aligned}
T^j_\lambda f(x) &= \int_{\mathbf{R}^n}e^{iQ_\lambda(|y|)}\psi_j(|y|)\frac{\Omega(y)}{|y|^n}f(x-y)dy \\
&= \int_{\mathbf{R}^n}e^{iQ_{2^j\circ\lambda}(|2^{-j}y|)}\psi(2^{-j}|y|)\frac{\Omega(y)}{|y|^n}f(x-y)dy\,.
\end{aligned} \tag{3.8}$$

There exists a constant $C_0 > 0$, such that $N(\lambda) \le C_0|\lambda|$ for any vector λ satisfying $N(\lambda) \ge 1$ (see [13, p. 797]). Then, by (3.8),

$$\begin{aligned}
&\sup_j \sup_{N(2^j\circ\lambda)>1} N(2^j\circ\lambda)^{\delta_0}|T_\lambda^j f(x)| \\
&\quad\le \sup_{a>0}\sup_{N(\lambda)>1} N(\lambda)^{\delta_0}\left|\int_{\boldsymbol{R}^n} e^{iQ_\lambda(|ay|)}\psi(a|y|)\frac{\Omega(y)}{|y|^n}f(x-y)dy\right| \\
&\quad\le C\sum_{l=0}^{\infty}2^{l\delta_0}\sup_{\substack{N(\lambda)\ge 2^l\\ a>0}}\left|\int_{\boldsymbol{R}^n} e^{iQ_\lambda(|ay|)}\psi(a|y|)\frac{\Omega(y)}{|y|^n}f(x-y)dy\right| \\
&\quad\le C\sum_{l=0}^{\infty}2^{l\delta_0}\sup_{\substack{|\lambda|\ge 2^l/C_0\\ a>0}}\left|\int_{\boldsymbol{R}^n} e^{iQ_\lambda(|ay|)}\psi(a|y|)\frac{\Omega(y)}{|y|^n}f(x-y)dy\right|.
\end{aligned} \tag{3.9}$$

If we can show that there is a $\delta_p > 0$ such that

$$\begin{aligned}
&\left(\int_{\boldsymbol{R}^n}\sup_{\substack{|\lambda|\ge 2^l/C_0\\ a>0}}\left|\int_{\boldsymbol{R}^n} e^{iQ_\lambda(|ay|)}\psi(a|y|)a^nK(ay)f(x-y)dy\right|^p dx\right)^{1/p} \\
&\quad\le C2^{-l\delta_p}\|f\|_{L^p},
\end{aligned} \tag{3.10}$$

then taking $\delta_0 = \delta_p/2$ and by (3.7) and (3.9), we have

$$\left\|\sup_\lambda |T_\lambda^+(f)|\right\|_{L^p} \le C\|f\|_{L^p}.$$

Thus, to complete the proof of Theorem 1.1, we just need to show inequality (3.10). It is easy to see that, to get (3.10), we need only to show for $t \ge 1/C_0$,

$$\left\|\sup_{\substack{|\lambda|\ge t\\ a>0}}\left|\int_{\boldsymbol{R}^n} e^{iQ_\lambda(|ay|)}\psi(a|y|)\frac{\Omega(y)}{|y|^n}f(\cdot-y)dy\right|\right\|_{L^p} \le Ct^{-\delta_p}\|f\|_{L^p}. \tag{3.11}$$

By a polar coordinate transformation, we have

$$\begin{aligned}
&\sup_{\substack{|\lambda|\ge t\\ a>0}}\left|\int_{\boldsymbol{R}^n} e^{iQ_\lambda(|ay|)}\psi(a|y|)\frac{\Omega(y)}{|y|^n}f(x-y)dy\right| \\
&\quad\le \int_{S^{n-1}}|\Omega(y')|\sup_{\substack{|\lambda|\ge t\\ a>0}}\left|\int_0^\infty e^{iQ_\lambda(ar)}\psi(ar)\frac{1}{r}f(x-ry')dr\right|d\sigma(y').
\end{aligned}$$

By the above inequality and Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_{\boldsymbol{R}^n}\sup_{\substack{|\lambda|\ge t\\ a>0}}\left|\int_{\boldsymbol{R}^n} e^{iQ_\lambda(|ay|)}\psi(a|y|)\frac{\Omega(y)}{|y|^n}f(x-y)dy\right|^p dx\right)^{1/p}\\
&\le\left[\int_{\boldsymbol{R}^n}\left(\int_{S^{n-1}}|\Omega(y')|\sup_{\substack{|\lambda|\ge t\\ a>0}}\left|\int_0^\infty e^{iQ_\lambda(ar)}\psi(ar)\frac{1}{r}f(x-ry')dr\right|d\sigma(y')\right)^p dx\right]^{1/p}\\
&\le\int_{S^{n-1}}|\Omega(y')|\left(\int_{\boldsymbol{R}^n}\sup_{\substack{|\lambda|\ge t\\ a>0}}\left|\int_0^\infty e^{iQ_\lambda(ar)}\psi(ar)\frac{1}{r}f(x-ry')dr\right|^p dx\right)^{1/p}d\sigma(y') \qquad (3.12)\\
&=\int_{S^{n-1}}|\Omega(y')|\left(\int_{L_{y'}^{\perp}}\int_{\boldsymbol{R}}\sup_{\substack{|\lambda|\ge t\\ a>0}}\left|\int_0^\infty e^{iQ_\lambda(ar)}\psi(ar)\right.\right.\\
&\qquad\left.\left.\times\frac{1}{r}f(z+(s-r)y')dr\right|^p dsdz\right)^{1/p}d\sigma(y'),
\end{aligned}$$

where for fixed $y'\in S^{n-1}$, $L_{y'}$ denotes the line through the origin containing y'. Thus for $x\in\boldsymbol{R}^n$, there are $s\in\boldsymbol{R}$ and $z\in L_{y'}^{\perp}$ such that $x=sy'+z$ and this decomposition is unique. Moreover, for fixed y' and $z\in L_{y'}^{\perp}$, denote $f(z+sy')$ by $f_{y',z}(s)$. It is obvious that

$$\begin{aligned}
&\int_{\boldsymbol{R}}\sup_{\substack{|\lambda|\ge t\\ a>0}}\left|\int_0^\infty e^{iQ_\lambda(ar)}\psi(ar)\frac{1}{r}f(z+(s-r)y')dr\right|^p ds\\
&\qquad\le\sum_{k=0}^{\infty}\int_{\boldsymbol{R}}\sup_{\substack{2^{k+1}t\ge|\lambda|\ge 2^k t\\ a>0}}\left|\int_0^\infty e^{iQ_\lambda(ar)}\psi(ar)\frac{1}{r}f_{y',z}(s-r)dr\right|^p ds.
\end{aligned}$$

Now, for $t\ge 1/C_0$, we define a maximal operator $\mathcal{R}_t$ by

$$\mathcal{R}_t(g)(s)=\sup_{\substack{2t\ge|\lambda|\ge t\\ a>0}}\left|\int_0^\infty e^{iQ_\lambda(ar)}\psi(ar)\frac{1}{r}g(s-r)dr\right|.$$

If we can show that there exists a $C>0$ such that, for $t\ge 1/C_0$ and $g\in L^p(\boldsymbol{R})$ $(1<p<\infty)$,

$$\|\mathcal{R}_t(g)\|_{L^p(\boldsymbol{R})}\le Ct^{-\delta_p}\|g\|_{L^p(\boldsymbol{R})}, \qquad (3.13)$$

then by (3.12),

$$\left(\int_{\boldsymbol{R}^n} \sup_{\substack{|\lambda|\ge t\\ a>0}} \left| \int_{\boldsymbol{R}^n} e^{iQ_\lambda(|ay|)}\psi(a|y|)\frac{\Omega(y)}{|y|^n} f(x-y)dy\right|^p dx\right)^{1/p}$$
$$\le \int_{S^{n-1}} |\Omega(y')| \left(\int_{L_{y'}^{\perp}} \sum_{k=0}^{\infty} \|\mathcal{R}_{2^k t}(f_{y',z}(\cdot))\|_{L^p(\boldsymbol{R})}^p dz\right)^{1/p} d\sigma(y')$$
$$\le Ct^{-\delta_p} \int_{S^{n-1}} |\Omega(y')| \left(\int_{L_{y'}^{\perp}} \sum_{k=0}^{\infty} 2^{-kp\delta_p} \int_{\boldsymbol{R}} |f_{y',z}(s)|^p ds dz\right)^{1/p} d\sigma(y')$$
$$\le Ct^{-\delta_p} \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\boldsymbol{R}^n)} .$$

Hence, to get (3.11), it suffices to prove (3.13). Note that ψ is a smooth function supported on $(1/4, 1)$. It is trivial that

$$\left|\int_0^\infty e^{iQ_\lambda(ar)}\psi(ar)\frac{1}{r}g(s-r)dr\right| \le 4a\int_{1/4a}^{1/a} |g(s-r)|dr \le CM(g)(s) ,$$

where M denotes the Hardy-Littlewood maximal operator on $\boldsymbol{R}$. Thus, for $1 < p < \infty$,

$$\|\mathcal{R}_t(g)\|_{L^p(\boldsymbol{R})} \le C\|g\|_{L^p(\boldsymbol{R})} , \tag{3.14}$$

where C is independent of t. If we can prove that, for some $\delta_2 > 0$,

$$\|\mathcal{R}_t(g)\|_{L^2(\boldsymbol{R})} \le Ct^{-\delta_2}\|g\|_{L^2(\boldsymbol{R})} \tag{3.15}$$

with C is independent of t, then (3.13) follows by using Marcinkiewicz interpolation theorem between (3.14) and (3.15) with $\delta_p = \min\{2/p, 2/p'\}\theta\delta_2$, where $0 < \theta < p/(2p-1)$.

We devote ourselves to the proof of (3.15) in the following. By the definition of $\mathcal{R}_t$, for fixed $s \in \boldsymbol{R}$, there are a nonzero vector $\lambda(s)$ in $\boldsymbol{R}^{d-1}$ satisfying $t \le |\lambda(s)| \le 2t$ and a positive number $a(s)$ such that

$$\left|\int_0^\infty e^{iQ_{\lambda(s)}(a(s)r)}\psi(a(s)r)\frac{1}{r}g(s-r)dr\right| \ge \frac{1}{2}\mathcal{R}_t(g)(s) . \tag{3.16}$$

For fixed vector valued function $\lambda(\cdot)$ and positive real valued function $a(\cdot)$, we define

$$\mathcal{L}_{\lambda,a}(g)(s) = \int_{\boldsymbol{R}} e^{iQ_{\lambda(s)}(a(s)r)}\psi(a(s)r)\frac{1}{r}g(s-r)dr .$$

Thus, by (3.16), to get (3.15) we just need to estimate the L^2 norm of $\mathcal{L}_{\lambda,a}(g)$. That is, we have to prove

$$\|\mathcal{L}_{\lambda,a}(g)\|_{L^2(\boldsymbol{R})} \le Ct^{-\delta_2}\|g\|_{L^2(\boldsymbol{R})} , \tag{3.17}$$

where C is independent of t and the choices of $\lambda(\cdot)$ and $a(\cdot)$.

For fixed $\lambda(\cdot)$ and $a(\cdot)$, $\mathcal{L}^*_{\lambda,a}$ denote the adjoint operator of $\mathcal{L}_{\lambda,a}$. Thus, $\mathcal{L}^*_{\lambda,a}$ can be represented as

$$\mathcal{L}^*_{\lambda,a}(h)(r) = \int_{\boldsymbol{R}} e^{-iQ_{\lambda(s)}(a(s)(s-r))}\psi(a(s)(s-r))\frac{1}{s-r}h(s)ds .$$

We consider the L^2 norm of $\mathcal{L}_{\lambda,a}\mathcal{L}^*_{\lambda,a}(g)$. It is easy to verify that

$$\mathcal{L}_{\lambda,a}\mathcal{L}^*_{\lambda,a}(g)(s) = \int_{\boldsymbol{R}} \mathcal{K}(s,u)g(u)du\,,$$

where

$$\begin{aligned}\mathcal{K}(s,u) &= \int_{\boldsymbol{R}} e^{iQ_{\lambda(s)}(a(s)r)}e^{-iQ_{\lambda(u)}(a(u)(u-s+r))}\psi(a(s)r)\frac{1}{r}\psi(a(u)(u-s+r))\frac{1}{u-s+r}dr \\ &= \left(e^{iQ_{\lambda(s)}(a(s)\cdot)}\psi(a(s)\cdot)\frac{1}{\cdot}\right) * \left(e^{-iQ_{\lambda(u)}(-a(u)\cdot)}\psi(-a(u)\cdot)\frac{1}{(-\cdot)}\right)(s-u).\end{aligned}$$

We claim that

$$\begin{aligned}|\mathcal{K}(s,u)| \le C\big\{&t^{-2\delta_2}a(s)\chi_{I_2}(a(s)(s-u)) + a(s)\chi_{E_{\lambda(s)}}(a(s)(s-u)) \\ &+ t^{-2\delta_2}a(u)\chi_{I_2}(a(u)(s-u)) + a(u)\chi_{E_{\lambda(u)}}(a(u)(s-u))\big\}\,,\end{aligned} \tag{3.18}$$

where $E_{\lambda(s)}$ and $E_{\lambda(u)}$ are subsets of $I_2 := (-2,2)$ satisfying $|E_{\lambda(s)}|, |E_{\lambda(u)}| \le t^{-4\delta_2}$ for $\delta_2 = (6d)^{-1}$. Once we verify (3.18), then (3.17) can be deduced from (3.18). In fact,

$$\begin{aligned}|\langle \mathcal{L}_{\lambda,a}\mathcal{L}^*_{\lambda,a}(g), \ell\rangle| &\le \int_{\boldsymbol{R}}\int_{\boldsymbol{R}} |\mathcal{K}(s,u)||g(u)||\ell(s)|duds \\ &\le Ct^{-2\delta_2}\int_{\boldsymbol{R}} |\ell(s)|a(s)\int_{|s-u|\le 2/a(s)} |g(u)|duds \\ &\quad + C\int_{\boldsymbol{R}} |\ell(s)|a(s)\int_{\boldsymbol{R}} \chi_{E_{\lambda(s)}}\big(a(s)(s-u)\big)|g(u)|duds \\ &\quad + Ct^{-2\delta_2}\int_{\boldsymbol{R}} |g(u)|a(u)\int_{|s-u|\le 2/a(u)} |\ell(s)|dsdu \\ &\quad + C\int_{\boldsymbol{R}} |g(u)|a(u)\int_{\boldsymbol{R}} \chi_{E_{\lambda(u)}}\big(a(u)(s-u)\big)|\ell(s)|dsdu \\ &\le Ct^{-2\delta_2}\int_{\boldsymbol{R}} |\ell(s)|M(g)(s)ds + C\int_{\boldsymbol{R}} |\ell(s)|\mathcal{M}_\varepsilon(g)(s)ds \\ &\quad + Ct^{-2\delta_2}\int_{\boldsymbol{R}} |g(u)|M(\ell)(u)du + C\int_{\boldsymbol{R}} |g(u)|\mathcal{M}_\varepsilon(\ell)(u)du\,,\end{aligned}$$

where $\varepsilon = t^{-4\delta_2}$. Using Hölder's inequality, the L^2 boundedness of M (see [12]) and Lemma 2.3, we get

$$|\langle \mathcal{L}_{\lambda,a}\mathcal{L}^*_{\lambda,a}g, \ell\rangle| \le Ct^{-2\delta_2}\|g\|_{L^2(\boldsymbol{R})}\|\ell\|_{L^2(\boldsymbol{R})}\,, \tag{3.19}$$

and (3.17) follows from (3.19). Thus, in order to finish the proof of Theorem 1.1, it remains to verify the claim (3.18).

For fixed s, u and function $a(\cdot), \lambda(\cdot)$, let $w = s-u$, $\mu = \lambda(u)$, $\nu = \lambda(s)$, $a_1 = a(u)$, $a_2 = a(s)$. Then, for fixed s, u, $\mathcal{K}(s,u)$ can be represented as

$$\mathcal{K}(s,u) = \int_{\boldsymbol{R}} e^{iQ_\nu(a_2 r)}e^{-iQ_\mu(a_1(r-w))}\psi(a_2 r)\frac{1}{r}\psi(a_1(r-w))\frac{1}{r-w}dr\,.$$

First we assume that $a_2 \geq a_1$. Thus, $h = a_1/a_2 \leq 1$. By rescaling by a_1, we obtain

$$\mathcal{K}(s, u) = \int_{\boldsymbol{R}} e^{iQ_\nu(r/h)} e^{-iQ_\mu(r-a_1w)} \psi(r/h) \frac{1}{r} \psi(r - a_1 w) \frac{a_1}{r - a_1 w} dr .$$

Hence, if we denote

$$\begin{aligned}\mathcal{F}_h^{\mu,\nu}(w) &= \int_{\boldsymbol{R}} e^{iQ_\nu(r/h)} e^{-iQ_\mu(r-w)} \psi(r/h) \frac{1}{r} \psi(r - w) \frac{1}{r - w} dr \\ &= \int_{\boldsymbol{R}} e^{iQ_\nu(r)} e^{-iQ_\mu(hr-w)} \psi(r) \frac{1}{r} \psi(hr - w) \frac{1}{hr - w} dr ,\end{aligned}$$

then we have

$$\mathcal{K}(s, u) = a_1 \mathcal{F}_h^{\mu,\nu}(a_1 w) .$$

Assume that, for $t \leq |\mu|, |\nu| \leq 2t$ and $0 < h \leq 1$, there is a measurable set E_μ in I_2 with $|E_\mu| \leq t^{-4\delta_2}$ such that

$$|\mathcal{F}_h^{\mu,\nu}(w)| \leq C(t^{-2\delta_2} \chi_{I_2}(w) + \chi_{E_\mu}(w)) . \tag{3.20}$$

Then when $a(s) \geq a(u)$,

$$\begin{aligned}|\mathcal{K}(s, u)| &\leq C(t^{-2\delta_2} a_1 \chi_{I_2}(a_1 w) + a_1 \chi_{E_\mu}(a_1 w)) \\ &= C[t^{-2\delta_2} a(u) \chi_{I_2}(a(u)(s - u)) + a(u) \chi_{E_{\lambda(u)}}(a(u)(s - u))] .\end{aligned}$$

By the symmetry of u and s, we can get similar inequality as above when $a(s) \leq a(u)$. Thus, (3.18) is proved under this assumption.

Following that, we just need to verify the existence of E_μ with the inequality (3.20). The discussion will be divided into two cases: h is near the origin and away from the origin.

CASE 1. $0 < h \leq \eta \ll 1$, where η will be chosen later. If we denote $\nu_1 = 0$, $\binom{k}{j} = k \cdot (k-1) \cdots (k-j+1)/j!$ and $\binom{k}{j} = 0$ if $k < j$, by a trivial calculation we have

$$\begin{aligned}Q_\nu(r) - Q_\mu(hr - w) &= \sum_{j=2}^{d} \nu_j r^j - \left[Q_\mu(-w) + \sum_{j=1}^{d} h^j r^j \sum_{k=2}^{d} \binom{k}{j} \mu_k (-w)^{k-j} \right] \\ &= \sum_{j=1}^{d} r^j \left(\nu_j - h^j \sum_{k=2}^{d} \binom{k}{j} \mu_k (-w)^{k-j} \right) - Q_\mu(-w) .\end{aligned} \tag{3.21}$$

If r and $hr - w$ are in $\operatorname{supp}(\psi) \subseteq \{1/4 < r \leq 1\}$, then we have $|w| \leq |hr - w| + hr \leq 1 + h \leq 2$ and

$$\begin{aligned}\sum_{j=1}^{d} \left| \nu_j - h^j \sum_{k=2}^{d} \binom{k}{j} \mu_k (-w)^{k-j} \right| &\geq \sum_{j=2}^{d} |\nu_j| - \sum_{j=1}^{d} h^j \sum_{k=2}^{d} \binom{k}{j} |\mu_k| |w|^{k-j} \\ &\geq \sum_{j=2}^{d} |\nu_j| - Ch \sum_{k=2}^{d} |\mu_k| .\end{aligned}$$

If η is chosen small enough, since $t \leq |\mu|, |\nu| \leq 2t$, we get

$$\sum_{j=1}^{d}\left|\nu_j - h^j \sum_{k=2}^{d}\binom{k}{j}\mu_k(-w)^{k-j}\right| \geq \sum_{j=2}^{d}|\nu_j| - C\eta\sum_{k=2}^{d}|\mu_k| \geq C\sum_{j=2}^{d}|\nu_j| \geq Ct\,.$$

By Lemma 2.1, we have

$$\left|\mathcal{F}_h^{\mu,\nu}(w)\right| \leq Ct^{-1/d}\chi_{I_2}(w)\,. \tag{3.22}$$

CASE 2. $\eta < h \leq 1$ and η is fixed now. We consider the term of degree 1 in r in the phase $Q_\nu(r) - Q_\mu(hr - w)$. Since there is no first order term in r in $Q_\nu(r)$, by (3.21), the first order term of the above is

$$-rh\sum_{k=2}^{d}k\mu_k(-w)^{k-1}\,.$$

Since $h > \eta$, by Lemma 2.1, we get

$$|\mathcal{F}_h^{\mu,\nu}(w)| \leq C\left|\sum_{k=2}^{d}k\mu_k(-w)^{k-1}\right|^{-1/d}\chi_{I_2}(w)\,.$$

We define

$$E_\mu = \left\{w \in I_2\,;\ \left|\sum_{k=2}^{d}k\mu_k(-w)^{k-1}\right| \leq \rho\right\},$$

and ρ will be chosen later. For $w \in (E_\mu)^c$, it is obvious that

$$|\mathcal{F}_h^{\mu,\nu}(w)| \leq C\rho^{-1/d}\chi_{I_2}(w)\,. \tag{3.23}$$

By Lemma 2.2, we obtain

$$|E_\mu| \leq C\left(\sum_{k=2}^{d}k|\mu_k|\right)^{-1/d}\rho^{1/d}\,.$$

Note that

$$\sum_{k=2}^{d}k|\mu_k| \geq \sum_{k=2}^{d}|\mu_k| = |\mu| \geq t\,.$$

Thus for $w \in E_\mu$, we have

$$|\mathcal{F}_h^{\mu,\nu}(w)| \leq C\chi_{E_\mu}(w)\,, \tag{3.24}$$

with $|E_\mu| \leq C(\rho/t)^{1/d}$.

Specially, we take $\rho = \bar{c}t^{1/3}$ with $\bar{c}$ appropriately small. Since $t \geq 1/C_0 > 0$ and $\delta_2 = 1/6d$, it follows from (3.22), (3.23) and (3.24) that

$$|\mathcal{F}_h^{\mu,\nu}(w)| \leq C(t^{-2\delta_2}\chi_{I_2}(w) + \chi_{E_\mu}(w))$$

with $|E_\mu| \leq t^{-4\delta_2}$, that is, the estimate (3.20) is satisfied for E_μ.

Thus, we complete the proof of Theorem 1.1.

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REFERENCES

[1] L. CARLESON, On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966), 135–157.
[2] L. COLZANI, Hardy spaces on spheres, Ph. D. Thesis, Washington University, St. Louis, 1982.
[3] W. CONNETT, Singular integrals near L^1, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Amer. Math. Soc. 35, Providence, R.I., 1979. 163–165.
[4] Y. DING AND H. LIU, Weighted L^p boundedness of Carleson type maximal operators, to appear in Proc. Amer. Math. Soc.
[5] D. FAN AND Y. PAN, Singular integral operators with rough kernels supported by subvarieties, Amer. J. Math. 119 (1997), 799–839.
[6] C. FEFFERMAN, Inequalities for strongly singular convolution operators, Acta Math. 124 (1970), 9–36.
[7] L. GRAFAKOS AND A. STEFANOV, Convolution Calderón-Zygmund singular integral operators with rough kernels, Analysis of Divergence: Control and Management of Divergent Processes, 119–143, Birkhauser, Boston-Basel-Berlin, 1999.
[8] R. HUNT, On the convergence of Fourier series, Orthogonal Expansions and Their Continuous Analogues (Proc. Cont. Edwardsville, Ill., 1967), 235–255, Southern Illinois Univ. Press, Carbondale Ill., 1968.
[9] E. PRESTINI AND P. SJÖLIN, A Littlewood-Paley inequality for the Carleson operator, J. Fourier Anal. Appl. 6 (2000), 457–466.
[10] F. RICCI AND G. WEISS, A characterization of $H^1(\Sigma_{n-1})$, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), 289–294, Amer. Math. Soc. 35, Providence, R.I., 1979.
[11] P. SJÖLIN, Convergence almost everywhere of certain singular integral and multiple Fourier series, Ark. Mat. 9 (1971), 65–90.
[12] E. M. STEIN, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, N.J., 1970.
[13] E. M. STEIN AND S. WAINGER, Oscillatory integrals related to Carleson's theorem, Math. Res. Lett. 8 (2001), 789–800.

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