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L^p ESTIMATES FOR SCHRÖDINGER OPERATORS WITH CERTAIN POTENTIALS

by Zhongwei SHEN (*)

0. Introduction.

In this paper we consider the Schrödinger differential operator

$$P = -\Delta + V(x) \quad \text{on} \quad \mathbb{R}^n, n \geq 3 \tag{0.1}$$

where $V(x)$ is a nonnegative potential. We will assume that V belongs to the reverse Hölder class B_q for some $q \geq n/2$. We are interested in the L^p boundedness of the operators $(-\Delta+V)^{i\gamma}, \nabla(-\Delta+V)^{-1/2}, \nabla(-\Delta+V)^{-1}\nabla$ and $\nabla^2(-\Delta+V)^{-1}$ where $\gamma \in \mathbb{R}$.

Note that a nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to $B_q (1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V dx\right) \tag{0.2}$$

holds for every ball B in $\mathbb{R}^n$ ([G], [M]).

One remarkable feature about the B_q class is that, if $V \in B_q$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on n and the constant C in (0.2), such that $V \in B_{q+\varepsilon}$ [G].

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We now state our main results in this paper.

THEOREM 0.3. — *Suppose $V \in B_q$ for some $q \geq n/2$. Then, for $1 < p \leq q$,*

$$\|\nabla^2(-\Delta + V)^{-1} f\|_p \leq C_p \|f\|_p$$

where C_p depends only on p, n and the constant in (0.2).

THEOREM 0.4. — *Suppose $V \in B_{n/2}$. Then, for $\gamma \in \mathbb{R}$, $(-\Delta + V)^{i\gamma}$ is a Calderón–Zygmund operator.*

It is well known that Calderón–Zygmund operators are bounded on L^p for $1 < p < \infty$.

THEOREM 0.5. — *Suppose $V \in B_q$ and $n/2 \leq q < n$. Then, for $1 < p \leq p_0$,*

$$\|\nabla(-\Delta + V)^{-1/2} f\|_p \leq C_p \|f\|_p$$

where $(1/p_0) = (1/q) - (1/n)$.

We remark that the ranges of p in Theorems 0.3 and 0.5 are optimal. This can be shown by considering the potential $V(x) = |x|^\alpha$ where $-2 < \alpha < 0$. See Section 7.

It follows easily from Theorem 0.5 that, if $V \in B_q$, $n/2 \leq q < n$,

$$\|(-\Delta + V)^{-1/2} \nabla f\|_p \leq C_p \|f\|_p \quad \text{for } p_0' \leq p < \infty \tag{0.6}$$

and

$$\|\nabla(-\Delta + V)^{-1} \nabla f\|_p \leq C \|f\|_p \quad \text{for } p_0' \leq p \leq p_0 \tag{0.7}$$

where $p_0' = p_0/(p_0 - 1)$.

THEOREM 0.8. — *Suppose $V \in B_n$. Then*

$$\nabla(-\Delta + V)^{-1/2}, (-\Delta + V)^{-1/2}\nabla \text{ and } \nabla(-\Delta + V)^{-1}\nabla$$

are Calderón–Zygmund operators.

We also obtain the L^p boundedness of the operators

$$V(-\Delta + V)^{-1}, \quad V^{1/2}(-\Delta + V)^{-1/2}, \quad V^{1/2}(-\Delta + V)^{-1}\nabla$$

and

$$V^{1/2}\nabla(-\Delta + V)^{-1}.$$

See Theorem 3.1, Theorem 5.10, Theorem 5.11 and Theorem 4.13 respectively. As a direct consequence of our L^p estimates, we have

COROLLARY 0.9. — *Suppose $V \in B_q$ for some $q \geq n/2$. Assume that $-\Delta u + Vu = f$ in $\mathbb{R}^n$. Then*

$$\|\nabla^2 u\|_p \leq C_p \|f\|_p \text{ for } 1 < p \leq q,$$
$$\|Vu\|_p \leq C\|f\|_p \text{ for } 1 \leq p \leq q,$$
$$\|V^{1/2}\nabla u\|_p \leq C\|f\|_p \text{ for } 1 \leq p \leq p_1$$

where $1/(p_1) = (3/2q) - (1/n)$ if $n/2 \leq q < n$; and $p_1 = 2q$ if $q \geq n$.

COROLLARY 0.10. — *Suppose $V \in B_q$ for some $q \geq n/2$. Assume that $-\Delta u + Vu = \operatorname{div} \vec{g}$ in $\mathbb{R}^n$. Then*

$$\|\nabla u\|_p \leq C_p \|\vec{g}\|_p \text{ for } p_0' \leq p \leq p_0 (1 < p < \infty \text{ if } q \geq n),$$
$$\|V^{1/2} u\|_p \leq C\|\vec{g}\|_p \text{ for } p_0' \leq p \leq 2q (1 \leq p \leq 2q \text{ if } q \geq n)$$

where $(1/p_0) = (1/q) - (1/n)$.

We now recall that an operator T taking $C_c^\infty(\mathbb{R}^n)$ into $L^1_{\text{loc}}(\mathbb{R}^n)$ is called a Calderón–Zygmund operator if

(a) T extends to a bounded linear operator on $L^2(\mathbb{R}^n)$,

(b) there exists a kernel K such that for every $f \in L_c^\infty(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy \quad \text{a.e. on } \{\operatorname{supp} f\}^c,$$

(c) the kernel K satisfies the Calderón–Zygmund estimates:

$$(0.11) \qquad \begin{cases} |K(x,y)| \leq \dfrac{C}{|x-y|^n}, \\ |K(x+h,y) - K(x,y)| \leq \dfrac{C|h|^\delta}{|x-y|^{n+\delta}}, \\ |K(x,y+h) - K(x,y)| \leq \dfrac{C|h|^\delta}{|x-y|^{n+\delta}} \end{cases}$$

for $x, y \in \mathbb{R}^n$, $|h| < |x-y|/2$ and for some $\delta > 0$. See [St2].

We remark that in his thesis, which inspired the work in this paper, J. Zhong [Z] studied the Schrödinger operator $-\Delta + V(x)$, assuming that V is a nonnegative polynomial in $\mathbb{R}^n$. He showed that, for $\gamma \in \mathbb{R}$, $(-\Delta + V)^{i\gamma}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-1/2}$ and $\nabla(-\Delta + V)^{-1}\nabla$ are Calderón–Zygmund operators with bounds depending only on the degree of V and the dimension n.

Related results can also be found in [HN] and [T]. In [HN], Helffer and Nourrigat considered the case of nonnegative polynomials. They proved the L^2 boundedness of $\nabla^2(-\Delta+V)^{-1}$ and $V^{1/2}\nabla(-\Delta+V)^{-1}$, based on a subelliptic estimate of Rothschild and Stein [RS]. In [T], the potential $V=|x|^2$ was considered in connection with the Hermite operator. Furthermore, it was pointed out by the referee that the L^p $(1<p<\infty)$ boundedness of the operator $(-\Delta+V)^{i\gamma}$ in Theorem 0.4 follows from a general result of W. Hebisch [H].

Note that, if V is a nonnegative polynomial, then

$$\max_{x\in B} V(x) \le C\left(\frac{1}{|B|}\int_B V(x)dx\right) \tag{0.12}$$

for every ball B in $\mathbb{R}^n$, where C depends only on the degree of V and n. It follows that V satisfies (0.2), i.e., $V\in B_q$ for all q, $1<q<\infty$, with the same constant as in (0.12). Hence, our Theorems 0.4 and 0.8 extend Zhong's results on the operator $(-\Delta+V)^{i\gamma}$, $\nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ to the general B_q class. Clearly, Theorem 0.3 implies that $\nabla^2(-\Delta+V)^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for all p, $1<p<\infty$, if V satisfies (0.12). But it seems that extra conditions are needed to assure that the kernel function for the operator $\nabla^2(-\Delta+V)^{-1}$ satisfies the Calderón–Zygmund estimate (0.11).

It is interesting to notice that, if $V(x)=|x_n-\varphi(x')|^\alpha$, $\alpha>0$, where $x'=(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}$ and $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ is a Lipschitz function, then V satisfies (0.12) with a constant depending only on n, α and $\|\nabla\varphi\|_\infty$. Also, if $V(x)=|x|^\alpha$, $\alpha q>-n$, then $V\in B_q$. In particular, $V\in B_{n/2}$ if $\alpha>-2$; and $V\in B_n$ if $\alpha>-1$.

The Schrödinger operator $-\Delta+V$ with nonnegative potentials is useful in the study of certain subelliptic operators. Indeed, by taking the partial Fourier transform in the t variable, the operator $-\Delta_x - V(x)\partial_t^2$ is reduced to $-\Delta_x+V(x)\xi^2$. See [Sm].

We briefly sketch the argument we will use in this paper. First, we note that, in the case that $V=P(x)$ is a nonnegative polynomial, the weight function

$$M(x,V)=\sum_{|\alpha|\le k}|\partial_x^\alpha P(x)|^{1/(|\alpha|+2)} \tag{0.13}$$

plays a key role. In (0.13), $k=$ degree of $P(x)$. See [Sm], [Z]. Here, for $V\in B_{n/2}$, we define the function $m(x,V)$ by

$$\frac{1}{m(x,V)}=\sup\left\{r>0:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)dy\le 1\right\}. \tag{0.14}$$

The function $m(x,V)$ was introduced in [Sh] for the potential V satisfying the condition (0.12) to study the Neumann problem for the operator

$-\Delta+V(x)$ in the region above a Lipschitz graph. Note that, if $r = \dfrac{1}{m(x,V)}$, then

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy = 1.$$

In the case that $V = P(x) \geq 0$ is a polynomial, it can be shown that

$$m(x,V) \sim M(x,V).$$

Next, we use a lemma due to C. Fefferman and D.H. Phong [F] to show that, if $V \in B_{n/2}$,

$$|\Gamma(x,y)| \leq \frac{C_k}{\{1+|x-y|m(x,V)\}^k}\cdot\frac{1}{|x-y|^{n-2}} \tag{0.15}$$

and, if $V \in B_n$,

$$|\nabla\Gamma(x,y)| \leq \frac{C_k}{\{1+|x-y|m(x,V)\}^k}\cdot\frac{1}{|x-y|^{n-1}} \tag{0.16}$$

for any $k > 0$, where $\Gamma(x,y)$ denotes the fundamental solution for the operator $-\Delta+V(x)$ in $\mathbb{R}^n$. We remark that similar approaches were used in [Sm] and [Z]. Also see [Sh].

Estimates (0.15) and (0.16) are essential to deal with the kernels of operators in the part where $|x-y| > \dfrac{1}{m(x,V)}$. For the part where $|x-y| \leq \dfrac{1}{m(x,V)}$, the key observation is that, if $V \in B_q$, $q > (n/2)$,

$$|\Gamma(x,y)-\Gamma_0(x,y)| \leq \frac{C\{|x-y|m(x,V)\}^{2-(n/q)}}{|x-y|^{n-2}} \tag{0.17}$$

where $\Gamma_0(x,y)$ is the fundamental solution for $-\Delta$ in $\mathbb{R}^n$.

Estimates like (0.15), (0.16) and (0.17) will enable us to control the operator $(-\Delta+V)^{i\gamma}, \nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ by the Hardy–Littlewood maximal function and the corresponding (maximal) singular integral operators associated with $-\Delta$.

Finally, we note that, to study $(-\Delta+V)^{i\gamma}$ and $\nabla(-\Delta+V)^{-1/2}$, by functional calculus, we actually need to deal with $\Gamma(x,y,\tau)$ and $\nabla\Gamma(x,y,\tau)$ where $\Gamma(x,y,\tau)$ is the fundamental solution for $-\Delta+V(x)+i\tau$, $\tau \in \mathbb{R}$. Moreover, if $V \in B_q$, $n/2 \leq q < n$, we do not have pointwise estimates for $\nabla\Gamma(x,y,\tau)$. But, these difficulties are overcome by our basic idea that, the operators associated with $-\Delta+V$ can be viewed as a perturbation of the corresponding operators associated with $-\Delta$ in the scale less than $\{m(x,V)\}^{-1}$.

The paper is organized as follows. In Section 1 we introduce the auxiliary function $m(x,V)$ and study its properties. We also state and prove the Fefferman–Phong Lemma (Lemma 1.9) under the assumption $V \in B_{n/2}$. In Section 2 we establish the estimate (0.15) on the fundamental solutions (Theorem 2.7). Section 3 is devoted to the proof of Theorem 0.3. In Section 4 we give the proof of Theorem 0.4. Theorem 0.5 is proved in Section 5, while Theorem 0.8 is proved in Section 6. A counterexample is given in Section 7.

Throughout this paper, unless otherwise indicated, we will use C and c to denote constants, which are not necessarily the same at each occurrence, which depend at most on the constant in (0.2) and the dimension n. By $A \sim B$, we mean that there exist constants $C > 0$ and $c > 0$, such that $c \le A/B \le C$.

Finally, the author would like to thank the referee for pointing out the relevance of Hebish's work, and for the valuable comments concerning the limitations on p in Theorem 0.3 and 0.5, which lead to the counterexample in Section 7.

1. The auxiliary function $m(x,V)$.

Most of the results in this section were proved in [Sh] under the assumption (0.12). The extension to the case $V \in B_q$, $q > n/2$ is fairly straightforward. For the sake of completeness we provide the proofs.

Throughout the section we will assume that $V \in B_q$ for some $q > n/2$.

It is well known that $V \in B_q$, $q > 1$ implies that $V(x)dx$ is a doubling measure,

$$\int_{B(x,2r)} V(y)dy \le C_0 \int_{B(x,r)} V(y)dy. \tag{1.1}$$

In fact, if $V \in B_q$, $q > 1$, then V is a Muckenhoupt A_∞ weight (e.g., see [St2]).

LEMMA 1.2. — *There exists $C > 0$ such that, for $0 < r < R < \infty$,*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \le C \left(\frac{R}{r}\right)^{\frac{n}{q}-2} \cdot \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

Proof. — By Hölder inequality,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} V dy \le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q dy\right)^{1/q}$$
$$\le \left(\frac{R}{r}\right)^{n/q}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} V^q dy\right)^{1/q}$$
$$\le C\left(\frac{R}{r}\right)^{n/q}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} V dy\right)$$

since $V \in B_q$.

The lemma then follows easily.

By Lemma 1.2 and the assumption $q > n/2$, we see that, for any $x \in \mathbb{R}^n$,

$$\lim_{r\to 0}\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = 0$$

and

$$\lim_{r\to \infty}\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = \infty.$$

DEFINITION 1.3. — *For $x \in \mathbb{R}^n$, the function $m(x,V)$ is defined by*

$$\frac{1}{m(x,V)} = \sup_{r>0}\left\{r\colon \frac{1}{r^{n-2}}\int_{B(x,r)} V dy \le 1\right\}.$$

Clearly, $0 < m(x,V) < \infty$ for every $x \in \mathbb{R}^n$ and if $r = \dfrac{1}{m(x,V)}$ then

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V dy = 1.$$

Moreover, by Lemma 1.2, if

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V dy \sim 1, \quad \text{then } r \sim \frac{1}{m(x,V)}.$$

The following lemma is very useful to us.

LEMMA 1.4. — *There exist $C > 0$, $c > 0$ and $k_0 > 0$ such that, for x, y in $\mathbb{R}^n$,*

(a) $$m(x,V) \sim m(y,V) \ \text{ if } \ |x-y| \le \frac{C}{m(x,V)},$$

(b) $$m(y,V) \le C\{1+|x-y|m(x,V)\}^{k_0} m(x,V),$$

(c) $$m(y,V) \ge \frac{c\, m(x,V)}{\{1+|x-y|m(x,V)\}^{k_0/(k_0+1)}}.$$

Proof. — Let $r = \dfrac{1}{m(x,V)}$. Suppose $|x-y| \le Cr$. Since Vdx is a doubling measure,

$$\frac{1}{r^{n-2}} \int_{B(y,r)} V dz \sim \frac{1}{r^{n-2}} \int_{B(x,r)} V dz = 1.$$

It follows that

$$m(y,V) \sim \frac{1}{r} = m(x,V).$$

To see part (b), suppose $|y-x| \sim 2^j r, j \ge 1$. Let $0 < r_1 < r$ and $2^k r_1 \sim 2^j r$. Then, by Lemma 1.2,

$$\begin{aligned}
\int_{B(y,r_1)} V dz &\le C(2^k)^{(n/q)-n} \int_{B(y,2^k r_1)} V dz \\
&\le C(2^k)^{(n/q)-n} \int_{B(y,2^j r)} V dz \\
&\le C(2^k)^{(n/q)-n} \int_{B(x,2^j r)} V dz \\
&\le C(2^k)^{(n/q)-n} . C_0^j . \int_{B(x,r)} V dz \text{ (by the doubling condition (1.1))} \\
&= C(2^k)^{(n/q)-n} \cdot C_0^j \cdot r^{n-2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{r_1^{n-2}} \int_{B(y,r_1)} V dz &\le C(2^k)^{(n/q)-n} \cdot C_0^j \cdot \left(\frac{r}{r_1}\right)^{n-2} \\
&\le C(2^{(n/q)-n} C_0)^j \cdot \left(\frac{r}{r_1}\right)^{(n/q)-2} \\
&\le 1/2 \text{ if } r_1 \le C_1^{-j} r \text{ and } C_1 \text{ is large .}
\end{aligned}$$

Hence, by Definition 1.3,

$$\frac{1}{m(y,V)} \ge C_1^{-j} r.$$

It follows that

$$m(y,V) \le C_1^j m(x,V) \le C\{1 + |x-y| m(x,V)\}^{k_0} m(x,V), k_0 = \log_2 C_1.$$

Finally, we need to show part (c). We may assume that

$$|x-y| \ge \frac{1}{m(y,V)},$$

for otherwise it follows easily from part (a).

By part (b),

$$\begin{aligned} m(x,V) &\le C\{1+|x-y|m(y,V)\}^{k_0} m(y,V) \\ &\le C|x-y|^{k_0} m(y,V)^{k_0+1}. \end{aligned}$$

Thus,

$$m(y,V) \ge \frac{c\, m(x,V)^{1/(k_0+1)}}{|x-y|^{k_0/(k_0+1)}} \ge \frac{c\, m(x,V)}{\{1+m(x,V)|x-y|\}^{k_0/(k_0+1)}}.$$

The proof is complete.

The following is an easy consequence of Lemma 1.4:

COROLLARY 1.5. — *There exist $C>0$, $c>0$ and $k_0>0$ such that, for any $x,y\in\mathbb{R}^n$,*

$$\begin{aligned} c\{1+|x-y|m(y,V)\}^{1/(k_0+1)} &\le 1+|x-y|m(x,V) \\ &\le C\{1+|x-y|m(y,V)\}^{k_0+1}. \end{aligned}$$

Using Hölder inequality and the B_q condition we see that

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-2}} dy \le \frac{C}{R^{n-2}} \int_{B(x,R)} V(y)dy. \tag{1.6}$$

Similarly, if $V\in B_n$, hence $V\in B_{n+\varepsilon}$,

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-1}} dy \le \frac{C}{R^{n-1}} \int_{B(x,R)} V(y)dy. \tag{1.7}$$

LEMMA 1.8. — *There exist $C>0$ and $k_0>0$ such that, if $Rm(x,V)\ge 1$, then*

$$\frac{1}{R^{n-2}} \int_{B(x,R)} V dy \le C\{Rm(x,V)\}^{k_0}.$$

Proof. — Let $r=\dfrac{1}{m(x,V)}$. Suppose $2^j r\le R<2^{j+1}r$, $j\ge 0$. Then, by the doubling condition (1.1),

$$\int_{B(x,R)} V dy \le C_0^{j+1} \int_{B(x,r)} V dy = C_0^{j+1} r^{n-2}.$$

It follows that

$$\begin{aligned} \frac{1}{R^{n-2}} \int_{B(x,R)} V dy &\le C_0^{j+1} \left(\frac{r}{R}\right)^{n-2} \le C_0 (C_0 2^{2-n})^j \\ &\le C(Rm(x,V))^{k_0}, k_0=\log_2 C_0+2-n. \end{aligned}$$

We end this section with a lemma due to C. Fefferman and D.H. Phong [F], which plays the crucial role in the estimates of the fundamental solution for the Schrödinger operator $-\Delta + V(x)$.

LEMMA 1.9 (C. Fefferman-D.H. Phong). — *Let $u \in C_c^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |u(x)|^2 m(x,V)^2 dx \le C\left\{\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx + \int_{\mathbb{R}^n} |u(x)|^2 V(x) dx\right\}.$$

Proof. — Fix $x_0 \in \mathbb{R}^n$, let $r_0 = \dfrac{1}{m(x_0, V)}$. Then

$$\int_B |\nabla u(x)|^2 dx \ge \frac{c}{r_0^{n+2}} \int_B \int_B |u(x) - u(y)|^2 dx dy$$

$$\int_B |u(x)|^2 V(x) dx \ge \frac{c}{r_0^n} \int_B \int_B |u(y)|^2 V(y) dx dy$$

where $B = B(x_0, r_0)$.

Adding two inequalities we obtain

$$\int_B |\nabla u|^2 dx + \int_B |u|^2 V dx \ge \frac{c}{r_0^n} \int_B \int_B \min_{y\in B}\left\{\frac{c_0}{r_0^2}, V(y)\right\} |u(x)|^2 dx dy$$

where $c_0 > 0$ is a constant to be determined later.

Recall that V is an A_∞ weight. Hence, there exists $\varepsilon > 0$ such that, for every ball B in $\mathbb{R}^n$,

$$\left|\left\{x \in B: V(x) \ge \frac{\varepsilon}{|B|}\int_B V dy\right\}\right| \ge \frac{1}{2}|B|.$$

Now, let $c_0 = \varepsilon |B(0,1)|^{-1}$, then

$$\int_B \min_{y\in B}\left\{\frac{c_0}{r_0^2}, V(y)\right\} dy \ge c r_0^{n-2}.$$

It follows that

$$\int_B |u(x)|^2 m(x,V)^2 dx \le \frac{C}{r_0^2}\int_B |u|^2 dx \le C\left\{\int_B |\nabla u|^2 dx + \int_B |u|^2 V dx\right\}$$

where we have used part (a) of Lemma 1.4. Moreover,

$$\int_{B(x_0,r_0)} |u(x)|^2 m(x,V)^{2+n} dx \le C\left\{\int_{B(x_0,r_0)} |\nabla u(x)|^2 m(x,V)^n dx \right.$$
$$\left. + \int_{B(x_0,r_0)} |u(x)|^2 V(x) m(x,V)^n dx\right\}.$$

To finish the proof, we integrate both sides of the above inequality in x_0 over $\mathbb{R}^n$ to obtain

$$\int_{\mathbb{R}^n} |u(x)|^2 m(x,V)^{2+n} \left(\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \right) dx$$
$$\leq C \left\{ \int_{\mathbb{R}^n} |\nabla u(x)|^2 m(x,V)^n \left(\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \right) dx \right.$$
$$\left. + \int_{\mathbb{R}^n} |u(x)|^2 V(x) m(x,V)^n \left(\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \right) \right\}.$$

Finally, note that, by part (a) of Lemma 1.4,

$$\int_{|x_0-x|<\frac{1}{m(x_0,V)}} dx_0 \sim \left(\frac{1}{m(x,V)} \right)^n .$$

The lemma then follows easily.

2. Estimates of fundamental solutions.

This section is devoted to the estimates of fundamental solutions for the operator $-\Delta + (V + i\tau)$ on $\mathbb{R}^n$ where $\tau \in \mathbb{R}$.

We will assume that $V \in B_q$ for some $q > n/2$ throughout this section.

LEMMA 2.1. — *Suppose $-\Delta u + (V + i\tau)u = 0$ in $B(x_0, 2R)$ for some $x_0 \in \mathbb{R}^n, R > 0$ and $\tau \in \mathbb{R}$. Then*

$$\sup\{|u(x)| : x \in B(x_0, R)\}$$
$$\leq \frac{C_k}{\{1 + R|\tau|^{1/2}\}^k \{1 + Rm(x_0,V)\}^k} \left\{ \frac{1}{R^n} \int_{B(x_0,2R)} |u|^2 dy \right\}^{1/2}$$

for any integer $k > 0$, where C_k depends only on k, n and the constant in the reverse Hölder inequality (0.2).

Proof. — Since

$$\Delta(|u|^2) = 2Re\Delta u \cdot \overline{u} + 2|\nabla u|^2 = 2V|u|^2 + 2|\nabla u|^2 \geq 0,$$

$|u|^2$ is subharmonic. It follows that

$$\sup\{|u(x)| : x \in B(x_0, R)\} \leq C \left(\frac{1}{R^n} \int_{B(x_0,3R/2)} |u|^2 dy \right)^{1/2} . \tag{2.2}$$

By Caccioppoli's inequality,

$$(2.3)\quad \int_{B(x_0,3R/2)} |\nabla u|^2 dx + \int_{B(x_0,3R/2)} |u|^2 V dx + |\tau| \int_{B(x_0,3R/2)} |u|^2 dx \le \frac{C}{R^2} \int_{B(x_0,2R)} |u|^2 dx.$$

Now, let $\eta \in C_0^\infty(B(x_0,3R/2))$ such that $\eta \equiv 1$ on $B(x_0,R)$ and $|\nabla \eta| \le C/R$. Applying Lemma 1.9 to the function $u\eta$, we obtain

$$\begin{aligned}\int_{B(x_0,R)} m(x,V)^2 |u|^2 dx &\le C\left\{ \int_{B(x_0,3R/2)} |\nabla u|^2 dx + \int_{B(x_0,3R/2)} |u|^2 V dx + \frac{1}{R^2}\int_{B(x_0,2R)} |u|^2 dx \right\} \\ &\le \frac{C}{R^2}\int_{B(x_0,2R)} |u|^2 dx\end{aligned}$$

where we used (2.3) in the last inequality.

Note that, by part (c) of Lemma 1.4, for $x \in B(x_0,R)$,

$$m(x,V) \ge \frac{c\, m(x_0,V)}{\{1+Rm(x_0,V)\}^{k_0/(k_0+1)}}.$$

It follows that

$$\int_{B(x_0,R)} |u|^2 dx \le C \cdot \frac{\{1+Rm(x_0,V)\}^{2k_0/(k_0+1)}}{\{m(x_0,V)R\}^2} \int_{B(x_0,2R)} |u|^2 dx.$$

Hence,

$$\int_{B(x_0,R)} |u|^2 dx \le \frac{C}{\{1+Rm(x_0,V)\}^{2/(k_0+1)}} \int_{B(x_0,2R)} |u|^2 dx.$$

Clearly, by repeating above argument, we have

$$(2.4)\quad \int_{B(x_0,3R/2)} |u|^2 dx \le \frac{C_k}{\{1+Rm(x_0,V)\}^k} \int_{B(x_0,2R)} |u|^2 dx \text{ for any } k>0.$$

Similarly, by Caccioppoli's inequality (2.3),

$$(2.5)\quad \int_{B(x_0,3R/2)} |u|^2 dx \le \frac{C_k}{(1+|\tau|^{1/2}R)^k} \int_{B(x_0,2R)} |u|^2 dx \quad \text{for any} \quad k>0.$$

The lemma then follows from (2.2), (2.4) and (2.5).

Let $\Gamma(x,y,\tau)$ denote the fundamental solution for the Schrödinger operator $-\Delta + (V(x)+i\tau), \tau \in \mathbb{R}$. Clearly,

$$\Gamma(x,y,\tau) = \Gamma(y,x,-\tau).$$

Since $V \geq 0$ and $V \in L^{n/2}_{\mathrm{loc}}$, it is well known that

$$|\Gamma(x,y,\tau)| \leq \frac{C}{|x-y|^{n-2}} \quad \text{for} \quad x,y \in \mathbb{R}^n \tag{2.6}$$

where C depends only on n.

Since $V \in B_{n/2}$ implies that $V \in B_q$ for some $q > n/2$, the following theorem follows easily from Lemma 2.1 and (2.6).

THEOREM 2.7. — *Suppose $V \in B_{n/2}$. Then, for any $x, y \in \mathbb{R}^n$,*

$$|\Gamma(x,y,\tau)| \leq \frac{C_k}{\{1+|\tau|^{1/2}|x-y|\}^k\{1+m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-2}}$$

where C_k is a constant depending only on n, k and the constant in (0.2).

COROLLARY 2.8. — *Suppose $V \in B_{n/2}$. Then there exists $C > 0$ depending on n and the constant in (0.2) such that, for every $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$,*

$$\|m(\cdot,V)^2(-\Delta+V)^{-1}f\|_p \leq C\|f\|_p.$$

Proof. — It follows from Theorem 2.7 that

$$\begin{aligned}\int_{\mathbb{R}^n} |\Gamma(x,y)|dy &\leq C\int_{\mathbb{R}^n} \frac{dy}{\{1+|x-y|m(x,V)\}^3|x-y|^{n-2}} \\ &\leq \frac{C}{m(x,V)^2}\end{aligned}$$

where $\Gamma(x,y) = \Gamma(x,y,0)$ is the fundamental solution for $-\Delta + V(x)$. Thus, if $f \in L^p(\mathbb{R}^n)$,

$$u(x) = (-\Delta+V)^{-1}f(x) = \int_{\mathbb{R}^n} \Gamma(x,y)f(y)dy,$$

we have

$$\begin{aligned}|u(x)| &\leq \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)|dy\right\}^{1/p'} \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)||f(y)|^p dy\right\}^{1/p} \\ &\leq \frac{C}{m(x,V)^{2/p'}} \left\{\int_{\mathbb{R}^n} |\Gamma(x,y)||f(y)|^p dy\right\}^{1/p}, \quad p' = \frac{p}{p-1}.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^n} |m(x,V)^2u(x)|^p dx \leq C\int_{\mathbb{R}^n} |f(y)|^p \left\{\int_{\mathbb{R}^n} m(x,V)^2|\Gamma(x,y)|dx\right\} dy.$$

Finally, note that, by part (b) of Lemma 1.4 and Theorem 2.7,

$$\int_{\mathbb{R}^n} m(x,V)^2|\Gamma(x,y)|dx \le C_k \int_{\mathbb{R}^n} \frac{(m(y,V))^2}{\{1+|x-y|m(y,V)\}^{k-2k_0}|x-y|^{n-2}}dx$$
$$\le C \quad \text{if we choose} \quad k = 2k_0+3.$$

Corollary 2.8 then follows.

Remark 2.9. — If V satisfies (0.12), then $V(x) \le Cm(x,V)^2$ a.e. on $\mathbb{R}^n$. It follows from Corollary 2.8 that, for $1 \le p \le \infty$,

$$\|V(-\Delta+V)^{-1}f\|_p \le C\|f\|_p.$$

Also, for $1 < p < \infty$, by the L^p boundedness of the Riesz transforms,

$$\begin{aligned}\|\nabla^2(-\Delta+V)^{-1}f\|_p &\le C\|\Delta(-\Delta+V)^{-1}f\|_p\\ &\le C\|V(-\Delta+V)^{-1}f\|_p + C\|f\|_p\\ &\le C\|f\|_p.\end{aligned}$$

3. The proof of Theorem 0.3.

In this section we give the proof of Theorem 0.3.

THEOREM 3.1. — *Suppose $V \in B_q$ for some $q \ge n/2$. Then, for $1 \le p \le q$,*

$$\|V(-\Delta+V)^{-1}f\|_p \le C\|f\|_p$$

where C depends only on n and the constant in (0.2).

Proof. — Let $f \in L^p(\mathbb{R}^n), 1 \le p \le q$ and

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x,y)f(y)dy.$$

We need to show that

$$\|Vu\|_p \le C\|f\|_p.$$

Write

$$u(x) = \int_{|y-x|<r} \Gamma(x,y)f(y)dy + \int_{|y-x|\ge r} \Gamma(x,y)f(y)dy = u_1(x) + u_2(x)$$

where $r = \dfrac{1}{m(x,V)}$.

By the properties of B_q class, $V \in B_{q_0}$ for some $q_0 > q$. We have

$$|u_1(x)| \le \int_{|y-x|<r} \frac{|f(y)|}{|x-y|^{n-2}} dy$$
$$\le Cr^{2-\frac{n}{q_0}} \left(\int_{|y-x|<r} |f(y)|^{q_0} dy \right)^{1/q_0}$$

where we have used Hölder inequality and the fact $q_0 > n/2$. Thus,

$$\int_{\mathbb{R}^n} |V(x)u_1(x)|^{q_0} dx \le C \int_{\mathbb{R}^n} \left\{ \int_{|y-x|<\frac{1}{m(x,V)}} |f(y)|^{q_0} dy \right\} V(x)^{q_0} m(x,V)^{n-2q_0} dx$$
$$= C \int_{\mathbb{R}^n} |f(y)|^{q_0} \left\{ \int_{|x-y|<\frac{1}{m(x,V)}} V(x)^{q_0}\, m(x,V)^{n-2q_0} dx \right\} dy.$$

Now, let $R = \dfrac{1}{m(y,V)}$. Then

$$\int_{|x-y|<\frac{1}{m(x,V)}} V(x)^{q_0} m(x,V)^{n-2q_0} dx \le CR^{2q_0-n} \int_{|x-y|\le CR} V(x)^{q_0} dx$$
$$\le CR^{2q_0} \left\{ \frac{1}{R^n} \int_{|x-y|\le CR} V(x) dx \right\}^{q_0}$$
$$\le C$$

where we used the part (a) of Lemma 1.4, (0.2) and Lemma 1.8.

Hence, we have proved that

$$\int_{\mathbb{R}^n} |V(x)u_1(x)|^{q_0} dx \le C \int_{\mathbb{R}^n} |f(x)|^{q_0} dx \quad \text{for some} \quad q_0 > q \ge n/2.$$

Next, note that, by (1.6),

$$\int_{\mathbb{R}^n} |V(x)u_1(x)| dx \le C \int_{\mathbb{R}^n} |f(y)| \left\{ \int_{|x-y|<\frac{1}{m(x,V)}} \frac{V(x)}{|x-y|^{n-2}} dx \right\} dy$$
$$\le C \int_{\mathbb{R}^n} |f(y)| dy.$$

Therefore, by interpolation,

$$\|Vu_1\|_p \le C\|f\|_p \quad \text{for} \quad 1 \le p \le q_0.$$

To finish the proof, we note that, by Theorem 2.7 and the Hölder inequality,

$$|u_2(x)| \le C \int_{|y-x|\ge r} \frac{|f(y)| dy}{\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}}$$
$$\le Cr^{2/p'} \left\{ \int_{|y-x|\ge r} \frac{|f(y)|^p dy}{\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}} \right\}^{1/p}$$

where $1 \le p \le q_0$ and $r = \dfrac{1}{m(x,V)}$.

Thus,

$$\int_{\mathbb{R}^n} |V(x)u_2(x)|^p dx \le C \int_{\mathbb{R}^n} |f(y)|^p \left\{ \int_{|y-x| \ge \frac{1}{m(x,V)}} \frac{|V(x)|^p dx}{m(x,V)^{2(p-1)}\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}} \right\} dy.$$

Now, fix $y \in \mathbb{R}^n$, let $R = \dfrac{1}{m(y,V)}$. By Lemma 1.4,

$$\begin{aligned}
&\int_{|y-x| \ge \frac{1}{m(x,V)}} \frac{|V(x)|^p dx}{m(x,V)^{2(p-1)}\{1+|x-y|m(x,V)\}^k |x-y|^{n-2}} \\
&\le C \int_{|y-x| \ge CR} \frac{|V(x)|^p dx}{R^{2(1-p)}(1+R^{-1}|x-y|)^{k_1}|x-y|^{n-2}}, \\
&\qquad \left(\text{where } k_1 = \frac{k-2(q-1)k_0}{k_0+1}\right) \\
&\le C \sum_{j=1}^{\infty} \frac{1}{(2^j R)^n} \int_{|x-y| \le 2^j R} V^p dx \cdot (2^j)^{-k_1+2} \cdot R^{2p} \\
&\le C \sum_{j=1}^{\infty} \left(\frac{1}{(2^j R)^n} \int_{|x-y| \le 2^j R} V(x) dx \right)^p \cdot (2^j)^{-k_1+2} \cdot R^{2p} \\
&\le C \sum_{j=1}^{\infty} \left(\frac{1}{R^{n-2}} \int_{|x-y| \le R} V(x) dx \right)^p \cdot C^j \cdot (2^j)^{-k_1+2} \\
&\le C \quad \text{if we choose } k \text{ sufficiently large.}
\end{aligned}$$

From this we have

$$\int_{\mathbb{R}^n} |V(x)u_2(x)|^p dx \le C \int_{\mathbb{R}^n} |f(x)|^p dx \quad \text{for} \quad 1 \le p \le q_0.$$

The theorem is then proved.

We are now in a position to give

The proof of Theorem 0.3. — Suppose $V \in B_q$ for some $q \ge n/2$. By Theorem 3.1

$$\|V(-\Delta+V)^{-1}f\|_p \le C\|f\|_p \quad \text{for} \quad 1 \le p \le q.$$

It follows that

$$\|\Delta(-\Delta+V)^{-1}f\|_p \le C\|f\|_p \quad \text{for} \quad 1 \le p \le q.$$

Then, by the L^p boundedness of Riesz transforms, for $1 < p \leq q$,

$$\|\nabla^2(-\Delta + V)^{-1} f\|_p \leq C_p \|\Delta(-\Delta + V)^{-1} f\|_p \leq C_p \|f\|_p.$$

4. The proof of Theorem 0.4.

In this section we give the proof of Theorem 0.4 stated in the Introduction. We also prove an L^p estimate for the operator $V^{1/2}\nabla(-\Delta+V)^{-1}$ in the section (Theorem 4.13).

By the functional calculus, we may write, for $\gamma \in \mathbb{R}$,

$$(-\Delta + V)^{i\gamma} = -\frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{i\gamma} (-\Delta + V + i\tau)^{-1} d\tau. \tag{4.1}$$

Thus,

$$\begin{aligned}(-\Delta + V)^{i\gamma} f(x) &= -\frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{i\gamma} (-\Delta + V + i\tau)^{-1} f(x) d\tau \\ &= \int_{\mathbb{R}^n} K(x,y) f(y) dy\end{aligned}$$

where

$$K(x,y) = -\frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{i\gamma} \Gamma(x,y,\tau) d\tau. \tag{4.2}$$

Note that, by Theorem 2.7,

$$|K(x,y)| \leq \frac{C_k e^{\pi|\gamma|/2}}{\{1 + |x-y| m(x,V)\}^k} \cdot \frac{1}{|x-y|^n}, \quad \text{for any } k > 0. \tag{4.3}$$

Let $\Gamma_0(x,y,\tau)$ denote the fundamental solution for the operator $-\Delta + i\tau$ in $\mathbb{R}^n$. It is well known that

$$\begin{cases} |\Gamma_0(x,y,\tau)| \leq \dfrac{C_k}{(1 + |\tau|^{1/2}|x-y|)^k} \cdot \dfrac{1}{|x-y|^{n-2}} \\ |\nabla_x \Gamma_0(x,y,\tau)| \leq \dfrac{C_k}{(1 + |\tau|^{1/2}|x-y|)^k} \cdot \dfrac{1}{|x-y|^{n-1}} \\ |\nabla_x^2 \Gamma_0(x,y,\tau)| \leq \dfrac{C_k}{(1 + |\tau|^{1/2}|x-y|)^k} \cdot \dfrac{1}{|x-y|^{n}} \end{cases} \tag{4.4}$$

where C_k is independent of $\tau \in \mathbb{R}$.

LEMMA 4.5. — *Suppose $V \in B_{n/2}$. Then there exists $C_k > 0$ such that*

$$|\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)| \leq \frac{C_k}{(1 + |\tau|^{1/2}|x-y|)^k} \cdot \frac{\{|x-y| m(x,V)\}^{2-(n/q_0)}}{|x-y|^{n-2}}$$

for $x, y \in \mathbb{R}^n$, $|x-y| \le \dfrac{1}{m(x,V)}$ *and for some* $q_0 > n/2$.

Proof. — Note that

$$-\Delta_x\{\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)\}+i\tau\{\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)\} = -V(x)\Gamma(x,y,\tau).$$

We have

$$\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau) = -\int_{\mathbb{R}^n}\Gamma_0(x,z,\tau)V(z)\Gamma(z,y,\tau)dz.$$

Now, let $R=|x-y|$ and suppose $R \le \dfrac{1}{m(x,V)}$. By Theorem 2.7,

$$\begin{aligned}
&|\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)| \\
&\le C_k\int_{\mathbb{R}^n}\frac{(1+|\tau|^{1/2}|x-z|)^{-k}(1+|\tau|^{1/2}|z-y|)^{-k}V(z)\,dz}{|x-z|^{n-2}\{1+m(y,V)|z-y|\}^k|z-y|^{n-2}} \\
&= C_k\int_{|z-x|<R/2} + C_k\int_{|z-y|<R/2} + C_k\int_{\substack{|z-x|\ge R/2\\ |z-y|\ge R/2}} \\
&= I_1+I_2+I_3.
\end{aligned}$$

Since $V \in B_{n/2}$, $V \in B_{q_0}$ for some $q_0 > n/2$. We obtain

$$\begin{aligned}
I_1 &\le \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-2}}\cdot\int_{B(x,R/2)}\frac{V(z)dz}{|z-x|^{n-2}} \\
&\le \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-2}}\cdot\frac{1}{R^{n-2}}\int_{B(x,R/2)}V(z)dz \\
&\le \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{\{Rm(x,V)\}^{2-(n/q_0)}}{R^{n-2}}
\end{aligned}$$

where we have used (1.6) in the second inequality and Lemma 1.2 in the third.

Similarly,

$$I_2 \le \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{\{Rm(x,V)\}^{2-(n/q_0)}}{R^{n-2}}.$$

Finally, we need to estimate I_3.

It is easy to see that

$$\begin{aligned}
I_3 &\le \frac{C_k}{(1+|\tau|^{1/2}R)^k}\int_{|z-y|\ge R/2}\frac{V(z)dz}{|z-y|^{2n-4}\{1+m(y,V)|z-y|\}^k} \\
&\le \frac{C_k}{(1+|\tau|^{1/2}R)^k}\left\{\int_{r>|z-y|\ge\frac{R}{2}}\frac{V(z)dz}{|z-y|^{2n-4}}+r^k\int_{|z-y|\ge r}\frac{V(z)dz}{|z-y|^{2n-4+k}}\right\}
\end{aligned}$$

where $r = \dfrac{1}{m(y,V)} \sim \dfrac{1}{m(x,V)}$.

Using Hölder inequality and the B_{q_0} condition (0.2), we obtain

$$\begin{aligned}
&\int_{r>|z-y|\geq R/2} \frac{V(z)dz}{|z-y|^{2n-4}} \\
&\qquad \leq C\left\{\int_{B(y,r)} V(z)^{q_0}dz\right\}^{1/q_0}\left\{\int_{R/2}^{r} t^{(n-1)-2(n-2)q_0'}dt\right\}^{1/q_0'} \\
&\qquad \leq C\left(\frac{R}{r}\right)^{2-(n/q_0)}\cdot\frac{1}{R^{n-2}}\cdot\left\{\int_{1/2}^{r/R} t^{(n-1)-2(n-2)q_0'}dt\right\}^{1/q_0'} \\
&\qquad \leq \frac{C\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}
\end{aligned}$$

where we have assumed, without loss of generality, that $(1/q_0) > (4-n)/n$ for $n \geq 3$, so $n - 2(n-2)q_0' < 0$.

Also, using the doubling condition and taking k sufficiently large, we see that

$$\begin{aligned}
r^k\int_{|z-y|\geq r}\frac{V(z)dz}{|z-y|^{2n-4+k}} &\leq Cr^k\sum_{j=1}^{\infty}\frac{1}{(2^jr)^{2n-4+k}}\int_{|z-y|\leq 2^jr} V(z)dz \\
&\leq Cr^{4-2n}\sum_{j=1}^{\infty}\frac{1}{(2^j)^{2n-4+k}}\cdot C_0^j\int_{|z-y|\leq r} V(z)dz \\
&\leq \frac{C}{r^{n-2}} \\
&\leq \frac{C\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}.
\end{aligned}$$

Thus,

$$I_3 \leq \frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{\{Rm(y,V)\}^{2-(n/q_0)}}{R^{n-2}}.$$

The proof is then finished since $m(y,V) \sim m(x,V)$ when $|x-y| \leq \dfrac{1}{m(x,V)}$.

We need another lemma before we carry out the proof of Theorem 0.4.

LEMMA 4.6. — *Suppose $V \in B_{q_0}$, $(n/2) < q_0 < n$. Assume that $-\Delta u + (V + i\tau)u = 0$ in $B(x_0, 2R)$. Then*

$$\left(\int_{B(x_0,R)} |\nabla u|^t dx\right)^{1/t} \leq CR^{(n/q_0)-2}\{1+Rm(x_0,V)\}^{k_0} \sup_{B(x_0,2R)} |u|$$

where $(1/t) = (1/q_0) - (1/n)$.

Proof. — Let $\eta \in C_0^\infty(B(x_0, 2R))$ such that $\eta = 1$ on $B(x_0, 3R/2)$, $|\nabla \eta| \le C/R$ and $|\nabla^2 \eta| \le C/R^2$.

Note that,

$$
\begin{aligned}
u(x)\eta(x) &= \int_{\mathbb{R}^n} \Gamma_0(x,y,\tau)\{-\Delta + i\tau\}(u\eta)(y)dy \\
&= \int_{\mathbb{R}^n} \Gamma_0(x,y,\tau)\{-Vu\eta - 2\nabla u \cdot \nabla \eta - u\Delta\eta\}dy \\
&= \int_{\mathbb{R}^n} \Gamma_0(x,y,\tau)\{-Vu\eta + u\Delta\eta\}dy \\
&\qquad + 2\int_{\mathbb{R}^n} \nabla_y \Gamma_0(x,y,\tau)\cdot(\nabla\eta)u dy.
\end{aligned}
\tag{4.7}
$$

Thus, for $x \in B(x_0, R)$,

$$
\begin{aligned}
|\nabla u(x)| &\le C \int_{B(x_0,2R)} \frac{V(y)|u(y)||\eta|}{|x-y|^{n-1}}dy + \frac{C}{R^{n+1}}\int_{B(x_0,2R)} |u(y)|dy \\
&\le C \sup_{B(x_0,2R)} |u| \cdot \int_{B(x_0,2R)} \frac{V(y)|\eta(y)|}{|x-y|^{n-1}}dy \\
&\qquad + \frac{C}{R^{n+1}}\int_{B(x_0,2R)} |u(y)|dy.
\end{aligned}
\tag{4.8}
$$

It then follows from the well known theorem on fractional integrals that

$$
\begin{aligned}
&\left(\int_{B(x_0,R)} |\nabla u(x)|^t dx\right)^{1/t} \\
&\le C \sup_{B(x_0,2R)} |u| \left(\int_{B(x_0,2R)} |V(x)|^{q_0} dx\right)^{1/q_0} + CR^{(n/q_0)-2} \sup_{B(x_0,2R)} |u| \\
&\le CR^{(n/q_0)-2} \sup_{B(x_0,2R)} |u| \left\{\frac{1}{R^{n-2}}\int_{B(x_0,2R)} V(x)dx + 1\right\}
\end{aligned}
$$

where $(1/t) = (1/q_0) - (1/n)$ and we have used (0.2) in the last inequality.

The desired estimate then follows from Lemma 1.8.

Remark 4.9. — If $V \in B_n$ and $-\Delta u + (V + i\tau)u = 0$ in $B(x_0, 2R)$, then

$$
\sup_{B(x_0,R)} |\nabla u| \le \frac{C}{R}\{1 + Rm(x_0, V)\}^{k_0} \cdot \sup_{B(x_0,2R)} |u|.
$$

Indeed, by (4.8) and (1.7), we have for $x \in B(x_0, R)$

$$\begin{aligned}
|\nabla u(x)| &\le C \sup_{B(x_0,2R)} |u| \cdot \frac{1}{R^{n-1}} \int_{B(x_0,2R)} V(y)dy + \frac{C}{R^{n+1}} \int_{B(x_0,2R)} |u(y)|dy \\
&\le \frac{C}{R} \sup_{B(x_0,2R)} |u| \left\{ \frac{1}{R^{n-2}} \int_{B(x_0,2R)} V(y)dy + 1 \right\} \\
&\le \frac{C}{R} \{1 + Rm(x_0, R)\}^{k_0} \cdot \sup_{B(x_0,2R)} |u|
\end{aligned}$$

where we used Lemma 1.8 in the last inequality.

Remark 4.10. — If $V \in B_{q_0}$ for some $q_0 > n$ and $-\Delta u + (V + i\tau)u = 0$ in $B(x_0, 2R)$, then, by (4.7) and the Calderón–Zygmund estimates,

$$\begin{aligned}
\|\nabla^2(u\eta)\|_{q_0} &\le C\|Vu\eta\|_{q_0} + CR^{(n/q_0)-2} \sup_{B(x_0,2R)} |u| \\
&\le C\left\{ \left(\int_{B(x_0,2R)} V^{q_0} dy \right)^{1/q_0} + R^{(n/q_0)-2} \right\} \sup_{B(x_0,2R)} |u| \\
&\le CR^{(n/q_0)-2} \left\{ \frac{1}{R^{n-2}} \int_{B(x_0,2R)} V(x)dx + 1 \right\} \sup_{B(x_0,2R)} |u|.
\end{aligned}$$

Hence, by Lemma 1.8,

$$\left(\int_{B(x_0,R)} |\nabla^2 u|^{q_0} dx \right)^{1/q_0} \le CR^{(n/q_0)-2}\{1 + Rm(x_0, V)\}^{k_0} \cdot \sup_{B(x_0,2R)} |u|.$$

We are now ready to give

The proof of Theorem 0.4. — Suppose $V \in B_{n/2}$. Then $V \in B_{q_0}$ for some $q_0 > n/2$. Note that

$$(-\Delta)^{i\gamma} f(x) = \int_{\mathbb{R}^n} K^0(x, y) f(y) dy$$

where

$$K^0(x, y) = -\frac{1}{2\pi} \int_{\mathbb{R}^n} (-i\tau)^{i\gamma} \Gamma_0(x, y, \tau) d\tau.$$

It follows easily from Lemma 4.5 that

$$|K(x,y) - K^0(x,y)| \le Ce^{|\gamma|\pi/2} \cdot \frac{\{|x-y|m(x,V)\}^{2-(n/q_0)}}{|x-y|^n} \tag{4.11}$$

for $x, y \in \mathbb{R}^n$ and $|x - y|m(x, V) \le 1$.

We now claim that

$$(4.12)\quad |(-\Delta+V)^{i\gamma}f(x)| \le |(-\Delta)^{i\gamma}f(x)| + \sup_{\varepsilon>0}\left|\int_{|y-x|>\varepsilon} K^0(x,y)f(y)dy\right| + Ce^{|\gamma|\pi/2}Mf(x)$$

where Mf is the Hardy–Littlewood maximum function of f.

Since $(-\Delta)^{i\gamma}$ is a Calderón–Zygmund operator, the boundedness of $(-\Delta+V)^{i\gamma}$ on $L^p(\mathbb{R}^n)$ for $1<p<\infty$ follows easily from (4.12). See [St2].

To see (4.12), we note that

$$\begin{aligned}(-\Delta+V)^{i\gamma}f(x) = (-\Delta)^{i\gamma}f(x) &+ \int_{|y-x|\le\frac{1}{m(x,V)}} \{K(x,y)-K^0(x,y)\}f(y)dy\\ &+ \int_{|y-x|>\frac{1}{m(x,V)}} K(x,y)f(y)dy\\ &+ \int_{|y-x|>\frac{1}{m(x,V)}} K^0(x,y)f(y)dy.\end{aligned}$$

This can be justified by using

$$(-\Delta+V)^{i\gamma}f(x) \equiv \lim_{\varepsilon\to 0^-}(-\Delta+V)^{i\gamma+\varepsilon}f(x)$$

and Lemma 4.5. By (4.11) and (4.3) it is not very hard to see that the second and the third term above are bounded in absolute value by $Ce^{\gamma\pi/2}Mf(x)$, while the last term is bounded by

$$\sup_{\varepsilon>0}\left|\int_{|y-x|>\varepsilon} K^0(x,y)f(y)dy\right|$$

(4.12) is proved.

To finish the proof we need to show that the kernel $K(x,y)$ satisfies (0.11) for some $\delta>0$.

First, by (4.3),

$$|K(x,y)| \le \frac{C}{|x-y|^n}.$$

Next, we fix $x_0, y_0 \in \mathbb{R}^n, h\in\mathbb{R}^n$ and $|h|<|x_0-y_0|/4$. Let $R=|x_0-y_0|/4$ and $u(x)=\Gamma(x,y_0,\tau)$. It follows from the imbedding theorem of Morrey (see [GT], p. 163) and Lemma 4.6 that, for $(1/t)=(1/q_0)-(1/n)$,

$$\begin{aligned}|u(x_0+h)-u(x_0)| &\le C|h|^{1-(n/t)}\left(\int_{B(x_0,R)}|\nabla u|^t dx\right)^{1/t}\\ &\le C\left(\frac{|h|}{R}\right)^{2-(n/q_0)} \sup_{B(x_0,2R)}|u|\cdot\{1+Rm(x_0,V)\}^{k_0}\\ &\le C\left(\frac{|h|}{R}\right)^{2-(n/q_0)}\cdot\frac{1}{(1+|\tau|^{1/2}|x_0-y_0|)^3}\cdot\frac{1}{|x_0-y_0|^{n-2}},\end{aligned}$$

where we used Theorem 2.7 in the last inequality.

Thus, we have proved that, for $x, y \in \mathbb{R}^n$, $h \in \mathbb{R}^n$ and $|h| < |x-y|/4$,

$$|\Gamma(x+h, y, \tau) - \Gamma(x, y, \tau)| \le \frac{C}{\{1+|\tau|^{1/2}|x-y|\}^3} \cdot \frac{|h|^\delta}{|x-y|^{n-2+\delta}}$$

where $\delta = 2 - (n/q_0) > 0$. Clearly, this estimate also holds for $|x-y|/4 \le |h| < |x-y|/2$. It then follows from (4.2) that

$$|K(x+h, y) - K(x, y)| \le \frac{C|h|^\delta}{|x-y|^{n+\delta}}$$

whenever $x, y, h \in \mathbb{R}^n$ and $|h| < |x-y|/2$.

The estimate of $K(x, y+h) - K(x, y)$ can be carried out in the same manner.

The proof is complete.

We close this section with an L^p estimate for the operator $V^{1/2}\nabla(-\Delta+V)^{-1}$.

THEOREM 4.13. — *Suppose $V \in B_{q_0}$ for some $q_0 \ge n/2$. Then, for $1 \le p \le p_0$,*

$$\|V^{1/2}\nabla(-\Delta+V)^{-1}f\|_p \le C\|f\|_p$$

where $(1/p_0) = (3/(2q_0)) - (1/n)$ if $n/2 \le q_0 < n$ and $p_0 = 2q_0$ if $q_0 \ge n$.

Proof. — Suppose $V \in B_{q_0}$ for some $q_0 \ge n/2$. Then $V \in B_{q_1}$ for some $q_1 > q_0$.

Let

$$Tf(x) = V(x)^{1/2} \int_{\mathbb{R}^n} \nabla_x \Gamma(x, y) f(y) dy.$$

The adjoint of T is given by

$$T^* f(x) = \int_{\mathbb{R}^n} \nabla_y \Gamma(y, x) V^{1/2}(y) f(y) dy.$$

By duality, it suffices to show that

$$\|T^* f\|_p \le C\|f\|_p \quad \text{for} \quad p_0' \le p \le \infty. \tag{4.14}$$

To this end, we let $r = \dfrac{1}{m(x, V)}$ and consider the case $n/2 \le q_0 < q_1 < n$. We choose t and p_1 such that $(1/t) = (1/q_1) - (1/n)$, $(1/p_1) = 1 - (3/2q_1) + (1/n)$. Thus,

$$\frac{1}{t} + \frac{1}{2q_1} + \frac{1}{p_1} = 1.$$

Hence, by Hölder inequality,

$$
\begin{aligned}
|T^* f(x)| &\le \sum_{j=-\infty}^{+\infty} \int_{2^{j-1}<|y-x|\le 2^j r} |\nabla_y \Gamma(y,x)| V^{1/2}(y)|f(y)|dy \\
&\le \sum_{j=-\infty}^{+\infty} \left(\int_{2^{j-1}r<|y-x|\le 2^j r} |\nabla_y \Gamma(y,x)|^t dy \right)^{1/t} \\
&\qquad \left(\int_{|y-x|\le 2^j r} V^{q_1} dy \right)^{1/2q_1} \left(\int_{|y-x|\le 2^j r} |f(y)|^{p_1} dy \right)^{1/p_1}.
\end{aligned}
$$

It follows from Lemma 4.6 and Theorem 2.7 that

$$
\left(\int_{2^{j-1}r<|y-x|\le 2^j r} |\nabla_y \Gamma(y,x)|^t dy \right)^{1/t} \le C_k \cdot \frac{(2^j r)^{(n/q_1)-n}}{(1+2^j)^k}.
$$

Thus,

$$
\begin{aligned}
|T^* f(x)| &\le C_k \sum_{j=-\infty}^{+\infty} \frac{(2^j r)}{(1+2^j)^k} \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V^{q_1} dy \right\}^{1/2q_1} \\
&\qquad \cdot \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} |f(y)|^{p_1} dy \right\}^{1/p_1} \\
&\le C_k \left\{ M(|f|^{p_1})(x) \right\}^{1/p_1} \sum_{j=-\infty}^{+\infty} \frac{(2^j r)}{(1+2^j)^k} \\
&\qquad \cdot \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{1/2}.
\end{aligned}
$$

Note that, by Lemma 1.2, if $j \le 0$,

$$
(2^j r) \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{1/2} \le C(2^j)^{1-(n/2q_1)}
$$

and, by doubling condition (1.1), if $j > 0$,

$$
(2^j r) \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{1/2} \le C(2^j)^{k_0} \quad \text{for some } k_0 > 0.
$$

Hence, choosing k sufficiently large, we obtain

$$
|T^* f(x)| \le C\{M(|f|^{p_1})(x)\}^{1/p_1}.
$$

It follows that

$$
\|T^* f\|_p \le C\|f\|_p \quad \text{for } p_1 < p \le \infty.
$$

(4.14) then follows since $p_0' > p_1$.

Finally, we consider the case $q_1 \geq n$. Clearly we may then assume $q_1 > n$ because of self improvement of the B_q class.

It is not difficult to see that Remark 4.9 implies

$$|\nabla_y \Gamma(x,y)| \leq \frac{C_k}{\{1+|x-y|m(x,V)\}^k} \cdot \frac{1}{|x-y|^{n-1}}, \quad \text{if } V \in B_{q_1}, q_1 > n.$$

Thus,

$$\begin{aligned} |T^* f(x)| &\leq C_k \sum_{j=-\infty}^{+\infty} \frac{1}{(1+2^j)^k} \cdot \frac{1}{(2^j r)^{n-1}} \cdot \int_{B(x,2^j r)} V^{1/2}(y)|f(y)|dy \\ &\leq C_k \sum_{j=-\infty}^{+\infty} \frac{2^j r}{(1+2^j)^k} \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V^{q_1} dy \right\}^{1/2q_1} \\ &\qquad \cdot \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} |f(y)|^{p_1} dy \right\}^{1/p_1} \end{aligned}$$

where $(1/p_1) = 1 - (1/2q_1)$.

Therefore, as in the first case, we have

$$\begin{aligned} &|T^* f(x)| \\ &\leq C_k \{M(|f|^{p_1})(x)\}^{1/p_1} \sum_{j=-\infty}^{+\infty} \frac{(2^j r)}{(1+2^j)^k} \left\{ \frac{1}{(2^j r)^n} \int_{B(x,2^j r)} V(y) dy \right\}^{1/2} \\ &\leq C \{M(|f|^{p_1})(x)\}^{1/p_1}. \end{aligned}$$

Note that $p_1 = (2q_1)' < (2q_0)' = p_0'$. (4.14) follows from the boundedness of the Hardy–Littlewood maximal function as before.

The proof is complete.

5. The proof of Theorem 0.5.

By functional calculus, we may write

$$(5.1) \qquad (-\Delta + V)^{-1/2} = -\frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{-1/2} (-\Delta + V + i\tau)^{-1} d\tau.$$

Thus,

$$(5.2) \qquad \nabla(-\Delta + V)^{-1/2} f(x) = \int_{\mathbb{R}^n} K_1(x,y) f(y) dy$$

where

$$K_1(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}}(-i\tau)^{-1/2}\nabla_x\Gamma(x,y,\tau)d\tau. \tag{5.3}$$

Let

$$Tf(x) = \int_{\mathbb{R}^n} K_1(y,x)f(y)dy. \tag{5.4}$$

To show Theorem 0.5, by duality, it is equivalent to prove that

$$\|Tf\|_p \le C_p\|f\|_p \quad \text{for} \quad p_0' \le p < \infty \tag{5.5}$$

where $p_0' = p_0/(p_0-1)$ and $(1/p_0) = (1/q) - (1/n)$. We write

$$\begin{aligned} Tf(x) = &\int_{|y-x|>r} K_1(y,x)f(y)dy \\ &+ \int_{|y-x|\le r}\{K_1(y,x) - K_1^0(y,x)\}f(y)dy \\ &+ \int_{|y-x|\le r} K_1^0(y,x)f(y)dy \end{aligned} \tag{5.6}$$

where $r = \dfrac{1}{m(x,V)}$ and $K_1^0(x,y)$ is the kernel for the operator $\nabla(-\Delta)^{-1/2}$.

LEMMA 5.7. — *Suppose $V \in B_{q_1}$ for some q_1, $(n/2) < q_1 < n$. Then*

$$\left|\int_{|y-x|>r} K_1(y,x)f(y)dy\right| \le C\left\{M(|f|^{p_1'})(x)\right\}^{1/p_1'}$$

where $r = \dfrac{1}{m(x,V)}$ and $(1/p_1') = 1 - (1/p_1) = 1 - (1/q_1) + (1/n)$.

Proof. — First, we fix $x_0, y_0 \in \mathbb{R}^n$. Let $u(y) = \Gamma(y,x_0,\tau)$ and $R = |x_0 - y_0|/4$. It follows from (4.8) that

$$|\nabla u(y_0)| \le C\int_{B(y_0,R)} \frac{V(y)|u(y)|}{|y-y_0|^{n-1}}dy + \frac{C}{R^{n+1}}\int_{B(y_0,2R)}|u(y)|dy.$$

Hence, by Theorem 2.7,

$$\begin{aligned} |\nabla_y\Gamma(y_0,x_0,\tau)| \le &\frac{C_k}{(1+|\tau|^{1/2}R)^k\{1+m(x_0,V)R\}^k} \\ &\cdot\left\{\frac{1}{R^{n-2}}\int_{B(y_0,R)}\frac{V(y)dy}{|y-y_0|^{n-1}} + \frac{1}{R^{n-1}}\right\}. \end{aligned}$$

Thus, by (5.3),

$$|K_1(y_0,x_0)| \le \frac{C_k}{\{1+m(x_0,V)R\}^k}\left\{\frac{1}{R^{n-1}}\int_{B(y_0,R)}\frac{V(y)}{|y-y_0|^{n-1}}dy + \frac{1}{R^n}\right\}.$$

Now, let $(1/p_1) = (1/q_1) - (1/n)$. For $j \geq 1$ integer, we use the fractional integral theorem to obtain

$$\begin{aligned}\left\{\int_{2^{j-1}r<|y-x_0|\leq 2^j r} |K_1(y,x_0)|^{p_1} dy\right\}^{1/p_1} & \\ \leq \frac{C_k}{(2^j)^k}\left\{\frac{1}{(2^j r)^{n-1}}\left(\int_{|y-x_0|\leq 2^{j+1}r} V^{q_1} dy\right)^{1/q_1} + (2^j r)^{(n/p_1)-n}\right\} & \\ \leq \frac{C_k}{(2^j)^k}\left\{(2^j r)^{(n/p_1)-n}\cdot C_0^j + (2^j r)^{(n/p_1)-n}\right\} & \\ \leq \frac{C}{2^j}\cdot (2^j r)^{-n/p_1'} \quad \text{if } k \text{ is sufficiently large,} &\end{aligned}$$

where $r = \dfrac{1}{m(x_0,V)}$ and we have used the reverse Hölder inequality (0.2) and the doubling condition (1.1).

Finally, by Hölder inequality,

$$\begin{aligned}\left|\int_{|y-x_0|>r} K_1(y,x_0)f(y)dy\right| &\leq \sum_{j=1}^{\infty}\int_{2^{j-1}r<|y-x_0|\leq 2^j r} |K_1(y,x_0)||f(y)|dy \\ &\leq \sum_{j=1}^{\infty}\left\{\int_{2^{j-1}r<|y-x_0|\leq 2^j r} |K_1(y,x_0)|^{p_1}dy\right\}^{1/p_1} \\ &\quad\cdot\left\{\int_{|y-x_0|\leq 2^j r} |f(y)|^{p_1'}dy\right\}^{1/p_1'} \\ &\leq C\left\{M(|f|^{p_1'})(x_0)\right\}^{1/p_1'}\sum_{j=1}^{\infty}\frac{1}{2^j} \\ &= C\left\{M(|f|^{p_1'})(x_0)\right\}^{1/p_1'}.\end{aligned}$$

LEMMA 5.8. — *Suppose $V \in B_{q_1}$ for some q_1, $n/2 < q_1 < n$. Then*

$$\left|\int_{|y-x|\leq r}\{K_1(y,x) - K_1^0(y,x)\}f(y)dy\right| \leq C\left\{M(|f|^{p_1'})(x)\right\}^{1/p_1'}$$

where $r = \dfrac{1}{m(x,V)}$ and $(1/p_1') = 1 - (1/p_1) = 1 - (1/q_1) + (1/n)$.

Proof. — First, we fix $x_0, y_0 \in \mathbb{R}^n$ such that $|x_0 - y_0| < \dfrac{1}{m(x_0,V)}$. Let $r = \dfrac{1}{m(x_0,V)}$ and $R = |x_0 - y_0|/4$. We claim that

$$(5.9)\qquad \begin{aligned}&|K_1(y_0,x_0)-K_1^0(y_0,x_0)|\\&\quad\le \frac{C}{R^{n-1}}\left\{\int_{B(y_0,R)}\frac{V(z)dz}{|z-y_0|^{n-1}}+R^{-1}\left(\frac{R}{r}\right)^{2-(n/q_1)}\right\}.\end{aligned}$$

Assume (5.9) for a moment, we give the proof of the lemma.

Let $j\le 0$ be an integer. It follows from the theorem on fractional integrals and (5.9) that, for $(1/p_1)=(1/q_1)-(1/n)$,

$$\begin{aligned}&\left\{\int_{2^{j-1}r<|y-x_0|\le 2^j r}|K_1(y,x_0)-K_1^0(y,x_0)|^{p_1}dy\right\}^{1/p_1}\\&\quad\le \frac{C}{(2^jr)^{n-1}}\left\{\left(\int_{B(x_0,2r)}V^{q_1}dz\right)^{1/q_1}+(2^j)^{2-(n/q_1)}(2^jr)^{(n/p_1)-1}\right\}\\&\quad\le \frac{C}{(2^jr)^{n-1}}\cdot r^{(n/q_1)-2}\\&\quad= C(2^j)^{2-(n/q_1)}(2^jr)^{-n/p_1'}\end{aligned}$$

where we have used (0.2) in the last inequality.

Now, by Hölder inequality,

$$\begin{aligned}&\int_{|y-x_0|\le r}|K_1(y,x_0)-K_1^0(y,x_0)||f(y)|dy\\&\quad\le\sum_{j=-\infty}^{0}\left(\int_{2^{j-1}r<|y-x_0|\le 2^jr}|K_1(y,x_0)-K_1^0(y,x_0)|^{p_1}dy\right)^{1/p_1}\\&\qquad\qquad\cdot\left(\int_{|y-x_0|\le 2^jr}|f(y)|^{p_1'}dy\right)^{1/p_1'}\\&\quad\le C\left\{M(|f|^{p_1'})(x_0)\right\}^{1/p_1'}\sum_{j=-\infty}^{0}(2^j)^{2-(n/q_1)}\\&\quad= C\left\{M(|f|^{p_1'})(x_0)\right\}^{1/p_1'}\end{aligned}$$

since $2-(n/q_1)>0$.

It remains to prove (5.9).

To this end, recall that

$$\Gamma(y,x_0,\tau)-\Gamma_0(y,x_0,\tau)=-\int_{\mathbb{R}^n}\Gamma_0(y,z,\tau)V(z)\Gamma(z,x_0,\tau)dz$$

(see the proof of Lemma 4.5). Thus,

$$\begin{aligned}
&|\nabla_y\Gamma(y_0,x_0,\tau)-\nabla_y\Gamma(y_0,x_0,\tau)|\\
&\quad\le\int_{\mathbb{R}^n}|\nabla_y\Gamma_0(y_0,z,\tau)|V(z)|\Gamma(z,x_0,\tau)|dz\\
&\quad\le C_k\int_{\mathbb{R}^n}\frac{(1+|\tau|^{1/2}|y_0-z|)^{-k}(1+|\tau|^{1/2}|z-x_0|)^{-k}V(z)\,dz}{|y_0-z|^{n-1}\{1+m(x_0,V)|z-x_0|\}^k|z-x_0|^{n-2}}\\
&\quad=C_k\int_{|z-y_0|<R}+C_k\int_{|z-x_0|<R}+C_k\int_{\substack{|z-x_0|\ge R\\|z-y_0|\ge R}}\\
&\quad=J_1+J_2+J_3
\end{aligned}$$

where $R=|x_0-y_0|/4$.

Clearly,

$$J_1\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-2}}\int_{B(y_0,R)}\frac{V(z)}{|z-y_0|^{n-1}}dz$$

and

$$\begin{aligned}
J_2&\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-1}}\int_{B(x_0,R)}\frac{V(z)}{|z-x_0|^{n-2}}dz\\
&\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-1}}\cdot\left(\frac{R}{r}\right)^{2-(n/q_1)}
\end{aligned}$$

where $r=\dfrac{1}{m(x_0,V)}$ and we have used (1.6) and Lemma 1.2 in the estimates of J_2.

Finally, note that

$$\begin{aligned}
J_3&\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R}\int_{|z-y_0|\ge R}\frac{V(z)dz}{|z-y_0|^{2n-4}\{1+m(x_0,V)|z-x_0|\}^k}\\
&\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-1}}\cdot\left(\frac{R}{r}\right)^{2-(n/q_1)}.
\end{aligned}$$

(See the estimate of I_3 in the proof of Lemma 4.5.) Thus, we have proved that

$$\begin{aligned}
&|\nabla_y\Gamma(y_0,x_0,\tau)-\nabla_y\Gamma_0(y_0,x_0,\tau)|\\
&\quad\le\frac{C_k}{(1+|\tau|^{1/2}R)^k}\cdot\frac{1}{R^{n-2}}\left\{\int_{B(y_0,R)}\frac{V(z)}{|z-y_0|^{n-1}}dz+\frac{1}{R}\cdot\left(\frac{R}{r}\right)^{2-(n/q_1)}\right\}.
\end{aligned}$$

(5.9) now follows easily from (5.3) and above estimate by integration.

The proof is complete.

With Lemmas 5.7 and 5.8, the proof of Theorem 0.5 is easy.

The proof of Theorem 0.5. — Suppose $V \in B_q$ for some q, $n/2 \leq q < n$. Then $V \in B_{q_1}$ where $q < q_1 < n$.

It follows from (5.6), Lemmas 5.7 and 5.8 that

$$|Tf(x)| \leq C\left\{M(|f|^{p_1'})(x)\right\}^{1/p_1'} + 2\sup_{\varepsilon>0}\left|\int_{|y-x|>\varepsilon} K_1^0(y,x)f(y)dy\right|$$

where $(1/p_1') = 1 - (1/p_1) = 1 - (1/q_1) + (1/n)$. Hence, by the standard Calderón–Zygmund theory,

$$\|Tf\|_p \leq C_p\|f\|_p \quad \text{for} \quad p_1' < p < \infty.$$

(5.5) now follows since $p_1' < p_0'$. Therefore,

$$\|\nabla(-\Delta + V)^{-1/2}f\|_p \leq C_p\|f\|_p \quad \text{for} \quad 1 < p \leq p_0.$$

THEOREM 5.10. — *Suppose $V \in B_q$ for some $q \geq n/2$. Then*

$$\|V^{1/2}(-\Delta + V)^{-1/2}f\|_p \leq C\|f\|_p \quad \textit{for} \quad 1 \leq p \leq 2q.$$

Proof. — Suppose $V \in B_q$ for some $q \geq n/2$. Then $V \in B_{q_0}$ for some $q_0 > n/2$.

Note that, by (5.1) and Theorem 2.7,

$$|V^{1/2}(-\Delta + V)^{-1/2}f(x)| \leq C_k \int_{\mathbb{R}^n} \frac{V(x)^{1/2}|f(y)|dy}{\{1 + m(y,V)|x-y|\}^k|x-y|^{n-1}}.$$

Let

$$Sf(x) = \int_{\mathbb{R}^n} \frac{V(x)^{1/2}f(y)dy}{\{1 + m(y,V)|x-y|\}^k|x-y|^{n-1}}.$$

It follows from the proof of Theorem 4.13 that

$$|S^*f(x)| \leq C\{M(|f|^{(2q_0)'})(x)\}^{1/(2q_0)'}$$

where S^* denotes the adjoint of S. Hence,

$$\|S^*f\|_p \leq C\|f\|_p \text{ for } (2q_0)' < p \leq \infty.$$

So $\|Sf\|_p \leq C\|f\|_p$ for $1 \leq p < 2q_0$ by duality. The theorem then follows easily.

THEOREM 5.11. — *Suppose $V \in B_q$ for some q and $n/2 \leq q < n$. Then, for $p_0' \leq p \leq 2q$,*

$$\|V^{1/2}(-\Delta + V)^{-1}\nabla f\|_p \leq C\|f\|_p \tag{5.12}$$

where $p_0' = p_0/(p_0-1)$ and $(1/p_0) = (1/q) - (1/n)$. Moreover, if $V \in B_q$ for some $q \geq n$, then (5.12) holds for $1 \leq p \leq 2q$.

Proof. — Note that

$$V^{1/2}(-\Delta + V)^{-1}\nabla = V^{1/2}(-\Delta + V)^{-1/2} \cdot (-\Delta + V)^{-1/2}\nabla.$$

The theorem then follows easily from Theorem 5.10 and Theorem 0.5 except for the case $p = 1$ and $q \geq n$. But, if $q \geq n$, we have

$$\begin{aligned}|V^{1/2}(-\Delta + V)^{-1}\nabla f(x)| &\leq \int_{\mathbb{R}^n} V(x)^{1/2}|\nabla_y\Gamma(x,y)|\,|f(y)|dy \\ &\leq C_k \int_{\mathbb{R}^n} \frac{V(x)^{1/2}|f(y)|dy}{\{1 + m(y,V)|x-y|\}^k|x-y|^{n-1}}.\end{aligned}$$

From this it is not hard to see that $\|V^{1/2}(-\Delta + V)^{-1}\nabla f\|_1 \leq C\|f\|_1$.

6. The proof of Theorem 0.8.

This section is devoted to the proof of Theorem 0.8. We will assume that $V \in B_n$ throughout the section.

From Remark 4.9 and Theorem 2.7 it is easy to see that

(6.1) $$\begin{aligned}&|\nabla_x\Gamma(x,y,\tau)| + |\nabla_y\Gamma(x,y,\tau)| \\ &\quad\leq \frac{C_k}{\{1 + |\tau|^{1/2}|x-y|\}^k\{1 + m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^{n-1}}.\end{aligned}$$

Also, since $\nabla_y\Gamma(x,y,\tau)$ is a solution to the equation $-\Delta u + (V + i\tau)u = 0$ in $\mathbb{R}^n \setminus \{y\}$ as a function of x, by the same token,

(6.2)
$$|\nabla_x\nabla_y\Gamma(x,y,\tau)| \leq \frac{C_k}{\{1 + |\tau|^{1/2}|x-y|\}^k\{1 + m(x,V)|x-y|\}^k} \cdot \frac{1}{|x-y|^n}.$$

Recall that

$$\nabla(-\Delta + V)^{-1/2}f(x) = \int_{\mathbb{R}^n} K_1(x,y)f(y)dy$$

where $K_1(x,y)$ is given by (5.3).

We may also write

(6.3) $$\nabla(-\Delta + V)^{-1}\nabla f(x) = \int_{\mathbb{R}^n} K_2(x,y)f(y)$$

where

(6.4) $$K_2(x,y) = -\nabla_x\nabla_y\Gamma(x,y),\ \Gamma(x,y) = \Gamma(x,y,0).$$

Clearly, by (6.1) and (6.2)

$$|K_\ell(x,y)| \le \frac{C_k}{\{1+|x-y|m(x,V)\}^k} \cdot \frac{1}{|x-y|^n} \quad \text{for } \ell = 1,2. \tag{6.5}$$

We now are in a positive to give

The proof of Theorem 0.8. — Suppose $V \in B_n$. Then $V \in B_{q_0}$ for some $q_0 > n$.

The boundedness of $\nabla(-\Delta+V)^{-1/2}$, $(-\Delta+V)^{-1/2}\nabla$ and $\nabla(-\Delta+V)^{-1}\nabla$ on $L^2(\mathbb{R}^n)$ is well known for any nonnegative potential V. We only need to show that the kernel $K_\ell(x,y)$ satisfies the Calderón–Zygmund condition (0.11). The result for $(-\Delta+V)^{-1/2}\nabla$ follows by duality.

We will give the details for the estimate

$$|K_2(x+h,y) - K_2(x,y)| \le \frac{C|h|^\delta}{|x-y|^{n+\delta}}, \delta > 0 \tag{6.6}$$

where $x, y \in \mathbb{R}^n$, $|h| < |x-y|/2$. The other estimates follow in a similar manner or follow from (6.5).

To see (6.6), we fix $x_0, y_0 \in \mathbb{R}^n$ and $h \in \mathbb{R}^n, |h| < |x_0 - y_0|/4$. Let $u(x) = \nabla_y \Gamma(x, y_0)$ and $R = |x_0 - y_0|/4$. It then follows from the imbedding theorem of Morrey and Remark 4.10 that

$$\begin{aligned}
|K_2(x_0+h,y_0) - &K_2(x_0,y_0)| \\
&= |\nabla u(x_0+h) - \nabla u(x_0)| \\
&\le C|h|^{1-(n/q_0)} \left\{ \int_{B(x_0,R)} |\nabla^2 u|^{q_0} dx \right\}^{1/q_0} \\
&\le C\left(\frac{|h|}{R}\right)^{1-(n/q_0)} \cdot \frac{1}{R} \cdot \{1 + Rm(x_0,V)\}^{k_0} \sup_{B(x_0,2R)} |u| \\
&\le C\left(\frac{|h|}{R}\right)^{1-(n/q_0)} \cdot \frac{1}{R^n} \\
&= \frac{C|h|^\delta}{|x_0-y_0|^{n+\delta}}
\end{aligned}$$

where $\delta = 1 - (n/q_0) > 0$ and we have used (6.1) in the last inequality.

(6.6) is then proved for $|h| < |x_0 - y_0|/4$, hence for $|h| < |x_0 - y_0|/2$ by (6.5).

The proof is finished.

7. A counterexample.

We end this paper with a counterexample which shows that the ranges of p in Theorems 0.3 and 0.5 are optimal.

Consider the case

$$V(x) = \frac{1}{|x|^{2-\varepsilon}}$$

where $0 < \varepsilon < 2$. Let

$$v(x) = \sum_{m=0}^{\infty} \frac{(\frac{1}{\varepsilon})^{2m}|x|^{\varepsilon m}}{m!\Gamma(\frac{n-2}{\varepsilon} + m + 1)}.$$

A direct computation shows that v satisfies the equation

$$-\Delta v + \frac{1}{|x|^{2-\epsilon}} v = 0 \text{ in } \mathbb{R}^n.$$

Next, let $u = \phi v$ where $\phi \in C_0^\infty(\mathbb{R}^n)$ and $\phi = 1$ for $|x| \le 1$. Then

$$-\Delta u + Vu = g \text{ in } \mathbb{R}^n,$$

where $g = -2\nabla v \cdot \nabla\phi - v\Delta\phi \in C_0^\infty(\mathbb{R}^n)$.

Now, given any $q_0 > n/2$. Let $\varepsilon = 2 - (n/q_0)$. Then

$$V(x) = \frac{1}{|x|^{2-\epsilon}} = \frac{1}{|x|^{n/q_0}} \in B_p \text{ for any } p < q_0.$$

We claim that the estimate in Theorem 0.3 does not hold for $p = q_0$. For otherwise it would imply that

$$\left\| \frac{1}{|x|^{n/q_0}} u \right\|_{q_0} < \infty.$$

This contradicts the fact that $u(x) \sim 1$ as $x \to 0$.

Similarly, if $n/2 < q_0 < n$ and the estimate in Theorem 0.5 held for $p = p_0$ where $(1/p_0) = (1/q_0) - (1/n)$, it would follow that

$$\|\nabla u\|_{p_0} = \|\nabla(-\Delta + V)^{-1} g\|_{p_0} \le C\|(-\Delta + V)^{-1/2} g\|_{p_0} < \infty,$$

where we have used the fact that

$$|(-\Delta + V)^{-1/2} g(x)| \le C \int_{\mathbb{R}^n} \frac{|g(y)|}{|x-y|^{n-1}} dy \le \frac{C}{(1+|x|)^{n-1}}.$$

It is easy to see that this is not the case since $\nabla u(x) \sim |x|^{\varepsilon-1} = |x|^{1-(n/q_0)} = |x|^{-n/p_0}$ as $x \to 0$.

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