Math. Z. (2015) 280:621–642
DOI 10.1007/s00209-015-1440-9

Mathematische Zeitschrift

L^p Fourier multipliers on compact Lie groups

Michael Ruzhansky · Jens Wirth

Received: 24 August 2013 / Accepted: 12 January 2015 / Published online: 8 March 2015

Abstract In this paper we prove L^p Fourier multiplier theorems for invariant and also non-invariant operators on compact Lie groups in the spirit of the well-known Hörmander–Mikhlin theorem on $\mathbb{R}^n$ and its variants on tori $\mathbb{T}^n$. We also give applications to a-priori estimates for non-hypoelliptic operators. Already in the case of tori we get an interesting refinement of the classical multiplier theorem.

Keywords Multipliers · Compact Lie groups · Pseudo-differential operators

Mathematics Subject Classification Primary 43A22 · 43A77; Secondary 43A15 · 22E30

1 Introduction

In this paper we prove L^p multiplier theorems for invariant and then also for non-invariant operators on compact Lie groups. We are primarily interested in Fourier multipliers rather than in spectral multipliers.

The topic has been attracting intensive research for a long time. There is extensive literature providing criteria for central multipliers, see e.g. Weiss [23], Coifman and Weiss [5], Stein [22], Cowling [8], Alexopoulos [2], to mention only very few. There are also results for functions of the sub-Laplacian, for example on SU(2), see Cowling and Sikora [9].

The first author was supported by the EPSRC Leadership Fellowship EP/G007233/1. The second author was supported by the EPSRC Grant EP/E062873/1 and by DAAD for visits to Imperial College London in December 2010 and February 2011.

M. Ruzhansky (✉)
Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK
e-mail: m.ruzhansky@imperial.ac.uk

J. Wirth
Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart,
Pfaffenwaldring 57, 70569 Stuttgart, Germany
e-mail: jens.wirth@iadm.uni-stuttgart.de

The topic of the L^p-bounded multipliers has been extensively researched on symmetric spaces of noncompact type for multipliers corresponding to convolutions with distributions which are bi-invariant with respect to the subgroup, see e.g. Anker [3] and references therein. However, general results on compact Lie groups are surprisingly elusive. For the case of the group $\mathrm{SU}(2)$ a characterisation for operators leading to Calderon–Zygmund kernels in terms of certain symbols was given by Coifman and Weiss in [5] based on a criterion for Calderon–Zygmund operators from [4] (see also [6]). The proofs and formulations, however, rely on explicit formulae for representations and for the Clebsch–Gordan coefficients available on $\mathrm{SU}(2)$ and are not extendable to other groups. In general, in the case when we do not deal with functions of a fixed operator, it is even unclear in which terms to formulate criteria for the L^p-boundedness.

In this paper we prove a general result for arbitrary compact Lie groups G. This becomes possible based on the tools initiated and developed by the first author and Turunen in [17] and [15], in particular the development of the matrix valued symbols and the corresponding quantization relating operators and their symbolic calculus with the representation theory of the group. In view of the results in [17,18], pseudo-differential operators in Hörmander classes $\Psi^m(G)$ can be characterised in terms of decay conditions imposed on the matrix valued symbols using natural difference operators acting on the unitary dual $\widehat{G}$. From this point of view Theorem 2.1 provides a Mikhlin type multiplier theorem which reduces the assumptions on the symbol ensuring the L^p-boundedness of the operator. In Theorem 3.5 we give a refinement of this describing precisely the difference operators that can be used for making assumptions on the symbol. For example, if G is semi-simple, only those associated to the root system suffice, which appears natural in the context.

We give several applications of the obtained result. Thus, in Corollary 5.1 we give a criterion for the L^p-boundedness for a class of operators with symbols in the class $\mathscr{S}^0_\rho(G)$ of type $\rho \in [0, 1]$. Such operators appear e.g. with $\rho = \frac{1}{2}$ as parametrices for the sub-Laplacian or for the “heat” operator, see Example 2.6, or with $\rho = 0$ for inverses of operators $X + c$, with $X \in \mathfrak{g}$ and $c \in \mathbb{C}$, see Corollary 2.7 on general G and Example 2.8 on $\mathrm{SU}(2)$ and $\mathbb{S}^3$. We note that although operators $X + c$ are not locally hypoelliptic, we still get a-priori L^p-estimates for them as a consequence of our result.

We illustrate Theorem 3.5 in Remark 2.9 in the special case of the tori $\mathbb{T}^n$. In different versions of multiplier theorems on $\mathbb{T}^n$, one usually expects to impose conditions on differences of order $[\frac{n}{2}] + 1$ applied to the symbol. In Remark 2.9 we show that e.g. on $\mathbb{T}^2$ or $\mathbb{T}^3$, it is enough to make an assumption on only one second order difference of a special form applied to the symbol. In particular, this improves by now classical theorems on L^p-multipliers requiring n differences, see e.g. Nikolskii [12, Sect. 1.5.3].

In Theorem 5.2 we give an application to the L^p-estimates for general operators from $C^\infty(G)$ to $\mathcal{D}'(G)$, not necessarily invariant. This result is also a relaxation of the symbolic assumptions on the operator compared to those in the pseudo-differential classes. In Theorem 5.2 we give a condition for symbols based on the $(1, 0)$-type behaviour. Since the number of imposed conditions is finite, it can be extended further to (ρ, δ)-type conditions similarly to the case of multipliers in Sect. 5. In general, symbol classes of type (ρ, δ) for matrix symbols on compact Lie groups were introduced in [18]. These symbols also satisfy a suitable version of the functional calculus, see the authors’ paper [20].

In [1], Fourier multiplier theorems have been recently obtained for operators to be bounded from L^p to L^q for $1 < p \le 2 \le q < \infty$ in the setting of the compact Lie group $\mathrm{SU}(2)$. However, those results are different in nature as they explore only the decay rate of symbols rather than the much more subtle behaviour expressed in terms of difference operators in this paper.

The paper is organised as follows. In Sect. 2 we formulate the results with several application and give a number of examples. In Sect. 3 we introduce the necessary techniques and prove the results. In Sect. 4 we briefly discuss central multipliers and the meaning of the difference operator $\mathbb{A}$ in this case. Finally, in Sect. 5 we prove corollaries for operators with symbols in $\mathscr{S}^0_\rho(G)$ and for non-invariant operators.

Some of the results of this paper have been announced in [19] without proof.

2 Multiplier theorems on compact Lie groups

Let G be a compact Lie group with identity 1 and the unitary dual $\widehat{G}$. The following considerations are based on the group Fourier transform

$$\mathscr{F}\phi = \widehat{\phi}(\xi) = \int_G \phi(g)\xi(g)^* \mathrm{d}g, \quad \phi(g) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{trace}\left(\xi(g)\widehat{\phi}(\xi)\right) = \mathscr{F}^{-1}[\widehat{\phi}] \tag{2.1}$$

defined in terms of equivalence classes $[\xi]$ of irreducible unitary representations $\xi : G \to \mathrm{U}(d_\xi)$ of dimension (degree) d_ξ. The Peter–Weyl theorem on G implies in particular that this pair of transforms is inverse to each other and that the Plancherel identity

$$\|\phi\|_2^2 = \sum_{[\xi]\in\widehat{G}} d_\xi \|\widehat{\phi}(\xi)\|_{\mathrm{HS}}^2 =: \|\widehat{\phi}\|^2_{\ell^2(\widehat{G})} \tag{2.2}$$

holds true for all $\phi \in L^2(G)$. Here

$$\|\widehat{\phi}(\xi)\|_{\mathrm{HS}}^2 = \operatorname{trace}\left(\widehat{\phi}(\xi)\widehat{\phi}(\xi)^*\right)$$

denotes the Hilbert–Schmidt (Frobenius) norm of matrices. The Fourier inversion statement (2.1) is valid for all $\phi \in \mathcal{D}'(G)$ and the Fourier series converges in $C^\infty(G)$ provided ϕ is smooth. It is further convenient to denote

$$\langle\xi\rangle = \max\{1, \lambda_\xi\},$$

where λ_ξ^2 is the eigenvalue of the Casimir element (positive Laplace–Beltrami operator) acting on the matrix coefficients associated to the representation ξ. The Sobolev spaces can be characterised by Fourier coefficients as

$$\phi \in H^s(G) \iff \langle\xi\rangle^s \widehat{\phi}(\xi) \in \ell^2(\widehat{G}),$$

where $\ell^2(\widehat{G})$ is defined as the space of matrix-valued sequences such that the sum on the right-hand side of (2.2) is finite.

For an arbitrary continuous linear operator $A : C^\infty(G) \to \mathcal{D}'(G)$ we denote its Schwartz kernel as $K_A \in \mathcal{D}'(G \times G)$ and by a change of variables we associate the right-convolution kernel

$$R_A(g_1, g_2) = K_A\left(g_1, g_1^{-1}g_2\right).$$

Thus, at least formally, we write

$$A\phi(g_1) = \int_G K_A(g_1, g_2)\phi(g_2)\mathrm{d}g_2 = \int_G \phi(g_2) R_A\left(g_1, g_2^{-1}g_1\right)\mathrm{d}g_2 = \phi * R_A(g_1, \cdot).$$

Following the analysis in [15] we denote the partial Fourier transform of the right-convolution kernel with respect to the second variable as symbol of the operator,

$$\sigma_A(g,\xi) := \widehat{R}_A(g,\xi) = \int_G R_A(g,g')\xi(g')^* \mathrm{d}g' \quad \in \mathcal{D}'(G)\widehat{\otimes}_\pi \Sigma(\widehat{G}), \tag{2.3}$$

which is a distribution taking values in the set of moderate sequences of matrices

$$\Sigma(\widehat{G}) = \left\{\sigma : \xi \mapsto \sigma(\xi) \in \mathbb{C}^{d_\xi \times d_\xi} : \|\sigma(\xi)\|_{\mathrm{op}} \lesssim \langle\xi\rangle^N \quad \text{for some} N\right\}.$$

Here we are concerned with left-invariant operators, which means that $A \circ T_g = T_g \circ A$ for all the left-translations $T_g : \phi \mapsto \phi(g^{-1}\cdot)$. This implies that the kernel K_A satisfies the invariance

$$K_A(g_1, g_2) = K_A\left(g^{-1}g_1, g^{-1}g_2\right)$$

for all $g \in G$ and hence R_A is independent of the first argument. In consequence, also the symbol is independent of the first argument and we will write $\sigma_A(\xi)$ for it. In combination with Fourier inversion formula (2.1) this means that the operator A can be written as

$$A\phi(g) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{trace}\left(\xi(g)\sigma_A(\xi)\widehat{\phi}(\xi)\right). \tag{2.4}$$

By this formula we can assign operators $A = \mathrm{Op}(\sigma_A)$ to arbitrary sequences $\sigma_A \in \Sigma(\widehat{G})$. It follows[1] that

$$\sigma_A(\xi) = \xi(g)^*(A\xi)(g) = (A\xi)(g)\big|_{g=1} \tag{2.5}$$

is independent of g. We refer to operators of this form as noncommutative Fourier multipliers. The Plancherel identity (2.2) implies that the operator A is bounded on $L^2(G)$ if and only if $\sigma_A \in \ell^\infty(\widehat{G})$, where

$$\ell^\infty(\widehat{G}) = \left\{\sigma_A \in \Sigma(\widehat{G}) : \sup_{[\xi]\in\widehat{G}} \|\sigma_A(\xi)\|_{\mathrm{op}} < \infty\right\},$$

and $\|\cdot\|_{\mathrm{op}}$ is the operator norm on the unitary space $\mathbb{C}^{d_\xi}$. Note that there is also another version of the space $\ell^\infty(\widehat{G})$ which is realised as the weighted sequence space over Hilbert–Schmidt norms, we refer to [15, Sect. 10.3.3] for its properties.

We now define difference operators $Q \in \mathrm{diff}^\ell(\widehat{G})$ acting on sequences $\sigma \in \Sigma(\widehat{G})$ in terms of corresponding functions $q \in C^\infty(G)$, which vanish to (at least) ℓth order in the identity element $1 \in G$, and their interrelation with the group Fourier transform given by

$$Q\sigma = \mathscr{F}\left(q(g)\mathscr{F}^{-1}\sigma\right). \tag{2.6}$$

Note, that $\sigma \in \Sigma(\widehat{G})$ implies $\mathscr{F}^{-1}\sigma \in \mathcal{D}'(G)$ and therefore the multiplication with a smooth function is well-defined. The main idea of introducing such operators is that applying differences to symbols of Calderon–Zygmund operators brings an improvement in the behaviour of $\mathrm{Op}(Q\sigma)$ since we multiply the integral kernel of $\mathrm{Op}(\sigma)$ by a function vanishing on its singular set. Different collections of difference operators have been explored in [18] in the pseudo-differential setting.

[1] In fact, (2.5) can be taken as a definition of the symbol σ_A of A, from which (2.3) and (2.4) follow; see also Sect. 5.3.

Difference operators of particular interest arise from matrix-coefficients of representations. For a fixed irreducible representation ξ_0 we define the (matrix-valued) difference ${}_{\xi_0}\mathbb{D} = ({}_{\xi_0}\mathbb{D}_{ij})_{i,j=1,\dots,d_{\xi_0}}$ corresponding to the matrix elements of $\xi_0(g) - \mathrm{I}$, i.e. with

$$q_{ij}(g) = \xi_0(g)_{ij} - \delta_{ij},$$

δ_{ij} the Kronecker delta. If the representation is fixed, we omit the index ξ_0. For a sequence of difference operators of this type,

$$\mathbb{D}_1 = {}_{\xi_1}\mathbb{D}_{i_1 j_1}, \mathbb{D}_2 = {}_{\xi_2}\mathbb{D}_{i_2 j_2}, \dots, \mathbb{D}_k = {}_{\xi_k}\mathbb{D}_{i_k j_k},$$

with $[\xi_m] \in \widehat{G}$, $1 \le i_m, j_m \le d_{\xi_m}$, $1 \le m \le k$, we define

$$\mathbb{D}^\alpha := \mathbb{D}_1^{\alpha_1} \cdots \mathbb{D}_k^{\alpha_k}.$$

Among other things, it follows from [18] that an invariant operator A belongs to the usual Hörmander class of pseudo-differential operators $\Psi^0(G)$ defined by localisations if and only if its matrix symbol satisfies

$$\|\mathbb{D}^\alpha \sigma_A(\xi)\|_{\mathrm{op}} \le C_\alpha \langle\xi\rangle^{-|\alpha|} \tag{2.7}$$

for all multi-indices α and for all $[\xi] \in \widehat{G}$. From this point of view the following condition (2.8) is a natural relaxation from the L^p-boundedness of zero order pseudo-differential operators to a multiplier theorem.

Theorem 2.1 *Denote by $\varkappa$ be the smallest even integer larger than $\frac{1}{2}\dim G$. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be left-invariant. Assume that its symbol σ_A satisfies*

$$\|\mathbb{D}^\alpha \sigma_A(\xi)\|_{\mathrm{op}} \le C_\alpha \langle\xi\rangle^{-|\alpha|} \tag{2.8}$$

for all multi-indices α with $|\alpha| \le \varkappa$, and for all $[\xi] \in \widehat{G}$. Then the operator A is of weak type $(1,1)$ and L^p-bounded for all $1 < p < \infty$.

Remark 2.2 (a) The assumptions given in the theorem can be relaxed. For the top order difference we need only one particular difference operator. Moreover, for the lower order difference operators we only need differences associated to the root system if G is semi-simple, and to an extended root system for a general compact Lie group. Such a refinement will be given in Theorem 3.5 once we introduced the necessary notation.

(b) Additional symmetry conditions for the operator imply simplifications. Later on we will show how the assumptions can be weakened for central multipliers.

(c) We have to round up the number of difference conditions to even integers. This seems to be for purely technical reasons, but was already observed similarly in [23] for central multipliers.

(d) The conditions are needed for the weak type $(1,1)$ property. Interpolation allows to reduce assumptions on the number of differences for L^p-boundedness.

Before proceeding to the proof of the theorem, we will mention some applications. As first example let us consider the known case of the Riesz transform.

Example 2.3 Let us consider the partial Riesz transform

$$\mathcal{R}_Z = (-\Delta)^{-1/2} \circ Z$$

associated to a left-invariant vector-field $Z \in \mathfrak{g}$ on a Lie group G. For simplicity we assume that Z is normalised with respect to the Killing form on $\mathfrak{g}$. The Riesz transform is a left-invariant operator acting on $L^2(G)$ with symbol

$$\sigma_{\mathcal{R}_Z}(\xi) = (\lambda_\xi)^{-\frac{1}{2}} \sigma_Z(\xi),$$

$\sigma_Z(\xi) = (Z\xi)(1)$ the symbol of the left-invariant vector field, and by definition of the Laplacian as sum of squares we have

$$\|\sigma_{\mathcal{R}_Z}(\xi)\|_{\mathrm{op}} \leq 1.$$

Note here, that $\lambda_\xi = 0$ implies that $\xi = 0$ is the trivial representation and therefore also $\sigma_Z(\xi) = 0$ as vector fields annihilate constants. It follows from Corollary 4.10 that this operator extends to a bounded operator on all $L^p(G)$, $1 < p < \infty$ and is of weak type $(1, 1)$, recovering the well-known result in [22].

Remark 2.4 In [22, p. 58], Stein asked whether the Riesz transform $\mathcal{R}_Z$ as well as the Riesz potentials $(-\Delta)^{i\gamma}$ (γ real) are pseudo-differential operators on G. This is in fact true on all closed Riemannian manifolds. Indeed, if p_0 denotes the projection to the zero eigenspace of $-\Delta$, then we have the identity

$$(-\Delta)^z = (-\Delta + p_0)^z - p_0$$

for all complex z. The operator $(-\Delta + p_0)^z$ is pseudo-differential for $\Re z < -1$ by [21] and p_0 is smoothing, implying that $(-\Delta)^z$ are pseudo-differential of order $\Re z/2$. By calculus this extends to all $z \in \mathbb{C}$. In particular, the L^p boundedness in Example 2.3 also follows.

Example 2.5 Let $\rho \in [0, 1]$. We denote by $\mathscr{S}^0_\rho(G)$ the set of all $\sigma_A \in \Sigma(\widehat{G})$ satisfying symbol estimates of type ρ

$$\|\mathbb{D}^\alpha \sigma_A(\xi)\|_{\mathrm{op}} \leq C_\alpha \langle\xi\rangle^{-\rho|\alpha|}$$

for all multi-indices α. Let $A = \mathrm{Op}(\sigma_A)$ be the associated operator to such a symbol. Then A defines a bounded operator mapping $W^{p,r}(G) \to L^p(G)$ for

$$r \geq \varkappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|,$$

$\varkappa$ as in Theorem 2.1 and $1 < p < \infty$. See Corollary 5.1, where we give a refined version of this.

Example 2.6 The previous example applies in particular to the parametrices constructed in [18]. Following the notation from that paper, we consider the sub-Laplacian

$$\mathcal{L}_s = \mathrm{D}_1^2 + \mathrm{D}_2^2$$

on $\mathbb{S}^3$. It was shown that it has a parametrix from $\mathrm{Op}\,\mathscr{S}^{-1}_{1/2}(\mathbb{S}^3)$ and therefore $\mathcal{L}_s u \in L^p(\mathbb{S}^3)$ implies regularity for u. More precisely, the sub-elliptic estimate

$$\|u\|_{W^{p,1-|\frac{1}{p}-\frac{1}{2}|}(\mathbb{S}^3)} \leq C_p \|\mathcal{L}_s u\|_{L^p(\mathbb{S}^3)} \tag{2.9}$$

holds true for all $1 < p < \infty$.

Similarly, the "heat" operator

$$H = \mathrm{D}_3 - \mathrm{D}_1^2 - \mathrm{D}_2^2$$

on $\mathbb{S}^3$ has a parametrix from $\mathrm{Op}\,\mathscr{S}^{-1}_{1/2}(\mathbb{S}^3)$. Consequently, we also get the sub-elliptic estimate (2.9) with H instead of $\mathcal{L}_s$.

Similar examples can be given for arbitrary compact Lie groups G. Operators in Example 2.6 are locally hypoelliptic, but the following corollary applies to operators which are only globally hypoelliptic.

Corollary 2.7 *Let X be a left-invariant real vector field on G. Then there exists a discrete exceptional set $\mathscr{C} \subset \mathrm{i}\mathbb{R}$, such that for any complex number $c \notin \mathscr{C}$ the operator $X + c$ is invertible with inverse in* $\operatorname{Op} \mathscr{S}^0_0(G)$. *Consequently, the inequality*

$$\|f\|_{L^p(G)} \leq C_p \|(X+c)f\|_{W^{p,\varkappa|\frac{1}{p}-\frac{1}{2}|}(G)}$$

holds true for all $1 < p < \infty$ and all functions f from that Sobolev space.

We prove this corollary later, but now only give its refinement on $\mathrm{SU}(2)$.

Example 2.8 To fix the scaling on the Lie algebra $\mathfrak{su}(2)$, let (ϕ, θ, ψ) be the (standard) Euler angles on $\mathrm{SU}(2)$ and let $D_3 = \partial/\partial\psi$. Let X be a left-invariant vector field on $\mathrm{SU}(2)$ normalised so that $\|X\| = \|D_3\|$ with respect to the Killing norm. Then it was shown in [18] that $\mathrm{i}\mathscr{C} = \frac{1}{2}\mathbb{Z}$, and $X + c$ is invertible if and only if $\mathrm{i}c \notin \frac{1}{2}\mathbb{Z}$. For such c, the inverse $(X+c)^{-1}$ has symbol in $\mathscr{S}^0_0(\mathrm{SU}(2))$. The same conclusions remain true if we replace $\mathrm{SU}(2)$ by $\mathbb{S}^3$. In particular, we get that

$$\|f\|_{L^p(\mathbb{S}^3)} \leq C_p \|(X+c)f\|_{W^{p,2|\frac{1}{p}-\frac{1}{2}|}(\mathbb{S}^3)}$$

holds true for all $1 < p < \infty$ and all functions f from that Sobolev space. We note that this estimate is non-localisable since operators $X + c$ are locally non-invertible and also not locally sub-elliptic (unless $n = 1$).

Remark 2.9 The Hörmander multiplier theorem [11], although formulated in $\mathbb{R}^n$, has a natural analogue on the torus $\mathbb{T}^n$. The refinement in Theorem 3.5 on the top order difference brings a refinement of the toroidal multiplier theorem, at least for some dimensions. If $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the set Δ_0 in Remark 3.2 consists of $2n$ functions $\mathrm{e}^{\pm 2\pi \mathrm{i} x_j}$, $1 \leq j \leq n$. Consequently, we have that

$$\rho^2(x) = 2n - \sum_{j=1}^{n} \left(\mathrm{e}^{2\pi \mathrm{i} x_j} + \mathrm{e}^{-2\pi \mathrm{i} x_j}\right)$$

in (3.1), and hence

$$\mathbb{A}\sigma(\xi) = 2n\sigma(\xi) - \sum_{j=1}^{n} \left(\sigma(\xi + e_j) + \sigma(x - e_j)\right)$$

in (3.7), where $\xi \in \mathbb{Z}^n$ and e_j is its jth unit basis vector in $\mathbb{Z}^n$.

A (translation) invariant operator A and its symbol σ_A are related[2] by

$$\sigma_A(k) = \mathrm{e}^{-2\pi \mathrm{i} x\cdot k} \left(A\mathrm{e}^{2\pi \mathrm{i} x\cdot k}\right) = \left(A\mathrm{e}^{2\pi \mathrm{i} x\cdot k}\right)|_{x=0}$$

and

$$A\phi(x) = \sum_{k\in\mathbb{Z}^n} e^{2\pi \mathrm{i} x\cdot k} \sigma_A(k)\widehat{\phi}(k).$$

Thus, it follows from Theorem 3.5 that, for example on $\mathbb{T}^3$, a translation invariant operator A is weak (1,1) type and bounded on $L^p(\mathbb{T}^3)$ for all $1 < p < \infty$ provided that there is a constant $C > 0$ such that

[2] On the torus we can abuse the notation by writing $\sigma_A(k)$ for $\sigma_A(e_k)$ for $e_k(x) = \mathrm{e}^{2\pi \mathrm{i} x\cdot k}$. For the consistent development of the toroidal quantization of general operators on the tori see [14] or [15], with an earlier partial exposition in [13].

$$|\sigma_A(k)| \le C,$$
$$|k||\sigma_A(k+e_j) - \sigma_A(k)| \le C,$$

and

$$|k|^2|\sigma_A(k) - \frac{1}{6}\sum_{j=1}^{3}\big(\sigma_A(k+e_j) + \sigma_A(k-e_j)\big)| \le C, \tag{2.10}$$

for all $k \in \mathbb{Z}^3$ and all (three) unit vectors e_j, $j = 1, 2, 3$. Here in (2.10) we do not make assumptions on all second order differences, but only on one of them.

3 Proofs

The proof of Theorem 2.1 is divided into several sections. First we introduce the tools we need to prove Calderon–Zygmund type estimates for convolution kernels. Later on we show how to reduce the above theorem to a statement of Coifman and de Guzman, see [4] and also [6]. Finally, we use properties of the root system with finite Leibniz rules for difference operators to prove the refinement of Theorem 2.1 given in Theorem 3.5.

3.1 A suitable pseudo-distance on G

At first we construct a suitable pseudo-distance on the group G in terms of a minimal set of representations. We now define with $n = \dim G$

$$\rho^2(g) = n - \operatorname{trace} \operatorname{Ad}(g) = \sum_{\xi\in\Delta_0} (d_\xi - \operatorname{trace} \xi(g)), \tag{3.1}$$

where $\operatorname{Ad} : G \to \mathrm{U}(\mathfrak{g}) \simeq \mathrm{U}(\dim G)$ denotes the adjoint representation of the Lie group G and

$$\operatorname{Ad} = (\dim Z(G))1 \oplus \bigoplus_{\xi\in\Delta_0} \xi$$

is its Peter–Weyl decomposition into irreducible components. Here, 1 denotes the trivial one-dimensional representation. For simplicity we assume first that the group is semi-simple, i.e., that the centre $Z(G)$ of the group G is trivial. Later on we will explain the main modifications for the general situation, see Remark 3.2.

Note, that $\rho^2(g)$ is nonnegative by definition and smooth. At first we claim that ρ defines a pseudo-distance

$$d_\rho(g, h) = \rho(g^{-1}h).$$

Lemma 3.1 *The above defined function $\rho(g)$ satisfies*

(1) $\rho^2(g) \ge 0$ and $\rho^2(g) = 0$ if and only if $g = 1$ is the identity in G;
(2) ρ^2 vanishes to second order in $g = 1$;
(3) ρ^2 is a class function, in particular it satisfies $\rho^2(g^{-1}) = \rho^2(g)$ and $\rho^2(gh^{-1}) = \rho^2(h^{-1}g)$;
(4) $|\rho(gh^{-1}) - \rho(g)| \le C\rho(h)$ for some constant $C > 0$ and all $g, h \in G$;
(5) $\rho(gh^{-1}) \le C(\rho(g) + \rho(h))$ for some constant $C > 0$ and all $g, h \in G$.

Proof (1) At first we note that for any (not necessarily irreducible) unitary representation ξ trivially $|\operatorname{trace} \xi(g)| \le d_\xi$ and therefore $\Re(d_\xi - \operatorname{trace} \xi(g)) \ge 0$. Furthermore, trace $\xi(g) = d_\xi$ is equivalent to $\xi(g) = \mathrm{I}$. Therefore, $\rho(g) = 0$ implies that $\mathrm{Ad}(g) = \mathrm{I}$ and therefore $g \in Z(G)$, i.e., $g = 1$.

(2) Differentiating the identity $\xi(g)\xi(g)^* = \mathrm{I}$ twice at the identity element and denoting $\xi^*(g) = \xi(g)^*$ implies the equations

$$\xi'(1) + \xi^{*\prime}(1) = 0,$$
$$\xi''(1) + 2\xi'(1) \otimes \xi^{*\prime}(1) + \xi^{*\prime\prime}(1) = 0,$$

the first implying that $(\Re \operatorname{trace} \xi)'(1) = 0$, while the second one gives for each $v \in \mathfrak{g} = \mathrm{T}_1 G$ the quadratic form

$$(v, (\Re \operatorname{trace} \xi)''(1)v) = -\|\xi'(1)v\|^2_{\mathrm{HS}}.$$

Summing this over $\xi \in \Delta_0$ implies

$$(v, \operatorname{Hess} \rho^2(1)v) = -\sum_{\xi\in\Delta_0} \|\xi'(1)v\|^2_{\mathrm{HS}},$$

and, therefore, if $v \in \mathfrak{g}$ is such that the left-hand side vanishes, then $v \in \bigcap_\xi \ker \xi'(1)$. By $Z(G) = \{1\}$ and the definition of $\rho^2(g)$ this implies $v = 0$.

(3) Obvious by construction.

(4) We observe that both the left and the right hand side vanish exactly in $h = 1$ to first order. The existence of the constant C follows therefore just by compactness of G.

(5) follows directly by (4).

$\square$

Remark 3.2 If the centre of the group is non-trivial, we have to make a slight change to the definition of $\rho^2(g)$. We have to include $2 \dim Z(G)$ additional representations to the set Δ_0 defined by the choice of an isomorphism $Z(G) \simeq \mathbb{T}^\ell = \mathbb{R}^\ell/\mathbb{Z}^\ell$. For each coordinate θ_j we include both $\theta \mapsto \mathrm{e}^{\pm 2\pi \mathrm{i}\theta_j}$, suitably extended to the maximal torus and then to G. The statement of Lemma 3.1 remains true for both modifications. In the following we assume that Δ_0 and $\rho(g)$ are defined in this way. In general, for the statements below to be true, any extension of Δ_0 will work as long as the function $\rho^2(g)$ in (3.1) is the square of a distance function on G in a neighbourhood of the neutral element.

3.2 A special family of mollifiers

Let $\tilde\varphi \in C_0^\infty(\mathbb{R})$ be such that $\tilde\varphi \geq 0$, $\tilde\varphi(0) = 1$ and $\tilde\varphi^{(\ell)}(0) = 0$ for all $\ell \geq 1$. Then for $r > 0$ we define

$$\varphi_r(g) = c_r \tilde\varphi\left(r^{-1/n}\rho(g)\right), \qquad \int_G \varphi_r(g)\mathrm{d}g = 1, \tag{3.2}$$

the normalisation condition used to define c_r. As $r \to 0$ obviously $\varphi_r \to \delta_1 \in \mathcal{D}'(G)$. Let furthermore

$$\psi_r(g) := \varphi_r(g) - \varphi_{r/2}(g).$$

At first we check the conditions of Coifman–de Guzman [4] (modulo the obvious modifications) for these functions.

Lemma 3.3 *(1)* $\sup_g |\varphi_r(g)| \sim c_r \sim r^{-1}$ *as* $r \to 0$.

(2) $\|\varphi_r\|_2 \sim r^{-1/2}$ *as* $r \to 0$.

(3) $\varphi_r * \varphi_s = \varphi_s * \varphi_r$.

(4) $\int_{\rho(g)\geq t^{1/n}} \varphi_r(g)\mathrm{d}g \leq C_N\left(\frac{r}{t}\right)^N$ *for all* $N \geq 0$.
(5) $\int_G |\varphi_r(gh^{-1}) - \varphi_r(g)|\mathrm{d}g \leq C' \frac{\rho(h)}{r^{1/n}}$.

Proof (1) We can find a chart in the neighbourhood of the identity element such that $\rho(g) = |x|$ and $\mathrm{d}g = \nu(x)\mathrm{d}x$ for some smooth density ν with $\nu(0) \neq 0$. Then direct calculation yields for small r

$$c_r^{-1} = \int_G \tilde{\varphi}(r^{-1/n}|x|)\nu(x)\mathrm{d}x = \int_0^1 \tilde{\varphi}(r^{-1/n}s)s^{n-1}\int_{\mathbb{S}^{n-1}} \nu(s\theta)\mathrm{d}\theta\mathrm{d}s$$
$$\lesssim \int_0^1 \tilde{\varphi}(r^{-1/n}s)s^{n-1}\mathrm{d}s \sim r.$$

(2) follows from (1) by interpolation with the normalisation condition used.
(3) this follows from φ_r being a class function.
(4) Again direct computation of the left-hand side yields for sufficiently small r

$$c_r \int_{s\geq t^{1/n}} \tilde{\varphi}(r^{-1/n}s)s^{n-1}\int_{\mathbb{S}^{n-1}} \nu(s\theta)\mathrm{d}\theta\mathrm{d}s$$
$$\lesssim c_r \int_{s\geq t^{1/n}} \tilde{\varphi}(r^{-1/n}s)s^{n-1}\mathrm{d}s \sim F\left(\tfrac{t}{r}\right)$$

for a function $F \in C_0^\infty(\mathbb{R}_+)$, which implies in particular the desired estimate.
(5) Using that $\tilde{\varphi} \in C_0^\infty(\mathbb{R})$ the mean value theorem implies in combination with Lemma 3.1(4)

$$|\varphi_r(gh^{-1}) - \varphi_r(g)| = c_r|\tilde{\varphi}\left(r^{-1/n}\rho(gh^{-1})\right) - \tilde{\varphi}\left(r^{-1/n}\rho(g)\right)|$$
$$\lesssim c_r r^{-1/n}|\rho\left(gh^{-1}\right) - \rho(g)| \lesssim c_r r^{-1/n}\rho(h).$$

Furthermore, the first expression is non-zero for small r only if either of the terms is non-zero, which gives $\rho(g) \lesssim r^{1/n}$ or $\rho(gh^{-1}) \lesssim r^{1/n}$. This corresponds for small r to two balls of radius $r^{1/n}$, i.e., volume r. Integration over $g \in G$ implies the desired statement.

□

As ψ_r and ρ^n satisfy all assumptions of [4], we have the following criterion.

Criterion *Assume* $A : L^2(G) \to L^2(G)$ *is a left-invariant operator on* G *satisfying*

$$\int_G |A\psi_r(g)|^2\rho^{n(1+\epsilon)}(g)\mathrm{d}g \leq Cr^\epsilon \tag{3.3}$$

for some constants $\epsilon > 0$ *and* $C > 0$ *uniform in* r. *Then* A *is of weak type* $(1, 1)$ *and bounded on all* $L^p(G)$ *for* $1 < p < \infty$.

Later on we will need some more properties of the functions ψ_r. We collect them as follows

Lemma 3.4 *Let* $q \in C^\infty(G)$ *be a smooth function vanishing to order* t *in* 1. *Then*

$$\|q(g)\psi_r(g)\|_{H^{-s}} \leq C_{q,s}r^{\frac{t+s}{n}-\frac{1}{2}} \tag{3.4}$$

for all $s \in [0, 1 + \frac{n}{2}]$.

Proof Note that the statement is purely local in a neighbourhood of 1. For sufficiently small r we find local co-ordinates near 1 supporting $\psi_r(g)$ and satisfying $\rho(g) = |x|$. We write q as Taylor polynomial $q_N(g)$ of degree $t + N$ plus remainder $R_N(g) = \mathcal{O}(\rho^{t+N+1}(g))$ and decompose $q(g)\psi_r(g)$ accordingly. First, we observe

$$\begin{aligned}
\|q_N\psi_r\|^2_{H^{-s}} &\sim \int \langle\xi\rangle^{-2s}\big|q_N(\partial_\xi)\big(\widehat{\varphi}(r^{1/n}|\xi|) - \widehat{\varphi}(2r^{1/n}|\xi|)\big)\big|^2 \mathrm{d}\xi \\
&\lesssim r^{2t/n}\int_{r^{1/n}|\xi|\le 1} \langle\xi\rangle^{-2s} r^{2/n}|\xi|^2|\xi|^{n-1}\mathrm{d}|\xi| \\
&\quad + r^{2t/n}\int_{r^{1/n}|\xi|\ge 1} r^{-M/n}|\xi|^{-2s-M}|\xi|^{n-1}\mathrm{d}|\xi| \\
&\lesssim r^{\frac{2t+2}{n}}\langle\xi\rangle^{-2s+2+n}\Big|_0^{r^{-1/n}} + r^{\frac{2t-M}{n}}|\xi|^{-2s-M+n}\Big|_{r^{-1/n}}^{\infty} \\
&\lesssim r^{\frac{2t+2s-n}{n}}
\end{aligned}$$

where ξ is (abusing notation) the Fourier co-variable to x. The integral is split into $r^{1/n}|\xi| \lesssim 1$ and $r^{1/n}|\xi| \gtrsim 1$. Second, we consider the remainder and show that it is smaller. Indeed,

$$\|R_N\psi_r\|_{H^{-s}} \lesssim \|R_N\psi_r\|_2 \lesssim \|R_N\|_{L^\infty(\operatorname{supp}\psi_r)}\|\psi_r\|_2 \lesssim r^{(t+N+1)/n}r^{-1/2}$$

and choosing $N > s - 1$ the desired smallness follows.

Assumptions we had to make were $-2s+2+n \ge 0$, i.e., $s \le 1 + \frac{n}{2}$ and $n - 2s - M < 0$, i.e., $M > n - 2s$. Furthermore, we need $s \ge 0$. The lemma is proven. □

3.3 Difference operators and Leibniz rules

We recall the definition of difference operators before Theorem 2.1. For a fixed irreducible representation ξ_0 we define the (matrix-valued) difference

$${}_{\xi_0}\mathbb{D} = ({}_{\xi_0}\mathbb{D}_{ij})_{i,j=1,\dots,d_{\xi_0}}$$

corresponding to the symbol $\xi_0(g) - \mathrm{I}$. We denote by δ_{ij} the Kronecker delta, $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \ne j$. Thus, we have

$${}_{\xi_0}\mathbb{D}_{ij} = \mathscr{F}\left(\xi_0(g)_{ij} - \delta_{ij}\right)\mathscr{F}^{-1}.$$

If the representation is fixed, we omit the index ξ_0. As observed in [18] these difference operators satisfy the finite (two-term) Leibniz rule

$$\mathbb{D}_{ij}(\sigma\tau) = (\mathbb{D}_{ij}\sigma)\tau + \sigma(\mathbb{D}_{ij}\tau) + \sum_{k=1}^{d_{\xi_0}}(\mathbb{D}_{ik}\sigma)(\mathbb{D}_{kj}\tau) \tag{3.5}$$

for all sequences $\sigma, \tau \in \Sigma(\widehat{G})$. Iterating this, for a composition $\mathbb{D}^k$ of $k \in \mathbb{N}$ difference operators of this form we have

$$\mathbb{D}^k(\sigma\tau) = \sum_{|\gamma|,|\delta|\le k\le|\gamma|+|\delta|} C_{k\gamma\delta}\,(\mathbb{D}^\gamma\sigma)\,(\mathbb{D}^\delta\tau), \tag{3.6}$$

with the summation taken over all multi-indices $\gamma, \delta \in \mathbb{N}_0^{\ell^2}$, $\ell = d_{\xi_0}$, satisfying $|\gamma|, |\delta| \le k \le |\gamma| + |\delta|$,

$$\mathbb{D}^\gamma = \mathbb{D}_{11}^{\gamma_{11}}\mathbb{D}_{12}^{\gamma_{12}}\cdots\mathbb{D}_{\ell,\ell-1}^{\gamma_{\ell,\ell-1}}\mathbb{D}_{\ell\ell}^{\gamma_{\ell\ell}},$$

and where constants $C_{k\gamma\delta}$ may depend on a particular form of $\mathbb{D}^k$.

Denote by $\mathbb{A}$ the difference operator associated to the symbol $\rho^2(g)$ defined in (3.1),

$$\mathbb{A} = \mathscr{F}\,\rho^2(g)\,\mathscr{F}^{-1}. \tag{3.7}$$

By Lemma 3.1 (2), this is a second order difference operator, $\mathbb{A} \in \operatorname{diff}^2(\widehat{G})$, and in view of (3.1) it can be decomposed as

$$\mathbb{A} = -\sum_{\xi\in\Delta_0}\sum_{i=1}^{d_\xi} {}_\xi\mathbb{D}_{ii}. \tag{3.8}$$

Therefore, after summation of the Leibniz rules (3.5) we observe that

$$\mathbb{A}(\sigma\tau) = (\mathbb{A}\sigma)\tau + \sigma(\mathbb{A}\tau) - \sum_{\xi\in\Delta_0}\sum_{i,j=1}^{d_\xi}({}_\xi\mathbb{D}_{ij}\sigma)({}_\xi\mathbb{D}_{ji}\tau). \tag{3.9}$$

Iterating this, we observe that

$$\begin{aligned}\mathbb{A}^2(\sigma\tau) &= \mathbb{A}\Big((\mathbb{A}\sigma)\tau + \sigma(\mathbb{A}\tau) + \sum(\mathbb{D}\sigma)(\mathbb{D}\tau)\Big)\\ &= (\mathbb{A}^2\sigma)\tau + 2(\mathbb{A}\sigma)(\mathbb{A}\tau) + \sigma(\mathbb{A}^2\tau)\\ &\quad + \sum\Big((\mathbb{D}\mathbb{A}\sigma)(\mathbb{D}\tau) + (\mathbb{D}\sigma)(\mathbb{D}\mathbb{A}\tau) + (\mathbb{D}^2\sigma)(\mathbb{D}^2\tau)\Big).\end{aligned}$$

In the sum the orders of difference operators always add up to 4. Similar we obtain for higher orders m,

$$\mathbb{A}^m(\sigma\tau) = (\mathbb{A}^m\sigma)\tau + \sigma(\mathbb{A}^m\tau) + \sum_{\ell=1}^{2m-1}\sum_j (Q_{\ell,j}\sigma)(\tilde{Q}_{\ell,j}\tau). \tag{3.10}$$

for some difference operators $Q_{\ell,j} \in \operatorname{diff}^\ell(\widehat{G})$ and $\tilde{Q}_{\ell,j} \in \operatorname{diff}^{2m-\ell}(\widehat{G})$.

3.4 Proof of Theorem 2.1

We note that Theorem 2.1 follows from its refined version which we give as Theorem 3.5 below. Let Δ_0 be an extended root system as in Remark 3.2, and we define the family of first order difference operators associated to Δ_0 by

$$\mathscr{D}^1 = \{{}_{\xi_0}\mathbb{D}_{ij} = \mathscr{F}\left(\xi_0(g)_{ij} - \delta_{ij}\right)\mathscr{F}^{-1} : \ \xi_0 \in \Delta_0,\ 1 \le i, j \le d_{\xi_0}\},$$

where δ_{ij} is the Kronecker delta. We write $\mathscr{D}^k$ for the family of operators of the form $\mathbb{D}^\alpha = \mathbb{D}_1^{\alpha_1}\cdots\mathbb{D}_l^{\alpha_l}$, where $\mathbb{D}_1, \dots, \mathbb{D}_l \in \mathscr{D}^1$, and for multi-indices $\alpha = (\alpha_1, \dots, \alpha_l)$ of any length but such that $|\alpha| \le k$. We note that for even $\varkappa$, in view of (3.8), the difference operator $\mathbb{A}^{\varkappa/2}$ is a linear combination of operators in $\mathscr{D}^\varkappa$. In general, clearly $\mathscr{D}^k \subset \operatorname{diff}^k(\widehat{G})$.

Theorem 3.5 *Denote by $\varkappa$ be the smallest even integer larger than $\frac{1}{2}\dim G$. Let operator $A : C^\infty(G) \to \mathcal{D}'(G)$ be left-invariant. Assume that its symbol σ_A satisfies*

$$\|\mathbb{A}^{\varkappa/2}\sigma_A(\xi)\|_{\mathrm{op}} \le C\,\langle\xi\rangle^{-\varkappa}$$

as well as

$$\|\mathbb{D}^\alpha\sigma_A(\xi)\|_{\mathrm{op}} \le C_\alpha\langle\xi\rangle^{-|\alpha|}, \tag{3.11}$$

for all operators $\mathbb{D}^\alpha \in \mathscr{D}^{\varkappa-1}$, *and for all* $[\xi] \in \widehat{G}$. *Then the operator* A *is of weak type* $(1,1)$ *and* L^p*-bounded for all* $1 < p < \infty$.

For the proof it is enough to check (3.3) with appropriate ϵ. If we choose ϵ such that $n(1+\epsilon) = 4m$, $m \in \mathbb{N}$, the condition is equivalent to

$$\|\mathbb{A}^m(\sigma_A \widehat{\psi}_r)\|_{\ell^2(\widehat{G})} \lesssim r^{\frac{2m}{n}-\frac{1}{2}}. \tag{3.12}$$

Applying the Leibniz rule (3.10) to the left-hand side implies for fixed $[\xi] \in \widehat{G}$ that we have

$$\begin{aligned}\|\mathbb{A}^m(\sigma_A \widehat{\psi}_r)(\xi)\|_{\mathrm{HS}} &\le \|\mathbb{A}^m \sigma_A(\xi)\|_{\mathrm{op}} \|\widehat{\psi}_r(\xi)\|_{\mathrm{HS}} + \|\sigma_A(\xi)\|_{\mathrm{op}} \|\mathbb{A}^m \widehat{\psi}_r(\xi)\|_{\mathrm{HS}} \\ &\quad + \sum_{\ell=1}^{2m-1} \sum_j \|\langle\xi\rangle^\ell Q_{\ell,j} \sigma_A(\xi)\|_{\mathrm{op}} \|\langle\xi\rangle^{-\ell} \tilde{Q}_{\ell,j} \widehat{\psi}_r\|_{\mathrm{HS}}\end{aligned}$$

for certain differences $Q_{\ell,j} \in \mathrm{diff}^\ell(\widehat{G})$ of order ℓ arising from Leibniz rule and corresponding differences $\tilde{Q}_{\ell,j} \in \mathrm{diff}^{2m-\ell}(\widehat{G})$. Summing d_ξ times these inequalities over $[\xi] \in \widehat{G}$ and using the assumptions of Theorem 3.5, we can apply Lemma 3.4 in the form

$$\|\tilde{q}_{\ell,j} \psi_r\|_{H^{-\ell}} \lesssim r^{\frac{2m}{n}-\frac{1}{2}} \tag{3.13}$$

for $0 \le \ell \le 1 + \frac{n}{2}$. Under the assumption that $2m \le 2 + \frac{n}{2}$ this implies the desired estimate (3.12).

Remark 3.6 Note, that the number of difference conditions is $\varkappa = 2m$, where $\frac{n}{2} < \varkappa \le 2 + \frac{n}{2}$, as we have to assure that $\epsilon > 0$ and that Lemma 3.4 is applicable.

4 Applications to central multipliers

We turn to some applications of Theorem 2.1. First we collect some statements about central sequences $\sigma \in \Sigma(\widehat{G})$, $\sigma(\xi) = \sigma_\xi \mathrm{I}$. Particular examples of interest are defined in terms of d_ξ or λ_ξ or appear in connection with invariant multipliers on homogeneous spaces with respect to massive subgroups. For the sake of simplicity we assume in the sequel that σ_ξ is defined on the full weight lattice $\Lambda \subset \mathfrak{t}^*$ for the Cartan subalgebra $\mathfrak{t}$, and treat $\widehat{G}$ as subset of Λ, representations identified with their dominant highest weights. We refer to e.g. [10] for Weyl group, Weyl dimension and Weyl character formula. We will use a notion of difference operators on the weight lattice; difference operators of higher order are understood as iterates of first order forward differences on this lattice.

4.1 Some auxiliary statements on central sequences

First, we consider the sequence d_ξ of dimensions of representations. We extend the sequence d_ξ to the full weight lattice by Weyl's dimension formula (after fixing the set Δ_0^+ of positive roots).

Lemma 4.1 *The sequence* d_ξ *satisfies the polynomial bound*

$$d_\xi \lesssim \langle\xi\rangle^\ell, \qquad \ell = |\Delta_0^+| \le \tfrac{1}{2}(n - \operatorname{rank} G), \tag{4.1}$$

together with the hypoellipticity estimate

$$\frac{|\triangle_k d_\xi|}{|d_\xi|} \le C_k \langle\xi\rangle^{-k}, \qquad d_\xi \ne 0, \tag{4.2}$$

for any difference operator $\triangle_k$ of order k acting on the weight lattice.

Proof We recall first, that the dimension d_ξ can be expressed in terms of the heighest weights (for simplicity also denoted by the variable $\xi \in \Lambda \subset \mathfrak{t}^*$, $\mathfrak{t} = \mathrm{T}_1\mathcal{T}$ for $\mathcal{T} \subset G$ a maximal torus of G) by Weyl's dimension formula

$$d_\xi = \frac{\prod_{\alpha\in\Delta_0^+}(\xi+\rho,\alpha)}{\prod_{\alpha\in\Delta_0^+}(\rho,\alpha)}, \qquad \rho = \frac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha. \tag{4.3}$$

The sum goes over the positive roots $\alpha \in \Delta_0^+$, which form a subset of the set Δ_0 used before. Weyl's dimension formula directly implies (4.1) from $\langle\xi\rangle \sim 1+||\xi||$.

In order to prove (4.2) we consider first an arbitrary difference of first order of the form $\triangle_\tau d_\xi = d_{\xi+\tau} - d_\xi$ for a suitable lattice vector $\tau \in \Lambda$. Then, an elementary calculation shows that

$$\frac{\triangle_\tau d_\xi}{d_\xi} = \frac{\prod_{\alpha\in\Delta_0^+}\big((\xi+\rho,\alpha)+(\tau,\alpha)\big) - \prod_{\alpha\in\Delta_0^+}(\xi+\rho,\alpha)}{\prod_{\alpha\in\Delta_0^+}(\xi+\rho,\alpha)}$$

and, therefore, we see that the right-hand side indeed behaves like $\langle\xi\rangle^{-1}$ for all $d_\xi \neq 0$. The full statement follows in analogy. □

In the following we extend the family of characters χ_ξ from $[\xi] \in \widehat{G}$ to the full weight lattice using the Weyl character formula

$$j(\exp x)\chi_\xi(\exp x) = \sum_{\omega\in\mathcal{W}} \operatorname{sign}(\omega)\mathrm{e}^{2\pi\mathrm{i}(\omega(\xi+\rho),x)} \tag{4.4}$$

for $x \in \mathfrak{t} \subset \mathfrak{g}$ the Cartan subalgebra and $\mathcal{W}$ its Weyl group. As usual

$$j(\exp x) = \sum_{\omega\in\mathcal{W}} \operatorname{sign}(\omega)\mathrm{e}^{2\pi\mathrm{i}(\omega\rho,x)} \tag{4.5}$$

denotes the Weyl denominator. As $\chi_\xi : \mathcal{T} \to \mathbb{C}$ is invariant under the adjoint action it extends to a unique central function on the group G. We collect two properties of these functions related to averaging over orbits of the Weyl group.

Lemma 4.2 *Let $\mathcal{O}_\xi := \{\omega\xi : \xi \in \mathcal{W}\}$. Then*

$$\sum_{\xi'\in\mathcal{O}_\xi} \chi_{\xi'}(\exp x) = \sum_{\xi'\in\mathcal{O}_\xi} \mathrm{e}^{2\pi\mathrm{i}(\xi',x)} \tag{4.6}$$

and

$$\sum_{\xi'\in\mathcal{O}_\xi} \chi_{\xi'}(\exp x)\chi_{\xi_*}(\exp x) = \sum_{\xi'\in\mathcal{O}_\xi} \chi_{\xi_*+\xi'}(\exp x) \tag{4.7}$$

for any fixed pair $\xi, \xi_* \in \Lambda$. Furthermore,

$$\int_G \chi_{\xi_*}(g)\overline{\chi_\xi(g)}\mathrm{d}g = \begin{cases} \operatorname{sign}(\omega) & \exists\omega\in\mathcal{W} : \omega(\xi+\rho) = \xi^*+\rho, \\ 0 & \textit{otherwise}. \end{cases} \tag{4.8}$$

Proof Using Weyl character formula we obtain

$$j(\exp x)\sum_{\xi'\in\mathcal{O}_\xi} \chi_{\xi'}(\exp x) = \sum_{\xi'\in\mathcal{O}_\xi}\sum_{\omega\in\mathcal{W}} \operatorname{sign}(\omega)\mathrm{e}^{2\pi\mathrm{i}(\omega(\xi'+\rho),x)}$$

$$= \sum_{\omega\in\mathcal{W}} \mathrm{sign}(\omega)\Big(\sum_{\xi'\in\mathcal{O}_\xi} \mathrm{e}^{2\pi\mathrm{i}(\omega\xi',x)}\Big)\mathrm{e}^{2\pi\mathrm{i}(\omega\rho,x)}$$
$$= j(\exp x)\sum_{\xi'\in\mathcal{O}_\xi} \mathrm{e}^{2\pi\mathrm{i}(\xi',x)}$$

using that elements of $\mathcal{W}$ permute the orbit $\mathcal{O}_\xi$ and hence the first identity. Similarly we obtain

$$\begin{aligned}
(j(\exp x))^2 &\sum_{\xi'\in\mathcal{O}_\xi} \chi_{\xi_*}(\exp x)\chi_{\xi'}(\exp x)\\
&= \sum_{\xi'\in\mathcal{O}_\xi}\Big(\sum_{\omega\in\mathcal{W}} \mathrm{sign}(\omega)\mathrm{e}^{2\pi\mathrm{i}(\omega(\xi_*+\rho),x)}\Big)\Big(\sum_{\omega'\in\mathcal{W}} \mathrm{sign}(\omega')\mathrm{e}^{2\pi\mathrm{i}(\omega'(\xi'+\rho),x)}\Big)\\
&= \sum_{\omega\in\mathcal{W}}\sum_{\omega'\in\mathcal{W}}\Big(\sum_{\xi'\in\mathcal{O}_\xi} \mathrm{sign}(\omega)\mathrm{e}^{2\pi\mathrm{i}(\omega(\omega^{-1}\omega'\xi'+\xi_*+\rho),x)}\Big)\,\mathrm{sign}(\omega')\mathrm{e}^{2\pi\mathrm{i}(\omega'\rho,x)}\\
&= (j(\exp x))^2\sum_{\xi'\in\mathcal{O}_\xi} \chi_{\xi_*+\xi'}(\exp x).
\end{aligned}$$

Furthermore, (4.8) follows by Weyl integration formula,

$$\int \chi_{\xi_*}(g)\overline{\chi_\xi(g)}\mathrm{d}g = \frac{1}{|\mathcal{W}|}\sum_{\omega,\omega'\in\mathcal{W}} \mathrm{sign}(\omega\omega')\int_{\mathbb{R}^k/\mathbb{Z}^k} \mathrm{e}^{2\pi\mathrm{i}(\omega(\xi_*+\rho)-\omega'(\xi+\rho),x)}\mathrm{d}x$$

combined with the orthogonality relations of trigonometric functions and the fact that $\mathcal{W}$ acts simply and transitively on the chambers. □

Lemma 4.3 *Assume G is semi-simple. Then by* (4.6)

$$\mathrm{trace}\,\mathrm{Ad}(\exp x) - \mathrm{rank}\,G = \sum_{\xi\in\Delta_0}\sum_{\xi'\in\mathcal{O}_\xi} \mathrm{e}^{2\pi\mathrm{i}(\xi',x)} = \sum_{\xi\in\Delta_0}\sum_{\xi'\in\mathcal{O}_\xi} \chi_{\xi'}(\exp x). \tag{4.9}$$

For the following we assume that $\sigma\in\Sigma(\widehat{G})$ is central, $\sigma(\xi)=\sigma_\xi\mathrm{I}$. This corresponds to a distribution $\mathscr{F}^{-1}\sigma\in\mathcal{D}'(G)$ invariant under the adjoint action of the group. The following lemma explains the action of the difference operator $\mathbb{A}$ on σ. We understand $d_\xi\sigma_\xi$ as scalar sequence on the lattice of dominant weights extended by the action of the Weyl group

$$\sigma_{\xi'} = \sigma_\xi, \qquad \text{if}\ \ \exists\omega\in\mathcal{W}: \xi'+\rho = \omega(\xi+\rho)$$

and recall that d_ξ and χ_ξ behave odd

$$d_{\xi'} = \mathrm{sign}(\omega)d_\xi, \qquad \chi_{\xi'} = \mathrm{sign}(\omega)\chi_\xi, \qquad \text{if}\ \ \exists\omega\in\mathcal{W}: \xi'+\rho=\omega(\xi+\rho).$$

Lemma 4.4 *Assume G is semi-simple. Then there exists a second order difference operator $\triangle_2$ acting on the lattice of heighest weights such that*

$$d_\xi\mathbb{A}\sigma = \triangle_2(d_\xi\sigma_\xi)\mathrm{I} \tag{4.10}$$

holds true.

Proof It suffices to prove the formula for elementary sequences σ_ξ which are $1/d_{\xi_*}$ for some dominant $\xi = \xi_*$ and 0 otherwise. Then $\mathbb{A}\sigma$ is the Fourier transform of $\rho^2(g)\chi_{\xi_*}(g)$, which in turn can be calculated based on equation (4.7) and (4.9),

$$\mathscr{F}[\mathbb{A}\sigma] = (\dim G - \operatorname{trace}\operatorname{Ad})\chi_{\xi_*} = \sum_{\xi\in\Delta_0}\Big(\delta_\xi\chi_{\xi_*} - \sum_{\xi'\in\mathcal{O}_\xi}\chi_{\xi'}\chi_{\xi_*}\Big)$$
$$= \sum_{\xi\in\Delta_0}\Big(\delta_\xi\chi_{\xi_*} - \sum_{\xi'\in\mathcal{O}_\xi}\chi_{\xi_*+\xi'}\Big) = \mathscr{F}[(d_\xi^{-1}\triangle_2(d_\xi\sigma_\xi))\mathrm{I}]$$

with $\delta_\xi = |\mathcal{O}_\xi|$ and the difference operator

$$\triangle_2\tau_\xi = \sum_{\xi'\in\Delta_0}\Big(\delta_{\xi'}\tau_\xi - \sum_{\xi''\in\mathcal{O}_{\xi'}}\tau_{\xi-\xi''}\Big)$$

acting on the weight lattice Λ. Near the walls of the Weyl chamber we made use of the particular extension of σ_ξ. The difference operator $\triangle_2$ annihilates linear functions on the lattice and is therefore of second order. □

Example 4.5 On the group $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ we obtain for $\mathbb{A} \sim d_1 - \operatorname{trace} t^1$ (in the notation of [15]) that central sequences σ^ℓ satisfy (4.10) with $\triangle_2\sigma^\ell = 2\sigma^\ell - \sigma^{\ell-1} - \sigma^{\ell+1}$, which is (up to sign) the usual second order difference on $\frac{1}{2}\mathbb{Z}$.

Remark 4.6 The statement of Lemma 4.4 extends to arbitrary compact groups. The additional representations used to define $\rho^2(g)$ give more summands adding up to another second order difference operator on the lattice.

Remark 4.7 N. Weiss used in [23] the remarkably similar looking function

$$\gamma(\exp\tau) = \sum_{\omega\in\mathcal{W}} \mathrm{e}^{2\pi\mathrm{i}(\omega\rho,\tau)} - |\mathcal{W}|,$$

$\mathcal{W}$ the Weyl group and again ρ the Weyl vector, in place of our distance function $\rho^2(g) = \dim G - \operatorname{trace}\operatorname{Ad}(g)$. This function seems to simplify the treatment of central multipliers (as the associated difference operator δ acts in a much simpler way on central sequences), but it does not allow the use of a finite Leibniz rule which is important for our proof in the non-central case. It is remarkable that $\delta d_\xi = 0$.

4.2 Functions of the Laplacian

We say a bounded function f defined on a normed linear space V has an asymptotic expansion at ∞,

$$f(\eta) \sim \sum_{k=0}^{\infty} f_k(\eta), \qquad |\eta| \to \infty \tag{4.11}$$

if there exist functions $f_k(\eta)$, homogeneous of order k for large η, such that

$$|f(\eta) - \sum_{k=0}^{N} f_k(\eta)| \le C_N(1+|\eta|)^{-N} \tag{4.12}$$

holds true for certain constants C_N. We fix a maximal torus $\mathcal{T}$ of G and denote by $\mathfrak{t}^*$ the dual of its Lie algebra.

Lemma 4.8 *Assume $f : \mathfrak{t}^* \to \mathbb{C}$ is bounded, even under the action of the Weyl group,*

$$f(\xi) = f(\xi') \quad \text{if } \xi' + \rho = \omega(\xi + \rho) \text{ for some } \omega \in \mathcal{W},$$

and has an asymptotic expansion into smooth components at ∞, and denote by $f(\xi)$ its restriction to the weight lattice $\Lambda \subset \mathfrak{t}^*$. Then the central sequence $f(\xi)\mathrm{I}$ defines an L^p-bounded multiplier on G for all $1 < p < \infty$.

Remark 4.9 It is enough to assume the asymptotic expansion up to fixed finite order $\varkappa$ as in Theorem 2.1.

Proof We identify $\mathfrak{t}^*$ with $\mathbb{R}^t$, $t = \operatorname{rank} G$, which is the space V in definition (4.11).

In a first step let $f_k(\eta)$ be smooth and homogeneous of degree $-k$ on $|\eta| \geq 1$. Then $f_k \in S^{-k}(\mathbb{R}^t)$ and by the arguments of [15, Theorem 4.5.3] we immediately get that the restriction of f to the lattice belongs to the symbol class $\mathscr{S}_1^{-k}(\mathcal{T})$.

Furthermore, lattice differences preserve $\mathcal{O}\big((1+|\eta|)^{-N}\big)$ for any N. Therefore, choosing N in dependence on the order of the difference we immediately see that the restriction of f to the lattice belongs to $\mathscr{S}_1^0(\mathcal{T})$.

In order to obtain the L^p-boundedness we follow the proof of Theorem 2.1. Note that ψ_r is defined in terms of the pseudo-distance ρ and therefore central. Hence $\widehat{\psi}_r(\xi)$ is a central sequence (also denoted by $\widehat{\psi}_r(\xi)$ for the moment and extended evenly to the full lattice) and thus by Lemma 4.4 in combination with Lemma 4.1 we obtain the desired bounds for the HS-norm of

$$\mathbb{A}(f(\xi)\widehat{\psi}_r(\xi)) = \frac{1}{d_\xi}\triangle_2(d_\xi f(\xi)\widehat{\psi}_r(\xi))$$

and for corresponding higher differences with respect to $\mathbb{A}$. □

Corollary 4.10 *Assume $f : \mathbb{R}_+ \to \infty$ has an asymptotic expansion up to order $\varkappa$ into homogeneous components at ∞. Then $f(-\Delta)$ is bounded on $L^p(G)$ for $1 < p < \infty$.*

Proof This follows from the fact that

$$\lambda_\xi^2 = ||\xi + \rho||^2 - ||\rho||^2$$

is even and has the desired asymptotic expansion in ξ. This implies that $f(\lambda_\xi^2)$ also has an asymptotic expansion, see Remark 4.9, and one-dimensionality allows one to choose the components of the expansion as smooth functions. □

Remark 4.11 Coifman and G. Weiss showed in [7] that central multipliers correspond to $L^p(G)$-bounded operators if $\mathscr{D}(d_\xi \sigma_\xi)$ is an $L^p(\mathcal{T})$-bounded multiplier on the corresponding lattice, where $\mathscr{D}$ is the product of elementary (backward) differences $\triangle_{-\alpha}$ corresponding to the positive roots $\alpha \in \Delta_0^+$.

5 Applications to non-central operators

In this section we give applications to invariant and non-invariant operators. Difference operators $\mathbb{D}^\alpha$ in this section correspond to those in Theorem 2.1 for simplicity of the formulations. However, in Remark 5.3 we explain that those associated to the extended root system analogously to those in Theorem 3.5 will suffice.

5.1 Mapping properties of operators of order zero.

As a second main example we consider operators associated to symbols $\mathscr{S}^0_\rho(G)$ of type $\rho \in [0, 1]$, i.e. matrix symbols for which

$$\|\mathbb{D}^\alpha \sigma_A(\xi)\|_{\mathrm{op}} \leq C_\alpha \langle \xi \rangle^{-\rho|\alpha|},$$

holds for all α and all $[\xi] \in \widehat{G}$, and ask for mapping properties of such operators within Sobolev spaces over $L^p(G)$. Such symbol classes appear naturally as parametrices for non-elliptic operators, see Example 2.6 and Corollary 2.7. We now give a refined version of a multiplier theorem for such operators:

Corollary 5.1 *Let $\rho \in [0, 1]$ and let $\varkappa$ be the smallest even integer larger than $\frac{1}{2}\dim G$. Assume that A is a left-invariant operator on G with matrix symbol σ_A satisfying*

$$\|\mathbb{D}^\alpha \sigma_A(\xi)\|_{\mathrm{op}} \leq C_\alpha \langle \xi \rangle^{-\rho|\alpha|} \quad \textit{for all} \quad |\alpha| \leq \varkappa \tag{5.1}$$

and all $[\xi] \in \widehat{G}$. Then A is a bounded operator mapping the Sobolev space $W^{p,r}(G)$ into $L^p(G)$ for $1 < p < \infty$ and

$$r = \varkappa(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Proof The proof follows by interpolation from two end point statements, the trivial one for $p = 2$ and the fact that $\langle\xi\rangle^{-\varkappa(1-\rho)}\sigma_A(\xi)$ defines an operator of weak type $(1, 1)$ on $L^1(G)$. The latter follows from Theorem 2.1 in combination with Leibniz rule (3.6) for difference operators,

$$\|\mathbb{D}^\alpha \langle\xi\rangle^{-\varkappa(1-\rho)}\sigma_A(\xi)\|_{\mathrm{op}} \lesssim \sum_{\ell,m\leq|\alpha|\leq\ell+m} \langle\xi\rangle^{-\varkappa(1-\rho)-\ell-m\rho} \lesssim \langle\xi\rangle^{-\varkappa+(\varkappa-|\alpha|)\rho}$$

which can be estimated by $\langle\xi\rangle^{-|\alpha|}$ whenever $|\alpha| \leq \varkappa$. □

Similar to Remark 5.3, Corollary 5.1 remains true if in (5.1) we take only the single difference $\mathbb{A}$ of order $\varkappa$ and only those differences that are associated to the extended root system Δ_0 for $|\alpha| \leq \varkappa - 1$, if we apply Theorem 3.5 instead of Theorem 2.1 in the proof.

We also note that the variable coefficient version $\mathscr{S}^m_{\rho,\delta}(G)$ of these classes $\mathscr{S}^m_\rho(G)$, especially the class $\mathscr{S}^m_{1,\frac{1}{2}}(G)$, played an important role in the proof of the sharp Gårding inequality on compact Lie groups in [16].

5.2 Proof of Corollary 2.7

Let X be left-invariant vector field on the group G with $\sigma_X(\xi) = (X\xi)(1)$ as its symbol. We assume[3] that the bases of the representation spaces are chosen such that $\sigma_X(\xi)$ is diagonal for all $[\xi] \in \widehat{G}$. Let further $[\eta] \in \widehat{G}$ be a fixed representation with associated differences $\mathbb{D}_{ij} = {}_\eta\mathbb{D}_{ij}$. Then for some τ_{ij} we have

$$\mathbb{D}_{ij}\sigma_X = (X\eta_{ij})(1) I_{d_\xi\times d_\xi} = \tau_{ij} I_{d_\xi\times d_\xi}$$

as can be seen immediately on the Fourier side and follows from $\mathscr{F}\delta_1 = I_{d_\xi\times d_\xi}$. By our choice of representation spaces, $\tau_{ij} = 0$ for $i \neq j$ and $\sum_j \tau_{jj} = 0$. The latter one is just

[3] This can always be arranged by diagonalising symmetric matrices; see also [15, Remark 10.4.20].

another formulation of the fact that the derivatives of the character $\chi_\eta(x) = \operatorname{trace} \eta(x)$ vanish in the identity element 1. Now

$$\sigma_{X+c}(\xi) = \sigma_X(\xi) + cI$$

is invertible for all ξ, whenever $c \notin \operatorname{spec}(-X) \subset \mathrm{i}\mathbb{R}$. For such c the Leibniz rule (3.5) for $\mathbb{D}_{ij}$ implies

$$0 = (\mathbb{D}_{ij}\sigma_{X+c}^{-1})\sigma_{X+c} + \tau_{ij}\sigma_{X+c}^{-1} + \sum_{k=1}^{d_\eta} \tau_{kj}(\mathbb{D}_{ik}\sigma_{X+c}^{-1}),$$

so that

$$\mathbb{D}_{ij}\sigma_{X+c}^{-1} = 0$$

for $i \neq j$ and $c + \tau_{jj} \notin \operatorname{spec}(-X)$, and

$$\mathbb{D}_{jj}\sigma_{X+c}^{-1} = -\tau_{jj}\sigma_{X+c}^{-1}\left(\sigma_{X+c+\tau_{jj}}\right)^{-1} = -\tau_{jj}\left(\sigma_X + cI\right)^{-1}\left(\sigma_X + (c+\tau_{jj})I\right)^{-1}.$$

Using this recursion formula we see that $\sigma_{X+c}^{-1} \in \mathscr{S}_0^0(G)$ provided all appearing matrix inverses exist, which means

$$c \notin \operatorname{spec}(-X) - \mathrm{i}\mathbb{N}[\tau_{11}, \ldots, \tau_{ll}],$$

where the latter stands for the set of all linear combinations of $\tau_{11}, \ldots, \tau_{ll}$ with integer coefficients. Outside this exceptional set of parameters by Corollary 5.1 we conclude the L^p-estimate

$$\|f\|_{L^p(G)} \leq C_p \|(X+c)f\|_{W^{p,\varkappa|\frac{1}{p}-\frac{1}{2}|}(G)}$$

for all $1 < p < \infty$.

5.3 Non-invariant pseudo-differential operators

The result for multipliers implies the L^p-boundedness for non-invariant operators if we assume sufficient regularity of the symbol. Again, such a result is an extension of the L^p-boundedness of pseudo-differential operators.

Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a linear continuous operator (not necessarily invariant). Following [15], we define its matrix symbol

$$\sigma_A : G \times \widehat{G} \to \bigcup_{[\xi]\in\widehat{G}} \mathbb{C}^{d_\xi \times d_\xi}$$

so that for each $(x, [\xi]) \in G \times \widehat{G}$, the matrix $\sigma_A(x,\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ is given by

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x).$$

In particular, for the left-invariant operators we have (2.5). Consequently, it was shown in [15] that such symbols are well-defined on $G \times \widehat{G}$ and that the operator A can be quantised as

$$A\phi(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{trace}(\xi(x)\sigma_A(x,\xi)\widehat{\phi}(\xi)).$$

We also have the relation (2.3) in this setting.

Let ∂_{x_j}, $1 \leq j \leq n$, be a collection of left invariant first order differential operators corresponding to some linearly independent family of the left-invariant vector fields on G. We denote $\partial_x^\beta := \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$. In [15], and completed in [18], it was shown that the Hörmander class $\Psi^m(G)$ of pseudo-differential operators on G defined by localisations can be characterised in terms of the matrix symbols. In particular, we have $A \in \Psi^m(G)$ if and only if its matrix symbol σ_A satisfies

$$\|\partial_x^\beta \mathbb{D}^\alpha \sigma_A(x,\xi)\|_{\mathrm{op}} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|}$$

for all multi-indices α, β, for all $x \in G$ and $[\xi] \in \widehat{G}$. For the L^p-boundedness it is sufficient to impose such conditions up to finite orders as follows, extending Theorem 2.1 to the non-invariant case:

Theorem 5.2 *Denote by $\varkappa$ be the smallest even integer larger than $\frac{n}{2}$, n the dimension of the group G. Let $1 < p < \infty$ and let $l > \frac{n}{p}$ be an integer. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a linear continuous operator such that its matrix symbol σ_A satisfies*

$$\|\partial_x^\beta \mathbb{D}^\alpha \sigma_A(x,\xi)\|_{\mathrm{op}} \leq C_{\alpha,\beta} \langle \xi \rangle^{-|\alpha|} \tag{5.2}$$

for all multi-indices α, β with $|\alpha| \leq \varkappa$ and $|\beta| \leq l$, for all $x \in G$ and $[\xi] \in \widehat{G}$. Then the operator A is bounded on $L^p(G)$.

Remark 5.3 The modifications similar to other formulations of multiplier theorems regarding the choice of difference operators remain true in a straightforward way. For example, it is enough to impose difference conditions $\mathbb{D}^\alpha$ in (5.2) only with respect to the (extended) root system. Thus, in analogy with Theorem 3.5, the conclusion of Theorem 5.2 remains true if we impose

$$\|\partial_x^\beta \mathbb{A}^{\varkappa/2} \sigma_A(x,\xi)\|_{\mathrm{op}} \leq C \langle \xi \rangle^{-\varkappa}$$

as well as (5.2) only for $\mathbb{D}^\alpha \in \mathscr{D}^{\varkappa-1}$, for all $|\beta| \leq l$. Similarly, Corollary 5.1 can be extended to the general (non-invariant) case.

Proof Let $Af(x) = (f * r_A(x))(x)$, where

$$r_A(x)(y) = R_A(x,y)$$

denotes the right-convolution kernel of A. Let

$$A_y f(x) := (f * r_A(y))(x),$$

so that $A_x f(x) = Af(x)$. Then

$$\|Af\|_{L^p(G)}^p = \int_G |A_x f(x)|^p \,\mathrm{d}x \leq \int_G \sup_{y\in G} |A_y f(x)|^p \,\mathrm{d}x.$$

By an application of the Sobolev embedding theorem we get

$$\sup_{y\in G} |A_y f(x)|^p \leq C \sum_{|\alpha|\leq l} \int_G |\partial_y^\alpha A_y f(x)|^p \,\mathrm{d}y.$$

Therefore, using the Fubini theorem to change the order of integration, we obtain

$$\begin{aligned}
\|Af\|^p_{L^p(G)} &\leq C \sum_{|\alpha|\leq l} \int_G \int_G |\partial_y^\alpha A_y f(x)|^p \,\mathrm{d}x\,\mathrm{d}y \\
&\leq C \sum_{|\alpha|\leq l} \sup_{y\in G} \int_G |\partial_y^\alpha A_y f(x)|^p \,\mathrm{d}x \\
&= C \sum_{|\alpha|\leq l} \sup_{y\in G} \|\partial_y^\alpha A_y f\|^p_{L^p(G)} \\
&\leq C \sum_{|\alpha|\leq l} \sup_{y\in G} \|f \mapsto f * \partial_y^\alpha r_A(y)\|^p_{\mathcal{L}(L^p(G))} \|f\|^p_{L^p(G)} \\
&\leq C\|f\|^p_{L^p(G)},
\end{aligned}$$

where the last inequality holds due to Theorem 2.1. □

References

1. Akylzhanov, R., Nursultanov, E., Ruzhansky, M.: Hardy-Littlewood inequalities and Fourier multipliers on SU(2). arXiv:1403.1731
2. Alexopoulos, G.: Spectral multipliers on Lie groups of polynomial growth. Proc. Am. Math. Soc. **120**, 973–979 (1994)
3. Anker, J.-P.: L_p Fourier multipliers on Riemannian symmetric spaces of the noncompact type. Ann. of Math. **132**, 597–628 (1990)
4. Coifman, R., de Guzman, M.: Singular integrals and multipliers on homogeneous spaces. Rev. Un. Mat. Argent. **25**, 137–143 (1970)
5. Coifman, R., Weiss, G.: Multiplier transformations of functions on SU(2) and Σ_2. Rev. Un. Mat. Argent. **25**, 145–166 (1970)
6. Coifman, R., Weiss, G.: Analyse Harmonique Non-Vommutative Sur Certains Espaces Homogenes, Lecture Notes in Mathematics, vol. 242. Springer, Berlin Heidelberg (1971)
7. Coifman, R., Weiss, G.: Central multiplier theorems for compact Lie groups. Bull. Am. Math. Soc. **80**, 124–126 (1973)
8. Cowling, M.: Harmonic analysis on semigroups. Ann. Math. **117**, 267–283 (1983)
9. Cowling, M., Sikora, A.: A spectral multiplier theorem for a sublaplacian on SU(2). Math. Z. **238**, 1–36 (2001)
10. Fegan, H.D.: Introduction to Compact Lie Groups, Series in Pure Mathematics, vol. 13. World Scientific, Singapore (1991)
11. Hörmander, L.: Estimates for translation invariant operators in L^p spaces. Acta Math. **104**, 93–140 (1960)
12. Nikolskii, S.M.: Approximation of functions of several variables and imbedding theorems (Russian), 2nd edn. Nauka, Moscow (1977)
13. Ruzhansky, M., Turunen, V.: On the Fourier analysis of operators on the torus, modern trends in pseudo-differential operators. Oper. Theory Adv. Appl. **172**, 87–105 (2007)
14. Ruzhansky, M., Turunen, V.: Quantization of pseudo-differential operators on the torus. J. Fourier Anal. Appl. **16**, 943–982 (2010)
15. Ruzhansky, M., Turunen, V.: Pseudo-Differential Operators and Symmetries. Birkhäuser, Basel (2010)
16. Ruzhansky, M., Turunen, V.: Sharp Gårding inequality on compact Lie groups. J. Funct. Anal. **260**, 2881–2901 (2011)
17. Ruzhansky, M., Turunen, V.: Global quantization of pseudo-differential operators on compact Lie groups, SU(2) and 3-sphere. Int. Math. Res. Not. (2012). Article ID rns122, 58 pp. doi:10.1093/imrn/rns122
18. Ruzhansky, M., Turunen, V., Wirth, J.: Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity. J. Fourier Anal. Appl. **20**, 476–499 (2014)

19. Ruzhansky, M.V., Wirth, J.: On multipliers on compact Lie groups. Funct. Anal. Appl. **47**(1), 72–75 (2013)
20. Ruzhansky, M., Wirth, J.: Global functional calculus for operators on compact Lie groups. J. Funct. Anal. **267**, 144–172 (2014)
21. Seeley, R.T.: Complex powers of an elliptic operator. Singular integrals. In: Proceedings of the Symposium on Pure Mathematics, Chicago, Ill., 1966, pp. 288–307. American Mathematical Society, Providence, RI (1967)
22. Stein, E.M.: Topics in Harmonic Analysis Related to the Littlewood–Paley Theory, vol. 63 of Annals of Mathematics Studies. Princeton University Press, Princeton (1970)
23. Weiss, N.: L^p estimates for bi-invariant operators on compact Lie groups. Am. J. Math. **94**, 103–118 (1972)