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Michael Helmuth Ruge

Louisiana State University and Agricultural & Mechanical College

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L_p REGULARITY AND EXTRAPOLATION

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by
Michael H. Ruge
Vordiplom, Universität Kaiserslautern, Federal Republic of Germany, 1982
Master of Science, Louisiana State University at Baton Rouge, U.S.A., 1986
Diplom, Universität Kaiserslautern, Federal Republic of Germany, 1988
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ABSTRACT.

The goal of this thesis was to isolate classes of bounded linear operators in $L_p(I)$ which on the one hand still have some of the well-known and useful properties of positive operators, but which on the other hand are large enough to include some important classes of operators (e.g. the Hilbert transform and the singular operators derived from it) that cannot be dominated by positive operators.

In Chapter I, we study as a first class of this kind the L_p *regular* operators. By definition such operators map equiintegrable sets in $L_p(I)$ into equiintegrable sets in $L_p(I)$ and sets compact in measure into sets compact in measure. We show that with respect to duality and pertubation theory they have properties similar to positive operators.

In Chapter II, we study *strongly* L_p *regular* operators as the class of operators, which preserves growth restrictions of L_p functions (formulated in terms of nonincreasing rearrangements of functions). We show that such operators can be extended to bounded linear operators on certain Lorentz and Marcinkiewicz spaces. Many important operators in analysis are in this class since we can show that all *interpolated* operators are strongly L_p regular.

Chapter III contains some representation theorems for linear operators in $L_p(I)$ by kernels of distributions, which are motivated by the representation of positive operators by stochastic kernels.

INTRODUCTION.

This research is motivated by the following well-known and useful properties of a regular[1)] operator T in $L_p(I)$, $1 \leq p < \infty$:

1) For any $0 \leq h \in L_p(I)$, there exists a $0 \leq g \in L_p(I)$ such that $|f| \leq h$ implies $|Tf| \leq g$ (cf **(Schä II)**).

2) There is a density $g \in L_1(I)$ such that $g^{-1/p}Tg^{1/p}$ extends to a bounded linear operator in $L_r(gd\mu)$ for any $1 \leq r \leq \infty$, i.e. T can be *extrapolated* to L_1 and L_∞ spaces (cf [**Wei III**]).

3) There is a stochastic kernel $(\mu_x)_{x \in I}$ of signed measures on I such that for all $f \in L_p(I)$:

$$Tf(x) = \int_I f d\mu_x \text{ a.e.,}$$

and the modulus $|T| : L_p(I) \to L_p(I)$ of T is given by

$$|T|f(x) = \int_I f d|\mu_x| \text{ a.e. (cf [\textbf{Arv}]).}$$

Each of these properties gives a characterisation of regular operators. Hence, it is clear that e.g. the Hilbert transform and other singular integral operators, which are not regular, cannot have these properties.

In this study, we look for conditions similar to **1)**, **2)**, **3)**, but somewhat less restrictive so that they are satisfied by useful operators like the Hilbert transform which do *not* meet the above conditions.

. . . .

[1)] $T : L_p(I) \to L_p(I)$ is called *regular* if there exists a positive operator $S : L_p(I) \to L_p(I)$ with $|Tf| \leq S|f|$ for all $f \in L_p(I)$.

In Chapter I, we consider the following weaker version of **1)** for an operator $T : L_p(I) \to L_p(I)$:

1') If $A \subset L_p(I)$ is an equiintegrable subset[2], then $T(A) \subset L_p(I)$ is equiintegrable.

While **1')** is a selfdual property, we show in Section I.2. that T has **1')** if and only if (iff) its dual $T' : L_q(I) \to L_q(I)$, $1/p + 1/q = 1$, maps sets compact in measure into sets compact in measure. Operators with this property and **1')** we call L_p *regular*. Then, in Section I.3., we extend a recent result on regular Fredholm pertubations to this much larger class of L_p regular operators. Indeed, every bounded linear operator $T : L_p(I) \to L_p(I)$, $1 \leq p < 2$, satisfies **1')**, and every operator obtained by interpolation is L_p regular (Section I.2.). For $p = 1$, we get a particularly complete characterisation of L_p regular operators in terms of the representation **3)** (see Section I.1.).

In order to obtain a version of the extrapolation result **2)**, we introduce *strongly* L_p *regular* operators in Chapter II as bounded linear opertators $T : L_p(I) \to L_p(I)$, $1 \leq p < \infty$, satisfying:

2') For any $0 \leq h \in L_p(I)$, there exists a $0 \leq g \in L_p(I)$ such that $f^\star \leq h^\star$ implies $(Tf)^\star \leq g^\star$ where $f^\star$ is the nonincreasing rearrangement[3] of f.

This condition is stronger than **1')** - in Section II.1. we construct a compact operator without **1')** -, but still weak enough so that interpolated operators and singular integral operators are strongly L_p regular (see Section II.8.).

. . . .

[2] For the definition of *equiintegrable subset in* $L_p(I)$, see Appendix A.

[3] For the Definition of the *nonincreasing rearrangement*, see Appendix B.

In Section II.6., we show that a strongly L_p regular operator extends to a bounded operator on appropriately chosen Lorentz and Marcinkievicz spaces, i.e. a weaker version of **2)** still holds. Further extrapolation results are given in Section II.7.

In Chapter III, we give representations of L_p operators resembling **3)**, but with the measures (μ_x) replaced by various kinds of distributions. This raises the question whether (strongly) L_p regular operators can be understood in terms of the distribution appearing in their representation (just as regular operators are singled out by the fact that the μ_x's are measures) - but this question remains open.

We need various auxiliary results on equiintegrable sets, rearrangements and regular functions which we collect in the two appendices. For the convenience of the reader, we also include some results on basic sequences in $L_p(I)$.

CHAPTER I.

L_p REGULAR OPERATORS.

1. Definitions and Examples.

In this chapter, let $I := (0,1)$ with (normalized) Lebesque measure μ, and assume $1 \leq p < \infty$, unless indicated otherwise. As usual, $\frac{1}{p'} + \frac{1}{p} = 1$.

Definition 1.1. A linear and bounded operator $T : L_p(I) \to L_p(I)$ is called L_p **regular** if it satisfies the following two conditions:
(P1) T maps equiintegrable subsets of $L_p(I)$ into equiintegrable sets in $L_p(I)$.
(P2) If $A \subset L_p(I)$ is norm-bounded and compact in measure, then $T(A) \subset L_p(I)$ is compact in measure.

A non-empty set $A \subset L_p(I)$ is called **equiintegrable** if for any $\epsilon > 0$, there exists a $c > 0$ such that

$$\int_{|f|>c} |f|^p d\mu < \epsilon$$

for any $f \in A$.

A non-empty set $A \subset L_p(I)$ is called **compact in measure** if for any sequence $(f_n) \subset A$, there exists a subsequence $(f_{n_j}) \subset (f_n)$ and a function $f \in A$ with $f_{n_j} \to f$ in measure.

If $1 < p < \infty$, **(P2)** is equivalent to the following: T preserves convergence in measure, i.e. if $(f_n) \subset L_p(I)$ is bounded and converges to 0 in measure, then $(Tf_n) \subset L_p(I)$ converges to 0 in measure.

Elementary examples for L_p regular operators, $1 \leq p < \infty$, are finite-dimensional and compact operators. Also, these properties are always fulfilled for certain p (see Theorem 2.2.).

The class of L_p regular operators is rather large, as the following examples show.

Example 1.2. Every **regular operator** $T : L_p(I) \to L_p(I)$, $1 < p < \infty$, i.e. every operator for which there is a positive operator $S : L_p(I) \to L_p(I)$ with $|Tf| \leq Sf$ for all $f \in L_p^+(I)$ (cf §*IV.1., p 229* of **(Schä II)**), in particular every **positive operator** is L_p regular. (For $p = 1$, see Example 1.6.)

Proof: To see that T satisfies **(P1)**, choose an equiintegrable set $M \subset L_p(I)$. Then, for all $\epsilon > 0$, there is a $0 < C < \infty$ such that $M \subset B_C + \epsilon U_{L_p}$, where $B_C := \{f \in L_p(I) : |f| \leq C\}$ and U_{L_p} is the unit ball of $L_p(I)$. Then

$$T(A) \subset \{f : |f| \leq C|T|1\} + \epsilon \|T\| U_{L_p},$$

where $|T|$ is the modulus of T (cf §*IV.1., p 229* of **(Schä II)**). It follows that $T(A)$ is equiintegrable in $L_p(I)$. That T also satisfies **(P2)**, now follows from the Duality Theorem 2.1. in the next section, since $T' : L_{p'}(I) \to L_{p'}(I)$ is also regular.

Example 1.3. The **Hilbert transform** $T : L_p(\Gamma) \to L_p(\Gamma)$, $1 < p < \infty$, is L_p regular. Here we define the Hilbert transform on the unit circle Γ by

$$H^\Gamma f(s) := \lim_{\epsilon \to 0+} H_\epsilon^\Gamma f(s),$$

where $s \in (-\pi, \pi]$ and

$$H_\epsilon^\Gamma f(s) := -\frac{1}{\pi} \int_{\pi > |t| \geq \epsilon} \frac{f(s-t)}{2\tan(t/2)} dt = -\frac{1}{\pi} \int_{\pi > |s-t| \geq \epsilon} \frac{f(t)}{2\tan((s-t)/2)} dt.$$

Since it is well-known (cf *Theorem II.2.4., p 117*, and *Theorem 2.6., p 118* of **(Tor)**) that the Hilbert transform on the unit circle Γ is a bounded linear operator on $L_p(\Gamma)$, $1 < p < \infty$, it follows from the interpolation result (Theorem 2.3.) in the next section that the Hilbert transform is L_p regular. Of course, this result implies that a large class of singular integral operators are also L_p regular.

Next we give some examples of L_p operators which are not L_p regular.

Example 1.4. Let (r_n) and (h_n) denote the Rademacher functions and the Haar system as defined in Appendix A.2. If (h'_n) denotes the Haar functions normalized in $L_{p'}(I)$ where $1 < p < 2$, then (h'_n) forms an unconditional basis for $L_{p'}(I)$ (cf Appendix A.2.). By Lemma A.2. and Khintchine's Inequality (cf Appendix A.2.), we have that

$$Tf = \sum \left(\int f h'_n d\mu \right) r_n$$

defines a bounded linear operator $T : L_p(I) \to L_p(I)$. Since $h'_n \to 0$ in measure and $Th'_n = r_n$, it is clear that T violates **(P2)**. The dual operator $T' : L_{p'}(I) \to L_{p'}(I)$ maps r_n into h'_n for all n, and therefore does not satisfy **(P1)**.

Example 1.5. Denote by $F : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$ the **Fourier transform** on the real line $(-\infty, \infty)$, i.e. for $f \in L_2(-\infty, \infty) \cap L_1(-\infty, \infty)$, let

$$Ff(t) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{-itx} dx$$

(here dx stands for the Lebeque measure) with the isometric extension to all of $L_2(-\infty, \infty)$ given by Plancherel's Theorem.

Fix $f \in L_1(-\infty,\infty) \cap L_2(-\infty,\infty)$, $\|f\|_2 = 1$, supp $f \subset K$ where $K \subset (-\infty,\infty)$ is compact. Set $g(t) := Ff(t)$ for $t \in (-\infty,\infty)$.

Define $(f_n) \subset L_1(-\infty,\infty) \cap L_2(-\infty,\infty)$ by $f_n(t) := f(t-n)$. Then $g_n(t) := Ff_n(t) = Ff(t)\, e^{-int}$ t-a.e. (cf § *VI.1., p 121* of **(Kat)**). Therefore, $F^{-1}g_n = f_n$, thus $F\overline{g_n} = F(\overline{Ff_n}) = \overline{f_n}$ (cf *Theorem VI.1.11., p 125* of **(Kat)**). Here, $\bar{z}$ denotes the complex conjugate to z.

Observe that $|g_n| = |g|$ and $\|f_n \chi_\Omega\| \to 0$ as $n \to \infty$ for any bounded set $\Omega \subset (-\infty,\infty)$. Thus (g_n) is equiintegrable in $L_2(-\infty,\infty)$, and (f_n) converges to 0 in measure.

Choose $\zeta \in L_1(-\infty,\infty)$, $0 < \zeta < \infty$ on $(-\infty,\infty)$, and consider the isometry $J : L_2(-\infty,\infty) \to L_2[(-\infty,\infty),\zeta dx]$ given by $f \to f \circ \zeta^{-1}$. Note that $L_2[(-\infty,\infty),\zeta dx]$ is isomorphic to $L_2(I)$ if $I := (0,1)$.

Then $T := JFJ^{-1}$ does not have **(P1)** and **(P2)**.

Indeed, for $(\tilde{f}_n)$ and $(\tilde{g}_n)$ given by $\tilde{f}_n := Jf_n$ and $\tilde{g}_n := Jg_n$, we see that $T\tilde{f}_n = \tilde{g}_n$ and $T\overline{\tilde{g}_n} = \overline{\tilde{f}_n}$. But $(\tilde{f}_n)$ converges to 0 in measure, while $(\tilde{g}_n)$ is equiintegrable in $L_2[(-\infty,\infty),\zeta dx]$.

Example 1.6. Every bounded linear operator $T : L_1(I) \to L_1(I)$ can be written as $T = T^d + T^a$ where T^a and T^d are operators of the form

$$T^a f(x) = \sum_{n=1}^{\infty} a_n(x) f(\sigma_n(x))$$

(here $a_n : I \to (-\infty,\infty)$, $\sigma_n : I \to I$ are Borel functions such that for μ-almost all (μ-a.a.) $x \in I$: $|a_n(x)| \geq |a_{n+1}(x)|$, $\sum_{n=1}^{\infty} |a_n(x)| < \infty$ and $\sigma_n(x) \neq \sigma_m(x)$ for $m \neq n$) and

$$T^d f(x) = \int f(y) d\nu_x(y)$$

(here $(\nu_x)_{x \in X}$ is a kernel of diffuse measures on I). For more details, see *Proposition 2.6.* and *Theorem 6.2.* of [**Wei IV**]. As in Example 1.2., one can show that T always has **(P1)**. With respect to **(P2)**, we have that

1) T^a always has **(P2)**.

2) T^d has **(P2)** iff the image of the unit ball U in $L_p(I)$ is compact in measure. In short: T is L_1 regular iff $T^d(U)$ is compact in measure. Furthermore:

3) There are integral operators without **(P2)**, and not every L_1 operator T with $T(U)$ compact in measure is an integral operator.

4) Let μ be a diffuse measure on T such that the Fourier coefficients $\hat{\mu}(n)$ do not converge to 0. Then T does not have **(P2)**.

Note that every L_1 operator which maps $L_\infty(I)$ into $L_\infty(I)$ defines an L_p regular operator $T : L_p(I) \to L_p(I)$ for all $1 < p < \infty$ (e.g. by Example 1.2.).

Proof: 1) For **(P2)**, it is enough to demonstrate that if (f_i) is bounded in $L_1(I)$ and $f_i(y) \to 0$ for all $y \in I$, then $T^a f_i(x) \to 0$ for μ-a.a. $x \in I$.

This certainly holds when the sum in the definition of T^a is finite. To reduce the general case to such finite sums, we choose by Egoroff's Theorem $E_n \subset I$, $E_n \subset E_{n+1} \subset \ldots$ with $\mu(I - \cup E_n) = 0$ and $\sum_{k=m}^{\infty} |a_k(x)| \to 0$ as $m \to \infty$ uniformly on each E_n. Now it follows that $Tf_i(y) \to 0$ in measure for $i \to \infty$ on each E_n.

2) If $T^d(U)$ is compact in measure, then **(P2)** is obviously satisfied. On the other hand, let $(f_n) \subset L_1(I)$ be a normalized sequence such that (Tf_n) is not compact in measure. If we can find a normalized $(g_n) \subset L_1(I)$ with $\mu(\text{supp } g_n) \to 0$ and $\|T^d g_n - T^d f_n\| \to 0$, then **(P2)** cannot hold for T. This is a consequence of the following claim.

Claim: For every $f \in L_1(I)$ and $\epsilon > 0$, there is a function $g \in L_1(I)$ with $\mu(\text{supp } g) \leq \epsilon$ and $\|T^d f - T^d g\| \leq \epsilon$.

Proof: It follows from the proof of the Lemma in **[Wei V]** that the claim holds for functions of the form $f = \mu(A)^{-1}\chi_A$. For a general function $f \in L_p(I)$, choose a simple function $\tilde{f} = \sum_{i=1}^{n} \frac{a_i}{\mu(A_i)}\chi_{A_i}$ with $\|f - \tilde{f}\| \leq \frac{\epsilon}{2\|T\|}$. For each i, there is a function g_i with $\mu(\text{supp } g_i) \leq \frac{\epsilon}{n}$, supp $g_i \subset A_i$ and

$$\|T^d g_i - T^d(\mu(A_i)^{-1}\chi_{A_i})\| \leq \frac{\epsilon}{2\|T\|}.$$

Then $g := \sum_{i=1}^{n} a_i g_i$ has the desired properties.

3) In order to find an integral operator T in $L_1(I)$ with $T(U)$ not compact in measure, we first choose a quotient map S of l_1 onto the span-closure R of the Rademacher functions (see Appendix A for their definition, also consult **(Lin)**). If P is a further quotient map of $L_1(I)$ onto l_1, then $T := S \circ P : L_1(I) \to L_1(I)$ is weakly compact, since R is isomorph to l_2 (see Khintchine's Inequality, Lemma A.2. of the Appendix), and therefore an integral operator (cf *Section III.2., p 67* of **(Die)**). But $T(U) = U_R$ is not compact in measure, since it contains the Rademacher functions.

To find the second operator, we choose a subspace X of $L_1(I)$ whose unit ball is compact in measure, but does not have the Radon-Nikodym property (cf **[Bou II]**). Then there is an operator $T : L_1(I) \to X \subset L_1(I)$ which is not representable, and therefore not an integral operator (cf *Section III.2* of **(Die)**), although $T(U) \subset U_X$ is compact in measure.

4) If $|\hat{\mu}(n_k)| \geq C > 0$, then $T(e^{in_k t}) = \hat{\mu}(n_k)e^{in_k t}$ forms a subspace of $T(U)$ which is not compact in measure. Hence T does not have **(P2)** by Part 2).

Let us denote by B_p the Banach algebra of all bounded linear operators $T : L_p(I) \to L_p(I)$ and by R_p the subset of all L_p regular operators.

Theorem 1.7. R_p, $1 \leq p < \infty$, is a norm-closed subalgebra of B_p which is **full** in the sense that for an invertible $T \in R_p$, we have that $T^{-1} \in R_p$.

Proof: It is easy to see that R_p is a subalgebra. R_p is closed in the operator norm since a bounded sequence $(f_n) \subset L_p(I)$ is equiintegrable (converges to 0 in measure) if for every $\epsilon > 0$, there is an equiintegrable sequence $(g_n) \subset L_p(I)$ (a sequence (g_n) which converges to 0 in measure) with $\|g_n - f_n\| \leq \epsilon$.

Finally, assume that T is invertible and L_p regular. If $S := T^{-1}$ does not satisfy **(P1)**, then there is a normalized equiintegrable sequence $(f_n) \subset L_p(I)$ such that (Sf_n) is not equiintegrable in $L_p(I)$. By Lemma A.1. of the Appendix, we can write a subsequence of (Sf_n) - call it again (Sf_n) - as a sum of an equiintegrable sequence $(g_n) \subset L_p(I)$ and a disjoint sequence (h_n), where $\|g_n\|$ does not converge to 0. Then $f_n = TSf_n = Tg_n + Th_n$, where (Tg_n) still is equiintegrable and (Th_n) converges to 0 in measure. Since (f_n) also converges to 0 in measure, it follows from Lemma A.2. Part A) that $\|Tg_n\| \to 0$. Since T is invertible, we obtain the contradiction $\|g_n\| \to 0$.

It can be shown similarly that T satisfies **(P2)**. For $p > 1$, this also follows from the duality result in Theorem 2.1., since $T' : L_{p'}(I) \to L_{p'}(I)$ is invertible and $L_{p'}$ regular.

Remark 1.8. That the algebra R_p is full, in particular implies that the spectrum of $T \in R_p$ is the same with respect to R_p as with respect to B_p. It is well-known (cf *Sect. IV.1., p 231* of **(Schä II)**) that this is not true for regular operators and the algebra of regular operators considered in *Sect. IV.1.* of **(Schä II)**. This also indicates that the L_p regular operators are a meaningfull extension of the classical regular operators: The resolvent operator of a regular operator may not be regular but is at least L_p regular.

2. Duality and Interpolation.

In this section, we show that for $1 < p < \infty$, L_p regularity is a self-dual property (Theorem 2.1.) and automatically holds for interpolated operators (Theorem 2.3.). Some of these properties were already used to find the examples in Section 1.

Theorem 2.1. Assume $T : L_p(I) \to L_p(I)$ is linear and bounded. Then: T has **(P1)**, iff $T' : L_{p'}(I) \to L_{p'}(I)$ has **(P2)**. T has **(P2)**, iff T' has **(P1)**.

Proof: i) Assume T has **(P1)**, but T' does not possess **(P2)**. Then we may assume without loss of generality (wlog) that there is $(f'_n) \subset L_{p'}(I)$, $\|f'_n\| = 1$ such that (f'_n) converges to 0 in measure, but $(T'f'_n)$ does not.

By Lemma A.1. of the Appendix, we may wlog write $T'f'_n = g'_n + h'_n$ where (g'_n) is an equiintegrable and (h'_n) a pairwise disjoint sequence in $L_{p'}(I)$. Since $(T'f'_n)$ is not equintegrable, we have $0 < \limsup \|g'_n\|$. Thus, wlog, we may assume $g'_n \not\equiv 0$ for all n and $0 < \lim \|g'_n\| < \infty$.

Using the Hahn-Banach theorem, we obtain $(g_n) \subset L_p(I), \|g_n\| = 1$ with $g_n(g'_n) = \|g'_n\|$.

As before, utilizing Lemma A.1. of the Appendix, we obtain wlog an equiintegrable sequence (U_n) and a disjoint sequence (u_n) in $L_p(I)$ such that for all n, U_n and u_n have disjoint support and $g_n = U_n + u_n$. The bounded- and disjointness of (u_n) imply its convergence to 0 in measure; in particular we have that $u_n(g'_n) \to 0$ (cf Lemma A.2. of the Appendix). Also, since $\|g'_n\| = U_n(g'_n)+u_n(g'_n)$, we can assume wlog, possibly by taking a subsequence of (U_n), that there is a $\delta > 0$ such that $U_n(g'_n) \geq \delta$ for all n.

Since by **(P1)**, (TU_n) is equiintegrable and (f'_n) a bounded sequence converging to 0 in measure, by Lemma A.2. of the Appendix, we have $f'_n(TU_n) \to 0$. Since (h'_n) is a disjoint and bounded sequence while (U_n) is equiintegrable, again by Lemma A.2. of the Appendix, we have $h'_n(U_n) \to 0$.

On the other hand, we see that $f'_n(TU_n) = g'_n(U_n) + h'_n(U_n)$. This forces $U_n(g'_n) = g'_n(U_n) \to 0$. This contradicts $U_n(g'_n) \geq \delta$.

ii) Now suppose that T has **(P2)**, but T' does not possess **(P1)**. Again, we then may assume that there exists a sequence (f'_n) which is equiintegrable in $L_{p'}(I)$, but $(T'f'_n)$ is not; furthermore (cf Lemma A.1. of the Appendix) $T'f'_n = g'_n+h'_n$ where (g'_n) is equiintegrable in $L_{p'}(I)$, (h'_n) is a disjoint sequence with $h'_n \not\equiv 0$ for all n and $\lim \|h'_n\| \geq c$ for some $c > 0$.

As in Part i), applying the Hahn-Banach Theorem to (h'_n) and utilizing Lemmata A.1 and A.2. of the Appendix, we obtain a normalized sequence $(h_n) \subset L_p(I)$, an equiintegrable sequence $(U_n) \subset L_p(I)$ and a disjoint sequence $(u_n) \subset L_p(I)$ such that for any n and some $\delta > 0$: $h_n(h'_n) = \|h'_n\|$, $h_n = U_n+u_n$ and $u_n(h'_n) \geq \delta$ (since $U_n(h'_n) \to 0$ as $n \to \infty$).

Furthermore, by **(P2)** and Lemma A.2, we have $f'_n(Tu_n) \to 0$. But also $f'_n(Tu_n) = g'_n(u_n) + h'_n(u_n)$. This forces $h'_n(u_n) \to 0$. This contradicts $u_n(h'_n) \geq \delta$.

The reverse implications follow by considering the dual operators.

Theorem 2.2. Assume $T : L_p(I) \to L_p(I)$ is a bounded and linear operator.

A) T always has **(P1)** if $1 \leq p < 2$.

B) T always has **(P2)** if $2 < p < \infty$.

The Fourier transform (cf Example 1.5.) shows that neither Part A) nor Part B) hold for $p = 2$.

Proof: A) Let $(f_n) \subset L_p(I)$ be equiintegrable. Since (f_n) is weakly compact, we may assume $\|f_n\| = 1$ and $f_n \to 0$ weakly. By *Prop. 1.a.12., p 7* of **(Lin I)**, there exists a subsequence of (f_n) - call it again (f_n) - which is basic.

If $p = 1$, then the equiintegrability of (f_n) is equivalent to its weak (sequential) precompactness (cf *Sect. IV.2., Theorem 1* of **(Die)**). Since T is bounded, it is also weakly continuous, and (Tf_n) weakly converges to 0. Thus (Tf_n) is equiintegrable.

Let $1 < p < 2$, and assume that (Tf_n) is not equiintegrable. By Lemma A.3. and Lemma A.4. of the Appendix, we see that for $T \not\equiv 0$:

$$c(\sum |\alpha_n|^p)^{1/p} \geq \|T\|^{-1} \|\sum \alpha_n T f_{(n)}\| \geq \|T\|^{-1} c'(\sum |\alpha_n|^p)^{1/p}$$

for some c, $c' > 0$ and some subsequence $(f_{(n)}) \subset (f_n)$.

But this implies that $(f_{(n)})$ is equivalent to the unit vector basis of l_p. By Lemma A.4. of the Appendix, $(f_{(n)})$ cannot be equiintegrable, a contradiction.

B) Let $2 < p < \infty$. Assume that there is a normalized sequence (f_n) which converges to 0 in measure, but (Tf_n) does not converge to 0 in measure.

Furthermore, wlog, suppose that (Tf_n) does not contain any subsequences converging to 0 in measure.

By selecting a subsequence again, we may assume that $(Tf_n) \subset M_p^\epsilon$ for some $\epsilon > 0$ (cf Lemma A.4. of the Appendix). Thus by *Theorem 3., p 166* of **[Kad]**, (Tf_n) is equivalent to the unit vector basis of l_2.

Since (f_n) contains a subsequence - call it again (f_n) - equivalent to the unit vector basis of l_p, we see that

$$c'(\sum |\alpha_n|^2)^{1/2} \leq \|T\| \|\sum \alpha_n f_n\| \leq c\,(\sum |\alpha_n|^p)^{1/p}$$

for some c, $c' > 0$. This would imply $l_p \subset l_2$ with $p > 2$ which is impossible.

Theorem 2.3. Assume $1 \leq p_1 < p_2 \leq \infty$ and $T : L_{p_i}(I) \to L_{p_i}(I)$, $i = 1, 2$ be linear and bounded. For any p, $p_1 < p < p_2$, T is a bounded linear operator from $L_p(I)$ into $L_p(I)$ which in addition is L_p regular.

Proof: T is an operator from $L_p(I)$ into $L_p(I)$ by the Riesz Interpolation Theorem (cf *Theorem IV.1.7., p 192* of **(Ben)**). According to the previous duality theorem, it suffices to show that T satisfies **(P2)**.

To this end, assume there is a bounded sequence $(f_n) \subset L_p(I)$ converging to 0 in measure such that (Tf_n) does not converge to 0 in measure. An application of Hölder's inequality shows that $\|f_n\|_{L_q(I)} \to 0$ for any $q < p$. Choose q with $p > q > p_1$. The boundedness of T as an operator from $L_q(I)$ into $L_q(I)$ then implies $\|Tf_n\|_{L_q(I)} \to 0$.

But convergence in norm induces convergence in measure in $L_q(I)$ and $L_p(I)$.

3. L_p Regular Operators as Fredholm Pertubations.

Although the class of L_p regular operators is much larger than the class of regular operators, it still shares some of the operator-theoretic properties of regular operators. For example, we show in this section that an L_p regular admissable Fredholm pertubation is compact (Corrollary 3.4.). This extends a result of **[Cas]** on regular operators.

The main step to obtain 3.4. is the following characterization of strictly singular operators.

Definition 3.1. An operator $T : L_p(I) \to L_p(I)$ is called **strictly singular** if $T|_M$ is not an isomorphic embedding for any infinite-dimensional subspace M of $L_p(I)$.

In Hilbert spaces and in the sequence spaces l_p, $1 \leq p < \infty$, the class of strictly singular operators coincides with the class of compact operators (cf *Theorem 5.2.2., p 82* of **(Pie I)**), but in $L_p(I)$, $p \neq 2$, there are strictly singular operators which are not compact (cf §*5.3.* of **(Pie I)**). We shall now show that these examples cannot be L_p regular.

Theorem 3.2. An L_p regular, bounded linear operator $T : L_p(I) \to L_p(I)$, $1 < p < \infty$, is strictly singular if and only if it is compact.

Proof: It is well-known that every compact operator is strictly singular. (Indeed, if $T|_M$ is compact and an isomorphism of M onto $T(M)$, then M has to be finite-dimensional.)

Now consider the case where $T : L_p(I) \to L_p(I)$ is strictly singular with $p > 2$. Assume T is not compact. Then we can select an unconditional normalized basis $(f_n) \subset L_p(I)$ weakly converging to 0 and satisfying for some $\delta > 0$ and all n: $\|Tf_n\| \geq \delta$. Then (Tf_n) weakly converges to 0.

Applying Lemma A.1. of the Appendix, we may write wlog $f_n = g_n + h_n$ where (g_n) is an equiintegrable, (h_n) is a disjoint sequence and for all n, g_n and h_n are pairwise disjoint.

By **(P1)** and **(P2)**, we have that (Tg_n) is equiintegrable and (Th_n) converges to 0 in measure. Thus (Th_n) converges weakly to 0, since (Th_n) is bounded. Also, as $Tg_n = Tf_n - Th_n$, (Tg_n) converges weakly to 0.

i) Assume first that for some $d > 0$, we have that $\|Tg_n\| \geq d$ for all n. Applying Lemma A.4. of the Appendix, we see that for some $\epsilon > 0$:

$$(Tg_n) \subset M_p^{\epsilon}(I) := \{f \in L_p(I) : \mu\{|f| \geq \epsilon\|f\|\} \geq \epsilon\},$$

where μ denotes (normalized) Lebesque measure. By *Theorem 3., p 166* of **[Kad]**, we may assume wlog that (Tg_n) is equivalent to the unit vector basis of l_2. The above condition on the equiintegrable sequence (g_n), i.e. the fact that $\|g_n\| \geq d/\|T\|$, together with an application of Lemma A.2. of the Appendix imply wlog that (g_n) is equivalent to the unit vector basis of l_2.

Setting $M := \overline{\text{span}}\ [g_n]$, we see that $T|_M$ is an isomorphism, and thus T cannot be strictly singular.

ii) We may therefore assume wlog that (Tg_n) converges to 0 in norm and also that for some $d > 0$: $\|Th_n\| \geq d$, for $Th_n = Tf_n - Tg_n$.

Thus Lemma A.3. of the Appendix can be applied to (Th_n): We obtain wlog that (Th_n) is equivalent to the unit vector basis of l_p.

On the other hand, since (h_n) is a disjoint sequence weakly converging to 0, satisfying for all n: $\|h_n\| \geq d/\|T\|$, we may assume wlog that (h_n) is equivalent to the unit vector basis of l_p.

Thus a contradiction is immediate for $M := \overline{\text{span}}\,[h_n]$.

The case $1 < p < 2$ is now quickly settled via the Duality Theorem 2.1., since by **[Wei I]**, T' is strictly singular.

The case $p = 2$ follows from the fact quoted above that in a Hilbert space compact and strictly singular operators always coincide. (If T is not compact, then there is a normalized sequence $(f_n) \in L_2(I)$ with $f_n \to 0$ weakly and $\inf \|Tf_n\| > 0$. Now both (f_n) and (Tf_n) contain subsequences which are equivalent to the unit vector basis of l_2.)

Definition 3.3. A) A bounded, linear operator $S : L_p(I) \to L_p(I)$ is called a **Fredholm operator** if the dimension of its kernel and the codimension of its range are both finite.
B) A bounded, linear operator $T : L_p(I) \to L_p(I)$ is an **(admissable) Fredholm pertubation** if for every Fredholm operator $S : L_p(I) \to L_p(I)$ the sum $T + S$ still is a Fredholm operator.

It was shown in **[Wei I]** that $T : L_p(I) \to L_p(I)$ is an admissable Fredholm pertubation iff T is strictly singular. Theorem 3.2. now gives:

Corollary 3.4. An L_p regular operator T is an admissable Fredholm pertubation iff T is compact.

Recall that every Fredholm operator $S : L_p(I) \to L_p(I)$ has a **Fredholm**

inverse, i.e. a (Fredholm) operator $T : L_p(I) \to L_p(I)$ such that $I - ST$ is compact.

Theorem 3.5. Every Fredholm inverse of an L_p regular Fredholm operator is also L_p regular.

Proof: We only show **(P2)** since **(P1)** is similar to Theorem 1.7.

Let (f_n) be a bounded sequence in $L_p(I)$, $1 < p < \infty$, converging to 0 in measure. We have to show that (Sf_n) converges to 0 in measure.

By Lemma A.1., write (Sf_n), wlog, as the sum of an equiintegrable sequence (g_n) and a disjoint sequence (h_n). Write $TS = I + K_1$ and $ST = I + K_2$ where K_i, $i = 1, 2$, are compact operators.

Since (f_n) is bounded and converges to 0 in measure, we have $(K_i f_n)$ converges to 0 in norm.

Applying T to $Sf_n = g_n + h_n$ gives $Tg_n = K_1 f_n + f_n - Th_n$. Since all the terms on the right converge to 0 in measure, we see that (Tg_n) converges to 0 in measure.

Since T has **(P1)** and **(P2)**, (Tg_n) also is equiintegrable and by Lemma A.2. Part A) of the Appendix, we get $\|Tg_n\| \to 0$. But $f_n = Tg_n - K_1 f_n - Th_n$ and

$$(*) \qquad Sf_n = STg_n - SK_1 f_n - STh_n,$$

thus $STh_n = h_n + K_2 h_n$. Hence (STh_n) converges to 0 in measure, and from (*), we see that (Sf_n) converges to 0 in measure.

Remark 3.6. Theorem 3.5. implies that the essential spectrum of an L_p regular Fredholm operator S equals the spectrum of its equivalence class $\hat{S}$ in the quotient algebra R_p/K_p modulo the compact operators K_p in $L_p(I)$. Again, the relation between the essential spectrum of S and spectra with respect to quotients of the classical regular operators is much more complicated. In particular, the essential resolvent of a regular operator consists not necessarily of regular operators, but according to Theorem 3.5., these operators still are L_p regular.

4. L_p Regularity in Terms of Rearrangements.

The following theorem provides an equivalent characterization of Condition **(P1)**.

Theorem 4.1. Assume $T : L_p(I) \to L_p(I)$, $1 < p < \infty$, is a bounded and linear operator. Then T satisfies Condition **(P1)** if and only if one of the following equivalent conditions holds:

(R1) For any sequence $(f_n) \subset L_p(I)$ satisfying $f_n^* \leq f$ for some nonincreasing nonnegative function $f \in L_p(I)$ and all n, there exist a *sub*sequence $(f_{(k)}) \subset (f_n)$ and a nonincreasing function $g \in L_p(I)$ with $(Tf_{(k)})^* \leq g$ for any k.

(R2) For any sequence $(f_n) \subset L_p(I)$ satisfying $f_n^{**} \leq f$ for some nonincreasing nonnegative function $f \in L_p(I)$ and all n, there exist a *sub*sequence $(f_{(k)}) \subset (f_n)$ and a nonincreasing function $g \in L_p(I)$ with $(Tf_{(k)})^{**} \leq g$ for any k.

(R3) If $A \subset L_p(I)$ is a set such that $\{f^* : f \in A\}$ is norm-compact, then $\{(Tf)^* : f \in A\}$ is norm-compact in $L_p(I)$.

In Condition **(R1)**, **(R2)** or **(R3)**, we may require that the functions f or g (or both) are regular, or we may replace any of them by their second rearrangements.

Proof: (P1) $\Longrightarrow$ (R1) If $f_n^* \leq f \in L_p(I)$, then (f_n) and, by assumption, (Tf_n) are equiintegrable sequences in $L_p(I)$. By Lemma B.6., i) $\Longrightarrow$ iii) of the Appendix, there is a subsequence (Tf_{n_k}) and a function $g \in L_p(I)$ with $(Tf_{n_k})^* \leq g$.

(R1) $\Longrightarrow$ (P1) Let $M \subset L_p(I)$ be equiintegrable. For any sequence $(Tf_n) \subset T(M)$, by Lemma B.6., i) $\Longrightarrow$ iii), there is a subsequence $(f_{n_k}) \subset (f_n)$ and a function $f \in L_p(I)$ with $(f_{n_k})^* \leq f$. By assumption, there is a subsequence - say $(f_{n_l}) \subset (f_{n_k})$ - and a function $g \in L_p(I)$ with $(Tf_{n_l})^* \leq g$. Now Lemma B.2., iii) $\Longrightarrow$ i) of the Appendix implies that $T(M)$ is equiintegrable.

(P1) $\Longrightarrow$ (R2) follows in the same manner using Lemma B.2., i) $\Longrightarrow$ iv).

(P1) $\Longrightarrow$ (R3) directly follows from Lemma B.2., i) $\Longrightarrow$ ii).

(R1) $\Longleftarrow$ (R2) $\Longleftarrow$ (R3) follows as in Lemma B.6.

By Lemma B.7., it is clear that in Condition **(R1)** or **(R2)**, regularity for any of the functions may or may not be required. Since for an L_p function f with $1 < p < \infty$, we have that $f \leq f^{**}$ and $\|f^{**}\| \leq D_p \|f\|$ with $D_p := \frac{p}{p-1}$ (cf Remark B.4. of the Appendix), we may also replace f or g (or both) by their second rearrangements in either of the conditions **(R1)** or **(R2)**.

There is also an easy reformulation of **(P2)** in terms of rearrangements:

Proposition 4.2. Let $T : L_p(I) \to L_p(I)$, $1 < p < \infty$, be bounded and linear. Then **(P2)** is equivalent to:

(P2') If (f_n) is a bounded sequence in $L_p(I)$ and $f_n^*(x) \to 0$ for all $x \in I$, then $(Tf_n)^*(x) \to 0$ for all $x \in I$.

Proof: It follows from the definition of f^* that $f_n^*(x) \to 0$ for all x iff $f_n \to 0$ in measure.

CHAPTER II.

EXTRAPOLATION.

5. Strongly L_p Regular Operators.

Unless indicated otherwise, the same assumptions on p and p' hold as in Chapter I, i.e. $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ where $q := p' = \infty$ if $p = 1$. In this chapter, with the exception of Section 8, I denotes either the interval $(0,1)$ or $(0,\infty)$.

Definition 5.1. A bounded linear operator $T : L_p(I) \to L_p(I)$ is called **strongly L_p regular** if:

(S1) For every nonincreasing nonnegative function $g \in L_p(I)$, there exists a nonincreasing nonnegative function $h \in L_p(I)$ such that $f \in L_p(I)$, $f^\star \leq g$ implies $(Tf)^\star \leq h$.

Remark 5.2. In our proofs, we shall need the following equivalent variations of Condition **(S1)**:

(S2) For every nonincreasing nonnegative function $g \in L_p(I)$, there exists a nonincreasing nonnegative function $h \in L_p(I)$ such that $f \in L_p(I)$, $f^{\star\star} \leq g$ implies $(Tf)^{\star\star} \leq h$.

(S3) There is a constant $c > 0$, depending only on $T : L_p(I) \to L_p(I)$, such that for every nonincreasing nonnegative function $g \in L_p(I)$, $\|g\| \leq 1$, there exists a nonincreasing nonnegative function $h \in L_p(I)$, $\|h\| \leq c$ such that $f \in L_p(I)$, $f^{\star\star} \leq g$ implies $(Tf)^{\star\star} \leq h$.

Furthermore, in Condition **(S1)**, **(S2)** or **(S3)**, we may require that the functions f or g (or both) are regular, or we may replace any of them by their second rearrangements.

Proof: The equivalence of the conditions **(S1)** and **(S2)**, as well as the last sentence of the remark are shown as in Theorem 4.1. Thus let us establish that Condition **(S2)** induces the validity of Condition **(S3)**.

To this end, assume that there is a sequence $(f_m) \subset L_p(I)$ of nonincreasing nonnegative functions with $\|f_m\| = 1$ such that (α_m) given by $\alpha_m := \inf \{ \|h\| : h \in L_p(I)$ is nonincreasing and nonnegative, and $f \in L_p(I)$, $f^{**} \leq f_m$ implies $(Tf)^{**} \leq h \}$ does not stay bounded. Wlog, we may assume that $\alpha_m \geq 4^m$.

Set $F := \sum 2^{-m} f_m$. Thus $F \in L_p(I)$. Applying **(R2)** on the nonincresing nonnegative function F, we obtain a $G \in L_p(I)$ and such that $f \in L_p(I)$, $f^{**} \leq F$ implies $(Tf)^{**} \leq G$. Since $f_m \leq 2^m F$ for all m, by the definition of (α_m), we see that $\|2^m G\| \geq 4^m$ or $\|G\| \geq 2^m$ for any m. This contradicts $G \in L_p(I)$.

Remark 5.3. By Section I.4. every strongly L_p regular operator on $I := (0,1)$ satisfies **(P1)**. But there are L_p regular - even compact - operators which are not strongly L_p regular as the following example shows. (For further examples of strongly L_p regular operators see Section 8.)

Example 5.4. For $n \geq 0$, $1 \leq p < \infty$ set

$$\beta_n := 2^{n/p}\,(n+1)^{-1/p}\,[\ln(n+2)]^{-1/p}.$$

Let $I := (0.1)$ and $F_n : I \to [0,\infty)$, $n > 0$, be given by

$$F_n(x) := \begin{cases} \beta_{n-1} & \text{if } 2^{-n-1} < x \leq 2^{-n}; \\ 0 & \text{otherwise.} \end{cases}$$

$(F_n) \subset L_p(I)$ is a disjoint sequence converging to 0 in norm.

Let (r_n) denote the set of Rademacher functions and

$$P : L_p(I) \to \overline{\operatorname{span}}\,[r_n] \subset L_p(I)$$

be the canonical projection. It is well-known that P is a bounded and linear operator, e.g. Khintchine's Inequality (see Remark A.2. of the Appendix). We now define the linear operator $L : \overline{\operatorname{span}}\,[r_n] \to L_p(I)$ by $Lr_n = F_n$.

Finally, define $T := L \circ P : L_p(I) \to \overline{\operatorname{span}}\,[g_n]$.

Claim: T is compact.

We can write T as $T = \chi_{(0,2^{-n}]}T + \chi_{(2^{-n},1)}T$ where the second operator is finite-dimensional (its range is in the span of $F_1, \ldots, F_{2^n-1}$) and the first operator satisfies $\|\chi_{(0,2^{-n}]}T\| \to 0$ as $n \to \infty$.

Indeed, for $f = \sum a_n r_n$, we get

$$\|\chi_{(0,2^{-n}]}Tf\|^p = \|\sum_{k=n}^{\infty} a_k F_k\|^p = \sum_{k=n}^{\infty} |a_k|^p \|F_k\|^p.$$

i) If $1 \leq p < 2$, an application of Hölder's inequality gives with $A := 2/(2-p)$ that

$$\leq (\sum_{k=n}^{\infty} |a_k|^2)^{p/2} \, (\sum_{k=n}^{\infty} \|F_k\|^{Ap})^{p/A} \leq \|f\| (\sum_{k=n}^{\infty} k^{-A}(\ln(k+1))^{-A})^{p/A}.$$

ii) If $2 \leq p < \infty$, we estimate

$$\leq \sup_{k \geq n} \|F_k\| \, (\sum_{k=n}^{\infty} |a_k|^2)^{p/2} \leq n^{-1/p} \|f\|^p.$$

In any case, we showed that $\|\chi_{(0,2^{-n}]}T\| \to 0$ for $n \to \infty$, and T is compact.

Claim: For $g \equiv 1$, Condition **(S1)** cannot be met.

Indeed, for any $n > 0$, we have $r_n^\star = g \equiv 1$, $(Tr_n)^\star = F_n^\star$ and $F_n^\star(x) = \beta_{n-1}$ for $0 < x < 2^{-n-1}$. Since

$$\int_{2^{-n-1}}^{2^{-n}} \beta_n^p \, dx = 1/2\,(n+1)^{-1}\ln(n+2),$$

and

$$\sum_{n>0} \int_{2^{-n-1}}^{2^{-n}} \beta_n^p \, dx = \infty,$$

any h as in **(S1)** would have to satisfy $h(x) \geq \beta_n$ for a.e. $2^{-n-1} < x \leq 2^{-n}$. Therefore, such h cannot lie in $L_p(I)$.

Thus T is not strongly L_p regular.

6. Extrapolation into Lorentz and Marcinkiewicz Spaces.

For the remainder of this chapter, if $1 \leq k \leq \infty$, let $\| \, \|_k$ denote the norm on $L_k(I)$ for $I := (0,1)$ or $(0,\infty)$. Also, it is understood that if $I := (0,1)$, then any condtition involving behaviour of a function at ∞ must be modified in an appropriate manner or omitted.

Under slight assumptions on g and h, it is possible to express Condition **(S2)** in terms of Marcinkiewicz spaces. We shall need the following definitions.

Definition 6.1. A nondecreasing function $\phi : I \to (0,\infty)$ satisfying $\phi(0+) := \lim_{x\to 0+} \phi(x) = 0$ is called **quasiconcave** if for $t > 0$, the function $\phi(t)/t$ is nonincreasing (cf *p 47* of **(Kre)**). We define the **concave majorant** $\tilde{\phi}$ of a quasiconcave function ϕ by

$$\tilde{\phi}(t) := \sup \{ \sum_{i=1}^{\infty} \lambda_i \phi(t_i) \ : \ \lambda_i \geq 0, \ \sum_{i=1}^{\infty} \lambda_i = 1, \ \sum_{i=1}^{\infty} \lambda_i t_i = t \}.$$

Then $\frac{1}{2}\tilde{\phi} \le \phi \le \tilde{\phi}$ (cf *p 49* of **(Kre)**). If $\psi(f)$ given by $\psi(f)(t) := tf(t)$ for $t > 0$ and some measurable function f on I is a quasiconcave function, then its concave majorant is denoted by $\tilde{\psi}(f)$. We set $\psi(f)(0+) := \lim_{t\to 0+} \psi(f)(t)$,
ψits concave majorant is denoted by $\psi(f)$. We set $\psi(f)(0+) := \lim_{t\to 0+} \psi(f)(t)$,
$\psi(f)(\infty) := \lim_{t\to\infty} \psi(f)(t)$ and $f_\star(t) := t/f(t)$.

$\phi(0+) := \lim_{x\to 0+} \phi(x) = 0$ and $\phi(\infty) := \lim_{x\to\infty} \phi(x) = \infty$ is defined to be the collection of all measurable function $x = x(t)$ on $I := (0,1)$ or $(0,\infty)$ satisfying

$$\|x\|_{\Lambda(\phi)} := \int_0^\infty x^\star(t)\ d\phi(t) < \infty,$$

where $x^\star$ denotes the first rearrangement of x on I (see Appendix B). $\Lambda(\phi)$ is a separable Banach space under $\|\ \|_{\Lambda(\phi)}$ (see **(Kre)**, *pp 107-115*). Under these assumptions on ϕ, the dual space is the **Marcinkiewicz space** $M(\phi)$ given by the norm

$$\|x\|_{M(\phi)} := \sup_{h\in I} \frac{1}{\phi(h)} \int_0^h x^\star(s)\ ds\ .$$

Remark 6.2. If $I := (0,1)$ or $(0,\infty)$ and $\phi(t) \equiv \phi_\alpha(t) := t^{1-1/\alpha}$ for some $1 < \alpha < \infty$ and any $0 < t < 1$, then according to *p 220* of **(Ben)**, we see that $M(L^\alpha) \equiv M(\phi_\alpha) = L^{\alpha,\infty}(I)$, and its dual space is $\Lambda(L^{\alpha'}) \equiv \Lambda(\phi_\alpha) = L^{\alpha',1}(I)$.

Here, $L^{p,q}(I)$, $1 \le p \le q$, $I := (0,1)$ or $(0,\infty)$, denotes the Banach space of all measurable functions f for which the norm

$$\|f\|_{p,q} = \begin{cases} [\int_I (t^{1/p} f^\star(t))^q t^{-1} dt]^{1/q} & \text{if } 1 \le q < \infty \\ \sup_I\ [t^{1/p} f^\star(t)] & \text{if } q = \infty \end{cases}$$

is finite. Note that for $q > p$, the above expression only defines a quasinorm, but that there is an actual norm equivalent to it (cf *Definition 2.b.8., p 142* of **(Lin II)**). For $p = q$, it is clear that $L^{p,q}(I)$ coincides with $L_p(I)$.

For $1 < p \leq \infty$ and $1 \leq q \leq \infty$, in the definition of the norm of $L^{p,q}(I)$, we may replace the first rearrangement f^* of the function f by its second rearrangement f^{**} without altering the Banach space (up to isomorphism). Furthermore, the dual space of $L^{p,q}(I)$ for $1 < p < \infty$ and $1 \leq q \leq \infty$ is isomorphic to $L^{p',q'}(I)$. (See **(Ben)**, *Lemma 4.5., p 219* and *Corollary 4.8., p 221.*)

Lemma 6.3. Let $T : L_p(I) \to L_p(I)$ be a bounded linear operator. Let $g, h \in L_p(I)$ be regular functions satisfying Condition **(S2)** for this operator T. If $I := (0, \infty)$, also assume $\psi(g)(\infty) = \psi(h)(\infty) = \infty$. Then
(S4) T defines a bounded operator from $M(\psi(g)) \equiv M(\psi(g^{**}))$ to $M(\psi(h)) \equiv M(\psi(h^{**}))$.

Proof: The regularity of g implies that $g^{**} \leq M[g]\ g$. Since $h \leq h^{**}$, we see that T satisfies the following: $f^{**} \leq g^{**}$, $f \in L_p(I)$ implies $(Tf)^{**} \leq M[g]\ h^{**}$ where $M[g]$ denotes the constant of regularity for g (cf Remark B.4. of the Appendix).

Let f_i denote either g^{**} or h^{**}. Then $\psi(f_i)$ are quasiconcave functions, since for example $\psi(g^{**})(t) = tg^{**}(t) = \int_0^t g(s)ds$ is nondecreasing, while $g^{**}(t)$ is nonincreasing. Furthermore, the concave majorants satisfy $\tilde{\psi}(f_i)(0+) = 0$ and $\tilde{\psi}(f_i)(\infty) = \infty$.

If $\|f\|_{M(\tilde{\psi}(g^{**}))} \leq 1$, then for any $t > 0$:

$$f^{**}(t) \leq \frac{1}{t}\tilde{\psi}(g^{**})(t) \leq \frac{2}{t}\psi(g^{**})(t)$$

or $f^{**} \leq 2g^{**}$. Condition **(S2)** then implies $(Tf)^{**} \leq 2M[g]\ h^{**}$, i.e.

$$\|Tf\|_{M(\tilde{\psi}(h^{**}))} \leq 2M[g].$$

According to Remark B.4. of the Appendix, we see that $g \leq g^{**} \leq M[g]\, g$. Therefore, we have $\psi(g) \leq \psi(g^{**}) \leq M[g]\, \psi(g)$. Looking at the Marcinkiewicz norms, we see that $M(\psi(g))$ and $M(\psi(g^{**}))$ are equivalent. Furthermore, from Definition 6.1., we have $\frac{1}{2}\tilde{\psi}(g) \leq \psi(g) \leq \tilde{\psi}(g)$. Thus $M(\tilde{\psi}(g))$ and $M(\psi(g))$ are isomorphic.

Clearly, the same holds for $M(\tilde{\psi}(h))$, $M(\psi(h))$ and $M(\psi(h^{**}))$.

The following technical lemma will be needed in the sequel.

Lemma 6.4. Assume that $T : L_p(I) \to L_p(I)$ is strongly L_p regular. Define $h : L_1^\star(I) \to L_1^\star(I)$ by

$$h(g) := \sup\{(Tf)^{**} : f \in L_p(I), f^{**} \leq (g^{1/p})^{**}\}$$

(for the definition of $L_1^\star(I)$, see Lemma B.7. of the Appendix). Then:

i) there is a constant $C > 0$ such that $\|h(g)\|_p \leq C\ \|g\|_1^{1/p}$;

ii) if $g_1 \leq g_2$ with $g_i \in L_1^\star(I)$, then $h(g_1) \leq h(g_2)$;

iii) if $g_i \uparrow g$ with $g_i, g \in L_1^\star(I)$, then $h(g_i) \uparrow h(g)$.

Proof: $h : L_p^\star(I) \to L_p^\star(I)$ is well-defined by **(S2)** and the last sentence of Remark 5.2. Thus Part i) is an immediate consequence of **(S3)** of Remark 5.2. Also, Part ii) is clear.

To show Part iii), consider $h^s(g) := \sup\{(Tf)^{**} : f \in L_p(I)$ is a step function, $f^{**} \leq (g^{1/p})^{**}\}$.

Claim: For any $g \in L_1^\star(I)$, we have $h^s(g) = h(g)$.

Proof: Fix $g \in L_1^\star(I)$. Given $f^{**} \leq (g^{1/p})^{**}$, choose a sequence of step functions $(f_n) \subset L_p(I)$ satisfying $f_n^{**} \leq f^{**}$ and $\|f - f_n\|_p \to 0$ as $n \to \infty$.

Then $\|Tf - Tf_n\|_p \to 0$, and thus

$$(Tf)^{**} = \lim_{n\to\infty}(Tf_n)^{**} \le h^s(g).$$

Thus, $h(g) \le h^s(g)$. Since clearly $h(g) \ge h^s(g)$, we have the *claim*.

It therefore suffices to show Part iii) for $h^s(g)$. Furthermore, since $g_i \uparrow g$ implies $h(g_i) \le h(g)$ by Part ii), it is enough to show that for any $\epsilon > 0$, there exists N such that

$$\|h^s(g_n)\|_p \ge \|h^s(g)\|_p - \epsilon$$

for any $n > N$.

By Fatou's Lemma, we can choose N such that $\|g^{1/p} - g_n^{1/p}\|_p \le \epsilon$ for any $n > N$.

Assume $f \in L_p(I)$ is a step function with $f^{**} \le (g^{1/p})^{**}$. Then $f^{**} \le (h_1 + h_2)^{**}$ where $h_1 := (g_n^{1/p})^*$ and $h_2 := (g^{1/p} - g_n^{1/p})^*$ are nonnegative and nonincreasing. Indeed, this follows from the fact that $(k+l)^{**} \le (k^* + l^*)^{**}$ (i.e. $\int_0^t (k+l)^* d\mu \le \int_0^t (k^* + l^*) d\mu$) for any functions $k, l \in L_p(I)$. Just set $k := g_n^{1/p}$ and $l := g^{1/p} - g_n^{1/p}$.

We may now apply *Theorem III.7.7., p 173* of **(Ben)** to obtain step functions $f_1, f_2 \in L_p(I)$ with $f = f_1 + f_2$ and $f_i^{**} \le h_i^{**}$ for $i = 1, 2$. Thus,

$$(Tf)^{**} = (T(f_1 + f_2))^{**} \le (Tf_1)^{**} + (Tf_2)^{**}$$

$$\le h^s(h_1^p) + h^s(h_2^p) = h^s(g_n) + h^s([g^{1/p} - g_n^{1/p}]^p),$$

i.e.

$$h^s(g) \le h^s(h_1^p) + h^s(h_2^p) = h^s(g_n) + h^s([g^{1/p} - g_n^{1/p}]^p).$$

Therefore, for some constant $C > 0$ and $n > N$,

$$\|h^s(g)\|_p \leq \|h^s(g_n)\| + C\ \|g^{1/p} - g_n^{1/p}\|_p \leq h^s(g_n) + C\ \epsilon,$$

This implies $\|h^s(g)\|_p \leq \|h^s(g_n)\|_p + C\ \epsilon$ which completes the proof.

Lemma 6.5. If $T : L_p(I) \to L_p(I)$, $1 < p < \infty$, is strongly L_p regular, and its dual $T' : L_q(I) \to L_q(I)$ is strongly L_q regular, then the following holds for some $C > 0$ and some nonincreasing positive function $h \in L_1(I)$:
if $f \in L_p(I)$ and $f^{**} \leq h^{1/p}$, then $(Tf)^{**} \leq Ch^{1/p}$; and,
if $f \in L_q(I)$ and $f^{**} \leq h^{1/q}$, then $(T'f)^{**} \leq Ch^{1/q}$.

Proof: Given $g \in L_1^\star(I)$, let $h(g)$ be defined as in Lemma 6.4., and $h_\epsilon(g^{1/p}) \in L_p(I)$ denote the regular function obtained when Lemma B.8. of the Appendix is applied to $f \equiv h(g) \in L_p^\star(I)$, i.e. $h_\epsilon(g^{1/p})$ stands for $h_\epsilon(h(g))$ in Lemma B.8. This regular function then satisfies

$$h_\epsilon(g^{1/p}) \geq h(g),$$

$$D[h_\epsilon(g^{1/p})] \leq \frac{\epsilon/p + 1}{1+\epsilon},$$

and

$$\|h_\epsilon(g^{1/p})\|_p \leq (1+\epsilon)C_1\ \|g\|_1^{1/p},$$

where C_1 is the constant obtained in Part i) of Lemma 6.4.

The same may be done for the dual operator $T' : L_q(I) \to L_q(I)$: For any $\epsilon > 0$ and any $g \in L_1^\star(I)$, by Lemma 6.4. and Lemma B.8. of the Appendix, there is a constant $C_2 \equiv C_2(q, T) > 0$ and a regular function $H_\epsilon(g^{1/q}) \equiv h_\epsilon(H(g)) \in L_q^\star(I)$ such that

$$H_\epsilon(g^{1/q}) \geq H(g) := \sup\{(T'f)^{**} : f \in L_q(I), f^{**} \leq (g^{1/q})^{**}\},$$

$$D[H_\epsilon(g^{1/q})] \leq \frac{\epsilon/q+1}{1+\epsilon},$$

and

$$\|H_\epsilon(g^{1/q})\|_q \leq (1+\epsilon)C_2\ \|g\|_1^{1/q}.$$

Claim: The functions $h_\epsilon(g^{1/p})$ and $H_\epsilon(g^{1/q})$ for any $\epsilon > 0$ satisfy:

α) $g_1 \leq g_2$, where $g_i \in L_1^\star(I)$, implies $h_\epsilon(g_1^{1/p}) \uparrow h_\epsilon(g_2^{1/p})$ and $H_\epsilon(g_1^{1/q}) \uparrow H_\epsilon(g_2^{1/q})$; even

β) $g_n \uparrow g$, where $g, g_i \in L_1^\star(I)$, implies $h_\epsilon(g_n^{1/p}) \uparrow h_\epsilon(g^{1/p})$ and $H_\epsilon(g_n^{1/q}) \uparrow H_\epsilon(g^{1/q})$.

Proof: We shall show the claim for $h_\epsilon(g^{1/p})$; the other case is similar.

α) If $g_1 \leq g_2$, then $h(g_1) \leq h(g_2)$ by Lemma 6.4. Part ii). By Lemma B.8. Part iii) of the Appendix, we have that

$$h_\epsilon(g_1^{1/p}) \equiv h_\epsilon(h(g_1)) \leq h_\epsilon(h(g_2)) \equiv h_\epsilon(g_2^{1/p}).$$

β) By Lemma 6.4. Part iii), we have that $h(g_i) \uparrow h(g)$ if $g_i \uparrow g$. As for Part α), Part β) now follows from Lemma B.8. Part iv) of the Appendix.

Set $S_\epsilon(g) := \frac{1}{4}\{(1+\epsilon)^{-p}C_1^{-p}[h_\epsilon(g^{1/p})]^p + (1+\epsilon)^{-q}C_2^{-q}[H_\epsilon(g^{1/q})]^q\}$. Then $\|S_\epsilon(g)\|_1 \leq \frac{1}{2}\|g\|_1$.

Let $\Delta > 0$ be given.

Pick any (nonincreasing strictly positive) regular function $g_0 \in L_1(I)$ with $D[g_0] \leq \min\{\Delta, 1\}$, $g_0(0+) = \infty$ and $\|g_0\|_1 = \frac{1}{2}$. If $I := (0,\infty)$, we can also arrange $\psi(g_0^{1/p})(\infty) = \infty$ and $\psi(g_0^{1/q})(\infty) = \infty$. Furthermore, once we fix $1 < l < \infty$, we may require $\psi(g_0^l)(0+) = \infty$, too. According to *Theorem 1.1.7., p 6* of **[Rug]**, we see that $g_0^{1/m}$ is regular for any $1 < m < \infty$.

We define a sequence $(g_n) \subset L_1(I)$ of nonincreasing positive functions by $g_{n+1} := g_0 + S_\epsilon(g_n)$. Then $g_{n+1} \geq g_n$ and $\|g_n\|_1 \leq 1$, as one can check inductively:

$$g_{n+1} = g_0 + S_\epsilon(g_n) \geq g_0 + S_\epsilon(g_{n-1}) = g_n,$$

(this is a consequence of Part α) above) and

$$\|g_{n+1}\| \leq \|g_0\| + \|S_\epsilon(g_n)\| \leq \|g_0\| + \frac{1}{2}\|g_n\|.$$

Using Part β) above and Fatou's Lemma, it follows that the sequence (g_n) converges a.e. to a nonincreasing positive function $g_\epsilon \in L_1(I)$ satisfying $g_\epsilon(0+) = \infty$, $g_\epsilon = g_0 + S_\epsilon(g_\epsilon)$. Also, if $I := (0, \infty)$, then $\psi(g_\epsilon^{1/p})(\infty) = \infty$ and $\psi(g_\epsilon^{1/q})(\infty) = \infty$.

First Major Claim: We now wish to demonstrate that for any $1 < m < \infty$, $\Delta > 0$, we can choose $\epsilon \equiv \epsilon(m, \Delta) > 0$ such that for any $g \in L_1^*(I)$, the function $[S_\epsilon(g)]^{1/m}$ is regular with $D[[S_\epsilon(g)]^{1/p}] \leq q + \Delta$ and $D[[S_\epsilon(g)]^{1/q}] \leq p + \Delta$.

Proof: We shall apply the regularity statements of Lemma B.8. Part v) of the Appendix on the functions $[h_\epsilon(g^{1/p})]^{p/m}$ and $[H_\epsilon(g^{1/q})]^{q/m}$ for appropriate $\epsilon > 0$. The following fact is needed:

For any $\infty > n > 0$ we can find a constant $c_n > 0$ such that for all $a, b \geq 0$, we have

$$(*) \qquad c_n^{-1}(a+b)^n \leq a^n + b^n \leq c_n(a+b)^n.$$

Taking $n := m$, $1 < m < \infty$ in (*), by Lemma B.8. Part v) of the Appendix (here $0 < k < p$ translates to $0 < p/m < p$ or $1 < m < \infty$, and

$0 < k < q$ to $0 < q/m < q$ or $1 < m < \infty$), for any $g \in L_1^*(I)$ and any

$$\epsilon > \epsilon_m \equiv \epsilon_m(p,q) := \max\{\frac{p/m-1}{1-1/m}, \frac{q/m-1}{1-1/m}, 0\}$$

$$= \frac{1}{m-1}\max\{p-m, q-m, 0\}$$

we obtain a constant $c \equiv c(p,q,m,T)$ such that

$$c^{-1}\{[h_\epsilon(g^{1/p})]^{p/m} + [H_\epsilon(g^{1/q})]^{q/m}\}$$

$$\leq [S_\epsilon(g)]^{1/m} \leq c\ \{[h_\epsilon(g^{1/p})]^{p/m} + [H_\epsilon(g^{1/q})]^{q/m}\}.$$

Thus, $[S_\epsilon(g)]^{1/m}$ is regular.

Claim: We have

$$D[[S_\epsilon(g)]^{1/m}] \leq \frac{1}{m}\frac{\epsilon + \max\{p,q\}}{\epsilon+1}.$$

Proof: Since

$$D[h_\epsilon(g^{1/p})] \leq \frac{\epsilon/p+1}{1+\epsilon},$$

we have that

$$D[h_\epsilon(g^{1/p})^{p/m}] \leq \frac{p}{m}\frac{\epsilon/p+1}{1+\epsilon} = \frac{1}{m}\frac{\epsilon+p}{\epsilon+1}$$

(cf Lemma B.8. Part v) of the Appendix). Similarly, we obtain

$$D[H_\epsilon(g^{1/q})^{q/m}] \leq \frac{q}{m}\frac{\epsilon/q+1}{1+\epsilon} = \frac{1}{m}\frac{\epsilon+q}{\epsilon+1}.$$

The claim now follows from

$$D[[S_\epsilon(g)]^{1/m}] \leq \max\{D[h_\epsilon(g^{1/p})^{p/m}], D[H_\epsilon(g^{1/q})^{q/m}]\}.$$

Thus, if $\epsilon \to \infty$, then $D[[S_\epsilon(g)]^{1/m}] \to \frac{1}{m}$ or $M[[S_\epsilon(g)]^{1/m}] \to \frac{1}{1-1/m}$. Therefore, given $\Delta > 0$, taking $m = p$ in the above inequality, we can pick $\epsilon > \epsilon_m$ so large such that $M[[S_\epsilon(g)]^{1/p}] \leq q + \Delta$. Similarly, taking $m = q$ in the above inequality, we can choose $\epsilon > 0$ so large such that $M[[S_\epsilon(g)]^{1/q}] \leq p + \Delta$, too.

This proves the first major claim.

Second Major Claim: We now want to show that for any $1 < m < \infty$, there is $\epsilon = \epsilon(m, \Delta) > 0$ such that

$$g_\epsilon^{1/m} = [g_0 + S_\epsilon(g_\epsilon)]^{1/m}$$

is regular with $M[g_\epsilon^{1/p}] \leq q + \Delta$ and $M[g_\epsilon^{1/q}] \leq p + \Delta$.

Proof: Since $g_0^{1/m}$ is regular, by the above, we obtain that $g_0^{1/m} + [S_\epsilon(g)]^{1/m}$ is regular. Taking $n := 1/m$ in (*), we see that $[g_0 + S_\epsilon(g)]^{1/m}$ is regular for any $1 < m < \infty$ and any $\epsilon > \epsilon_m$.

Furthermore, since $D[g_0] \leq \min\{\Delta, 1\}$, we have for $\Delta > 0$ (sufficiently small) that

$$M[g_0^{1/p}] \leq \frac{1}{1 - \Delta/p} \leq q + \Delta$$

and

$$M[g_0^{1/q}] \leq \frac{1}{1 - \Delta/q} \leq p + \Delta.$$

Therefore, if $\epsilon > \epsilon_m$ is sufficiently large, we have for any $g \in L_1^*(I)$ that

$$M[[g_0 + S_\epsilon(g)]^{1/p}] \leq q + \Delta$$

and

$$M[[g_0 + S_\epsilon(g)]^{1/q}] \leq p + \Delta.$$

This holds in particular if we choose $g \equiv g_\epsilon$. Therefore, for $1 < m < \infty$ and $\epsilon > \epsilon_m$ sufficiently large we obtain that

$$g_\epsilon^{1/m} = [g_0 + S_\epsilon(g_\epsilon)]^{1/m}$$

is regular with

$$M[g_\epsilon^{1/p}] = M[[g_0 + S_\epsilon(g_\epsilon)]^{1/p}] \leq q + \Delta$$

and

$$M[g_\epsilon^{1/q}] = M[[g_0 + S_\epsilon(g_\epsilon)]^{1/q}] \leq p + \Delta.$$

This is the second major claim.

Finally, if $f^{**} \leq g_\epsilon^{1/p} \leq (g_\epsilon^{1/p})^{**}$, $f \in L_p(I)$, then

$$(Tf)^{**} \leq h_\epsilon(g_\epsilon^{1/p}) \leq 4^{1/p}(1+\epsilon)C_1\, g_\epsilon^{1/p},$$

and, if $f^{**} \leq g_\epsilon^{1/q} \leq (g_\epsilon^{1/q})^{**}$, $f \in L_q(I)$, then

$$(Tf)^{**} \leq H_\epsilon(g_\epsilon^{1/q}) \leq 4^{1/q}(1+\epsilon)C_2\, g_\epsilon^{1/q}.$$

Choosing $C := \max\{4^{1/p}C_1,\ 4^{1/q}C_2\}\ (1+\epsilon)$, shows that for $h := g_\epsilon$:
If $f \in L_p(I)$ and $f^{**} \leq h^{1/p}$, then $(Tf)^{**} \leq Ch^{1/p}$; and
if $f \in L_q(I)$ and $f^{**} \leq h^{1/q}$, then $(T'f)^{**} \leq Ch^{1/q}$.

This proves Lemma 6.5.

Applying Lemma 6.3. to T and $h^{1/p}$ as g and h in **(S4)**, and then again to T' and $h^{1/q}$ as g and h in **(S4)**, we obtain:

Theorem 6.6. If $T : L_p(I) \to L_p(I)$, $1 < p < \infty$, is strongly L_p regular, and $T' : L_q(I) \to L_q(I)$ is strongly L_q regular, then there exists a nonincreasing positive function $h \in L_1(I)$ such that: T extends to a bounded map

$$\text{from } M(\psi(h^{1/p})) \text{ into } M(\psi(h^{1/p}))$$

and

$$\text{from } \Lambda(\tilde{\psi}([h^{1/q}]^{**})) \text{ into } \Lambda(\tilde{\psi}([h^{1/q}]^{**})).$$

Furthermore, for any $\Delta > 0$ and any $1 < l < \infty$, we can choose this function $h \in L_1(I)$ such that

i) we have $h(0+) = \infty$, $\psi(h^l)(0+) = \infty$, and, if $I := (0, \infty)$,

$$\psi(h^{1/p})(\infty) = \psi(h^{1/q})(\infty) = \infty;$$

ii) $h^{1/p}$ and $h^{1/q}$ are regular with constants of regularity $M[h^{1/p}] \le q + \Delta$ and $M[h^{1/q}] \le p + \Delta$.

Remark 6.7. If we only know that T is strongly L_p regular without any assumption on its dual T', then a simplified version of our argument gives a *one-sided* extrapolation result:

there is a regular function $g \in L_p(I)$ with $\psi(g)$ concave such that T defines a bounded map from $M(\psi(g))$ into $M(\psi(g))$.

7. Further Extrapolation Results.

7.1. Boyd Indices and h-Numbers: Let $I := (0, \infty)$. A Banach space $(E, \|\ \|)$ of measurable functions on I is called **symmetric** if $f \in E$ and $|g| \le |f|$ a.e., implies that $g \in E$ and $\|g\| \le \|f\|$, and if g is equimeasurable with f, then $g \in E$ and $\|g\| = \|f\|$. This is equivalent to: If $f \in E$ and $f^\star \ge g^\star$, then $g \in E$ and $\|f\| \ge \|g\|$.

The **lower and upper dilation exponents** of E are given by

$$\alpha_E := \lim_{s \to 0+} \frac{\ln \|D_s\|}{\ln s}$$

and

$$\beta_E := \lim_{s \to \infty} \frac{\ln \|D_s\|}{\ln s},$$

where $\|D_s\|$ denotes the norm of the linear operator $D_s : E \to E$ given by $D_s f(t) := f(t/s)$ for $s, t \in (0, \infty)$.

Their recipicals are called the **upper and lower Boyd indices** denoted as $p_E := 1/\alpha_E$ and $q_E := 1/\beta_E$.

Given a measurable, everywhere finite function h on $I := (0, \infty)$, define for $s > 0$

$$\underline{h}(s) := \sup_{t \in I} \frac{h(t/s)}{h(t)} \quad \text{and} \quad \underline{\underline{h}}(s) := \inf_{t \in I} \frac{h(t/s)}{h(t)} .$$

If the following limits exist:

$$\underline{h}_0 := \lim_{s \to 0+} \frac{\ln s}{\ln \underline{h}(s)} \quad \text{and} \quad \underline{\underline{h}}_0 := \lim_{s \to 0+} \frac{\ln s}{\ln \underline{\underline{h}}(s)},$$

also,

$$\underline{h}_\infty := \lim_{s \to \infty} \frac{\ln s}{\ln \underline{h}(s)} \quad \text{and} \quad \underline{\underline{h}}_\infty := \lim_{s \to \infty} \frac{\ln s}{\ln \underline{\underline{h}}(s)},$$

we refer to them as the **h-numbers** $(\underline{h}_0, \underline{\underline{h}}_0, \underline{h}_\infty, \underline{\underline{h}}_\infty)$ of the function h. In general, they satisfy

$$\underline{h}_0 \le \underline{\underline{h}}_0 \quad \text{and} \quad \underline{h}_\infty \le \underline{\underline{h}}_\infty .$$

7.2. ***T*-Admissability:** For the remainder of this section, the symbol $T : L_p(I) \to L_p(I)$, $1 < p < \infty$, is reserved for a strongly L_p regular operator whose dual $T' : L_q(I) \to L_q(I)$ is strongly L_q regular, while $h \in L_1(I)$ stands for a function which is obtained when Theorem 6.6. is applied on T and T', i.e. a function such that: T defines a bounded map

$$\text{from } M(\psi(h^{1/p})) \text{ into } M(\psi(h^{1/p})),$$

and also

$$\text{from } \Lambda(\tilde{\psi}([h^{1/q}]^{**})) \text{ into } \Lambda(\tilde{\psi}([h^{1/q}]^{**})).$$

Furthermore, we shall assume that $h \in L_1(I)$ meets all the technical assumptions of Theorem 6.6. for some $\Delta > 0$ and $l > 1$. Such a function h will be called T-**admissable**.

The following theorem does not make any assumptions on $l > 1$.

Theorem 7.3. Let $I := (0, \infty)$. Assume h is T-admissable with its h-numbers satisfying

$$p\underline{h}_\infty > (q\underline{\underline{h}}_0)'.$$

(In particular, this forces $p\underline{h}_\infty > 1$ and $q\underline{\underline{h}}_0 > 1$.)

Furthermore, assume that

$$\frac{1}{M[h^{1/p}]} + \frac{1}{M[h^{1/q}]} \geq 1.$$

Then T defines a bounded map from any symmetric space E into E whose Boyd indices (p_E, q_E) satisfy

$$p\underline{h}_\infty > p_E \geq q_E > (q\underline{\underline{h}}_0)'.$$

In particular, for any $1 \leq r \leq \infty$, T defines a bounded map from $L^{k,r}(I)$ into $L^{k,r}(I)$, and thus also from $L_k(I)$ into $L_k(I)$, provided that

$$p\underline{h}_\infty > k > (q\underline{\underline{h}}_0)'.$$

Proof: Set

$$\phi_0(t) := \psi_0(t) := ([h^{1/p}]^{**})^{-1}(t) = \frac{t}{\int_0^t h^{1/p}(s)ds}$$

and

$$\phi_1(t) := \psi_1(t) := t[h^{1/q}]^{**}(t) = \int_0^t h^{1/q}(s)ds$$

for $t \in (0, \infty)$. Clearly,

$$\phi_1(0+) = \psi_1(0+) = 0.$$

Furthermore,

$$\phi_0(0+) = \psi_0(0+) = 0$$

by L'Hôpital's Rule. Also,

$$\phi_0(\infty) = \psi_0(\infty) = \infty,$$

since $(h^{1/p})^{**} \in L_p(I)$, and

$$\psi_1(\infty) = \psi_1(\infty) = \infty,$$

since $\int_0^t h^{1/q}(s)\, ds \geq th^{1/q}(t) = \psi(h^{1/q})(t)$.

Claim: $\phi_i \equiv \psi_i$, $i = 1, 2$, are quasiconcave functions.

Proof: Clearly, $\phi_1(t)$ is nondecreasing, while $\phi_0(t)/t$ is nonincreasing. Furthermore,

$$(\phi_1(t)/t)' = t^{-1}\left\{-(h^{1/q})^{**}(t) + h^{1/q}(t)\right\} \leq 0,$$

and thus $\phi_1(t)/t$ is nonincreasing. Finally,

$$(1/\phi_1(t))' = t^{-1}\left\{-(h^{1/p})^{**}(t) + h^{1/p}(t)\right\} \leq 0,$$

and thus $\phi_1(t)$ is nondecreasing.

Claim: $\phi_0\phi_1^{-1}$ is nonincreasing over $I := (0, \infty)$.

Proof: Differentiating the function $\phi_0\phi_1^{-1}$ shows that $(\phi_0\phi_1^{-1})' \leq 0$ iff

$$1 \leq \frac{h^{1/p}(t)}{(h^{1/p})^{**}(t)} + \frac{h^{1/q}(t)}{(h^{1/q})^{**}(t)}$$

for all $t \in I$. This is satisfied if

$$\frac{1}{M[h^{1/p}]} + \frac{1}{M[h^{1/q}]} \geq 1.$$

Since $\Lambda(\tilde{\phi}_0) \subset M((\phi_0)_*)$ and $\Lambda(\tilde{\phi}_1) \subset M((\phi_1)_*)$ (cf *p 130* of (**Kre**)), we see that T extends to a bounded map from $\Lambda(\tilde{\phi}_0)$ into $M((\psi_0)_*)$, and from $\Lambda(\tilde{\phi}_1)$ into $M((\psi_1)_*)$.

Thus all the assumptions of *Theorem 6.1., p 129* of (**Kre**) are met: We obtain that T extends to a bounded map from any symmetric space E into $E_1 := \{f \in L_1(I) + L_\infty(I) : f^{**} \in E\} \subset E$ (for the definition of $L_1(I) + L_\infty(I)$ see *Section 3.3., pp 15 -16* of (**Kre**), also see *p 125* of (**Kre**)), if the lower and upper dilation exponents (α_E, β_E) of E satisfy $\gamma_1 > \beta_E \geq \alpha_E > \delta_0$.

Here

$$\delta_0 := \lim_{s \to \infty} \frac{\ln \|D_s\|_{\Lambda(\tilde{\phi}_0) \to \Lambda(\tilde{\phi}_0)}}{\ln s} = \lim_{s \to \infty} \frac{\ln \underline{\tilde{\phi}}_0(1/s)}{\ln s}$$

and

$$\gamma_1 := \lim_{s \to 0+} \frac{\ln \|D_s\|_{\Lambda(\tilde{\phi}_1) \to \Lambda(\tilde{\phi}_1)}}{\ln s} = \lim_{s \to 0+} \frac{\ln \underline{\tilde{\phi}}_1(1/s)}{\ln s}.$$

Indeed, note that according to *pp 53* and *99* of (**Kre**), we obtain for a *concave* function ϕ with $\phi(0+) = 0$ and $\phi(\infty) = \infty$ that the norm of the dilation operator $D_s : \Lambda(\phi) \to \Lambda(\phi)$ satisfies $\|D_s\|_{\Lambda(\phi) \to \Lambda(\phi)} = \underline{\phi}(1/s)$.

We have that

$$\frac{1}{2M[h^{1/p}]}\,\underline{h}(s)^{1/p} \leq \|D_s\|_{\Lambda(\tilde{\phi}_0)\to\Lambda(\tilde{\phi}_0)} = \underline{\tilde{\phi}}_0(1/s) \leq 2M[h^{1/p}]\,\underline{h}(s)^{1/p}$$

and

$$\frac{1}{2M[h^{1/q}]}\,s\,\underline{\underline{h}}(s)^{-1/q} \leq \|D_s\|_{\Lambda(\tilde{\phi}_1)\to\Lambda(\tilde{\phi}_1)} = \underline{\tilde{\phi}}_1(1/s) \leq 2M[h^{1/q}]\,s\,\underline{\underline{h}}(s)^{-1/q}.$$

Indeed, if $\phi_0(t) = ([h^{1/p}]^{**})^{-1}(t)$ is *concave* (with $\phi_0(0+) = 0$ and $\phi_0(\infty) = \infty$), then, for example, $\|D_s\|_{\Lambda(\phi_0)\to\Lambda(\phi_0)} = \underline{\phi}_0(1/s)$

$$= \sup_{t\in(0,\infty)} \frac{([h^{1/p}]^{**})^{-1}(ts)}{([h^{1/p}]^{**})^{-1}(t)} \leq M[h^{1/p}] \sup_{t\in(0,\infty)} \frac{h^{-1/p}(ts)}{h^{-1/p}(t)} = M[h^{1/p}]\,\underline{h}^{1/p}(s).$$

The factor 2 occurs because ϕ_0 is only assumed to be quasiconcave.

Similarly, one proves the opposite inequality:

$$\frac{1}{2M[h^{1/p}]}\,\underline{h}(s)^{1/p} \leq \|D_s\|_{\Lambda(\tilde{\phi}_0)\to\Lambda(\tilde{\phi}_0)}$$

as well as the inequalities in the case of $\tilde{\phi}_1$.

Finally,

$$\delta_0 = \lim_{s\to\infty} \frac{\ln \underline{\tilde{\phi}}_0(1/s)}{\ln s} = \lim_{s\to\infty} \frac{\ln \underline{h}^{1/p}(s)}{\ln s} = \frac{1}{p}\frac{1}{\underline{h}_\infty},$$

while

$$\gamma_1 = \lim_{s\to 0+} \frac{\ln \underline{\tilde{\phi}}_1(1/s)}{\ln s} = \lim_{s\to 0+} \frac{\ln\{s\underline{\underline{h}}^{-1/q}(s)\}}{\ln s} = 1 - \frac{1}{q}\frac{1}{\underline{\underline{h}}_0}.$$

Taking recipicals, gives the claim in terms of Boyd indices.

The last statement then follows immediately from the fact that for any $1 < k < \infty$ and $1 \leq r \leq \infty$, the lower and upper Boyd indices of $L^{k,r}(I)$ are k

(cf *Section 2.b., p 142* of **(Lin II)**), and that $L^{k,k}(I)$ coincides with $L_k(I)$ (cf Remark 6.2.).

Remark 7.4. Assume $h \in L_1(I)$ meets the assumptions of Theorem 7.3. From *Formula (1.24), p54* of **(Kre)**, a computation as in the proof of Theorem 7.3. shows that the h-numbers $(\underline{h}_0, \underline{\underline{h}}_0, \underline{h}_\infty, \underline{\underline{h}}_\infty)$ of a T-admissable function h on $I := (0, \infty)$ always satisfy

$$1/p \leq \underline{h}_\infty \leq \underline{h}_0 \leq \infty \text{ and } 1/q \leq \underline{\underline{h}}_0 \leq \underline{\underline{h}}_\infty \leq \infty.$$

Indeed, in this proof, we saw $\delta_0 = \frac{1}{p\underline{h}_\infty}$. The lower dilation exponent to the same quasiconcave function, namely to $([h^{1/p}]^{**})^{-1}(t)$, computes as $\gamma_0 = \frac{1}{p\underline{h}_0}$. *Formula (1.24)* of **(Kre)** states $0 \leq \gamma_0 \leq \delta_0 \leq 1$.

A similar argument works for the quasiconcave function $t\ ([h^{1/q}]^{**})(t)$.

Remark 7.5. From Theorem 6.6., it is clear that we can select a sequence of T-admissable $h_n \in L_1(I)$ with

$$D[h_n^{1/p}] \downarrow \frac{1}{p}, \quad D[h_n^{1/q}] \downarrow \frac{1}{q},$$

and thus also

$$\frac{1}{M[h_n^{1/p}]} + \frac{1}{M[h_n^{1/q}]} \uparrow 1.$$

The following theorem takes this into account.

Theorem 7.6. Assume h is T-admissable for some $1 < l < \infty$ as in the technical assumptions of Theorem 6.6. with $I := (0, 1)$. Furthermore, suppose that for some $0 \leq \epsilon < \frac{1}{pl}$,

$$\frac{1}{M[h^{1/p}]} + \frac{1}{M[h^{1/q}]} \geq 1 - \epsilon,$$

and also require that

$$\min\{m[h^{1/p}], m[h^{1/q}]\} \geq \frac{1}{1-\epsilon},$$

where

$$m[W] := \inf_{x\in I} \frac{1}{xW(x)} \int_0^x W(t)\, dt.$$

Then T extends to a bounded map

from E into $\tilde{E} := \{f \in L_1(I) + L_\infty(I) : \|t^{-\epsilon} f^{**}(t)\|_E < \infty\}$,

provided

$$p\underline{h}_\infty > p_E \gtrsim q_E > (q\underline{h}_0)'.$$

Proof: For x, y real, define for $t \in I := (0, \infty)$,

$$f_x^y(t) := \begin{cases} t^x & \text{if } t \leq 1 \\ t^y & \text{if } t > 1. \end{cases}$$

Note that this function is continuous on I, in particular at $t = 1$.

Set

$$\overline{\psi}_0 := f_0^{-\epsilon}\phi_0,$$

$$\overline{\psi}_1 := f_\epsilon^0 \phi_1,$$

$$\overline{\phi}_0 := f_{-\epsilon}^0 \phi_0,$$

and

$$\overline{\phi}_1 := f_0^\epsilon \phi_1,$$

where $\phi_i \equiv \psi_i$, $i = 1, 2$, are defined as in the proof of Theorem 7.3.

Claim: These four functions $\overline{\phi}_i$ and $\overline{\psi}_i$, $i = 1, 2$, are quasiconcave with

$$\overline{\phi}_i(0+) = \overline{\psi}_i(0+) = 0$$

and

$$\overline{\phi}_i(\infty) = \overline{\psi}_i(\infty) = \infty.$$

Furthermore, $\overline{\phi}_0\overline{\phi}_1^{-1}$ is nonincreasing over $I := (0, \infty)$.

Proof: As in the proof of Theorem 7.3.,

$$\overline{\psi}_0(0+) = \overline{\psi}_1(0+) = \overline{\phi}_1(0+) = 0.$$

By L'Hôpital's Rule,

$$\overline{\phi}_0(\infty) = (1-\epsilon)\lim_{t\to 0+}\frac{t^{-\epsilon}}{h^{1/p}(t)} = (1-\epsilon)\lim_{t\to 0+}\frac{t^{1/(pl)-\epsilon}}{[t^{1/l}h(t)]^{1/p}} = 0,$$

since $\epsilon < 1/(pl)$ and $\lim_{t\to 0+} t^{1/l}h(t) = [\psi(h^l)(0+)]^{1/l} = \infty$.

As in the proof of Theorem 7.3.,

$$\overline{\psi}_1(\infty) = \overline{\phi}_0(\infty) = \overline{\phi}_1(\infty) = \infty.$$

By l'Hôpital's Rule,

$$\overline{\phi}_0(\infty) = (1-\epsilon)\lim_{t\to\infty}\frac{t^{1/p-\epsilon}}{[th(t)]^{1/p}} = \infty,$$

since $\epsilon < 1/p$ and $h \in L_1(I)$.

The functions $\overline{\phi}_i(t)$, $i = 1, 2$, $t \in I$, are nondecreasing.

Indeed, as in the proof of Theorem 7.3., $\overline{\psi}_1(t)$ and $\overline{\phi}_1(t)$ are nondecreasing.

For $t \le 1$, as in the proof of Theorem 7.3., $\overline{\psi}_0(t)$ is nondecreasing. For $t > 1$, we see that $\overline{\psi}_0'(t) \ge 0$, since

$$1 - \epsilon \ge \frac{1}{m[h^{1/p}]}.$$

For $t \geq 1$, as in the proof of Theorem 7.3., $\overline{\phi}_0(t)$ is nondecreasing. For $t < 1$, we see that $\overline{\phi}_0'(t) \geq 0$, since

$$1 - \epsilon \geq \frac{1}{m[h^{1/p}]}.$$

The functions $\overline{\phi}_i(t)/t$, $i = 1, 2$, $t \in I$, are nonincreasing.

Indeed, we easily conclude as in the proof of Theorem 7.3. that $\overline{\psi}_0(t)/t$ and $\overline{\phi}_0(t)/t$ are nonincreasing.

For $t \geq 1$, we see as in the proof of Theorem 7.3. that $\overline{\psi}_1(t)/t \equiv \psi_1(t)/t$ is nonincreasing. For $t < 1$, $\overline{\psi}_1(t)/t$ is nonincreasing, since $(\overline{\phi}_1(t)/t)' \leq 0$ if

$$\epsilon \leq 1 - \frac{1}{m[h^{1/q}]}.$$

For $t \leq 1$, we see as in the proof of Theorem 7.3. that $\overline{\phi}_1(t)/t \equiv \phi_1(t)/t$ is nonincreasing. For $t > 1$, $\overline{\phi}_1(t)/t$ is nonincreasing, since $(\overline{\phi}_1(t)/t)' \leq 0$ if

$$\epsilon \leq 1 - \frac{1}{m[h^{1/q}]}.$$

Thus ϕ_i, $i = 1, 2$, are quasiconcave functions.

The function $\overline{\phi}_0\overline{\phi}_1^{-1}$ is nonincreasing over $I := (0, \infty)$.

Indeed, for any $t \in I$, we see that

$$\overline{\phi}_0(t)\overline{\phi}_1^{-1}(t) = \frac{t^{1-\epsilon}}{\int_0^t h^{1/p}(s)\, ds \ \int_0^t h^{1/q}(s)\, ds}.$$

Differentiating as in the proof of Theorem 7.3., then shows that $(\overline{\phi}_0\overline{\phi}_1^{-1})' \leq 0$ if

$$\frac{1}{M[h^{1/p}]} + \frac{1}{M[h^{1/q}]} \geq 1 - \epsilon.$$

This proves the claim.

Note that $f_0^{-\epsilon}, f_\epsilon^0 \leq 1$. Thus $\overline{\psi}_i \leq \psi_i$ or $(\overline{\psi}_i)_\star \geq (\psi_i)_\star$. Therefore, $1/(\overline{\psi}_i)_\star \leq 1/(\psi_i)_\star$, and $M((\psi_i)_\star) \subset M((\overline{\psi}_i)_\star)$.

Note that $f_{-\epsilon}^0, f_0^\epsilon \geq 1$. Thus $\overline{\phi}_i \geq \phi_i$ or $1/\overline{\phi}_i \leq 1/\phi_i$. Therefore, $M(\overline{\phi}_i) \supset M(\phi_i)$, and by duality, $\Lambda(\tilde{\overline{\phi}}_i) \subset \Lambda(\tilde{\phi}_i)$.

Since T extends to a bounded map from $\Lambda(\tilde{\phi}_i)$ into $M(\psi_i)$ (cf proof of Theorem 7.3.), the above continuous embeddings show that T defines a bounded map from $\Lambda(\tilde{\overline{\phi}}_i)$ into $M((\overline{\phi}_i)_\star)$ for $i = 1, 2$.

Furthermore, we see that $\overline{\psi}_0 \overline{\psi}_1^{-1} = \overline{\phi}_0 \overline{\phi}_1^{-1}$. Also,

$$\kappa(t) := \overline{\psi}_0(t)\overline{\phi}_0^{-1}(t) = f_0^{-\epsilon}(t)\phi_0(t)(f_\epsilon^0)^{-1}(t)\phi_0^{-1}(t) = f_{-\epsilon}^\epsilon(t) = t^{-\epsilon}.$$

Therefore, when we apply *Theorem 6.1., p 129* of **(Kre)**, we see that T extends to a bounded map from any symmetric space E into $\tilde{E}$ where

$$\|f\|_{\tilde{E}} := \|t^{-\epsilon} f^{\star\star}(t)\|_E,$$

if the upper and lower dilation exponents (α_E, β_E) of E satisfy

$$\overline{\delta}_0 < \alpha_E \leq \beta_E < \overline{\gamma}_1.$$

Here $\overline{\delta}_0$ and $\overline{\gamma}_1$ compute as follows. Note that for $s \geq 1$, we have that

$$s^{-\epsilon} \leq \underline{f}_{-\epsilon}^0(1/s) = \sup_{t \in (0,\infty)} \frac{f_{-\epsilon}^0(ts)}{f_{-\epsilon}^0(t)} \leq 1,$$

while for $s \leq 1$, we have

$$1 \geq \underline{f}_0^\epsilon(1/s) = \sup_{t \in (0,\infty)} \frac{f_0^\epsilon(ts)}{f_0^\epsilon(t)} \geq s^\epsilon.$$

Therefore, as in the proof of Theorem 7.3., $\overline{\delta}_0 = \lim_{s\to\infty} \frac{\ln \tilde{\underline{\phi}}_0(1/s)}{\ln s}$

$$= \lim_{s\to\infty} \frac{\ln\{\tilde{\underline{\phi}}_0 \underline{f}^0_{-\epsilon}\}(1/s)}{\ln s} \le \lim_{s\to\infty} \frac{\ln \tilde{\underline{\phi}}_0(1/s)}{\ln s} = \lim_{s\to\infty} \frac{\ln \underline{h}^{1/p}(s)}{\ln s} = \frac{1}{p}\frac{1}{\underline{h}_\infty} = \delta_0,$$

while $\overline{\gamma}_1 = \lim_{s\to 0+} \frac{\ln \tilde{\underline{\phi}}_1(1/s)}{\ln s}$

$$= \lim_{s\to 0+} \frac{\ln\{\tilde{\underline{\phi}}_1 \underline{f}^\epsilon_0\}(1/s)}{\ln s} \ge \lim_{s\to 0+} \frac{\ln\{s\underline{\underline{h}}^{-1/q}(s)\}}{\ln s} = 1 - \frac{1}{q}\frac{1}{\underline{\underline{h}}_0} = \gamma_1.$$

In fact, $\gamma_1 + \epsilon \ge \overline{\gamma}_1 \ge \gamma_1$ and $\delta_0 \ge \overline{\delta}_0 \ge \delta_0 - \epsilon$. Here δ_0 and γ_1 are defined as in the proof of Theorem 7.3. This shows the claim.

Remark 7.7. Suppose that $h \in L_p(I)$, $\epsilon \ge 0$ are as in Theorem 7.6. For $E := L_k(I)$ with $p\underline{h}_\infty > k > (q\underline{\underline{h}}_0)'$ and $k < 1/\epsilon$ (omit if $\epsilon = 0$), we see by Remark 6.2. that

$$\|f\|_E^k = \int_I [t^{-\epsilon} f^{**}(t)]^k \, dt = \int_I [t^{1/k-\epsilon} f^{**}(t)]^k t^{-1} \, dt = \|f\|^k_{L_{\frac{1}{1/k-\epsilon},k}(I)}$$

Therefore, T extends to a bounded map from $L_k(I)$ into $L_{\frac{1}{1/k-\epsilon},k}(I)$.

Finally, let us look at a particular function $h \in L_1(I)$ for $I := (0,1)$. This will be important at the end of the next section (Remark 8.10.).

Proposition 7.8. If h is T-admissable for $I := (0,1)$ where the function $h : (0,1) \to (1,\infty)$ is given as

$$h(t) := t^{-1/\gamma} \text{ for some } 1 < \gamma < \infty,$$

then, for $1 \le r \le \infty$,

$$T \text{ extends to a bounded map from } L^{\tilde{p},r}(0,1) \text{ into } L^{\tilde{p},r}(0,1),$$

provided $\tilde{p}$ lies strictly between γp and $(\gamma q)'$.

Furthermore, if $\gamma > 2\max\{1/p, 1/q\}$, then T defines a bounded map from $L_2(0,1)$ into $L_2(0,1)$.

Proof: We observe that, if h is T-admissable, then the operator T extends to a bounded map from $L^{(\gamma q)',1}(0,1)$ into $L^{(\gamma q)',1}(0,1)$, and from $L^{\gamma p,\infty}(0,1)$ into $L^{\gamma p,\infty}(0,1)$, i.e. is an operator of weak types $((\gamma q)', (\gamma q)')$ and $(\gamma p, \gamma p)$. Here, we use the continuous imbedding $L^{k,\infty}(0,1) \subset L^{k,1}(0,1)$ for $1 < k < \infty$. The Theorem now follows from the Marcinkiewicz Interpolation Theorem (cf *Theorem 4.13., p 225* of **(Ben)**), since $\gamma p \neq (\gamma q)'$ always.

Also, we see that $\gamma p > 2$ and $(\gamma q)' < 2$, if $\gamma > 2\max\{1/p, 1/q\}$.

8. Examples of Strongly L_p Regular Operators.

We shall show here that the Hilbert transform and the Calderon operator are strongly L_p regular. Since the Hilbert transform is the basic building block for many singular integral operators and the Calderon operator is the typical operator of weak type, they give raise to many concrete examples of strongly L_p regular operators.

8.1. The Calderon Operator: Let $I := (0,1)$ or $(0,\infty)$. Consider the following integral operators defined for $0 < a \leq 1$ and $0 \leq b < 1$:

$$P_a f(t) := t^{-a} \int_0^t s^a f(s) \frac{ds}{s}$$

and

$$Q_b f(t) := t^{-b} \int_t^\infty s^b f(s) \frac{ds}{s}$$

for $t \in I$ and $f \in L_1(I) \cap L_\infty(I)$. (The upper limit of the second integral is to be replaced by 1, if $I := (0,1)$. We shall assume such modifications whenever necessary.)

The **Calderon operators** are then given by

$$S_{a,b} := P_a + Q_b.$$

Notice that for any nonnegative function $f \in L_1(I) \cap L_\infty(I)$, $S_{a,b}f$ is nonincreasing, since for example

$$\frac{d}{dt} P_a f(t) = -at^{-a} \int_0^t s^a f(s) \frac{ds}{s} + f(t)\, t^{-1},$$

and thus

$$\frac{d}{dt} S_{a,b} f(t) = -at^{-a} \int_0^t s^a f(s) \frac{ds}{s} - bt^{-b} \int_0^t s^b f(s) \frac{ds}{s} \leq 0.$$

Furthermore, if $0 \leq b < a \leq 1$, we may write

$$S_{a,b} f(t) = \int_0^\infty f(s) \zeta_t(s) ds,$$

where

$$\zeta_t(s) := \min \{\frac{s^{a-1}}{t^a}, \frac{s^{b-1}}{t^b}\}$$

is nonincreasing. Indeed, it is easy to see that

$$\zeta_t(s) = \begin{cases} \frac{s^{a-1}}{t^a} & \text{if } s \leq t \\ \frac{s^{b-1}}{t^b} & \text{if } s > t \end{cases},$$

since, for $0 \leq b < a \leq 1$, we have that $(s/t)^a \leq (s/t)^b$ if $0 < s \leq t$, while $(s/t)^a > (s/t)^b$ if $s > t > 0$.

Theorem 8.2. If $1/p < a \leq 1$ and $b < 1/p$, then the Calderon operator $S_{a,b}$ extends to a strongly L_p regular operator, and its dual $S'_{a,b} : L_q(I) \to L_q(I)$ to a strongly L_q regular operator.

Proof: Let $I := (0, \infty)$. Then P_a extends to a bounded linear operator $P_a : L_p(I) \to L_p(I)$ if $1/p < a \leq 1$, while $Q_b : L_p(I) \to L_p(I)$ is bounded for $0 \leq b < 1/p$ according to *Theorem III.5.15., p 150* of **(Ben)**.

Using Remark B.4. and the inequality

$$\int_0^\infty f(s)\eta(s)ds \leq \int_0^\infty f^\star(s)\eta(s)ds$$

where η is any nonincreasing nonnegative function (cf *Theorem I.2.2., p 44* of **(Ben)**), we see that for $f \in L_p(I)$:

$$(S_{a,b}f)^\star(t) = S_{a,b}f(t) \leq \int_0^\infty f^\star(s)\zeta_t(s)ds \leq \int_0^\infty f^{\star\star}(s)\zeta_t(s)ds,$$

if ζ_t is the function of 8.1.

Given $g \in L_p(I)$, where g is nonincreasing and nonnegative, set $h := 2(S_{a,b}(g^{\star\star}))^{\star\star} \in L_p(I)$. Then, if $0 \leq f$ and $f^{\star\star} \leq g$, we obtain that

$$(S_{a,b}f)^{\star\star} \leq 1/2\, h.$$

For any $f^{\star\star} \leq g$, we obtain (cf Remark B.4.)

$$(S_{a,b}f)^{\star\star} \leq (S_{a,b}f_+)^{\star\star} + (S_{a,b}f_-)^{\star\star} \leq h.$$

Here f_+ and f_- denote the positive and negative part of f, resp. Thus $S_{a,b}$ satisfies **(S2)** of Remark 5.2., and is strongly L_p regular.

Since $S'_{a,b} \equiv S_{1-b,1-a}$, according to *Definition III.5.14., p 150* of **(Ben)**, the strong L_q regularity of the dual follows by the same argument.

Finally, let $I := (0,1)$. If $S_{a,b}$ denotes the Calderon operator on $(0,\infty)$ while $\tilde{S}_{a,b}$ is the Calderon operator on $(0,1)$, we see that for any $f \in L_p(I)$:

$$\tilde{S}_{a,b}f = \chi_{(0,1)} S_{a,b}f,$$

and the result holds in this case, too.

8.3. Operators of Weak Type: Let $I := (0,1)$ or $(0,\infty)$. An operator defined from $L_\infty(I) \cap L_1(I)$ into the set of measurable functions is of **weak type** p, if there is a constant $M > 0$ such that for all $f \in L_1(I) \cap L_\infty(I)$:

$$(Tf)^*(t) \leq Mt^{-1/p} \int_0^\infty s^{1/p} f^{**}(s) \frac{ds}{s}.$$

In terms of Lorentz norms, for $1 < p < \infty$, this says: $\|Tf\|_{p,\infty} \leq M\|f\|_{p,1}$ (cf Remark 6.2.).

The next theorem states that the Marcinkiewicz Interpolation Theorem not only implies L_p boundedness, but also strong L_p regularity.

Theorem 8.4. Let T be of weak type q_1 and q_2 with $1 \leq q_1 < q_2 \leq \infty$. Then, for every p with $q_1 < p < q_2$, T defines a strongly L_p regular operator $T : L_p(I) \to L_p(I)$.

Proof: *Theorem IV.4.11, p 223* of **(Ben)** implies that T also is of joint weak type $(q_1, q_1; q_2, q_2)$, i.e. for $f \in L_1(I) \cap L_\infty(I)$, we get a constant $c > 0$ such that (cf *Definition III.5.4.* of **(Ben)**)

$$(Tf)^*(t) \leq c\, S_{1/q_1,1/q_2}(f^*)(t).$$

Since $1/q_1 > 1/p$ and $1 - 1/q_2 > 1 - 1/p$, it follows from the Theorem 8.2. that $S_{1/q_1,1/q_2}$ is strongly L_p regular, and thus so is T.

Since L_p bounded operators are in particular of weak type p, we get:

Corollary 8.5. Let $T : L_p(I) \to L_p(I)$ be a bounded linear operator for all $p \in (a,b)$ with $1 \leq a < b \leq \infty$. Then T is strongly L_p regular for all $p \in (a,b)$.

8.6. The Hilbert Transform: Let $I := (-\infty, \infty)$. For $f \in L_1(I) \cap L_\infty(I)$ define the **maximal Hilbert transform** H_{max} by

$$H_{max}f(s) := \sup_{\epsilon>0} |H_\epsilon f(s)|,$$

where

$$H_\epsilon f(s) := \frac{1}{\pi}\int_{|s-t|\geq\epsilon} \frac{f(t)}{s-t}dt = -\frac{1}{\pi}\int_{|t|\geq\epsilon} \frac{f(s-t)}{t}dt.$$

For $1 < p < \infty$, we see from *Theorem III.4.7., p 134* of **(Ben)** that for some $c > 0$ and any $f \in L_p(-\infty,\infty)$:

$$(H_{max}f)^* \leq c\, S_{1,0}(f^*),$$

where $S_{1,0}$ is the Calderon operator of 8.1. This, in paricular, implies the boundedness of the (maximal) Hilbert transform for $1 < p < \infty$. (See also *Theorem III.4.9., p139* of **(Ben)**.) Here we define the **Hilbert transform**

$$H : L_p(-\infty,\infty) \to L_p(-\infty,\infty)$$

as

$$Hf(s) := \lim_{\epsilon\to 0+} H_\epsilon f(s).$$

If $I := (-\pi, \pi]$, we define the **Hilbert transform** by

$$Hf(s) := \lim_{\epsilon\to 0+} H_\epsilon f(s),$$

where $s \in (-\pi, \pi]$ and

$$H_\epsilon f(s) := -\frac{1}{\pi}\int_{\pi>|t|\geq\epsilon}\frac{f(s-t)}{2\tan(t/2)}dt = -\frac{1}{\pi}\int_{\pi>|s-t|\geq\epsilon}\frac{f(t)}{2\tan((s-t)/2)}dt.$$

Theorem V.2.4. of **(Tor)** shows that H defines a bounded linear operator $H : L_p(I) \to L_p(I)$ in this case.

By appealing to Theorem 8.2. in the case of $I := (-\infty, \infty)$, or to Corollary 8.5. if $I := (-\pi, \pi]$, we get:

Theorem 8.7. Let $I := (-\infty, \infty)$ or $(-\pi, \pi]$, and $1 < p < \infty$. Then the Hilbert transform H defines a strongly L_p regular operator.

For the unit circle $I := \Gamma$, which we cannonically identify with $(-\pi, \pi]$, we denote by $\sum_{n=-\infty}^{\infty} \alpha_n e^{int}$ the usual Fourier expansion of a function $f \in L_p(\Gamma)$. The spaces

$$H_p(\Gamma) := \overline{\text{span}}\,\{e^{int} : n = 0, 1, 2, 3, ...\}$$

are called the **Hardy spaces** on Γ.

Corollary 8.8. Let $1 < p < \infty$. The projection

$$f(t) \equiv \sum_{n=-\infty}^{\infty} \alpha_n e^{int} \to Pf(t) = \sum_{n=0}^{\infty} \alpha_n e^{int}$$

of $L_p(\Gamma)$ onto the Hardy space $H_p(\Gamma)$ is strongly L_p regular.

Proof: It follows from *Proposition III.3.1.* and *Proposition III.6.2.* of **(Tor)** that the above projection is L_p bounded iff H is bounded on $L_p(-\pi, \pi]$.

Define the **Hardy-Littlewood maximal operator** M for a locally integrable function f by

$$(Mf)(x) := \sup_Q \frac{1}{\mu(Q)}\int_Q |f(t)|dt,$$

where the supremum extends over all subarcs $Q \subset \Gamma$, and μ or dt, resp., denote the normalized Lebesque measure on Γ. Here Γ is naturally identified with the interval $(-\pi, \pi]$. Then M extends to a bounded operator $M : L_p(\Gamma) \to L_p(\Gamma)$ for any $1 < p \leq \infty$. Note that M is *not* linear.

Remark 8.9. For $1 < p < \infty$, the Hardy-Littlewood operator on Γ satisfies **(S1)** of Definition 5.1.

Proof: It follows from *Exercise III.12.(b), p 177* of **(Ben)** that for some constant $C > 0$, the Hardy-Littlewood maximal operator satisfies $(Mf)^* \leq C\ f^{**}$ for any locally integrable function f, and thus for any $f \in L_p(\Gamma)$.

Remark 8.10. The following special case of the Calderon operator, namely $S \equiv S_{1,0}$, is encounterd rather often. Clearly, $S : L_p(I) \to L_p(I)$, $I := (0, \infty)$, is of the form

$$Sf(s) = \frac{1}{s}\int_0^s f(t)dt + \int_s^\infty \frac{f(t)}{t}dt$$

with the obvious modification if $I := (0,1)$.

If $I := (0,1)$, for any $1 < \gamma < \infty$, set

$$h_\gamma : (0,1) \to (0,\infty) \text{ as } h_\gamma(s) := s^{-1/\gamma}$$

(cf Proposition 7.8.). Then the Calderon operator on $I := (0,1)$ satisfies

$$Sh_\gamma(t) = \{\gamma + \frac{1}{1 - 1/\gamma}\}\ h_\gamma(t),$$

where $t \in (0,1)$.

For any $1 < \gamma < \infty$, by Remark 6.2. and Lemma 6.3., the Calderon operator defines a bounded map from $M(\psi(h_\gamma)) \equiv L^{\gamma,\infty}(0,1)$ into $L^{\gamma,\infty}(0,1)$.

Since $S \equiv S_{1,0}$ is a selfdual operator ($S'_{1,0} = S_{1,0}$ according to *Definition III.5.14., p 150* of **(Ben)**), S also extends to a bounded map from $\Lambda(\psi(g_\gamma)) \equiv L^{\gamma',1}(0,1)$ into $L^{\gamma',1}(0,1)$.

If $I := (0,\infty)$, consider

$$h_{\alpha,\beta}(t) := \begin{cases} t^{-1/\alpha} & \text{if } 0 < t \leq 1 \\ t^{-1/\beta} & \text{if } t > 1 \end{cases}$$

for $1 < \alpha < \infty$ and $0 < \beta < \infty$. A computation shows that the Calderon operator satisfies

$$Sh_{\alpha,\beta}(s) = \begin{cases} (\frac{1}{1-1/\alpha} + \alpha)s^{-1/\alpha} + (\beta - \alpha) & \text{if } s < 1 \\ (\frac{1}{1-1/\beta} + \beta)s^{-1/\beta} + (\frac{1}{1-1/\alpha} - \frac{1}{1-1/\beta})\frac{1}{s} & \text{if } s \geq 1. \end{cases}$$

i) Assume $0 < \beta < 1$ and $1 < \alpha < \infty$. Then $h_{\alpha,\beta} \in L_1(I)$. Furthermore, $Sh_{\alpha,\beta} \leq Ch_{\alpha,\beta}$ is satisfied for

$$C \equiv C(\alpha,\beta) := \max\{\frac{1}{1-1/\alpha} + \alpha, \frac{1}{1-1/\beta} + \beta\}.$$

Fix $1 < p < \infty$. Consider S as a bounded operator $S : L_p(0,\infty) \to L_p(0,\infty)$. If

$$1/p < \beta < 1,$$

then $\psi(h_{\alpha,\beta}^{1/p})$ is a (quasi)concave function with

$$\psi(h_{\alpha,\beta}^{1/p})(\infty) = \infty.$$

Claim: If $\beta > 1/p$, then $h_{\alpha,\beta}^{1/p}$ is a regular function.

Proof: For $t \leq 1$, we see that

$$\frac{1}{th_{\alpha,\beta}^{1/p}(t)} \int_0^t h_{\alpha,\beta}^{1/p}(s)\, ds = \frac{1}{1 - 1/(p\alpha)},$$

since $p > 1$ and thus $p\alpha > 1$, and

$$\frac{1}{t h_{\alpha,\beta}^{1/q}(t)} \int_0^t h_{\alpha,\beta}^{1/q}(s)\, ds = \frac{1}{1 - 1/(q\alpha)},$$

since $q > 1$ and thus $q\alpha > 1$.

Furthermore, for $t > 1$, we obtain that

$$\frac{1}{t h_{\alpha,\beta}(t)} \int_0^t h_{\alpha,\beta}(s)\, ds = \{\frac{1}{1-1/\alpha} - \frac{1}{1-1/\beta}\}\, t^{1/\beta-1} + \frac{1}{1-1/\beta}.$$

Thus, we need to require that $p\beta > 1$ so that $h_{\alpha,\beta}^{1/p}$ is a regular function.

Note that (if also $q\beta > 1$)

$$M[h^{1/p}] = \max\{\frac{1}{1-1/(p\alpha)}, \frac{1}{1-1/(p\beta)}\} = (p\beta)' > q$$

and

$$M[h^{1/q}] = \max\{\frac{1}{1-1/(q\alpha)}, \frac{1}{1-1/(q\beta)}\} = (q\beta)' > p.$$

Therefore, the second condition in Theorem 7.3. fails, since

$$\frac{1}{M[h^{1/p}]} + \frac{1}{M[h^{1/q}]} < 1/q + 1/p = 1.$$

Nevertheless, according to Lemma 6.3., we may conclude that S defines a bounded map from $M(\psi(h_{\alpha,\beta}^{1/p}))$ into $M(\psi(h_{\alpha,\beta}^{1/p}))$. Since $p > 1$ is arbitrary, and $h_{\alpha,\beta}^{1/p} \equiv h_{p\alpha,q\beta}$, we see that for any $0 < \beta < 1$ and $1 < \alpha < \infty$, S extends to a bounded map from $M(\psi(h_{\alpha,\beta}))$ into $M(\psi(h_{\alpha,\beta}))$, and by duality ($S$ is selfdual), from $\Lambda(\psi(h_{\alpha,\beta}))$ into $\Lambda(\psi(h_{\alpha,\beta}))$.

ii) Given $1 < \alpha < \infty$, setting $\beta := \alpha$, by Remark 6.2. and Lemma 6.3., we see that S defines a bounded map from $M(\psi(h_{\alpha,\alpha})) \equiv L^{\alpha,\infty}(0,\infty)$ into

$L^{\alpha,\infty}(0,\infty)$, and also from $\Lambda(\psi(h_{\alpha,\alpha})) \equiv L^{\alpha',1}(0,\infty)$ into $L^{\alpha',1}(0,\infty)$, since S is selfdual.

iii) Suppose now $1 < \alpha, \beta < \infty$. If we understand by

$$L^{\alpha',\beta',1} \equiv L^{\alpha',1}(0,1) \cap L^{\beta',1}(1,\infty)$$

the space of all measurable functions f on $(0,\infty)$ satisfying

$$\chi_{(0,1)} f \in L^{\alpha',1}(0,\infty) \text{ and } \chi_{(1,\infty)} f \in L^{\beta',1}(0,\infty),$$

and similarly define the space $L^{\alpha,\beta,\infty}$, then S defines a bounded map from $L^{\alpha',\beta',1}$ into $L^{\alpha',\beta',1}$, and from $L^{\alpha,\beta,\infty}$ into $L^{\alpha,\beta,\infty}$.

CHAPTER III.

REPRESENTATION OF L_p OPERATORS BY KERNELS OF DISTRIBUTIONS.

9. Definitions and Examples.

Many well-known operators in analysis have a useful representation by kernels of distributions.

Example 9.1. For every positive operator $T : L_p(X,\mu) \to L_p(Y,\nu)$, $1 \le p < \infty$, there is a kernel $(\mu_y)_{y\in Y}$ of measures on X such that for all $f \in L_p(X,\mu)$:

$$Tf(y) = \int f \, d\mu_y \quad \nu\text{-a.e.}$$

Example 9.2. The Hilbert transform H on $L_p(-\infty,\infty)$, $1 < p < \infty$, is given by

$$Hf(x) = C_x(f) \quad \mu\text{-a.e.}$$

for all $f \in L_p(-\infty,\infty)$, where $x \to C_x$ is the kernel of Cauchy's principle value distribution

$$C_x(f) = \lim_{\epsilon\to 0+} \frac{1}{\pi} \int_{\{|x-t|\ge\epsilon\}} \frac{f(t)}{x-t} dt$$

(cf *Sect. III.4., p 126* of **(Ben)**).

Example 9.3. For $f \in L_2(-\infty,\infty) \cap L_1(-\infty,\infty)$, we may write the Fourier transform

$$Ff(x) = E_x(f)$$

where $x \to E_x$ is the kernel of the distribution

$$E_x(g) = \int e^{ixy} g(y)dy.$$

In the case of the first example, we actually have a characterization of positive operators in terms of the representing kernel. This leads us to the following question.

Problem 9.4. Is it possible to characterize strongly L_p regular operators in terms of a representation

$$Tf(x) = D_x(f)$$

where D_x are distributions in an appropriate class?

Such a characterization would distinguish between Examples 9.1. and 9.2. (which are strongly L_p regular) and Example 9.3. (which is *not* strongly L_p regular).

At this point, we cannot answer the above question, but in the next section, we give a general representation theorem for L_p operators in terms of distributions, which may be considered as a first step in this direction.

This theorem is motivated by Lemma 9.5. below. But first some notations.

We shall assume $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let X be an open, bounded subset of R^N. μ will denote the Lebesque measure on X.

By $W_p^n(X)$ with $1 \leq p \leq \infty$ and $n \geq 1$ we denote the **Sobolev space** consisting of all functions for which the following norm is finite:

$$\|f\|_{n,p} := \sum_{|j| \leq n} \|D^j f\|.$$

Here the norm $\| \ \|$ denotes the appropriate L_p norm on X. We denote its topological dual by $W_{p'}^{-n}(X)$.

The Banach space $L_\infty(\mu, n, p)$ consists of all $f(t,x) \equiv f(t)(x)$ on $X \times X$ such that the following norm is finite:

$$|||f||| := \text{ess sup}_{t \in X} \|f(t)\|_{n,p}.$$

Similarly, one defines $L_\infty(\mu, -n, p')$.

With the space $D(X)$ of **test functions** we mean the space consisting of all C^∞ functions with compact support with its usual topology. The **distribution space** $D'(X)$ is its dual.

The following illustrates that if certain assumptions are made on a linear bounded operator $T : L_p(X) \to L_p(X)$, then its kernel can be described as a distributional derivative.

For simplicity, let $X := I := (0,1)$. Denote by $J : L_p(I) \to C(I)$ the bounded linear operator given by

$$Jf(t) := \int_0^t f d\mu$$

where $1 \leq p < \infty$. Set $\tilde{T} := J \circ T : L_p(I) \to C(I)$, and denote its dual by $\tilde{T}' : M(I) \to L_{p'}(I)$.

Lemma 9.5. If $\tilde{T} := J \circ T : L_p(I) \to C(I)$ maps $D(I)$ into $D(I)$, then

$$Tf(x) = (\frac{d}{dx} G(x))(f) \ x\text{-a.e.}$$

for any $f \in D(I)$, where $G(x) : I \to L_{p'}(I)$ for any $x \in I$ is given by

$$G(x)(y) = \tilde{T}' \delta_x(y).$$

Proof: Since $L_{p'}(I) \subset D'(I)$ and $\frac{d}{dx}\tilde{T}f(x) = Tf(x)$ in $D(I)$, we have for any $f, \zeta \in D(I)$ that

$$\begin{aligned} < Tf(x), \zeta(x) > &= < \frac{d}{dx}\tilde{T}f(x), \zeta(x) > = - < \tilde{T}f(x), \frac{d}{dx}\zeta(x) > \\ &= - << \delta_x, \tilde{T}f >, \frac{d}{dx}\zeta(x) > = - << \tilde{T}'\delta_x, f >, \frac{d}{dx}\zeta(x) > \\ &= - << G(x), f >, \frac{d}{dx}\zeta(x) > = < \frac{d}{dx} < G(x), f >, \zeta(x) > . \end{aligned}$$

Since $< G(\cdot), f > = \tilde{T}f(\cdot) \in D(I)$ and thus differentiable, we may apply *Remark A2.2., pp 148/49* of **(Gel I)** to obtain that

$$\frac{d}{dx} < G(x), f > = < \frac{d}{dx}G(x), f > .$$

Therefore,

$$< Tf(x), \zeta(x) > = << \frac{d}{dx}G(x), f >, \zeta(x) >$$

for any $\zeta \in D(I)$. This gives the claim.

In Section 10, we shall deal with bounded linear operators which map some Sobolev space $W_p^r(X)$ into $L_\infty(X)$. However, not all bounded linear operators have this property as the following example shows.

Example 9.6. There exist bounded linear operators $T : L_p(X) \to L_p(X)$ for which there is a $G : X \to L_p(X)$ such that $Th = G(h)$ for any $h \in L_p(X)$, but T does *not* map any Sobolev space $W_p^r(X)$ into $L_\infty(X)$.

Proof: Set $p := 2$, $X := I := (0, 1)$ and $I_n := [1/(n+1), 1/n)$ for $n > 0$. Then

$$I = \dot{\bigcup} I_n .$$

For each n, pick a function $g_n \in L_2(I) - L_\infty(I)$ and $f_n \in W_n^2(I)$ which both are supported in I_n and satisfy $\|f_n\|_{L_2(I)} = \|g_n\|_{L_2(I)} = 1$.

Define the linear operator $T_n : L_2(I) \to L_2(I)$ for $h \in L_2(I)$ by

$$T_n h := \int \chi_{I_N} h f_n d\mu \cdot g_n.$$

Since $T_n f_n = g_n$, the restriction of T_n to $L_2(I_n)$ does not map $W_2^n(I_n)$ into $L_\infty(I_n)$. Thus $T := \sum_{n=1}^{\infty} T_n$ as an operator from $L_2(I)$ into $L_2(I)$ is bounded and linear, but does not map any $W_2^r(I)$ into $L_\infty(I)$.

Define $G : I \to L_2(I)$ by $G(t) := \sum_{n=1}^{\infty} g_n(t) \cdot f_n$. An application of Lebesque Dominated Convergence Theorem demonstrates that $Th = G(h)$ a.e. for any $h \in L_p(I)$.

10. Kernels with Values in Sobolev Spaces.

Our theorem provides representation of any bounded linear operator on $L_p(\Omega)$ where Ω is an open, bounded subset of R^N.

Theorem 10.1. If $T : L_p(\Omega) \to L_p(\Omega)$, $1 < p < \infty$, is a bounded and linear operator, then there exists a $G \in L_\infty(\mu, -2N, p')$ (cf 9.4.) such that for any $f \in L_p(\Omega)$, the following holds in the Bochner sense:

$$T'f = \int G f d\mu.$$

Furthermore, for any $g \in W_p^{2N}(\Omega)$, we have $Tg(t) = G(t)(g)$ t-a.e.

Proof: Let $J : W_p^{2N}(\Omega) \to L_\infty(\Omega)$ denote the canonical imbedding. Since J is nuclear for $1 \leq p \leq \infty$ by *Theorem 3.c.5., p 186/87* of **(Kön)**, we can write

$$T \circ J : W_p^{2N}(X) \to L_\infty(X)$$

as

$$TJf = \sum_{n=0}^{\infty} \lambda_n \, f_n(f) \, g_n,$$

where

$$(\lambda_n) \in l_1, \ \|f_n\|_{-2N,p'} = 1, \text{ and } \|g_n\|_{L_\infty(\Omega)} = 1.$$

For $f \in U_{W_{2N,p}(\Omega)}$, the unit ball of $W_{2N,p}(\Omega)$, we have that

$$\|TJf\| \leq \sum |\lambda_n||g_n|.$$

Set $g := \sum |\lambda_n||g_n|$. Then $g \in L_p(\Omega)$ and $|Tf| \leq g$. Set

$$\Omega_n := \{t \in \Omega : g(t) \leq n\}$$

and

$$T_n := \chi_{\Omega_n} T|_{W_p^{2N}(\Omega)} = \chi_{\Omega_n}(T \circ J) : W_p^{2N}(\Omega) \to L_\infty(\Omega).$$

Since

$$\|T_n\|_{W_p^{2N}(\Omega) \to L_\infty(\Omega)} = \sup_{\|f\|_{2N,p}=1} \|\chi_{\Omega_n} \ |Tf| \ \|_{L_\infty(X)} \leq n,$$

we have that T_n is a bounded linear operator. Since W_p^{2N} is reflexive, we have that

$$T_n' : L_1(\Omega) \to W_{p'}^{-2N}(\Omega)$$

is a weakly compact operator. The Dunford-Pettis Theorem (cf *Lemma 11, p 75* of **(Die)**) thus provides for any n a

$$G_n : \Omega \to W_{p'}^{-2N}(\Omega)$$

such that for any $f \in L_1(\Omega)$:

$$T_n' f = \int_\Omega G_n(t) f(t) d\mu(t).$$

Also, the norm of T_n equals ess $\sup_{t\in\Omega} \|G_n(t)\|_{-2N,p'}$. Since for any $t \in \Omega_n$ and any $n < m$, we have that $G_n(t) = G_m(t)$ as distributions, we may define G by setting $G(t) = G_n(t)$ for $t \in \Omega_n$.

Claim: For $g \in W_p^{2N}(\Omega)$ and any n, we have $T_n g(x) = G_n(x)(g)$ x-a.e. on Ω.

First, it is clear that $G_n(x) = 0$ for any n and $x \in \Omega - \Omega_n$. Second, by *Theorem 6, p 47* of **(Die)**, we see that

$$(T_n' f)(g) = \int_\Omega G_n(t)(g) f(t) d\mu(t).$$

Therefore, we get for any measurable set $A \subset \Omega$:

$$\int_A T_n g(t) d\mu(t) = (T_n' \chi_A)(g) = \int_A G_n(t)(g) d\mu(t),$$

which is the claim.

Claim: For $g \in W_p^{2N}(\Omega)$ we have $Tg(x) = G(x)(g)$ x-a.e. on Ω.

By the previous claim, we have for any n and any measurable set $A_n \subset \Omega_n$ that

$$\int_{A_n} Tg(t) d\mu(t) = \int_{A_n} G(t)(g) d\mu(t).$$

$$\int_{A_n} Tg(t) d\mu(t) = \int_{A_n} G(t)(g) d\mu(t).$$

Since we can write every measurable set $A \subset \Omega$ as the disjoint union of $A_n \subset \Omega_n - \Omega_{n-1}$ where $n > 0$ and $\Omega_0 := \phi$, the claim follows from Lebesque Dominated Convergence Theorem.

Remark 10.2. We indicate an alterative proof of Theorem 10.1. which is based on a disintegration result of **[Edg]**.

Assume that the operator $T : L_p(\Omega) \to L_p(\Omega)$ maps $W_p^r(\Omega)$ into $L_\infty(\Omega)$. Let $T' : L_1(\Omega) \to W_p^{-r}(\Omega)$ denote the dual operator to $T|_{W_p^r(\Omega)}$. Set

$$A := \{ \frac{T'\chi_E}{\nu(E)} : E \in B(\Omega),\ \nu(E) > 0 \},$$

where $B(\Omega)$ denotes the collection of all Borel sets of Ω. Since

$$\| \frac{T'\chi_E}{\nu(E)} \|_{W_{p'}^{-r}(\Omega)} \leq \|T'\|_{L_1(\Omega)\to W_p^{-r}(\Omega)},$$

we see that A is weak* relative compact. Setting $m(E) := T'\chi_E$ for any $E \in B(\Omega)$, we see that m is a vector measure absolutely continuous with respect to ν.

Identifying S with Ω, F with $B(\Omega)$, λ with ν and V with $W_{p'}^{-r}(\Omega)$ in *Theorem 2.1., p 447* of **[Edg]**, we obtain a $G : \Omega \to W_{p'}^{-r}(\Omega)$ such that

$$T'\chi_E = \int_E G d\nu.$$

As in the above proof, we now obtain that

$$Th = G(h) \ \ \nu\text{-a.e.}$$

for all $h \in W_p^r(\Omega)$.

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APPENDIX A.

Equiintegrability and Basic Sequences.

In this appendix, we give some basic results on equiintegrable and pairwise disjoint sequences. These facts are essential for the proofs in Chapter I.

First, we show that any bounded sequence in $L_p(I)$, $I := (0,1)$ with (normalized) Lebesque measure μ, $1 \le p < \infty$, can be split into a disjoint sum of an equiintegrable and a disjoint sequence. (For the history and extensions of this device, consult [**Wei VI**].) Then we show how equiintegrable and disjoint sequences relate to basic sequences isomorphic to l_2 and l_p. The latter results are well-known in Banach space theory, but they are still scattered in the literature and we collect them here for the convenience of the reader.

A non-empty set $M \subset L_p(I)$ is called **equiintegrable** if for any $\epsilon > 0$, there exists a $c > 0$ such that

$$\int_{|f|>c} |f|^p d\mu < \epsilon$$

for any $f \in M$.

M is said to be **disjoint**, if for any two functions $f, g \in M$, there are disjoint measurable sets $A, B \subset I$ with $f|_A = 0$ a.e. and $g|_B = 0$ a.e.

Lemma A.1. Let $(f_n) \subset L_p(I)$, $I := (0,1)$, $1 \le p < \infty$. Set

$$\lambda(f_n) := \inf_m \limsup_n \|\chi_{\{|f_n| \ge m\}} f_n\|.$$

If $\lambda(f_n)$ is finite, there exist a subsequence $(f_{n_k}) \subset (f_n)$ and two sequences of functions (g_k) and (h_k) such that:

1) $f_{n_k} = g_k + h_k$ for all k, and g_k and h_k are disjoint;

2) (g_k) is an equiintegrable sequence in $L_p(I)$;

3) (h_k) is a sequence of disjointly supported functions.

Proof: If $\lambda(f_n) = 0$, then (f_n) is an equiintegrable sequence. We may therefore assume, wlog, that $\lambda(f_n) \neq 0$. We can find a subsequence $(f_{(i)}) \subset (f_n)$ such that $\lim_i \|\chi_{F_i} f_{(i)}\| = \lambda(f_n)$ where $F_i := \{\ |f_{(i)}| \geq i\ \}$. Set $g_i := f_{(i)} - \chi_{F_i} f_{(i)}$ and $h_i := \chi_{F_i} f_{(i)}$.

Claim: $\lambda(g_n) = 0$. Assume not. Then we can find a subsequence $(g_{i_j}) \subset (g_i)$, and sets $G_j := \{\ |g_{i_j}| \geq j\ \}$ such that $\lim_j \|\chi_{G_j} g_{i_j}\| =: \lambda_1 > 0$. Thus, $\|\chi_{\{|f_{(i_j)}| \geq j\}} f_{(i_j)}\| \geq \|\chi_{F_{i_j} \cup G_j} f_{(i_j)}\|$. Therefore,

$$\|\chi_{\{|f_{(i_j)}| \geq j\}} f_{(i_j)}\|^p \geq \|\chi_{F_{i_j}} f_{(i_j)}\|^p + \|\chi_{G_j} f_{(i_j)}\|^p \to \lambda(f_n)^p + \lambda_1^p > \lambda(f_n)^p,$$

a contradiction. Hence $\lambda(g_n) = 0$, and (g_n) is equiintegrable, as required in Part 2).

The (h_i) can be made disjoint using the following procedure: Reindexing $(f_{(i)})$, assume that $h_i = \chi_{F_i} f_i$, $F_i = \{|f_i| \geq i\}$ and wlog $\|h_i\| > 0$. Set $H_i := h_i / \|h_i\| = \chi_{F_i} f_i / \|\chi_{F_i} f_i\|$.

Since $\lambda(f_n) < \infty$, we have $\mu(F_i) \to 0$. Therefore, for any $0 < \epsilon < 1$, there exists $N(\epsilon)$ such that $\mu\{|H_i| \geq \epsilon\} \leq \mu(F_i) < \epsilon$ for any $i \geq N(\epsilon)$. In other words, (H_i) converges to 0 in measure. Furthermore, if $i \geq N(\epsilon)$ and $E := \{|H_i| \geq \epsilon\}$, then

$$\int_E |H_i|^p d\mu = \int_0^1 |H_i|^p d\mu - \int_{E^c} |H_i|^p d\mu = 1 - \int_{\{|H_i| \geq \epsilon\}} |H_i|^p d\mu > 1 - \epsilon.$$

Thus, for $\epsilon := 2^{-2}$, we can find n_1 such that $\mu(E_1) < 2^{-2}$ and

$$\int_{E_1} |H_{n_1}|^p d\mu > 1 - 2^{-2}$$

where $E_1 := \{|H_{n_1}| \geq 2^{-2}\}$.

Again, we can find $\tilde{n}_2 > n_1$ such that for $j \geq \tilde{n}_2$ we have that $\mu(E^j) < 2^{-3}$ and

$$\int_{E^j} |H_j|^p d\mu > 1 - 2^{-3}$$

where $E^j := \{|H_j| \geq 2^{-3}\}$. Since $\mu(E^j) \to 0$ as $j \to \infty$, we can find $n_2 \geq \tilde{n}_2$ such that

$$\int_{E^{n_2}} |H_{n_1}|^p d\mu < 2^{-3}.$$

Thus for $E_2 := E^{n_2}$, we see that $\mu(E_2) < 2^{-3}$,

$$\int_{E_2} |H_{n_2}|^p d\mu > 1 - 2^{-3}$$

and

$$\int_{E_1} |H_{n_1}|^p d\mu < 2^{-3}.$$

Inductively, we obtain a subsequence $(H_{(l)}) \subset (H_i)$ and a sequence (E_l) of measurable sets such that $\mu(E_l) < 2^{-l-1}$,

$$\int_{E_l} |H_{(l)}|^p d\mu > 1 - 2^{-l-1}$$

and

$$\int_{E_l} \sum_{k=1}^{l-1} |H_{(k)}|^p d\mu < 2^{-l-1}.$$

Set $A_l := E_l - \bigcup_{k=l+1}^{\infty} E_l$. (A_l) is a sequence of pairwise disjoint sets. Let $\tilde{h}_l := H_{(l)}\chi_{A_l}$. Then

$$\|\tilde{h}_l - H_{(l)}\|^p = \int_{A_l^c} |H_{(l)}|^p d\mu = \int_{E_l^c} |H_{(l)}|^p d\mu + \int_{E_l - A_l} |H_{(l)}|^p d\mu$$

$$\leq 2^{-l-1} + \sum_{k=l+1}^{\infty} \int_{E_k} |H_{(l)}|^p d\mu \ < \ 2^{-l-1} + \sum_{k=l+1}^{\infty} 2^{-k-1} \ < 2^{-l}.$$

Set $\underline{h}_l = \tilde{h}_l \ \|\chi_{F_{(l)}} f_{(l)}\|$. Then

$$\|\underline{h}_l - h_{(l)}\| = \|\chi_{F_{(l)}} f_{(l)}\| \ \|\tilde{h}_l - H_{(l)}\| \leq M \ 2^{-l} \to 0$$

where $M := \sup_l \|\chi_{F_{(l)}} f_{(l)}\| < \infty$, since $\lambda(f_n) < \infty$.

Thus $(\underline{h}_l)$ and $(\underline{g}_l)$ given by $\underline{g}_l := f_l - \underline{h}_l$ meet all parts of the Lemma.

Lemma A.2. A) Let $(f_n) \subset L_p(I)$, $1 \leq p < \infty$, be an equiintegrable sequence converging to 0 in measure. Then $\|f_n\| \to 0$.

B) Let (f_n) be a normalized sequence in $L_p(I)$, $1 < p < \infty$, which converges to 0 in measure. Let (g_n) be an equiintegrable sequence in $L_{p'}(I)$. Then $\int |f_n g_n| d\mu \to 0$.

Proof: A) i) Assume that even $f_n \to 0$ a.e. By the equiintegrability of (f_n), given any $\epsilon > 0$, we can find a constant $c > 0$ such that

$$\int_{\{|f_n|>c\}} |f_n|^p d\mu < \epsilon$$

for any n. Thus, if $f_n \to 0$ a.e., then by Fatou's Lemma

$$0 \leq \liminf_{n\to\infty} \int_I |f_n|^p d\mu \leq \epsilon + \int_{\{|f_n|\leq c\}} \limsup_{n\to\infty} |f_n|^p d\mu$$

$$\leq \epsilon + \int_I \limsup_{n\to\infty} |f_n|^p d\mu = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have the claim in this case.

ii) If $\|f_n\| \not\to 0$, then there is a subsequence $(f_{n'})$ of (f_n) with $\|f_{n'}\| > a$ for some $a > 0$. But since $f_{n'} \to 0$ in measure, there is a subsequence $(f_{n'_k})$ of $(f_{n'})$ with $f_{n'_k} \to 0$ a.e. By Part i), $\|f_{n'_k}\| \to 0$. Contradiction to $\|f_{n'}\| > a$.

B) By Hölder's inequality, $h_n := f_n g_n$ is an equiintegrable sequence in $L_1(I)$, and converges to 0 in measure. Now we may apply Part A).

We now turn to **basic sequences** (f_n) in $L_p(I)$, i.e. (f_n) is a basis of $\overline{\text{span}}[f_n]$. It is well-known (cf *Proposition 1.a.12.* of **(Lin I)**) that a sequence (f_n) with $0 < \inf \|f_n\| \leq \sup \|f_n\| < \infty$ and $f_n \to 0$ weakly has a subsequence which is a basic sequence.

A basic sequence (f_n) of a Banach space X is said to be **unconditional**, if for any $x \in \overline{\text{span}}[f_n]$, its basis expansion $\sum_{n=0}^{\infty} a_n f_n$ converges unconditionnally, i.e. $\sum_{n=0}^{\infty} \epsilon_n a_n f_n$ converges in X for any choice of $\epsilon_n \in \{0, 1\}$. For further details, consult *Section 1.d.* of **(Lin I)**.

Let $t \in I$. The sequence (r_n) of **Rademacher functions** is given by

$$r_0 \equiv 1,\ r_n(t) := \text{ sgn } \sin\,(2^n \pi t) \text{ for } n > 0,$$

while the **Haar system** is defined as $h_1 \equiv 1$ and, for $k = 0, 1, \ldots,\ l = 1, 2, \ldots, 2^k$:

$$h_{2^k+l}(t) = \begin{cases} 1 & \text{if} \quad (2l-2)\, 2^{-k-1} \leq t \leq (2l-1)\, 2^{-k-1} \\ -1 & \text{if} \quad (2l-1)\, 2^{-k-1} < t \leq 2l\, 2^{-k-1} \\ 0 & \text{otherwise.} \end{cases}$$

For the Rademacher functions, **Khintchine's Inequality** holds: For some constant $c = c(p) > 0$ and any square-summable complex sequence (a_n) we have

$$c^{-1}\,(\sum_{n=0}^{\infty} |a_n|^2)^{1/2} \leq \| \sum_{n=0}^{\infty} a_n r_n \| \leq c\,(\sum_{n=0}^{\infty} |a_n|^2)^{1/2},$$

where $1 \leq p < \infty$, and, as usual, $\| \ \|$ denotes the norm on $L_p(I)$.

If we normalize the Haar system in $L_p(I)$, $1 < p < \infty$, then they form an unconditional normalized basis in $L_p(I)$ (cf *Definition 1.a.4., p 3* of **(Lin I)**).

Lemma A.3. A) For any normalized unconditional basis $(f_n) \subset L_p(I)$ there exist a $c > 0$ such that

$$\text{if } 1 \le p \le 2: \quad \|\sum_n \alpha_n f_n\| \ge c\,(\sum_n |\alpha_n|^2)^{1/2},$$

$$\text{if } 2 \le p < \infty: \quad \|\sum_n \alpha_n f_n\| \le c\,(\sum_n |\alpha_n|^2)^{1/2}.$$

B) For any normalized unconditional basis $(f_n) \subset L_p(I)$ we have for some $c > 0$ that

$$\text{if } 1 \le p \le 2: \quad \|\sum_n \alpha_n f_n\| \le c\,(\sum_n |\alpha_n|^p)^{1/p}.$$

$$\text{if } 2 \le p < \infty: \quad \|\sum_n \alpha_n f_n\| \ge c\,(\sum_n |\alpha_n|^p)^{1/p}.$$

C) If $1 \le p < \infty$, then any disjoint normalized sequence $(f_n) \subset L_p(I)$ is equivalent to the unit vector basis in l_p.

Proof: Part A) easily follows from a result on *p 131* of **(Lin)**, while Part B) can be found on *p 4* of **[Joh]** or *p 209* of **[Ros]**. Part C) is well-known and easy to check:

$$\int |\alpha_n f_n|^p d\mu = \int \sum |\alpha_n|^p |f_n|^p d\mu = \sum |\alpha_n|^p.$$

Lemma A.4. Assume that $1 \le p < \infty$, and that $(f_n) \subset L_p(I)$ is a sequence converging weakly to 0 and satisfying

$$0 < \inf \|f_n\| \le \sup \|f_n\| < \infty.$$

Consider the following conditions:

A) A subsequence $(f_{n_k}) \subset (f_n)$ converges to 0 in measure.

B) $(f_n) \not\subset M_p^\epsilon$ for any $\epsilon > 0$ where $M_p^\epsilon := \{\, f \in L_p(I) : \, \mu\{|f| \geq \epsilon \|f\|\} \geq \epsilon \,\}$.

C) There exists a subsequence $(f_{n_k}) \subset (f_n)$ which is isomorphic to the unit vector basis of l_p.

D) There are $c > 0$ and a subsequence $(f_{n_k}) \subset (f_n)$ such that for any $(\alpha_k) \in l_p$:

$$\text{if } 1 \leq p < 2 : \; \|\sum_k \alpha_k f_{n_k}\| \leq c\,(\sum_k |\alpha_k|^p)^{1/p},$$

$$\text{if } 2 < p < \infty : \; \|\sum_k \alpha_k f_{n_k}\| \geq c\,(\sum_k |\alpha_k|^p)^{1/p}.$$

E) (f_n) is not equiintegrable.

F) There exist a subsequence $(f_{n_k}) \subset (f_n)$, a $\delta > 0$ and a sequence of disjoint sets (E_k) such that for all k:

$$\int_{E_k} |f_{n_k}|^p d\mu \geq \delta.$$

Then:

if $1 \leq p < 2$: A $\Longleftrightarrow$ B $\Longrightarrow$ C $\Longleftrightarrow$ D $\Longleftrightarrow$ E $\Longleftrightarrow$ F, and

if $2 < p < \infty$: A $\Longleftrightarrow$ B $\Longleftrightarrow$ C $\Longleftrightarrow$ D $\Longrightarrow$ E $\Longleftrightarrow$ F.

Note that if $1 \leq p < 2$, then Part C) does not imply Part B), and if $2 < p < \infty$, then Part E) does not imply Part D).

Proof: A $\Longleftrightarrow$ B, E $\Longleftrightarrow$ F and B $\Longrightarrow$ D are clear.

Other directions are based on known results: (We may assume, wlog, that $\|f_n\| = 1$.)

B $\Longrightarrow$ C: *Theorem 2, p 164* of **[Kad]**.

D $\Longrightarrow$ E: For $p = 1$, this follows from the fact that the unit vector basis of l_1 is not weakly compact, but equiintegrable sets are weakly compact in $L_1(I)$. For $1 < p < 2$, this follows from *Theorem 8* of **(Ros)**, and for $2 < p < \infty$, from *Theorem 3* of [**Kad**].

F $\Longrightarrow$ B if $1 \leq p < 2$: *Prop. I.1.15, p 21* of **(Lin)**, also *Lemma 1, p 4* and *Lemma 2, pp 12 - 13* of [**Joh**].

D $\Longrightarrow$ B if $2 < p < \infty$: *Lemma 1, p 4* of [**Joh**] and *Theorem 3, a $\Longleftrightarrow$ d, p 166* of [**Kad**].

The remark at the end of the Lemma can be demonstrated by considering a sequence (g_i) on $L_p(0,2)$ given by $g_i := f_i + r_i$ where (r_i) denote the Rademacher functions supported on $(0,1)$ while (f_i) is a normalized, disjoint sequence supported on $[1,2)$.

If $1 \leq p < 2$, then (g_i) satisfies Part C) since (f_i) does by Lemma A.3. Part C) and by Khintchine's inequality:

$$\begin{aligned} \|\sum \alpha_n g_n\|^p &= \|\sum \alpha_n f_n\|^p + \|\sum \alpha_n r_n\|^p \\ &\leq \sum |\alpha_n|^p + (\sum |\alpha_n|^2)^{p/2} \leq 2\sum |\alpha_n|^p, \end{aligned}$$

and

$$\|\sum \alpha_n g_n\|^p \geq \|\sum \alpha_n g_n \chi_{(0,1)}\|^p = \sum |\alpha_n|^p.$$

But also $(g_i) \subset M_p^\epsilon$ for any $0 < \epsilon \leq 1/2$.

If $2 < p < \infty$, then (g_n) satisfies Part F), since (f_i) does. But Part D) cannot be valid, since this would imply for some $c > 0$:

$$\sum |\alpha_n|^p \geq \|\sum \alpha_n g_n\|^p \geq \|\sum \alpha_n r_n\|^p \geq c\,(\sum |\alpha_n|^2)^{p/2}$$

for any $(\alpha_n) \in l_p$, which is impossible for $p > 2$.

APPENDIX B.

Rearrangements and Regular Functions.

In this appendix we collect some properties of rearrangements and regular functions that are related to equiintegrable sets and L_p operators. They will be useful in Section I.4. and Chapter II. Unless indicated otherwise, $I := (0,1)$ or $(0,\infty)$.

Definition B.1. A function $W : I \to (0,\infty)$ is called **regular** if it is strictly positive, left continuous and nonincreasing satisfying $W(1-) := \lim_{x\to 1-} W(x) > 0$ (lim exists because of left continuity) and $\int_0^1 W(t)dt < \infty$ such that its **constant of regularity** $M[W]$ is finite where

$$M[W] := \sup_{x\in I} \frac{1}{xW(x)} \int_0^x W(t)dt.$$

Then we define

$$D[W] := 1 - \frac{1}{M[W]}.$$

Furthermore, if W is regular and $I := (0,1)$, set the **infinitesimal constant of regularity** as

$$M(W) := \limsup_{x\to 0+} \frac{1}{xW(x)} \int_0^x W(t)dt$$

and

$$D(W) := 1 - \frac{1}{M(W)}.$$

Clearly, $M(W) \le M[W]$ and $D(W) \le D[W]$.

Note that if $I := (0, \infty)$, we do not require W to be integrable.

Remark B.2. A) Regularity on $I := (0,1)$ or $(0,\infty)$ is equivalent to requiring that

$$\sup_{x \in I} \int_0^x \frac{W(t)}{W(x-t)} dt < \infty$$

(cf *Theorem 1.1.2., p 3* of [**Rug**]).

B) Assume that $W \in L_p(I)$, $I := (0,1)$, is differentiable (in the usual sense) as a real-valued function and that the following limit exists:

$$L := \lim_{x \to 0+} \frac{-xW'(x)}{W(x)}.$$

Then:

i) If W is regular, then $0 < L < 1$ and the infinitesimal constant of regularity of W satisfies $D(W) = L$ (cf *Theorem 1.3.3., p 10* of [**Rug**]).

ii) If $L < 1$, then W is regular and $D(W) = L$ (cf *Lemma 1.3.6., p 12* of [**Rug**]).

iii) Under the assumptions of Part i) or Part ii), we have for any $\infty > p > 0$:

$$W^p \text{ is regular.} \iff p\, D(W) < 1.$$

Then, $D(W^p) = p\, D(W)$. (See *Theorem 1.3.4., p 10* of [**Rug**].)

C) Let $M > 0$, and $W, W_n : I \to (0,\infty)$, $I := (0,1)$ or $(0,\infty)$, be strictly positive, left continuous and nonincreasing functions. Assume that (W_n) is a sequence of integrable functions converging to W pointwise and in mean, whose (infinitesimal) constants of regularity are no larger than M. Then W is regular with (infinitesimal) constant of regularity at most M.

Definition B.3. We define the **(first) rearrangement** $f^\star : I \to (0,\infty)$ of a Lebesque measurable , a.e. finite function $f : I \to (0,\infty)$ by

$$f^\star(t) := \inf \{ \tau \ : \ n_f(\tau) < t \},$$

where

$$n_f(\tau) := \mu \{ t \ : \ |f(t)| > \tau \}$$

and μ denotes the Lebesque measure on I. We assume that $n_f(\tau)$ is finite for some $\tau \in (0,\infty)$. In a similar way, one defines the (first) rearrangement for a measurable, a.e. finite function on the real line $(-\infty,\infty)$ or the unit circle Γ.

Two measurable, a.e. finite functions f and g are said **equimeasurable** if $f^\star = g^\star$.

The **second rearrangement** $f^{\star\star} : I \to (0,\infty)$ at $t \in I$ is defined as the average of $f^\star$ over the interval $(0,t)$, i.e.

$$f^{\star\star}(t) := \frac{1}{t}\int_0^t f^\star(t)dt.$$

Remark B.4. It is clear from the definition that $f^\star$ always is nonincreasing and left continuous. For a nonincreasing nonnegative function f, we therefore see that $f(x) = f^\star(x)$ for all but possibly countably many $x \in I$, and f and $f^\star$ define the same equivalence class in $L_p(I)$.

We always have $f^\star \le f^{\star\star}$, and if the function f is regular, we have for all $x \in I$:

$$f^{\star\star}(x) \le M[f]\ f^\star(x).$$

For $f \in L_p(I), 1 < p < \infty$, we have that

$$\|f\| = \|f^\star\| \le \|f^{\star\star}\| \le q\ \|f\|,$$

where q denotes the conjugate index to p (cf *Theorem 4.9., p 139* of **(Ben)**). Our main reason for using the second rearrangement instead of the first, is the following inequality which does not hold in general for the first rearrangement: For two functions $f, g \in L_p(I)$, we have

$$(f+g)^{**}(x) \leq f^{**}(x) + g^{**}(x)$$

for all $x \in I$ (cf *§6.1., p 125* of **(Kre)**).

We shall need the following compactness principle.

Lemma B.5. If $(f_n) \subset L_p(I)$ is a sequence of nonincreasing nonnegative equiintegrable functions, then there exists a subsequence $(f_{(k)})$ of (f_n) and a nonincreasing function $f \in L_p(I)$ such that for any $0 < \epsilon < 1$:

$$\lim_{k \to \infty} \|f - f_{(k)}\| = 0.$$

Proof: Wlog, assume $I = (0, \infty)$. Helly's theorem (cf *Satz 3.2., p 247* of **(Vog)**) inductively provides functions $f^m \in L_p(1/m, \infty)$ for any $m > 0$ and subsequences

$$(f_{l_i^{(m)}}) \subset (f_{l_i^{(m-1)}}) \subset \ldots \subset (f_{l_i^{(1)}}) \subset (f_n),$$

such that for any $m > 0$ and all continuity points $x \in (1/m, \infty)$ of f^m:

$$f_{l_i^{(m)}}(x) \to f^m(x).$$

Since for $0 < l < k$, $f^k = f^l$ on $(1/l, \infty)$ except on an at most countable set, we may pointwise define a (measurable) nonincreasing function f which for any $m > 0$ equals f^m on $(1/m, \infty)$ except on a countable set.

Set $f_{(k)} := f_{l_k^{(k)}}$. Then $(f_{(k)})$ converges pointwise to the measurable function f. Since (f_n) is an equiintegrable sequence, we get from Lemma A.2 that $\chi_{[\epsilon,\infty)} f \in L_p(I)$ and

$$\|\chi_{[\epsilon,\infty)} f - f_{(k)}\| \to 0$$

for any $0 < \epsilon$. Also, the equiintegrability of (f_n) implies its boundedness, say $\|f_n\| \leq C$ for all n. It follows that $\|f^m\| \leq C$ for all m and $f \in L_p(I)$ by Fatou's lemma. Thus also $\|f - f_{(k)}\| \to 0$.

The second lemma deals with an alternate description of equiintegrability in terms of rearrangements.

Lemma B.6. The following are equivalent for $1 < p < \infty$:

i) $M \subset L_p(I)$ is equiintegrable.

ii) The set $\{f^\star : f \in M\}$ is norm-compact in $L_p(I)$.

iii) For any sequence $(f_n) \subset M$ there exist a *sub*sequence $(f_{(k)}) \subset (f_n)$ and a nonincreasing function $f \in L_p(I)$ such that $f^\star_{(k)} \leq f$ for any k.

iv) For any sequence $(f_n) \subset M$ there exist a *sub*sequence $(f_{(k)}) \subset (f_n)$ and a nonincreasing function $f \in L_p(I)$ such that $f^{\star\star}_{(k)} \leq f$ for any k.

Proof: i) $\Longrightarrow$ ii) follows from Lemma B.5.

ii) $\Longrightarrow$ iii) Given a sequence $(f_n) \subset M$, we choose by Part ii) a subsequence $(f_{(k)})$ such that $f^\star_{(k)} \to f \in L_p(I)$ in norm. Set $h_{(k)} := f^\star_{(k)} - f^\star_{(k)} \wedge f$ where $f^\star_{(k)} \wedge f$ denotes the pointwise infimum of $f^\star_{(k)}$ and f.

Since $\|f^\star_{(k)} - f\| \to 0$,

$$\lim_{k \to \infty} \|h_{(k)}\| = 0$$

and, by taking subsequences $(h_{(l)})$ again, we obtain that

$$\sum_l \|h_{(l)}\| < \infty.$$

Setting $F := f + \sum h_{(l)}$, $F \in L_p(I)$, we see that $(f_{(l)})$ and F have the disired properties of Part ii).

iii) $\Longrightarrow$ iv) $f^{\star}_{(k)} \leq f$ implies $f^{\star\star}_{(k)} \leq f^{\star\star} \in L_p(I)$.

iv) $\Longrightarrow$ i) If $M \subset L_p(I)$ were not equiintegrable, we could find $\epsilon > 0$ and a sequence $(f_n) \subset M$ such that

$$\|\chi_{\{|f_n|>n\}}\, f_n\| \geq \epsilon.$$

From Part iv), we obtain a subsequence $(f_{n_k}) \subset (f_n)$ and and an L_p function f such that $f^{\star}_{n_k} \leq f^{\star\star}_{n_k} \leq f$.

But then $\|\chi_{\{f>n_k\}}\, f\| \geq \epsilon$ for any k, contradicting $f \in L_p(I)$.

Lemma B.7. For any $1 < p \leq \infty$, $\epsilon > 0$, let $L^{\star}_p(I)$ denote the class of nonincreasing nonnegative functions $f \in L_p(I) - \{0\}$. Then, for any $\epsilon > 0$, there exists a map $W_\epsilon : L^{\star}_p(I) \to L^{\star}_p(I)$ such that for any $f \in L^{\star}_p(I)$:

i) $f \leq W_\epsilon(f)$,

ii) $\|W_\epsilon(f)\| \leq (1+\epsilon)\|f\|$,

iii) if $f_1 \leq f_2$ where $f_i \in L^{\star}_p(I)$, then $W_\epsilon(f_1) \leq W_\epsilon(f_2)$,

iv) if $f_n \uparrow f$ (i.e. $f_0 \leq f_1 \leq \ldots \leq f_i \leq \ldots \leq f$ and $f_n \to f$) where $f_i, f \in L^{\star}_p(I)$, then $W_\epsilon(f_n) \uparrow W_\epsilon(f)$,

v) $W_\epsilon(f)$ is regular and the following inequality holds for any $x \in I$:

$$\int_0^x W_\epsilon(f)(t)dt \leq (1 + \frac{1}{\epsilon})\; q\; xW_\epsilon(f)(x)$$

where q is the conjugate index to p ($q = 1$ if $p = \infty$).

Proof: Define $T : L_p(I) \to L_p(I)$ by $Tg(x) := \frac{1}{x}\int_0^x g(t)dt$. Then T is a bounded linear operator with

$$\|T\| = q = \begin{cases} \frac{p}{p-1} & \text{if } p < \infty \\ 1 & \text{if } p = \infty \end{cases}$$

for any $1 < p \leq \infty$ (cf *Lemma III.3.9., p 124* of **(Ben)** with $q = p = \lambda^{-1}$). Thus, we may define $S := \sum_{n=0}^{\infty}(\frac{T}{a})^n$ for any $a > q$ as a bounded linear operator in $L_p(I)$.

Set $W := W_\epsilon(f) := Sf$. Then $W(1-) > 0$ and $\int_0^1 W(t)dt < \infty$ are clear. Also, Part i) is satisfied.

Since $Tg = g^{**}$ for every nonincreasing nonnegative function $g \in L_p(I)$, we see that W is the sum of nonincreasing nonnegative functions and therefore nonincreasing itself.

Part iii) follows from $TW \leq aW$, i.e. since T and S are bounded ($a > q$):

$$aW = aSf = a\sum_{n=0}^{\infty}(\frac{T}{a})^n f \geq \sum_{n=0}^{\infty}\frac{T^{n+1}}{a^n}f = TW.$$

If we choose $a := (1 + \frac{1}{\epsilon})q$, then

$$\|W\| = \|Sf\| \leq \sum_{n=0}^{\infty}\|(\frac{T}{a})^n f\| \leq \sum_{n=0}^{\infty}(\frac{\epsilon}{\epsilon+1})^n\|f\| \leq (1+\epsilon)\|f\|.$$

Thus Part ii) is clear.

Part iv) follows from the fact that S is a positive operator. Part v) is immediate, since Part iv) shows $W_\epsilon(f_0) \leq W_\epsilon(f_1) \leq \ldots \leq W_\epsilon(f_i) \leq \ldots \leq W_\epsilon(f)$ and the continuity of S implies $W_\epsilon(f_i) \to W_\epsilon(f)$.

Lemma B.8. For any $1 < p \leq \infty$, $\epsilon > 0$, there is a map $h_\epsilon : L_p^\star(I) \to L_p^\star(I)$ such that for any $f \in L_p^\star(I)$:

i) $f \leq h_\epsilon(f)$, and $h_\epsilon(f)$ is the second rearrangement of a regular function,

ii) $\|h_\epsilon(f)\| \leq (1+\epsilon)q\ \|f\|$,

iii) if $f_1 \leq f_2$ where $f_i \in L_p^\star(I)$, then $h_\epsilon(f_1) \leq h_\epsilon(f_2)$,

iv) if $f_n \uparrow f$ where $f_i, f \in L_p^\star(I)$, then $h_\epsilon(f_n) \uparrow h_\epsilon(f)$,

v) for any $0 < k < p$ and any $\epsilon > \max(0, \frac{k-1}{1-k/p})$, $h_\epsilon^k(f) \equiv [h_\epsilon(f)]^k$ is regular, and the following inequalities hold:

$$M[h_\epsilon^k(f)] \leq \frac{\epsilon+1}{\epsilon\,(1-k/p)-k+1} \quad \text{and} \quad D[h_\epsilon^k(f)] \leq k\,\frac{\epsilon/p+1}{1+\epsilon}.$$

Proof: The identity $W(x) = x(W^{**})'(x) + W^{**}(x)$ $(x \in I)$, valid for any absolutely continuous, in particular any regular function $W \in L_p(I)$, demonstrates that for any $0 < k < \infty$ and $0 \leq c < 1$, the following are equivalent:

$$\int_0^x W(t)dt \leq (1-c)^{-1}\, xW(x)$$

$$\Longleftrightarrow$$

$$\frac{-x(W^{**})'(x)}{W^{**}(x)} \leq c$$

$$\Longleftrightarrow$$

$$\frac{-x[(W^{**})^k]'(x)}{(W^{**})^k(x)} \leq kc.$$

Set $h := W^{**}$. Then the last inequality is equivalent to

$$-x(h^k)'(x) \leq kc\, h^k(x).$$

For $0 < k \leq p$ we see that

$$\lim_{t\to 0+} th^k(t) = \lim_{t\to 0+} t[W^{**}]^k(t) = 0,$$

since $\chi_{(0,1)}W^{**} \in L_k(I)$ if $0 < k \leq p$. Therefore, assuming $kc < 1$ and $0 < k \leq p$, integrating

$$-x(h^k)'(x) \leq kc\, h^k(x)$$

by parts gives

$$\int_0^x h^k(t)dt \leq (1-kc)^{-1}\, xh^k(x).$$

Thus, if W is regular, then $h^k \equiv [W^{**}]^k$ for $0 < k \leq p$ is regular provided $kc < 1$ where c is determined by $M[W] = (1-c)^{-1}$.

Given $f \in L_p(I) - \{0\}$ and $\epsilon > 0$, let $W \equiv W_\epsilon(f)$ be the function obtained when Lemma B.7. is applied. Let $h \equiv h_\epsilon(f) := [W_\epsilon(f)]^{**}$. Then $h_\epsilon(f)(1-) > 0$ and $\int_0^1 h_\epsilon(f)(t)dt < \infty$ are clear. Parts i), iii) and iv) follow from the corresponding parts of Lemma B.7. and the properties of the second rearrangement, in particular its continuity. Part ii) is an immediate consequence of Lemma B.7. and Remark B.4.

Finally, we need to show Part v). For $1 < k < p$, setting

$$(1-c)^{-1} := (1+\frac{1}{\epsilon})q \geq M[W_\epsilon(f)]$$

shows

$$c = \frac{\epsilon/p+1}{\epsilon+1}.$$

Taking

$$\epsilon > \frac{k-1}{1-k/p} \quad (\iff k\frac{\epsilon/p+1}{\epsilon+1} < 1),$$

we see that

$$kc = \frac{k(\epsilon/p+1)}{\epsilon+1} < 1,$$

and thus $h_\epsilon^k(f) \equiv [h_\epsilon(f)]^k$ is regular with constant of regularity $M[h_\epsilon^k(f)]$ no larger than

$$(1-kc)^{-1} = (\frac{\epsilon(1-k/p)-k+1}{\epsilon+1})^{-1} = \frac{\epsilon+1}{\epsilon(1-k/p)-k+1}.$$

For $0 < k \leq 1$, any $\epsilon > 0$ will do. In this case, the regularity of $h_\epsilon^k(f)$ also follows from *Theorem 1.1.7., p 6* of [**Rug**].

Lemma B.9. For any $1 < p \leq \infty$, $f \in L_p(I)$, $\epsilon > 0$ and any bounded linear operator $T : L_p(I) \to L_p(I)$, there exists a nonincreasing function $F \in L_p(I)$ such that

i) $(TF)^{**} \leq (1+\frac{1}{\epsilon})\, q\, \|T\| \cdot F$,

ii) $\|F\| \leq (1+\epsilon)\|f\|$, and $f \leq F$,

where $\|\ \|$ denotes the norm on $L_p(I)$ and q is the conjugate index to p.

Proof: Assume $T \not\equiv 0$. Define inductively for any $n > 0$ the operators

$$T_n : L_p(I) \to L_p(I)$$

by $T_1 f := (Tf)^{**}$ and $T_{n+1} f := (T(T_n f))^{**}$. Then (T_n) is a sequence of bounded linear operators in $L_p(I)$, and a simple estimate shows that

$$\|T_n f\| \leq q^n \|T\|^n \|f\|,$$

where q denotes the conjugate index to $1 < p \leq \infty$. Set

$$F := \sum_{n=0}^{\infty} a^n T_{n+1} f,$$

where

$$a := \frac{\epsilon}{(1+\epsilon)\, q\, \|T\|}.$$

Then

$$(TF)^{**} = (\sum_{n=1}^{\infty} a^n T(T_n f))^{**} \leq \sum_{n=1}^{\infty} a^n (T(T_n f))^{**} = \sum_{n=2}^{\infty} a^{n-1} T_n f \leq \frac{1}{a} F,$$

and

$$\|F\| \leq \sum_{n=1}^{\infty} \frac{\epsilon^n \|T_n\|}{(1+\epsilon)^n q^n \|T\|^n} \|f\| \leq \sum_{n=1}^{\infty} (\frac{\epsilon}{1+\epsilon})^n \|f\| \leq (1+\epsilon)\|f\|.$$

VITA.

Michael Helmuth Ruge was born in Hagen/Westfalen, Federal Republic of Germany on the 13th of March 1962. He got his German Abitur from the Albert-Einstein-Gymnasium at Frankenthal (Pfalz) in 1980 and obtained his German Vordiplom at the Universität Kaiserslautern in 1982. He was awarded an M.S. in Mathematics at the Louisiana State University in 1986 and completed his German Diplom at the Universität Kaiserslautern in 1988. The title of his Diplomarbeit was *On Regularity and Rearrangements in* L_p *Spaces.* Besides that, he helped in revising the college textbook *Calculus with Analytic Geometry* by L.I. Holder.

He was a Fulbright Scholar from 1984 to 1986 and currently cochairs the Fulbright Scholar Association at LSU. Also, he is an active member of the Honor Society of Phi Kappa Phi. After graduation, he plans to join a major computer company.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Michael Ruge

Major Field: Mathematics

Title of Dissertation: L_p Regularity and Extrapolation

Approved:

Lutz Weis
Major Professor and Chairman

F. Glen Hensley
Dean of the Graduate School

EXAMINING COMMITTEE:

John A. Hildebrant

William A. Adkins

G. Ferreyra

James R. Rutherford

Frank R. Groves Jr.

Date of Examination:

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