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L_p-THEORY FOR A CLASS OF SINGULAR ELLIPTIC DIFFERENTIAL OPERATORS

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1. INTRODUCTION

Let Ω be an arbitrary connected domain in the n-dimensional Euclidean space R_n. Let $\varrho(x)$ be a weight function,

$$\varrho(x) \in C^\infty(\Omega)\,, \quad \varrho(x) > 0\,,$$
$$\varrho(x) \to \infty \quad \text{for} \quad x \to \partial\Omega \quad \text{or} \quad |x| \to \infty\,,$$
$$|D^\gamma \varrho(x)| \leqq c\varrho^{1+|\gamma|}(x)^*)\,.$$

Examples. (a) In every bounded domain there exists a function $\varrho(x)$ such that

$$c_1\, d(x) \leqq \varrho^{-1}(x) \leqq c_2\, d(x)\,, \quad 0 < c_1 < c_2$$

holds. $d(x)$ is the distance of $x \in \Omega$ from the boundary of $\partial\Omega$.

(b) $\Omega = R_n$,

$$\varrho(x) = (1 + |x|^2)^{\varkappa/2} \quad \text{or} \quad \varrho(x) = \exp\,(1 + |x|^2)^{\varkappa/2}\,; \quad \varkappa > 0\,.$$

We consider singular elliptic differential operators A,

$$Au = \sum_{|\alpha| \leqq 2m} a(x)\, D^\alpha u\,.$$

Besides the (singular) ellipticity condition we assume the growth conditions

$$D^\gamma\, a_\alpha(x) = O(\varrho^{\varkappa_{|\alpha|}+|\gamma|})\,; \quad \varkappa_l = \nu\,\frac{2m - l}{2m} + \mu\,\frac{l}{2m}\,; \quad (l = 0, \ldots, 2m)\,;$$

*) We denote all unimportant constants by $c, c', c'', \ldots, c_1, c_2, \ldots$

ν and μ are real numbers; $\nu \geqq 0$; $\nu > \mu + 2m$; $|\gamma| \geqq 0$. The exact definition is given in Section 2.2.

Let be

$$S_{\varrho(x)} = \{f \mid f \in C^\infty(\Omega),\ \sup_{x\in\Omega} \varrho^l(x) \left|D^\alpha f(x)\right| < \infty \ \text{ for all } \alpha \text{ and } l = 0, 1, 2, \ldots\},$$

$$\|f\|_{W_p{}^\lambda} = \left(\int_{\Omega\times\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\lambda p}}\, dx\, dy + \|f\|^p_{L_p}\right)^{1/p}; \quad 1 < p < \infty; \quad 0 < \lambda < 1;$$

$$W_p^0 = L_p,$$

and

$$W^s_{p,\sigma,\tau}(\Omega) = \{f \mid f \in D'(\Omega),\ \|f\|_{W^s{}_{p,\sigma,\tau}} = (\sum_{|\alpha|=[s]} \|\varrho^{\sigma/p} D^\alpha f\|^p_{W^{\{s\}}{}_p} + \|\varrho^{\tau/p} f\|^p_{L_p})^{1/p} < \infty\},$$

$$0 \leqq s = [s] + \{s\},\ [s] \text{ integer},\ 0 \leqq \{s\} < 1;\ \tau > \sigma + ps.$$

The main results of this paper are:

(a) $$S_{\varrho(x)}(\Omega) \subset L_p(\Omega) \quad \text{iff} \quad \exists a > 0,\quad \varrho^{-a} \in L_1(\Omega).$$

($\subset$ always means a continuous embedding of the left space into the right space). If such a number a exists then $S_{\varrho(x)}(\Omega)$ is a nuclear space isomorphic to s, the space of rapidly decreasing sequences.

(b) $A - \lambda E$ is an isomorphic map from

$$S_{\varrho(x)}(\Omega) \text{ onto } S_{\varrho(x)}(\Omega)$$

and from

$$W^{2m+s}_{p,\varkappa+p\mu(1+s/2m),\varkappa+p\nu(1+s/2m)}(\Omega) \text{ onto } W^s_{p,\varkappa+p\mu s/2m,\varkappa+p\nu s/2m}(\Omega).$$

$s \geqq 0$, $\varkappa$ is an arbitrary real number. λ is a complex number with $\operatorname{Re}\lambda \leqq c$. (There exists $a > 0$ such that $\varrho^{-a}(x) \in L_1(\Omega)$).

The exact formulations of both the assumptions and the results follow in the next sections.

Locally convex spaces of the type $S_{\varrho(x)}(\Omega)$ are considered in [3, 4, 5] also in connection with special operators of the described type (selfadjoint, acting in $L_2(\Omega)$). The spaces $W^s_{p,\sigma,\tau}(\Omega)$ are introduced in [6]. We developed in [6] an interpolation theory for these spaces which is the basis for some results proved here. Further, we obtain an improvement of a structure theorem for the spaces $W^s_{p,\sigma,\tau}(\Omega)$; $\tau > \sigma + sp$; $1 < p < \infty$. In [6] we showed that all these spaces have a Schauder basis, $W^s_{p,\sigma,\tau}(\Omega)$ is isomorphic to l_p for $s \neq$ integer. Now we obtain, moreover that $W^{2m}_{p,\sigma,\tau}(\Omega)$, $m = 0, 1, 2, \ldots$; is isomorphic to $L_p((0, 1))$.*)

*) By other methods it is possible to prove that all the spaces $W^m_{p,\sigma,\tau}(\Omega)$; $m = 0, 1, 2, \ldots$; are isomorphic to $L_p((0, 1))$.

2. DEFINITION

2.1. The weight function $\varrho(x)$. Let Ω be an arbitrary connected (bounded or unbounded) domain in the n-dimensional Euclidean space R_n. $\partial\Omega$ denotes the boundary. $C^\infty(\Omega)$ is the set of all complex infinitely differentiable functions. We consider a weight functions $\varrho(x)$,

$$\varrho(x) \in C^\infty(\Omega)\,; \quad \varrho(x) > 0 \quad \text{for} \quad x \in \Omega\,; \tag{1}$$

for all multiindices γ there exists $c_\gamma > 0$ with

$$|D^\gamma \varrho(x)| \leqq c_\gamma\, \varrho^{1+|\gamma|}(x) \tag{2}$$

for $x \in \Omega$; for all $K > 0$ there exist $\varepsilon_k > 0$ and $r_k > 0$ with

$$\varrho(x) > K \quad \text{for} \quad d(x) \leqq \varepsilon_k \quad \text{or} \quad |x| \geqq r_k \; (x \in \Omega)\,. \tag{3}$$

$d(x)$ is the distance of the point $x \in \Omega$ from the boundary $\partial\Omega$. We considered weight functions of such type in [6], Section 3.5, example 2. We write

$$\Omega^{(j)} = \{x \mid x \in \Omega;\ \varrho(x) < 2^j\}\,; \quad j = N\,, \quad N + 1, \ldots\,; \quad (\Omega^{(N)} \neq \emptyset)\,.$$

In [6] we showed

$$d(\partial\Omega^{(j)},\ \partial\Omega^{(j+1)}) \geqq c\,.\,2^{-j}\,, \tag{4}$$

$c > 0$ is independent of j. $d(\partial\Omega^{(j)}, \partial\Omega^{(j+1)})$ is the distance between the boundarie $\partial\Omega^{(j)}$ and $\partial\Omega^{(j+1)}$.

Let us describe an important example for weight functions $\varrho(x)$. Let Ω be a bounded domain, $d(x)$ denotes again the distance of $x \in \Omega$ from the boundary. In [6], Section 3.5, we mentioned the existence of weight functions $\varrho(x)$ with the desired properties such that

$$c_1\, d(x) \leqq \varrho^{-1}(x) \leqq c_2\, d(x)\,; \quad 0 < c_1 < c_2\,,$$

$\varrho^{-1}(x)$ is a *"general distance function"*. We mentioned in the introduction other simple examples of weight functions $\varrho(x)$.

2.2. Operators of the type $A^{(m)}_{\mu,\nu}$. Let m be an integer; $m = 1, 2, \ldots$; μ and ν are real numbers; $\nu > \mu + 2m$;

$$\varkappa_l = \frac{1}{2m}\,[\nu(2m - l) + \mu l]\,; \quad l = 0, 1, \ldots, 2m\,. \tag{5}$$

A is said to be an operator of the type $A^{(m)}_{\mu,\nu}$ if

$$A\mu = \sum_{l=0}^{m} \sum_{|\alpha|=2l} \varrho^{\varkappa_{2l}}(x)\, b_\alpha(x)\, D^\alpha u + \sum_{|\beta|<2m} a_\beta(x)\, D^\beta u\,. \tag{6}$$

$b_\alpha(x)$, $a_\beta(x)$ are infinitely differentiable real functions, $D^\gamma b_\alpha(x)$ bounded in Ω for all γ and all α; $|\alpha| = 2l$; $l = 0, \ldots, m$. There exists a positive number c so that for all $\xi = (\xi_1, \ldots, \xi_n) \in R_n$, $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$ and all $x \in \Omega$

(7a) $$(-1)^m \sum_{|\alpha|=2m} b_\alpha(x)\, \xi^\alpha \geqq c|\xi|^{2m}\,, \quad b_{(0,\ldots,0)}(x) \geqq c\,,$$

(7b) $$(-1)^l \sum_{|\alpha|=2l} b_\alpha(x)\, \xi^\alpha \geqq 0\,; \quad l = 1, \ldots, m-1\,;$$

holds (ellipticity-condition).

For all γ and all β ($|\beta| < 2m$); is

(8) $$D^\gamma a_\beta(x) = o(\varrho^{\varkappa|\beta| + |\gamma|}(x))\,.$$

(This means: For all $\varepsilon > 0$ there exists an integer $j_0(\varepsilon)$ such that

$$|D^\gamma a_\beta(x)| \leqq \varepsilon\, \varrho^{\varkappa|\beta| + |\gamma|}(x) \quad \text{for} \quad x \in \Omega - \Omega^{(j)}\,; \quad j \geqq j_0(\varepsilon))\,.$$

Let us describe a few simple examples.

(a) Let Ω be a bounded domain. Let $\varrho^{-1}(x)$ be a general distance function. Then

$$Au = \varrho^\mu(x)\,(-\Delta)^m\, u + \varrho^\nu(x)\, u\,; \quad \nu > \mu + 2m\,;$$

is an $A^{(m)}_{\mu,\nu}$-operator. If $\partial\Omega \in C^\infty$ we may assume $\varrho^{-1}(x) = d(x)$ near the boundary.

(b) Let $\Omega = R_n$. It is easy to see that

$$Au = (1 + |x|^2)^{\eta_1}\,(-\Delta)^m\, u + (1 + |x|^2)^{\eta_2}\, u\,; \quad \eta_2 > \eta_1\,;$$

is an $A^{(m)}_{\mu,\nu}$-operator. (We choose $\varrho(x) = (1 + |x|^2)^\varkappa$ with a suitable positive number $\varkappa$.)

3. PROPERTIES OF THE OPERATORS $A^{(m)}_{\mu,\nu}$

3.1. Powers of $A^{(m)}_{\mu,\nu}$. Lemma 3.1. *Let A be an operator of the type $A^{(m)}_{\mu,\nu}$. Then A^k is an operator of the type $A^{(km)}_{k\mu,k\nu}$; $k = 1, 2, \ldots$*

Proof. Assume that the lemma is true for $k = 1, \ldots, j$. Let

(9) $$A^j u = \sum_{l=0}^{jm} \sum_{|\alpha|=2l} \varrho^{\varkappa^{(j)}_{2l}}(x)\, b^{(j)}_\alpha(x)\, D^\alpha u + \sum_{|\beta|<2mj} a^{(j)}_\beta(x)\, D^\beta u$$

with the properties of the coefficients described in Section 2.2. Particularly

$$\varkappa^{(j)}_l = \nu\,\frac{2mj - l}{2m} + \mu\,\frac{l}{2m}\,; \quad l = 0, \ldots, 2mj\,.$$

It is

$$\varkappa_l^{(j)} + \varkappa_s = \varkappa_{l+s}^{(j+1)}$$

(6), (9), and the last relation show that the "main part" of $A^{j+1}u = A^j(Au)$ has the right structure, (7) is true. Using

$$|D^\gamma \varrho^\varkappa(x)| \leqq c\, \varrho^{\varkappa+|\gamma|}(x)$$

and

$$(10) \qquad \varkappa_l^{(j)} + \varkappa_s + |\gamma| = \varkappa_{l+s}^{(j+1)} + |\gamma| < \varkappa_{l+s-|\gamma|}^{(j+1)} \quad \text{for} \quad 0 < |\gamma| \leqq l + s$$

we obtain that the "perturbation part" has also the desired structure.

3.2. The spaces $W^l_{p,\sigma,\tau}(\Omega)$. In the next section we shall prove an a-priori estimate. For this purpose we introduce the spaces $W^l_{p,\sigma,\tau}(\Omega)$. Let l be an integer; $l = 1, 2, \ldots$; let σ and τ be real numbers; $l < p < \infty$, $\tau > \sigma + pl$. $D'(\Omega)$ denotes the complex distributions in Ω. We write

$$W^l_{p,\sigma,\tau}(\Omega) = \{f \mid f \in D'(\Omega),\ \|f\|_{W^l{}_{p,\sigma,\tau}} = \Big(\sum_{|\alpha|=l} \|\varrho^{\sigma/p} D^\alpha f\|^p_{L_p(\Omega)} + \|\varrho^{\tau/p} f\|^p_{L_p(\Omega)}\Big)^{1/p} < \infty\}\,.$$

These Banach spaces are introduced in [6]. Further we write

$$L_{p,\sigma}(\Omega) = W^0_{p,\sigma,\sigma}(\Omega) = \{f \mid f \in D'(\Omega),\ \|f\|_{L_{p,\sigma}} = \|\varrho^{\sigma/p} f\|_{L_p} < \infty\}\,.$$

Let us recall some properties proved in [6]. $\Omega^{(j)}$ has the same meaning as in Section 2.1. We write

$$\Omega_j = \Omega^{(j+2)} - \Omega^{(j-1)}\,; \quad j = N+1,\ N+2,\ \ldots\,; \quad \Omega_N = \Omega^{(N+2)}\,.$$

There exists a set of functions $\{\psi_j(x)\}_{j=N}^\infty$ with

$$0 \leqq \psi_j(x) \leqq 1\,; \quad \psi_j(x) \in C_0^\infty(\Omega_j)\,; \quad \sum_{j=N}^\infty \psi_j(x) \equiv 1 \quad \text{for} \quad x \in \Omega\,;$$

$$(11) \qquad |D^\gamma \psi_j(x)| \leqq c\, 2^{j|\gamma|}\,; \quad j = N,\ N+1,\ \ldots\,; \quad 0 \leqq |\gamma| < \infty\,,$$

(c is independent of j and $|\gamma|$). We cover Ω_j with balls,

$$(12) \qquad \Omega_j \subset \bigcup_{l=1}^{N_j} K_l^{(j)} \subset \Omega_{j-1} \cup \Omega_{j+1}\,; \quad K_l^{(j)} = \{x \mid |x - x_{j,l}| \leqq c\, 2^{-j}\}\,,$$

c is a suitable positive number independent of j, see (4).*) Now we choose systems

*) By suitable choice of $K_l^{(j)}$ there exists a number L such that $\bigcap_{m=1}^{L} K_{l_m}^{(j)} = \emptyset$; $j = N$, $N + 1, \ldots$; L is independent of j; $l_r \neq l'$.

$\{\varphi_l^{(j)}(x)\}_{l=1}^{N_j}$; $j = N, N+1, \ldots$; with

$$0 \leqq \varphi_l^{(j)}(x) \leqq 1\,; \quad \varphi_l^{(j)}(x) \in C_0^\infty(K_l^{(j)})\,; \quad \sum_{l=1}^{N_j} \varphi_l^{(j)}(x) = 1 \quad \text{for} \quad x \in \Omega_j\,;$$

$$(13) \qquad \left|D^\gamma \varphi_l^{(j)}(x)\right| \leqq c\, 2^{j|\gamma|}\,; \quad j = N, N+1, \ldots; \quad l = 1, \ldots, N_j\,; \quad |\gamma| \geqq 0\,.$$

The method developed in [6] shows that it holds

$$(14) \qquad W^l_{p,\sigma,\tau}(\Omega) = \{f \mid f \in D'(\Omega);\ \|f\|^*_{W^l{}_{p,\sigma,\tau}} =$$
$$= [\sum_{j=N}^{\infty} \sum_{m=1}^{N_j} (2^{j\sigma}\|\psi_j\, \varphi_m^{(j)} f\|^p_{W^l{}_p(R_n)} + 2^{j\tau}\|\psi_j\, \varphi_m^{(j)} f\|^p_{L_p(R_n)})]^{1/p} < \infty\}\,.$$

($f(x) = 0$ for $x \notin \Omega$). The norms $\|f\|_{W^l{}_{p,\sigma,\tau}}$ and $\|f\|^*_{W^l{}_{p,\sigma,\tau}}$ are equivalent. In [6], Theorem 3.2, we proved that $C_0^\infty(\Omega)$ is a dense subset in these spaces. The following lemma will be helpful for the further considerations:

Lemma 3.2. *Let l be an integer; α a multindex; $0 < |\alpha| < l$, σ and τ real numbers; $\tau > \sigma + pl$; $1 < p < \infty$. Then there exists a positive number c with*

$$\left(\int_\Omega \varrho^\varkappa(x)\, |D^\alpha f(x)|^p\, dx\right)^{1/p} \leqq c\|f\|_{W^l{}_{p,\sigma,\tau}}\,;$$
$$\varkappa \leqq \tau\, \frac{l - |\alpha|}{l} + \sigma\, \frac{|\alpha|}{l}\,; \quad f \in W^l_{p,\sigma,\tau}(\Omega)\,.$$

Proof. It is

$$\int_\Omega \varrho^\varkappa(x)\, |D^\alpha f(x)|^p\, dx \leqq c \sum_{j=N}^{\infty} \sum_{m=1}^{N_j} 2^{j\varkappa} \|D^\alpha(\psi_j\, \varphi_m^{(j)} f)\|^p_{L_p(R_n)} \leqq$$
$$\leqq c' \sum_{j=N}^{\infty} \sum_{m=1}^{N_j} (2^{j\sigma}\|\psi_j\, \varphi_m^{(j)} f\|^p_{W^l{}_p(R_n)})^{|\alpha|/l}\, (2^{j\tau}\|\psi_j\, \varphi_m^{(j)} f\|^p_{L_p(R_n)})^{(l-|\alpha|)/l} \leqq c''\|f\|^p_{W^l{}_{p,\sigma,\tau}}\,.$$

We used (14). This proves the lemma.

3.3. A-priori estimate. The basis for the further considerations is the following a-priori estimate.

Lemma 3.3. *Let A be an operator of the type $A^{(m)}_{\mu,\nu}$, $\nu \geqq 0$. Let $\varkappa$ be a real number. Then there exist three numbers c_1, c_2, and c_3; $c_2 > c_1 > 0$; c_3 real, that*

$$(15) \qquad c_2\|u\|_{W^{2m}{}_{p,\varkappa+p\mu,\varkappa+p\nu}} \geqq \|Au - \lambda u\|_{L_{p,\varkappa}} \geqq c_1\|u\|_{W^{2m}{}_{p,\varkappa+p\mu,\varkappa+p\nu}}$$

for $\operatorname{Re} \lambda \leqq c_3$ (λ *complex*) *holds.* $u \in W^{2m}_{p,\varkappa+p\mu,\varkappa+p\nu}(\Omega)$.

Proof. We use the functions $\psi_j(x)$, $\varphi_k^{(j)}(x)$, the balls $K_k^{(j)}$ (with the centre $x_{j,k}$), and the equivalent norm (14) introduced in the last section. First we assume that $u \in W^{2m}_{p,\varkappa+p\mu,\varkappa+p\nu}(\Omega)$ vanishes outside of a fixed ball $K_k^{(j)}$. It is

$$Au - \lambda u = Bu + Cu + Du\,,$$

$$Bu = \sum_{l=0}^{m} \sum_{|\alpha|=2l} \varrho^{\varkappa_{2l}}(x_{j,k})\, b_\alpha(x_{j,k})\, D^\alpha u - \lambda u\,,$$

$$Cu = \sum_{l=0}^{m} \sum_{|\alpha|=2l} \left[\varrho^{\varkappa_{2l}}(x)\, b_\alpha(x) - \varrho^{\varkappa_{2l}}(x_{j,k})\, b_\alpha(x_{j,k})\right] D^\alpha u\,,$$

$$Du = \sum_{|\beta|<2m} a_\beta(x)\, D^\beta u\,.$$

We denote the Fourier transformation in $S'(R_n)$ (the set of tempered distributions) by F, the inverse Fourier transformation is F^{-1}. Then

$$\|Bu\|^p_{L_{p,\varkappa}} \geqq c\, \varrho^\varkappa(x_{j,k}) \int_{R_n} \left|\sum_{l=0}^{m} \sum_{|\alpha|=2l} \varrho^{\varkappa_{2l}}(x_{j,k})\, b_\alpha(x_{j,k})\, D^\alpha u - \lambda u\right|^p \mathrm{d}x\,,$$

$c > 0$. ($u(x) = 0$ for $x \notin \Omega$). We use the transformation

$$x = \varrho^{-(\nu-\mu)/2m}(x_{j,k})\, y\,, \quad u(x) = v(y) \tag{16}$$

and we obtain with the help of $\varkappa_{2l} + 2l(\nu - \mu)/2m = \nu$

$$\|Bu\|^p_{L_{p,\varkappa}} \geqq \varrho^{\varkappa+\nu p - n(\nu-\mu)/2m}(x_{j,k})\,.$$
$$\cdot \left\|F^{-1}\left(\sum_{l=0}^{m} \sum_{|\alpha|=2l} (-1)^l\, b_\alpha(x_{j,k})\, \xi^\alpha - \lambda \varrho^{-\nu}(x_{j,k})\right) Fv\right\|^p_{L_p(R_n)}\,,$$

$c > 0$. (7) shows that

$$\gamma(\xi) = (1 + |\xi|^2)^m \left(\sum_{l=0}^{m} \sum_{|\alpha|=2l} (-1)^l\, b_\alpha(x_{j,k})\, \xi^\alpha - \lambda \varrho^{-\nu}(x_{j,k})\right)^{-1} \tag{17}$$

and

$$\lambda \varrho^{-\nu}(x_{j,k}) \left(\sum_{l=0}^{m} \sum_{|\alpha|=2l} (-1)^l\, b_\alpha(x_{j,k})\, \xi^\alpha - \lambda \varrho^{-\nu}(x_{j,k})\right)^{-1}$$

are multipliers in the sense of Michlin-Hörmander, see [1]. Re $\lambda \leqq 0$. With the help of Hörmander's multiplier theorem [1], 2.5, it follows

$$\left\|F^{-1}\left(\sum_{l=0}^{m} \sum_{|\alpha|=2l} (-1)^l\, b_l(x_{j,k})\, \xi^\alpha - \lambda \varrho^{-\nu}(x_{j,k})\right) Fv\right\|_{L_p(R_n)} \geqq$$
$$\geqq c\left(\left\|F^{-1}(1 + |\xi|^2)^m\, Fv\right\|_{L_p(R_n)} + |\mathrm{Re}\,\lambda|\, \varrho^{-\nu}(x_{j,k})\, \|v\|_{L_p(R_n)}\right).$$

$c > 0$ is independent of λ, j, and k. Using again the transformation (16) we obtain

$$(18) \qquad \|Bu\|_{L_{p,\varkappa}}^p \geqq c\varrho^{\varkappa+\nu p-n(\nu-\mu)/2m}(x_{j,k}) \cdot$$
$$\cdot \Big(\sum_{|\alpha|=2m} \|D^\alpha v\|_{L_p(R_n)}^p + \|v\|_{L_p(R_n)}^p + |\mathrm{Re}\,\lambda|^p \varrho^{-\nu p}(x_{j,k}) \|v\|_{L_p(R_n)}^p \Big) \geqq$$
$$\geqq c'\Big(\varrho^{\varkappa+\mu p}(x_{j,k}) \sum_{|\alpha|=2m} \|D^\alpha u\|_{L_p(R_n)}^p + \varrho^{\varkappa+\nu p}(x_{j,k}) \|u\|_{L_p(R_n)}^p +$$
$$+ |\mathrm{Re}\,\lambda|^p \varrho^{\varkappa}(x_{j,k}) \|u\|_{L_p(R_n)}^p\Big) \geqq c'' \|u\|_{W^{2m}{}_{p,\varkappa+p\mu,\varkappa+p\nu}}^p + c''' |\mathrm{Re}\,\lambda|^p \|u\|_{L_{p,\varkappa}}^p$$

c, c', c'', and c''' are positive numbers. Now we estimate $\|Cu\|_{L_{p,\varkappa}}$. (2) shows

$$|\varrho^{\varkappa_{2l}}(x)\, b_\alpha(x) - \varrho^{\varkappa_{2l}}(x_{j,k})\, b_\alpha(x_{j,k})| \leqq c\, 2^{j(\varkappa_{2l}+1)} |x - x_{j,k}|\,;$$

$x \in K_k^{(j)}$. Now we choose the number c in (12) sufficiently small (but independent of j). Then we obtain with the help of Lemma 3.2

$$(19) \qquad \|Cu\|_{L_{p,\varkappa}}^p \leqq \varepsilon \sum_{l=0}^{m} \sum_{|\alpha|=2l} 2^{j\varkappa+j\varkappa_{2l}p} \|D^\alpha u\|_{L_p(R_n)}^p \leqq$$
$$\leqq \varepsilon' \sum_{|\alpha|\leqq 2m} \int_\Omega \varrho^{\varkappa+\varkappa_{|\alpha|}p}(x)\, |D^\alpha u(x)|^p \,\mathrm{d}x \leqq \varepsilon'' \|u\|_{W^{2m}{}_{p,\varkappa+\mu p,\varkappa+\nu p}}^p,$$

ε'' is an arbitrary positive number.

Finally we estimate $\|Du\|_{L_{p,\varkappa}}$. If ω is a bounded domain, $\bar\omega \subset \Omega$, the well-known formula reads

$$\|u\|_{W^{2m-1}{}_p(\omega)} \leqq \varepsilon \|u\|_{W^{2m}{}_p(\omega)} + c_\varepsilon \|u\|_{L_p(\omega)} \leqq \varepsilon' \|u\|_{W^{2m}{}_{p,\varkappa+\mu p,\varkappa+\nu p}(\Omega)} + c'_{\varepsilon'} \|u\|_{L_{p,\varkappa}(\Omega)}\,.$$

With the help of this estimate, the assumption (8) and Lemma 3.2 we obtain

$$(20) \qquad \|Du\|_{L_{p,\varkappa}} \leqq \varepsilon \|u\|_{W^{2m}{}_{p,\varkappa+\mu p,\varkappa+\nu p}} + c_\varepsilon \|u\|_{L_{p,\varkappa}},$$

ε is an arbitrary positive number. c_ε is independent of j and k. (18), (19), and (20) show that

$$(21) \qquad \|Au - \lambda u\|_{L_{p,\varkappa}}^p \geqq c\|u\|_{W^{2m}{}_{p,\varkappa+p\mu,\varkappa+p\nu}}^p + \big(c'|\mathrm{Re}\,\lambda|^p - c''\big) \|u\|_{L_{p,\varkappa}}^p$$

holds; c, c', and c'' are positive numbers. $\gamma^{-1}(\xi)$, see formula (17), is also a multiplier. Using $\nu \geqq 0$ and Hörmander's multiplier theorem [1], we obtain

$$\|Bu\|_{L_{p,\varkappa}}^p \leqq c\|u\|_{W^{2m}{}_{p,\varkappa+\mu p,\varkappa+\nu p}}^p\,.$$

By means of this estimate, (19) and (20) we obtain the left hand side of (15). This proves the lemma for the particular function u. Now we consider a general function $u \in W^{2m}_{p,\varkappa+p\mu,\varkappa+p\nu}(\Omega)$. Then

$$u = \sum_{j=N}^{\infty} \sum_{k=1}^{N_j} \psi_j\, \varphi_k^{(j)} u\,.$$

(21) shows (we assume $c'|\mathrm{Re}\,\lambda|^p - c'' > 0$)

$$c\|u\|^p_{W^{2m}_{p,\varkappa+p\mu,\varkappa+p\nu}} + (c'|\mathrm{Re}\,\lambda|^p - c'')\,\|u\|^p_{L_{p,\varkappa}} \leqq \tag{22}$$

$$\leqq c'' \sum_{j=N}^{\infty} \sum_{k=1}^{N_j} \|A(\psi_j\,\varphi_k^{(j)}u) - \lambda\psi_j\,\varphi_k^{(j)}u\|^p_{L_{p,\varkappa}}\,.$$

It is

$$A(\psi_j\varphi_k^{(j)}u) - \lambda\psi_j\varphi_k^{(j)}u =$$

$$= \psi_j\varphi_k^{(j)}(Au - \lambda u) + \sum_{0\leqq|\beta|<2m;\,1\leqq|\alpha|\leqq 2m-|\beta|} c_{\beta,\alpha}(x)\, D^\alpha(\psi_j\varphi_k^{(j)})\, D^\beta u\,.$$

For $|\alpha| \geqq 1$ it is $\varkappa_{|\alpha|+|\beta|} + |\alpha| < \varkappa_{|\beta|}$ (see (10)). From this relation it follows that

$$c_{\beta,\alpha}(x)\, D^\alpha(\psi_j\varphi_k^{(j)}) = O(\varrho^{\varkappa_{|\alpha|+|\beta|}}\varrho^{|\alpha|}) = O(\varrho^{\varkappa_{|\beta|}-\delta})\,,$$

$\delta > 0$. Analogously to (19), (20) we obtain

$$\sum_{j=N}^{\infty} \sum_{k=1}^{N_j} \|A(\psi_j\varphi_k^{(j)}u) - \lambda\psi_j\varphi_k^{(j)}u\|^p_{L_{p,\varkappa(\Omega)}} \leqq \tag{23}$$

$$\leqq c\|Au - \lambda u\|^p_{L_{p,\varkappa(\Omega)}} + c \sum_{|\beta|<2m} \int_\Omega \varrho^{\varkappa+p\varkappa_{|\beta|}-\delta'}|D^\beta u|^p\,dx \leqq$$

$$\leqq c\|Au - \lambda u\|^p_{L_{p,\varkappa(\Omega)}} + \varepsilon\|u\|^p_{W^{2m}_{p,\varkappa+p\mu,\varkappa+p\nu}\Omega} + c_\varepsilon\|u\|^p_{L_{p,\varkappa(\Omega)}}\,,$$

$\delta' > 0$; ε is an arbitrary positive number. Choosing $|\mathrm{Re}\,\lambda|$ sufficiently large we obtain the right hand side of (15) from (22) and (23). The left hand side of (15) follows in a similar way.

4. THE SPACES $S_{\varrho(x)}(\Omega)$

4.1. Definition and inclusion property. Ω is again an arbitrary connected domain in R_n and $\varrho(x)$ is the weight function defined in Section 2.1. $C^\infty(\Omega)$ is the set of all complex, in Ω infinitely differentiable functions. We write

$$S_{\varrho(x)}(\Omega) = \{f \mid f \in C^\infty(\Omega),\ \|f\|_{l,\alpha} = \sup_{x\in\Omega} \varrho^l(x)\,|\,D^\alpha f(x)| < \infty \tag{24}$$

$$\text{for all } l = 0, 1, 2, \ldots \text{ and all multiindices } \alpha\}\,.$$

$S_{\varrho(x)}(\Omega)$ is a (F)-space (a complete separable locally convex space equipped with a countable set of semi-norms). In [3, 4, 5] we introduced the (F)-space $S_{q(x)}(\Omega)$, (we have to replace in (24) $\varrho(x)$ by $q(x)$). $q(x)$ is a weight function such that

$$q(x) \in C^\infty(\Omega)\,, \quad q(x) \geqq c > 0\,, \tag{25}$$

$\exists\sigma$, $0 \leqq \sigma < \frac{1}{2}$ so that

(26) $$|D^{\gamma} q(x)| \leqq c_{\gamma} q^{1+\sigma|\gamma|}(x)$$

(27) $$q(x)\, \mathrm{d}^2(x) \geqq C > 0 \quad \text{for} \quad x \in \Omega$$

($d(x)$ is the distance for $x \in \Omega$ from the boundary $\partial\Omega$).

(28) $$\exists a > 0 \quad \text{with} \quad q^{-\alpha}(x) \in L_1(\Omega)\,.$$

We choose

$$q(x) = \varrho^{\varkappa}(x)\,, \quad \varkappa > 2\,.$$

It is easy to see that (25) and (26) are true ($\sigma = \varkappa^{-1}$). Further,

$$S_{q(x)}(\Omega) = S_{\varrho(x)}(\Omega)\,.$$

We shall show that (27) is also true. Let $x \in \Omega^{(j+1)} - \Omega^{(j)}$, see Section 2.1. Then formula (4) implies

$$q(x)\, d^2(x) \geqq c\, 2^{j\varkappa}\, 2^{-2j} \geqq c' > 0\,.$$

This proves (27). Now we discuss (28).

Lemma 4.1. *Let be* $1 < p < \infty$.

(29) $$S_{\varrho(x)}(\Omega) \subset L_p(\Omega)$$

holds if and only if

(30) $$\exists a > 0 \quad \text{such that} \quad \varrho^{-\alpha}(x) \in L_1(\Omega)\,.$$

Proof. It is easy to see that (29) is a consequence of (30). We prove the opposite implication. First we assume the existence of a positive number b so that

$$|\Omega^{(j+1)} - \Omega^{(j)}| \leqq b^j\,; \quad j = N,\, N+1,\, \ldots$$

holds. Let $2^a > b$. It follows

$$\int_{\Omega} \varrho^{-a}(x)\, \mathrm{d}x \leqq \sum_{j=N}^{\infty} \int_{\Omega^{(j+1)} - \Omega^{(j)}} \varrho^{-a}(x)\, \mathrm{d}x + c \leqq c + c' \sum_{j=N}^{\infty} 2^{-ja} b^j < \infty\,.$$

Now we assume that (30) is untrue. The last consideration shows the existence of a set of numbers $j_1, j_2, \ldots,$ and

$$0 < a_1 < a_2 < \ldots < a_l < \ldots\,, \quad a_l \to \infty \quad \text{for} \quad l \to \infty\,,$$

with

(31) $$|\Omega^{(j_l+1)} - \Omega^{(j_l)}| > a_l^{j_l}\,; \quad j_{l+1} - j_l \geqq k\,,$$

k a given positive integer. Let

$$u(x) = a_l^{-j_l p^{-1}} \quad \text{for} \quad x \in \Omega^{(j_l+2)} - \Omega^{(j_l-1)}\,; \quad l = 1, 2, \ldots\,;$$
$$u(x) = 0 \ \text{otherwise} \quad (x \in \Omega)\,.$$

Now we use Sobolev's smoothness-method, see [2]. We write

$$v(x) = (u(x))_{c2^{-j_l}} \quad \text{for} \quad x \in \Omega^{(j_l+3)} - \Omega^{(j_l-2)}\,,$$
$$v(x) = 0 \quad \text{otherwise} \quad (x \in \Omega)\,.$$

$c\,2^{-j_l}$ is the radius of the method [2]. c is sufficiently small, see (4). k is sufficiently large, see (31). It is $v \in C^\infty(\Omega)$ and

$$|\varrho^m(x)\, D^\alpha\, v(x)| \leqq c\, 2^{j_l m}\, 2^{j_l|\alpha|}\, a_l^{-j_l p^{-1}} = \left(\frac{a_l^{p-1}}{2^{m+|\alpha|}}\right)^{-j_l}$$

for $x \in \Omega^{(j_l+3)} - \Omega^{(j_l-2)}$. The last estimate shows

$$v \in S_{\varrho(x)}(\Omega)\,.$$

On the other hand, it is

$$\int_\Omega |v(x)|^p \, dx \geqq \sum_{l=1}^{\infty} \int_{\Omega^{(j_l+1)}-\Omega^{(j_l)}} |v(x)|^p \, dx \geqq \sum_{l=1}^{\infty} a_l^{-j_l} |\, \Omega^{(j_l+1)} - \Omega^{(j_l)}| = \infty\,.$$

Hence

$$v \notin L_p(\Omega)\,.$$

This shows that (29) is untrue. This proves the lemma.

4.2. Isomorphic property. We denote by s the nuclear space of rapidly decreasing sequences. This means

$$s = \{\xi \mid \xi = (\xi_j)_{j=1}^\infty,\ \xi_j \text{ complex},\ \|\xi\|_l = \sup_j j^l |\xi_j| < \infty \ \text{for} \ l = 0, 1, 2, \ldots\}\,.$$

Theorem 4.2. *Let*

$$\varrho^{-a}(x) \in L_1(\Omega)$$

for a suitable positive number a. Then $S_{\varrho(x)}(\Omega)$ is isomorphic to s.

Proof. We use the results of [5]. Let

$$(32) \qquad (Af)(x) = -\Delta f(x) + \varrho^\varkappa(x) f(x)\,, \quad D(A) = C_0^\infty(\Omega)\,.$$

$\varkappa > 2$. In [5], pp. 301–302 we showed that A is an essential selfadjoint operator acting in $L_2(\Omega)$.

$$(33) \qquad D(\bar{A}^\infty) = \bigcap_{k=1}^{\infty} D(\bar{A}^k) = S_{\varrho(x)}(\Omega)\,.$$

This includes the equivalence of the topologies, where the locally convex space $D(\bar{A}^\infty)$ is equipped with the norms $\|\bar{A}^l u\|_{L_2}$, $l = 0, 1, 2, \ldots$ (We use the above mentioned fact that $q(x) = \varrho^\varkappa(x)$, $\varkappa > 2$ is a function $q(x)$ of the type considered in [5].) $\bar{A}$ is a positive-definite operator with a pure point spectrum. Let

$$N(\lambda) = \sum_{\lambda_j < \lambda} 1 \tag{34}$$

be the number of eigenvalues smaller than λ (including their multiplicity). In [5], p. 292 we noted that $D(\bar{A}^\infty)$ (and hence also $S_{\varrho(x)}(\Omega)$) is isomorphic to s if and only if

$$c_1 \lambda^{\tau_1} \leqq N(\lambda) \leqq c_2 \lambda^{\tau_2} \tag{35}$$

holds for suitable positive numbers c_1, c_2, τ_1 and τ_2. Now we prove (35). Let K be a ball, $\bar{K} \subset \Omega$. We write

$$Bu = -\Delta u + d\,; \quad D(B) = \{u \mid u \in W_2^2(K),\ u|_{\partial K} = 0\}\,,$$

$d = 0$. It is well known that B is a positive-definite selfadjoint operator with a pure point spectrum acting in $L_2(K)$. Let $N_B(\lambda)$ be the analogous function to $N(\lambda)$, formula (34). The well known eigenvalue distribution for B and Courant's maximum-minimum principle show for sufficiently large d

$$c\lambda^{n/2} = N_B(\lambda) \leqq N(\lambda)\,, \quad c > 0\,. \tag{36}$$

This proves the first inequality of (35). Now we prove the other inequality of (35). Let λ be a given positive number. We determine an integer σ_λ such that

$$\varrho^\varkappa(x) > \lambda \quad \text{for} \quad x \in \Omega - \Omega^{(\sigma_\lambda)}\,.$$

For instance

$$\sigma_\lambda = [c \log \lambda]\,, \tag{37}$$

$c > 0$ sufficiently large, independent of λ, $\lambda \geqq \lambda_0$. Now we cover $\Omega^{(\sigma_\lambda)}$ with cubes

$$Q_\nu = \{x \mid x = (x_j)_{j=1}^n,\ |x_j - x_{j,\nu}| < \tfrac{1}{2}\}\,. \tag{38}$$

We estimate the number of the needed cubes. It is

$$|\Omega^{(\sigma_\lambda)}| = \int_{\Omega^{(\sigma_\lambda)}} \varrho^{-a}(x)\, \varrho^a(x)\, \mathrm{d}x \leqq c\, 2^{a\sigma_\lambda} \leqq c' \lambda^\mu\,.$$

We used (30). By means of (4) it is not difficult to show that it is sufficient to consider $c'' \lambda^{\tilde{\mu}}$, $\tilde{\mu} > 0$ cubes of the type (38). Let $\tilde{N}(\lambda)$ be the function for the eigenvalue distribution for the Neumann problem for $-\Delta$ in the unit cube. It is

$$\tilde{N}(\lambda) \leqq c\lambda^{n/2} \quad \text{for} \quad \lambda \geqq \lambda_0\,.$$

Courant's maximum-minimum principle implies now

$$N(\lambda) \leqq c''\lambda^{\bar{\mu}}\,\tilde{N}(\lambda) \leqq c'''\lambda^{\bar{\mu}+n/2} \quad \text{for} \quad \lambda \geqq \lambda_0 \,.$$

This proves the right hand inequality in (35). We remarked above that this is sufficient for the proof of the theorem.

4.3. Remark. Note that Theorem 4.2 is an affirmative answer to the problem 2 of [5], p. 310.

5. ISOMORPHIC PROPERTIES FOR OPERATORS OF THE TYPE $A^{(m)}_{\mu,\nu}$

5.1. A special case. In [5], p. 298, we remarked that

$$(39) \qquad A_0 u = -\Delta u + \varrho^{\varkappa}(x)\,u\,; \quad \varkappa > 2\,; \text{ (30) holds;}$$

is an isomorphic map from $S_{\varrho(x)}(\Omega)$ onto $S_{\varrho(x)}(\Omega)$. We need an extension of this result.

Lemma 5.1. *There exists such an operator A of the type $A^{(m)}_{\mu,\nu}$ that $A - \lambda E$ is an isomorphic map from $S_{\varrho(x)}(\Omega)$ onto $S_{\varrho(x)}(\Omega)$ for every real $\lambda \leqq -1$. $\varrho(x)$ is the described function such that* (1), (2), (3) *and* (30) *hold.*

Proof. We consider the operator A_0, formula (39). A_0 is an operator of the type $A^{(1)}_{0,\varkappa}$. We proved in [5], p. 301 that A_0^{mk} is an isomorphic map from $W^{2mk}_{2,0,2mk\varkappa}(\Omega)$ onto $L_2(\Omega)$, m and k are integers >0. Let

$$A_1 u = (-\Delta)^m u + \varrho^{\sigma_1}(x)\,u - \lambda\varrho^{\sigma_2}(x)\,u\,,$$

$\sigma_1 > 2m$, λ a real number, $\lambda \leqq -1$, σ_2 a real number. A_1 is an operator of the type $A^{(m)}_{0,\max(\sigma_1,\sigma_2)}$. We choose $\varkappa = (1/m)\max(\sigma_1, \sigma_2)$. A_0^{mk} and A_1^k are positive-definite operators acting in $L_2(\Omega)$ with the domain of definition $W^{2mk}_{2,0,2mk\varkappa}(\Omega)$. The same is true for

$$(40) \qquad B_\mu u = (1-\mu)\,A_0^{mk} + \mu A_1^k\,; \quad 0 \leqq \mu \leqq 1\,.$$

Lemma 3.3 shows that

$$(41) \qquad c_2\|u\|_{W^{2mk}{}_{2,0,2mk\varkappa}} \geqq \|B_\mu u\|_{L_2} \geqq c_1\|u\|_{W^{2mk}{}_{2,0,2mk\varkappa}}\,,$$

where c_1 and c_2 are positive numbers independent of μ. Now we assume that B_{μ_0} is an isomorphic map from $W^{2mk}_{2,0,2mk\varkappa}(\Omega)$ onto $L_2(\Omega)$. Then

$$\|B_{\mu_0}^{-1}\|_{L_2 \to W^{2mk}{}_{2,0,2mk\varkappa}} \leqq c_1^{-1}\,.$$

We consider the equation

$$B_\mu u = f \in L_2(\Omega)\,.$$

It is equivalent to

$$u + B_{\mu_0}^{-1}(B_\mu - B_{\mu_0})\,u = B_{\mu_0}^{-1} f \in \overset{\circ}{W}{}^{2mk}_{2,0,2mk\varkappa}(\Omega)\,. \tag{42}$$

(40) and (41) show

$$\|B_{\mu_0}^{-1}(B_\mu - B_{\mu_0})\|_{W^{2mk}{}_{2,0,2mk\varkappa} \to W^{2mk}{}_{2,0,2mk\varkappa}} < 1 \quad \text{for} \quad |\mu - \mu_0| \leqq c\,,$$

c is independent of μ_0. But then it follows by the standard argument that B_μ is an isomorphic map from $W^{2mk}_{2,0,2mk\varkappa}(\Omega)$ onto $L_2(\Omega)$. We start with $\mu_0 = 0$, $B_0 = A_0^{mk}$. After a finite number of steps we find that A_1^k is an isomorphic map from $W^{2mk}_{2,0,2mk\varkappa}(\Omega)$ onto $L_2(\Omega)$. Using the fact that $C_0^\infty(\Omega)$ is dense in $W^{2mk}_{2,0,2mk\varkappa}(\Omega)$ we obtain that

$$A_1^k,\ D(A_1^k) = W^{2mk}_{2,0,2m\varkappa k}(\Omega)$$

is the usual k-th power of the selfadjoint operator

$$A_1,\ D(A_1) = W^{2m}_{2,0,2m\varkappa}(\Omega)\,.$$

In [5] we proved

$$\bigcap_{k=1}^{\infty} D(A_1^k) = \bigcap_{k=1}^{\infty} W^{2mk}_{2,0,2m\varkappa k}(\Omega) = S_{\varrho(x)}(\Omega)$$

(including the topologies). This shows that A_1 is an isomorphic map from $S_{\varrho(x)}(\Omega)$ onto $S_{\varrho(x)}(\Omega)$. But then also $A = \varrho^{-\sigma_2}(x)\,A_1$ is an isomorphic map from $S_{\varrho(x)}(\Omega)$ onto $S_{\varrho(x)}(\Omega)$. This proves the lemma.

5.2. Isomorphic property in $S_{\varrho(x)}(\Omega)$. Theorem 5.2. *Let A be an operator of the type $A^{(m)}_{\mu,\nu}$, $\nu \geqq 0$. Then $A^{(m)}_{\mu,\nu} - \lambda E$ is an isomorphic map from $S_{\varrho(x)}(\Omega)$ onto $S_{\varrho(x)}(\Omega)$ for* Re $\lambda \leqq c$. *$\varrho(x)$ is the described function such that* (1), (2), (3) *and* (30) *hold.*

Proof. Let A_0 be the special operator determined in Lemma 5.1. Similarly to (40) we consider

$$B_\mu u = (1 - \mu)\,A_0^k + \mu A^k\,; \quad 0 \leqq \mu \leqq 1\,.$$

By repeating the arguments of the last lemma the theorem is proved. (We use Lemma 3.3 with $p = 2$ and $\varkappa = 0$).

5.3. Isomorphic property in $W^{2mk}_{p,\varkappa+kp\mu,\varkappa+kp\nu}(\Omega)$. Theorem 5.3. *Let $\varkappa$ be an arbitrary real number. $\varrho(x)$ is the described function such that* (1), (2), (3) *and* (30) *hold. Let A be an operator of the type $A^{(m)}_{\mu,\nu}$, $\nu \leqq 0$. Let*

$$D(A) = W^{2m}_{p,\varkappa+p\mu,\varkappa+p\nu}(\Omega)$$

be the domain of definition. A is considered in $L_{p,\varkappa}(\Omega)$, $1 < p < \infty$.

(a) *It is*

$$(43) \qquad D(A^k) = W^{2mk}_{p,\varkappa+pk\mu,\varkappa+pk\nu}(\Omega)\,; \quad k = 1, 2, \ldots$$

(b) *For* $\operatorname{Re}\lambda \leqq c$ *(with a suitable c),* $A - \lambda E$ *is an isomorphic map from* $W^{2m(k+1)}_{p,\varkappa+p\mu(k+1),\varkappa+p\nu(k+1)}(\Omega)$ *onto* $W^{2mk}_{p,\varkappa+p\mu k,\varkappa+p\nu k}(\Omega)$, $k = 1, 2, \ldots$

Proof. $C_0^\infty(\Omega)$ (and hence also $S_{\varrho(x)}(\Omega)$) is dense in $W^{2mk}_{p,\varkappa+pk\mu,\varkappa+pk\nu}(\Omega)$. Lemma 3.3 and Theorem 5.2 show that $A - \lambda E$ is an isomorphic map from $W^{2m}_{p,\varkappa+p\mu,\varkappa+p\nu}(\Omega)$ onto $L_{p,\varkappa}(\Omega)$, $\operatorname{Re}\lambda \leqq c$. Using again Theorem 5.2, Lemma 3.1, and Lemma 3.3, we obtain

$$D(A^k) \supset W^{2mk}_{p,\varkappa+pk\mu,\varkappa+pk\nu}(\Omega) \supset S_{\varrho(x)}(\Omega)\,.$$

But Lemma 3.3 and the one-to-one map $(A - \lambda E)^k$ show that (43) is true. (b) is an easy consequence of (a). This proves the theorem.

5.4. The spaces $W^s_{p,\sigma,\tau}(\Omega)$. We extend the definition of $W^l_{p,\sigma,\tau}(\Omega)$ given in Section 3.2. Let $0 < \eta < 1$. We write

$$\|f\|_{W^\eta_p(\Omega)} = \left(\int_{\Omega\times\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\eta p}}\,dx\,dy + \|f\|^p_{L_p(\Omega)}\right)^{1/p}; \quad 1 < p < \infty\,.$$

Let Ω be an arbitrary connected domain in R_n. $\varrho(x)$ is the weight function of Section 2.1. Let $0 < s = [s] + \{s\}$, $[s] =$ integer, $0 < \{s\} < 1$. We introduced in [6] the spaces $W^s_{p,\sigma,\tau}(\Omega)$, σ and τ real numbers, $\tau > \sigma + sp$,

$$W^s_{p,\sigma,\tau}(\Omega) = \{f \mid f \in D'(\Omega)\,,$$
$$\|f\|_{W^s_{p,\sigma,\tau}} = \Big(\sum_{|\alpha|=[s]} \|\varrho^{\sigma/p} D^\alpha f\|^p_{W_p^{\{s\}}(\Omega)} + \|\varrho^{\tau/p} f\|^p_{L_p(\Omega)}\Big)^{1/p} < \infty\}\,.$$

For $s =$ integer, $W^s_{p,\sigma,\tau}(\Omega)$ has the meaning of Section 3.2. For our purpose the following fact proved in [6] is important: Let $(A_0, A_1)_{\Theta,p}$ be the real interpolation method of Lions-Peetre, $1 < p < \infty$, $0 < \Theta < 1$. (A short description is given in [6].) Then

$$(44) \qquad (W^{m_1}_{p,\sigma_1,\tau_1}(\Omega), W^{m_2}_{p,\sigma_2,\tau_2}(\Omega))_{\Theta,p} = W^{m_1(1-\Theta)+m_2\Theta}_{p,\sigma,\tau}(\Omega)\,,$$

m_1 and m_2 integers, $m_1 = 0, 1, 2, \ldots, m_2 = 1, 2, \ldots$ (For $m_1 = 0$ we assume $\sigma_1 = \tau_1$ and $W^0_{p,\sigma_1,\tau_1}(\Omega) = L_{p,\sigma_1}(\Omega)$.) $0 < \Theta < 1$; $1 < p < \infty$,

$$(45) \qquad (\tau_1 - \sigma_1)\, m_2 = (\tau_2 - \sigma_2)\, m_1\,; \quad m_1(1 - \Theta) + m_2\Theta \neq \text{integer}\,;$$

$$(46) \qquad \tau = (1 - \Theta)\,\tau_1 + \Theta\tau_2;\ \sigma = \tau - (m_1(1 - \Theta) + m_2\Theta)\,\frac{\tau_2 - \sigma_2}{m_2}\,.$$

A proof is given in Theorem 4.3 [6] (more general cases can be found there, too).

Further, we denote by $[A_0, A_1]_\Theta$, $0 < \Theta < 1$, the complex interpolation method. It is

$$[W^{m_1}_{p,\sigma_1,\tau_1}(\Omega), W^{m_2}_{p,\sigma_2,\tau_2}(\Omega)]_\Theta = W^{m_1(1-\Theta)+m_2\Theta}_{p,\sigma,\tau}(\Omega) \tag{47}$$

for $1 < p < \infty$, $m_1(1 - \Theta) + m_2\Theta =$ integer, provided (45) and (46) hold. A proof is given in Theorem 4.3 of [6].

5.5. The main result. Theorem 5.5. *Let $\varkappa$ be an arbitrary real number. $\varrho(x)$ is the described function such that* (1), (2), (3) *and* (30) *hold. Let A be an operator of the type $A^{(m)}_{\mu,\nu}$, $\nu \geqq 0$. Then $A - \lambda E$ is an isomorphic map from*

$$W^{2m+s}_{p,\varkappa+p\mu(1+s/2m),\varkappa+p\nu(1+s/2m)}(\Omega) \quad \textit{onto} \quad W^{s}_{p,\varkappa+p\mu s/2m,\varkappa+p\nu s/2m}(\Omega)\ ;$$

$s \geqq 0$, $1 < p < \infty$. $\operatorname{Re} \lambda \leqq c$.

Proof. Theorem 5.3 and the general interpolation theory show that $A - \lambda E$ is an isomorphic map from

$$\left(W^{2m(k+2)}_{p,\varkappa+p\mu(k+2),\varkappa+p\nu(k+2)}(\Omega),\ W^{2m(k+1)}_{p,\varkappa+p\mu(k+1),\varkappa+p\nu(k+1)}(\Omega)\right)_{\Theta,p}$$

onto

$$\left(W^{2m(k+1)}_{p,\varkappa+p\mu(k+1),\varkappa+p\nu(k+1)}(\Omega),\ W^{2mk}_{p,\varkappa+p\mu k,\varkappa+p\nu k}(\Omega)\right)_{\Theta,p}$$

and similarly for the complex interpolation method. $k = 0, 1, 2, \dots$ Let $s = 2mk + 2m(1 - \Theta) = 2m(k + 1 - \Theta)$. It is easy to see that the condition (45) holds. (46) yields the desired indices. This proves the theorem.

5.6. Remark. In [6] we introduced also the spaces $H^s_{p,\sigma,\tau}(\Omega)$ (Lebesgue spaces with weights) and $B^s_{p,p,\sigma,\tau}(\Omega)$ (Besov spaces with weights). We do not repeat the definitions here, see [6]. The interpolation theory for these spaces developed in [6] shows that the following theorem is true.

Theorem 5.6. *Theorem 5.5 is true after replacing the W-spaces by the H-spaces or by the B-spaces.*

5.7. Remark. A special case of Theorem 5.5 is proved in [4], Theorems 7 and 8.

5.8. The structure of the spaces $W^s_{p,\sigma,\tau}(\Omega)$. l_p is the usual sequence space.

Theorem 5.8. *Let be $1 < p < \infty$; σ, τ are real numbers, $s \geqq 0$; $\tau > \sigma + sp$. Then*

$$W^s_{p,\sigma,\tau}(\Omega) \quad \textit{is isomorphic to } l_p,\ s \neq \textit{integer}\,, \tag{48}$$

and

(49) $$W^{2m}_{p,\sigma,\tau}(\Omega) \text{ isomorphic to } L_p((0,1)), \quad m = 0, 1, 2, \ldots$$

Proof. (48) is proved in [6], Theorem 7. Further it follows from the consideration in [6] that all the spaces $W^{2m}_{p,\sigma+\varkappa,\tau+\varkappa}(\Omega)$ are isomorphic to one another, see formula (14). $-\infty < \varkappa < \infty$. So we may assume without a loss of generality $\tau \geqq 0$. But then it follows from Theorem 5.3 that $W^{2m}_{p,\sigma,\tau}(\Omega)$ is isomorphic to $L_p(\Omega)$ and hence also isomorphic to $L_p((0, 1))$.

5.9. Remark. By other methods it is possible to show that (49) holds also for $m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

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