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ŁS condition for filled Julia sets in $\mathbb{C}$

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Abstract In this article we derive an inequality of Łojasiewicz–Siciak type for certain sets arising in the context of the complex dynamics in dimension 1. More precisely, if we denote by dist the Euclidean distance in $\mathbb{C}$, we show that the Green function G_K of the filled Julia set K of a polynomial such that $\mathring{K} \neq \emptyset$ satisfies the so-called ŁS condition $G_A \geq c \cdot \mathrm{dist}(\cdot, K)^{c'}$ in a neighborhood of K, for some constants $c, c' > 0$. Relatively few examples of compact sets satisfying the ŁS condition are known. Our result highlights an interesting class of compact sets fulfilling this condition. For instance, this is the case for the filled Julia sets of quadratic polynomials of the form $z \mapsto z^2 + a$, provided that the parameter a is parabolic, hyperbolic or Siegel. The fact that filled Julia sets satisfy the ŁS condition may seem surprising, since they are in general very irregular and sometimes they have cusps. However, we provide an explicit example of a curve which has a cusp and satisfies the ŁS condition. In order to prove our main result, we define and study the set of obstruction points to the ŁS condition. We also prove, in dimension $n \geq 1$, that for a polynomially convex and L-regular compact set of non-empty interior, these obstruction points are rare, in a sense which will be specified.

Keywords ŁS condition · Green function · Pluricomplex Green function · Complex dynamics · Filled Julia set · Potential theory

Mathematics Subject Classification 37F50 · 37F10 · 31C99 · 32U35

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1 Introduction

We call *pluricomplex Green function* G_A of a compact set $A \subset \mathbb{C}^n$, $n \geq 1$, the plurisubharmonic function defined as

$$G_A := \sup{}^*\left\{ v \in \mathrm{PSH}(\mathbb{C}^n) : \; v|_A \leq 0, \; v(z) \leq \frac{1}{2} \log\left(1 + \|z\|^2\right) + O(1) \right\},$$

where $\sup^*$ denotes the upper semi-continuous regularization of the upper envelope, and $\mathrm{PSH}(\mathbb{C}^n)$ denotes the set of plurisubharmonic functions in $\mathbb{C}^n$. The set A is called *L-regular* if G_A is continuous. In this case, the set $\{G_A = 0\}$ is the polynomially convex envelope $\hat{A}$ of A. We also consider, for an open bounded set $U \subset \mathbb{C}^n$, the *Green function of* $A \subset U$ *relative to* U defined by

$$G_{A,U} := \sup{}^*\left\{ v \in \mathrm{PSH}(U) : \; v \leq 1, \; v|_A \leq 0 \right\}.$$

The reader should pay attention to the fact that in [9] the Green function of A relative to U is defined as $G_{A,U} - 1$.

Let $U_a := \{G_A < a\}$ for $a \in \mathbb{R}^+ \backslash \{0\}$. If A is not pluripolar and $\hat{A} \subset U_a$, then a relation between G_A and G_{A,U_a} holding in U_a is given by Proposition 5.3.3 in [9]:

$$G_A = G_{A,U_a}. \tag{1}$$

A compact $A \subset \mathbb{C}^n$ is said to *satisfy the ŁS condition* if there exists an open set U containing it and two constants $c, c' > 0$ such that its pluricomplex Green function G_A verifies the following regularity condition :

$$\forall z \in U, \quad G_A(z) \geq c \cdot \mathrm{dist}(z, A)^{c'},$$

where dist denotes the Euclidean distance (see for instance [6] or [2]).

On a compact set $A \subset \mathbb{C}^n$ verifying the ŁS condition, as well as the HCP condition (i.e., the Hölderian continuity of G_A, for example a semialgebraic compact set), we have the rapid approximation property of continuous functions by polynomials. Relatively few examples of compacts satisfying the ŁS condition are known. Some examples are given in [13]. Let us also note that Pierzchała showed in [14] that a compact verifying the ŁS condition is polynomially convex. Białas and Kosek [3] construct such sets using holomorphic dynamics.

Along the same vein, we show that the so-called filled Julia sets in $\mathbb{C}$ satisfy the ŁS condition. More precisely, our main goal is to show the following result concerning the filled Julia set of a polynomial $f : \mathbb{C} \to \mathbb{C}$, i.e., the set of points $z \in \mathbb{C}$ whose orbit $(f^n(z))_n$ is bounded :

Theorem *The filled Julia set of a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree ≥ 2, if its interior is non-empty, satisfies the ŁS condition.*

Recall that a compact set in $\mathbb{C}$ is polynomially convex if and only if its complement is connected, so the filled Julia set of a polynomial is polynomially convex. The differentials operators ∂ and $\overline{\partial}$ will be understood in the sense of currents. Recall that a continuous function u from an open set of $\mathbb{C}^n$ into $\mathbb{R}$ is pluriharmonic (harmonic if $n = 1$) if and only if $\partial\overline{\partial}u = 0$ (see for example Theorem 2.28 in [11]).

In Sect. 2, we recall some definitions and elementary facts about holomorphic dynamics in one dimension, and we give a useful lemma concerning the regularity of filled Julia sets. More precisely, this lemma shows that the filled Julia set K of a polynomial of degree $d \geq 2$ with non-empty interior satisfies $\overline{\mathring{K}} = K$. In Sect. 3, we define in $\mathbb{C}^n$, $n \geq 1$, the set of obstruction points to the ŁS condition, and we prove that the complement of this set is big, in a sense which will be specified. We also study explicitly the ŁS condition on several examples of compact sets in $\mathbb{C}$. In particular, we provide an example of a curve which has a cusp and satisfies, however, the ŁS condition. Section 4 is devoted to the proof of the main theorem previously stated.

2 Dynamics in $\mathbb{C}$

We start by recalling some definitions related to one-dimensional holomorphic dynamics. Let us consider a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$.

We call *Fatou set* of f, denoted $\mathcal{F}$, the largest open subset in which the family of iterations f^n is equicontinuous.

The Julia set of f, denoted J, is the complement of $\mathcal{F}$ in $\mathbb{C}$. Let us note for what follows that J is not a polar set.

We call *filled Julia set of* f the set K of points $z \in \mathbb{C}$ whose orbit $(f^n(z))_n$ is bounded. Note that K is compact, as ∞ is a superattractive fixed point of f, hence belonging to $\mathcal{F}$. The complement of K is the basin of attraction of infinity. We have $\partial K = J$ and $G_K = G_J$.

There are many situations where the set K is of non-empty interior. Consider, for instance, the case where $f(z) = z^2 + a$ with $a \in \mathbb{C}$. By Sullivan's classification theorem (see, e.g., Theorem 2.1 in [4] or Theorem 3.2 of [12]), we can distinguish three cases where $\mathring{K} \neq \emptyset$. The first case is when a is chosen in the interior of the Mandelbrot set in such a way that f is *hyperbolic* in the sense of [4] p. 89. By Theorem 4.7 in [12], f is hyperbolic if and only if some iterate f^k of f has a fixed point $z_0 \in \mathbb{C}$ for which $|(f^k)'(z_0)| < 1$. The second case is when a is chosen on the boundary of the Mandelbrot set such that some iterate f^k of f has a fixed point $z_0 \in \mathbb{C}$ for which $(f^k)'(z_0)$ is a root of the unity. By Theorem 6.5.10 of [1] and Theorem 4.8 of [12], this corresponds to the *parabolic* case in the Sullivan's classification. By Theorem 4.8 of [12], the last case is when a is chosen on the boundary of the Mandelbrot set such that $\mathring{K}$ contains a *Siegel disk* and all its preimages in the sense of Definition 7.1.1 of [1].

We construct the subharmonic function $G : \mathbb{C} \to \mathbb{R}^+$, limit in L^1_{loc} of the sequence $(\log(1+|f^n|)/d^n)_n$ (see [8] for a general construction). It is known that G is continuous (and even Hölderian [10], see also Theorem 3.2 of [4]), harmonic in $\mathcal{F}$, that it verifies $G(z) = 0$ if and only if $z \in K$, and also that $G(z) - \log|z| = O(1)$ at infinity. By uniqueness, G is therefore the pluricomplex Green function of K (and of J). It satisfies by construction the invariance property

$$G \circ f = d \cdot G. \tag{2}$$

The measure $\frac{i}{\pi}\partial\overline{\partial}G$ is a probability measure of support exactly J (see, e.g., [7]). We will use the following preliminary lemma about filled Julia sets.

Lemma 1 *The filled Julia set K of a polynomial of degree $d \geq 2$ with non-empty interior satisfies $\overline{\mathring{K}} = K$.*

Proof Suppose, by contradiction, that there exists $x \in \partial K$ having a neighborhood U which does not intersect $\mathring{K}$. Then, there exists $n_0 \in \mathbb{N}$ such that $K \subset f^{n_0}(U)$ (see for example Theorem 4.2.5. of [1]). But this contradicts the fact that $f^{n_0}(U \cap K) \subset \partial K$. Thus, every open subset of $\mathbb{C}$ intersecting $J = \partial K$ also intersects $\mathring{K}$. In other words, $\overline{\mathring{K}} = K$. □

3 Study of the obstruction to the ŁS condition

For $n \geq 1$, let

$$O_c := \left\{z \in \mathbb{C}^n : \mathrm{dist}(z, A) < 1, G_A(z) < c \cdot \mathrm{dist}(z, A)^{1/c}\right\}. \tag{3}$$

Note that the sequence of open sets O_c is increasing with c for $c < 1$. The ŁS condition is satisfied by a compact non-pluripolar set $A \subset \mathbb{C}^n$, L-regular and polynomially convex, if and only if the set

$$I := \bigcap_{c>0} \overline{O_c} \subset \partial A$$

is empty. We call I *the set of obstruction points to the ŁS condition*.

Example 1 [3] If A is the union of two disks of radius 1, tangent to each other at the origin, then it does not satisfy the ŁS condition; the set of obstruction points to the ŁS condition is $I = \{0\} \neq \emptyset$.

Example 2 The previous set A is mapped by the function $g : z \to z^2$ onto a filled cardioid $\mathcal{C}$, and we have $g^{-1}(\mathcal{C}) = A$. We deduce from Theorem 5.3.1 of [9] that the set of obstruction points to the ŁS condition for $\mathcal{C}$ is $I = \{0\} \neq \emptyset$.

Example 3 For $\varepsilon \in]0, 1[$ fixed, consider the sets $L_\varepsilon := \{(1+i)t,\ t \in [-\varepsilon, \varepsilon]\}$, $L'_\varepsilon := \{(1-i)t,\ t \in [-\varepsilon, \varepsilon]\}$, and $X_\varepsilon := L_\varepsilon \cup L'_\varepsilon \subset B(0, 2\varepsilon)$. We show that X_ε satisfies the ŁS condition, i.e., $I = \emptyset$. Indeed, the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \frac{i}{2}z^2$ maps X_ε onto $[-1, 1]$. On the other hand, $g^{-1}([-1, 1]) = X_\varepsilon$. Theorem 5.3.1 of [9] implies $G_{X_\varepsilon} = G_{[-1,1]} \circ g$. Since the segment $[-1, 1]$ is convex, it satisfies the ŁS condition (see [6]). Moreover, it follows from Theorem 1 in [5] that $\forall z \in \mathbb{C}$,

$$\mathrm{dist}\,(g(z), [-1, 1]) \geq \frac{1}{4}|z|\mathrm{dist}\,(z, X_\varepsilon) \geq \frac{1}{4}\mathrm{dist}\,(z, X_\varepsilon)^2.$$

We deduce that X_ε also satisfies the ŁS condition.

Considering the previous examples, one may think that a cusp prevents the ŁS condition to hold. However, we show in the counter-example below that this is not the case.

Example 4 We will use here the notations of Example 3. Define $f : B(0, 2\varepsilon) \to f(B(0, 2\varepsilon))$ by $f(z) := e^z - z - 1$. We suppose that $\varepsilon > 0$ is sufficiently small such that f is a proper map with only one critical point at the origin. The curve $f(L_\varepsilon) = \{(\cos t - 1) + i(\sin t - t), t \in [-\varepsilon, \varepsilon]\}$ has a cusp at the origin, in the sense that

$$\lim_{t\to 0^+} \frac{\frac{\partial}{\partial t} f((1+i)t)}{\left\|\frac{\partial}{\partial t} f((1+i)t)\right\|} = -\lim_{t\to 0^-} \frac{\frac{\partial}{\partial t} f((1+i)t)}{\left\|\frac{\partial}{\partial t} f((1+i)t)\right\|} = -1.$$

However, we prove that the set $f(L_\varepsilon)$ satisfies the ŁS condition. Indeed, note that this set is invariant by conjugation. But $\forall z \in L_\varepsilon$, $f(\overline{z}) = \overline{f(z)}$, hence $f(L_\varepsilon) = f(L'_\varepsilon)$. Take $y \in f(X_\varepsilon)\backslash\{0\}$. Thanks to the previous considerations, there exist two preimages $x \in L_\varepsilon$ and $x' = \overline{x} \in L'_\varepsilon$ of y by f. Since the order of f at the origin is 2, for ε sufficiently small there is no other preimage of y in $B(0, 2\varepsilon)$. Thus, $f^{-1}(f(X_\varepsilon)) = X_\varepsilon$.

Take now $y \in f(B(0, 2\varepsilon)$, and $x \in B(0, 2\varepsilon)$ a preimage of y by f. By Proposition 4.5.14 in [9] we have

$$G_{f(X_\varepsilon), f(B(0,2\varepsilon))}(y) = G_{X_\varepsilon, B(0,2\varepsilon)}(x). \tag{4}$$

Let $x_0 \in X_\varepsilon$ which attains the distance $\text{dist}(X_\varepsilon, x)$. Recall from Example 3 that X_ε satisfies the ŁS condition. Together with (4), this leads to

$$G_{f(X_\varepsilon), f(B(0,2\varepsilon))}(y) \geq e \cdot \text{dist}(X_\varepsilon, x)^{\frac{1}{e}} = e \cdot \text{dist}(x_0, x)^{\frac{1}{e}},$$

for some $e > 0$ independent of y. The finite-increment theorem then implies the existence of some constant $e' > 0$, depending only on ε, such that

$$G_{f(X_\varepsilon), f(B(0,2\varepsilon))}(y) \geq e' \cdot \text{dist}(f(x_0), y)^{\frac{1}{e'}} \geq e' \cdot \text{dist}(f(X_\varepsilon), y)^{\frac{1}{e'}}.$$

Since the hypotheses of Proposition 5.3.3 of [9] are verified, the functions $G_{f(X_\varepsilon), B(0,2\varepsilon)}$ and $G_{f(X_\varepsilon)}$ are comparable in the sense of this Proposition. This allows us to conclude that $f(X_\varepsilon) = f(L_\varepsilon)$ fills the ŁS condition.

This example illustrates the fact that it is not so surprising that filled parabolic Julia sets can satisfy the ŁS condition, even if they have cusps.

The following result provides more insight into the structure of the complement of O_c. We prove it for any $n \geq 1$. Recall that, given an open set $U \subset \mathbb{C}^n$, a set $E \subset U$ is called *pluripolar* if for each $a \in E$ there exist a neighborhood $V \subset U$ of a and a plurisubharmonic function $v : V \to \mathbb{R} \cup \{-\infty\}$ such that $E \cap V \subset \{v = -\infty\}$.

Proposition 1 *Let $A \subset \mathbb{C}^n$, $n \geq 1$, be a non-pluripolar, L-regular and polynomially convex compact set. Suppose that the pluricomplex Green function G_A is pluriharmonic outside of A (harmonic if $n = 1$).*

Then, there exists $c_0 > 0$ such that $\forall c \in]0, c_0]$, ∂A is included in the boundary of the open set $\{z \in \mathbb{C}^n : G_A(z) > c \cdot dist(z, A)^{1/c}\}$.

Proof Let μ denote the positive measure $\dfrac{i}{\pi}\partial\overline{\partial}G_A \wedge \omega^{n-1}$ on $\mathbb{C}^n$, where

$$\omega := \frac{i}{2\pi}\partial\overline{\partial}\log(1 + \|z\|^2)$$

is the Fubini–Study form. Note that the support of the measure μ is exactly ∂A. Indeed, $supp(\mu) \subset \partial A$ since $\frac{i}{\pi}\partial\overline{\partial}G_A = 0$ in $\mathbb{C}^n \backslash \partial A$ by hypothesis. On the other hand, if there existed $x \in \partial A \backslash supp(\mu)$, then G_A would be (pluri)harmonic in a neighborhood of x, hence null in this neighborhood, which cannot happen because A is polynomially convex.

Let us suppose by contradiction that $\forall c_0 > 0,\ \exists c \in]0, c_0],\ \exists x \in \partial A,\ \exists r > 0,\ \ B(x, r) \cap \{z \in \mathbb{C}^n, G_A(z) > c \cdot \text{dist}(z, A)^{1/c}\} = \emptyset$.

Thus, we can take $c' \in \left]0, \dfrac{1}{4n}\right[$, $x' \in \partial A$, and $r' > 0$, such that

$$G_A(z) \leq c' \cdot \text{dist}(z, A)^{\frac{1}{c'}}, \quad \forall z \in B(x', r').$$

Denote $r_0 := \frac{r'}{2}$. Let us establish the following Chern–Levine–Nirenberg-type inequality : $\forall r < r_0, \forall x \in B(x', r_0) \cap \partial A$,

$$\mu\big(B(x, r)\big) \leq k \cdot r^{-2n} \sup_{B(x,2r)} G_A \leq c' \cdot k \cdot (2r)^{\frac{1}{c'} - 2n}, \tag{5}$$

for some constant $k > 0$ independent of r, r_0, x' and c'. Let indeed $\xi : \mathbb{C}^n \to \mathbb{R}^+$ be a positive test function $\equiv 1$ in $B(0, 1)$ and having its support in $B(0, 2)$. There exists a decreasing sequence $(G_n)_n$ of $\mathcal{C}^\infty$ plurisubharmonic functions converging toward G_A (Theorem 2.9.2 in [9]). Theorem 3.4.3 in [9] and Stokes' theorem imply that $\forall r < r_0$, $\forall x \in B(x', r_0) \cap \partial A$,

$$\begin{aligned}\mu\big(B(x,r)\big) &\leq \int_{\mathbb{C}^n} \xi\left(\frac{z_1}{r}, \dots, \frac{z_n}{r}\right) d\mu\,(z_1, \dots, z_n) \\ &= \lim_{m\to+\infty} \int_{\mathbb{C}^n} \frac{G_m}{r^{2n}} (\partial\overline{\partial}\xi)\left(\frac{z_1}{r}, \dots, \frac{z_n}{r}\right) \wedge \omega^{n-1}.\end{aligned}$$

Then, the monotone convergence theorem implies that $\forall r < r_0$, $\forall x \in B(x', r_0) \cap \partial A$,

$$\mu\big(B(x,r)\big) \leq \int_{\mathbb{C}^n} \frac{G_A}{r^{2n}} (\partial\overline{\partial}\xi)\left(\frac{z_1}{r}, \dots, \frac{z_n}{r}\right) \wedge \omega^{n-1} \leq k \cdot r^{-2n} \sup_{B(x,2r)} G_A,$$

where k depends only on the sum of the supremum norms of the coefficients of the differential form $\partial\overline{\partial}\xi$. Therefore, (5) holds.

With the notation $\nu := \dfrac{\mu}{\mu\big(B(x', r_0)\big)} \mathbf{1}_{B(x', r_0)}$, where $\mathbf{1}_{B(x', r_0)}$ is the characteristic function of $B(x', r_0)$, the measure ν is a probability measure, and we can rewrite (5) : $\forall r > 0$, $\forall x \in B(x', r_0) \cap \partial A$,

$$\nu\big(B(x,r)\big) \leq \frac{c' \cdot k}{\mu\big(B(x', r_0)\big)} \cdot (2r)^{\frac{1}{c'} - 2n}.$$

Then, by Frostman Lemma (see for example Lemma 10.2.1 in [1]), the Hausdorff dimension of $\partial A \cap B(x_0, r_0)$ is strictly greater than $2n$ for our choice $c' < \frac{1}{4n}$, which gives a contradiction. (Recall that Frostman Lemma ensures that, if m is a probability measure on a metric space E verifying $m\big(B(x,r)\big) < q \cdot r^\alpha$ for all $x \in E$, $r > 0$, with fixed $q > 0$, $\alpha > 0$, then the Hausdorff dimension of E is greater than α).

We thus conclude that $\exists c_0 > 0$, $\forall c \in]0, c_0]$, $\forall x \in \partial A$, $\forall r > 0$:

$$B(x,r) \cap \left\{ z \in \mathbb{C}^n,\, G_A(z) > c \cdot \operatorname{dist}(z, A)^{1/c} \right\} \neq \emptyset,$$

which proves the statement. □

4 Proof of the main theorem

We will need the following result of Poletsky (Corollary p. 170 in [15], see also [16], or Theorem 2.2.10 and Corollary 2.2.13 in [18]), generalized by Rosay [17]. Let U be a connected complex manifold of dimension $n \geq 1$. We denote by $\mathcal{H}_{z,U}$ the set of holomorphic functions $h : V_h \to U$ from a neighborhood V_h of $\overline{\Delta} = \{|z| \leq 1\} \subset \mathbb{C}$ (possibly depending on h) into U such that $h(0) = z$. We also denote by PSH(U) the set of plurisubharmonic functions defined on U.

Proposition 2 *Let $u : U \to \mathbb{R}$ be an upper semi-continuous function. With the previous notations, the function defined by*

$$\tilde{u}(z) := \frac{1}{2\pi} \inf_{f \in \mathcal{H}_{z,U}} \int_0^{2\pi} u\left(f\left(e^{i\theta}\right)\right) d\theta,$$

if it is not everywhere equal to $-\infty$, belongs to PSH(U) and verifies $\tilde{u} \leq u$. Moreover, this function $\tilde{u}$ is maximal among all the functions in PSH(U) verifying this inequality.

Remark 1 We deduce from Proposition 2 the following property of antisubharmonic functions, i.e., functions with subharmonic opposite. Let $B := B(a, r) \subset \mathbb{C}$ be an open ball, $u : \overline{B} \to \mathbb{R}$ a continuous function, antisubharmonic in B. Then, $\hat{u} : B \to \mathbb{R}$ is an harmonic function, with the same boundary values as u, in the sense that $\lim_{z \to z_0} \hat{u} = u(z_0)$ for $z_0 \in \partial B$.

Indeed, given a continuous function $g : \overline{B} \to \mathbb{R}$, denote by $\tilde{g} : B \to \mathbb{R}$ the solution of the Dirichlet problem in B with boundary condition $g_{|\partial B}$, that is to say, the unique continuous function defined on $\overline{B}$ which is harmonic in B and equal to g on ∂B. Then, $v := \max(\tilde{u}, \hat{u})$ is a subharmonic function with the same values as u on ∂B. Since u is antisubharmonic, we have $\tilde{u} \leq u$. Thus,

$$\hat{u} \leq v \leq u.$$

Since $\hat{u}$ is maximal among the subharmonic functions which are $\leq u$ in B and equal to u on ∂B, we conclude that $\hat{u} = v$, and hence $\tilde{u} = \hat{u}$.

Thanks to Theorem 3.1.4 in [9], the conclusion is the same if B is a ball in $\mathbb{C}^n$, when substituting the expression "harmonic function" by "maximal plurisubharmonic function," and the expression "antisubharmonic function" by "antiplurisubharmonic function".

Let $U \subset \mathbb{C}^n, n \geq 1$, be a bounded open set. Denote by λ the normalized Lebesgue measure on the unit circle $\partial\mathbb{U} \subset \mathbb{C}$. Denote also by $\Lambda_{z,U}$ the set of measures of the form $h_*\lambda(\cdot) := \lambda(h^{-1}(\cdot))$, where $h : V_h \to U$ is an holomorphic function defined in a neighborhood V_h (possibly depending on h) of the closed unit disk $\overline{\mathbb{U}}$, such that $h(0) = z$. Note that the Dirac measure δ_z belongs to $\Lambda_{z,U}$. (This corresponds to the case where the function h is constant, equal to z.) An immediate consequence of Proposition 2 is the following corollary, where $\mathbf{1}_G$ denotes the characteristic function of $G \subset \mathbb{C}^n$:

Corollary 1 *Let $U \subset \mathbb{C}^n$ be a bounded open set, and $A \subset U$ a L-regular non-pluripolar compact set satisfying $\overline{\mathring{A}} = A$. Then*

$$\frac{1}{2\pi} \inf_{f \in \mathcal{H}_{z,U}} \int_0^{2\pi} -\mathbf{1}_{\mathring{A}} \circ f(e^{i\theta}) \mathrm{d}\theta = - \sup_{\mu_z \in \Lambda_{z,U}} \mu_z(\mathring{A}) = G_{A,U}(z) - 1.$$

Recall that we denote by K the filled Julia set of a polynomial application $f : \mathbb{C} \to \mathbb{C}$ of degree ≥ 2, and $\mathrm{dist}(\cdot, \cdot)$ the Euclidean distance on $\mathbb{C}^n$. Let us prove the main result stated in the introduction :

Theorem 1 *Let $K \subset \mathbb{C}$ be the filled Julia set of a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$, of non-empty interior. Then, K satisfies the ŁS condition.*

Proof For $b \in \mathbb{R}^+ \backslash \{0\}$, denote $U_b := \{G_K < b\} \subset \mathbb{C}$. For $l \in \mathbb{R}^+ \backslash \{0\}$, denote also $K_l := \{z \in \mathbb{C} \mid \mathrm{dist}(z, K) \leq l\}$. Then choose $a > 0$ such that $K_2 \subset f^{-1}(U_a)$. Note that $f^{-1}(U_a) = U_{\frac{a}{d}} \subset\subset U_a$ by (2). Denote by $\mathcal{C}_a$ the annulus $U_a \backslash f^{-1}(U_a)$. There exists $\delta \in]0, 1[$ such that

$$G_{K_{2,U_a}} \geq \delta G_{K,U_a} \quad \text{on} \quad \mathcal{C}_a. \tag{6}$$

Take $c \in \left]0, \dfrac{\delta}{2a}\right[$, sufficiently small to have $\overline{O_c} \subset f^{-1}(U_a)$ and $\left(\frac{1}{c^2}\right)^c < 2$. We have $\forall \varepsilon \in]0, 2]$, $\forall y \in U_a$,

$$\begin{aligned} c \cdot \operatorname{dist}(y, K)^{\frac{1}{c}} &\geq \inf_{\mu_y \in \Lambda_{y,U_a}} \int_{U_a} c \cdot \operatorname{dist}(\cdot, K)^{\frac{1}{c}} \mathrm{d}\mu_y \\ &\geq \inf_{\mu_y \in \Lambda_{y,U_a}} \int_{U_a \setminus \mathring{K}_\varepsilon} c \cdot \operatorname{dist}(\cdot, K)^{\frac{1}{c}} \mathrm{d}\mu_y \\ &\geq \left(\min_{\overline{U_a \setminus K_\varepsilon}} c \cdot \operatorname{dist}(\cdot, K)^{\frac{1}{c}} \right) \inf_{\mu_y \in \Lambda_{y,U_a}} \int_{U_a \setminus \mathring{K}_\varepsilon} \mathrm{d}\mu_y \\ &= c\varepsilon^{\frac{1}{c}} G_{K_\varepsilon, U_a}(y). \end{aligned}$$

The first inequality comes from the fact that the Dirac measure δ_y belongs to Λ_{y,U_a}. The last inequality comes from Corollary 1, whose application is allowed by Lemma 1, and from Corollary 4.5.9 in [9]. Then taking $\varepsilon = \left(\frac{1}{c^2}\right)^c < 2$, we obtain in U_a:

$$c \cdot \operatorname{dist}(\cdot, K)^{\frac{1}{c}} \geq \frac{1}{c} G_{K_\varepsilon, U_a}. \tag{7}$$

Now suppose, by contradiction, that $O_c \neq \emptyset$ [see Eq. (3) for definition]. Recall that $c < \frac{\delta}{2a}$. Note that there exists a constant $e \in]0, 1[$ such that $\forall z \in \overline{U_a} \setminus \{\operatorname{dist}(\cdot, K) < 1\}$,

$$G_K(z) \geq e \cdot c \cdot \operatorname{dist}(z, K)^{\frac{1}{c}}. \tag{8}$$

Recall that the function $c \mapsto c \cdot \operatorname{dist}(x, K)^{\frac{1}{c}}$ is increasing. Up to diminishing c, we can thus suppose that $c < \frac{\delta e}{2a}$.

We can then choose $x \in O_c \setminus \{G_K < \frac{2ac^2}{\delta e} \operatorname{dist}(\cdot, K)^{\frac{1}{c}}\}$.

Let us control the growth of the iterates of f. Note that $z \in O_c$ implies a "slow growth" of $(f^n(z))_n$, in the sense that $\forall n \geq 1$ such that $f^n(z) \in \{\operatorname{dist}(\cdot, K) < 1\} \setminus O_c$, we have

$$\frac{1}{d^n} c \cdot \operatorname{dist}(f^n(z), K)^{\frac{1}{c}} \leq G_K(z) < c \cdot \operatorname{dist}(z, K)^{\frac{1}{c}},$$

and hence

$$\operatorname{dist}(f^n(z), K) < d^{nc} \operatorname{dist}(z, K). \tag{9}$$

Moreover, by a similar reasoning, Eq. (8) implies that for all $z \in O_c$ and $n \geq 1$ such that $f^n(z) \in \overline{U_a} \setminus \{\operatorname{dist}(\cdot, K) < 1\}$, we have

$$\operatorname{dist}(f^n(z), K) < \left(\frac{d^n}{e}\right)^c \operatorname{dist}(z, K). \tag{10}$$

Finally, by (9) and (10), for all $z \in O_c$ and $n \geq 1$ such that $f^n(z) \in \overline{U_a} \setminus O_c$, we have

$$\operatorname{dist}(f^n(z), K) < \left(\frac{d^n}{e}\right)^c \operatorname{dist}(z, K). \tag{11}$$

Conclusion Recall that we have chosen $x \in O_c \setminus \{G_K < \frac{2ac^2}{\delta e} \operatorname{dist}(\cdot, K)^{\frac{1}{c}}\}$. Since $U_a \setminus K = \bigcup_{i \geq 0} f^{-i}(\mathcal{C}_a)$ by (2), there exists $N > 0$ such that $f^N(x) \in \mathcal{C}_a$. Equations (11), (7), (6), (1), then (2), give

$$\begin{aligned}
\frac{c}{e}\cdot \operatorname{dist}(x,K)^{\frac{1}{c}} &\geq \frac{c}{d^N}\operatorname{dist}\left(f^N(x),K\right)^{\frac{1}{c}}\\
&\geq \frac{1}{cd^N}G_{K_\varepsilon,U_a}\circ f^N(x)\\
&\geq \frac{\delta}{cd^N}G_{K,U_a}\circ f^N(x)\\
&= \frac{\delta}{ca}G_K(x).
\end{aligned}$$

But this contradicts our assumption $x \notin \{G_K < \frac{2ac^2}{\delta e}\operatorname{dist}(x,K)^{\frac{1}{c}}\}$. We conclude that $O_c = \emptyset$. In other words, K satisfies the ŁS condition. □

Remark 2 We note that if f is assumed to be *hyperbolic*, that is to say if f do not have critical points in J, there exist a constant $b > 0$ and a neighborhood of K in which

$$\operatorname{dist}(f(\cdot),K) \geq b\cdot \operatorname{dist}(\cdot,K). \tag{12}$$

Indeed, it is sufficient to establish this inequality outside K. Let then V be a neighborhood of K in which $|f'| \geq a$ for some $a > 0$, let $z \in V\backslash K$, and $z_0 \in J$ such that $f(z_0) \in J$ achieves the distance $\operatorname{dist}(f(z),J)$. Then, Theorem 1 of [5] shows the existence of a constant $k > 0$ (depending only on the degree of f) and of a point $z_1 \in J = \partial K$, such that

$$\operatorname{dist}(f(z),K) = \operatorname{dist}(f(z),f(z_0)) \geq a\cdot k\cdot \operatorname{dist}(z,z_1) \geq a\cdot k\cdot \operatorname{dist}(z,K).$$

In the particular case where $b \geq 1$ in (12), we obtain a simpler proof of Theorem 1, and a more quantitative estimation for c in Eq. (3). Indeed, suppose $O_c \neq \emptyset$ with $O_c \subset\subset V$. We can choose $x \in O_c$ such that $f(x) \notin O_c$. Then, (11) together with (12) give

$$c > \frac{\log b}{\log \frac{d}{e}}.$$

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References

1. Beardon, A.F.: Iteration of Rational Functions. Springer, Berlin (1991)
2. Belghiti, M.T.: Éléments pour une théorie constructive des fonctions lisses. Thèse de doctorat d'État, Université Ibn Tofaïl, Kenitra (2004)
3. Białas-Cież, L., Kosek, M.: Iterated function systems and Łojasiewicz–Siciak condition of Green's function. Potential Anal. **34**, 207–221 (2011)
4. Carleson, L., Gamelin, T.W.: Complex Dynamics. Springer, Berlin (1993)
5. Dubinin, V.N.: On the finite-increment theorem for complex polynomials. Math. Notes **5**, 647–654 (2010)
6. Gendre, L.: Inégalités de Markov singulières et approximation des fonctions holomorphes de la classe M. Ph.D. thesis, Université Toulouse III (2005)
7. Guedj, V.: Propriétés ergodiques des applications rationnelles. In: Quelques aspects des systèmes dynamiques polynomiaux. Panor. Synthèses, No. 30, pp. 97–202. Soc. Math. France, Paris (2010)
8. Guedj, V., Sibony, N.: Dynamics of polynomial automorphisms of $\mathbb{C}$. Ark. Mat. **40**, 207–243 (2002)

9. Klimek, M.: Pluripotential Theory. Clarendon Press, Oxford (1991)
10. Kosek, M.: Hölder continuity property of fillin-in Julia sets in $\mathbb{C}^n$. Proc. Am. Math. Soc. **125**, 2029–2032 (1997)
11. Lelong, P., Gruman, L.: Entire Functions of Several Complex Variables. Springer, Berlin (1986)
12. McMullen, C.T.: Complex Dynamics and Renormalization. Princeton University Press, Princeton (1994)
13. Pierzchała, R.: An estimate for the Siciak extremal function: subanalytic geometry approach. J. Math. Anal. Appl. **430**, 755–776 (2015)
14. Pierzchała, R.: Markov's inequality in the o-minimal structure of convergent power series. Adv. Geom. **12**, 647–664 (2012)
15. Poletsky, E.A.: Plurisubharmonic functions as solutions of variational problems. In: Several Complex Variables and Complex Geometry (Santa Cruz, CA, 1989), Part. 1, pp. 163–171. American Math. Soc. (1991)
16. Poletsky, E.A.: Holomorphic currents. Indiana Univ. Math. J. **42**, 85–144 (1993)
17. Rosay, J.P.: Poletsky theory of disks on holomorphic manifolds. Indiana Univ. Math. J. **52**, 157–169 (2003)
18. Stout, E.L.: Polynomial Convexity. Springer, Berlin (2007)