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# An L-Shaped Array for Estimating 2-D Directions of Wave Arrival

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**Abstract**—For the estimation of two-dimensional (2-D) directions of plane wave arrival, an L-shaped array of sensors has 37% better accuracy potential than the conventional cross array. Computationally efficient maximum likelihood algorithm for the L-shaped array is available to achieve its full potential.

## I. INTRODUCTION

UNIFORM linear arrays (ULA's) for estimation of wave arrival have been studied extensively [1]–[3]. It is understood that the ULA can provide only the one-dimensional (1-D) information of wave arrival. For two-dimensional (2-D) wave arrival information, a 2-D array of sensors is needed. A conventional 2-D array is the cross array, which consists of two ULA's connected orthogonally at the middle of each array. The cross array is important because of its simple structure so that the computationally efficient maximum likelihood (ML) algorithm [2], [3] can be designed to achieve its full potential. (Note that the Cramer–Rao bound (CRB) of the cross array can be shown to be the same as the CRB of each ULA in the cross array so that the ML algorithm for the ULA is also optimum for the cross array.) On the other hand, another conventional array called circular array has a much more complicated structure so that there is no efficient ML algorithm available for multiple wave direction finding. In this paper, we present another simple structured 2-D array, called the L-shaped array. The L-shaped array consists of two ULA's connected orthogonally at the one end of each ULA.

In Section II, we show that the Cramer–Rao bounds (CRB's) of the estimated wave directions based on the L-shaped array are about 37% smaller than those for the cross array. CRB indicates the accuracy potential because it is the (reachable) lower bound on variance of any unbiased estimate [4].

In Section III, an efficient ML algorithm is developed utilizing the ULA structure inherent in the L-shaped array.

## II. L-SHAPED ARRAY

To appreciate the accuracy potential and the unique structure of the L-shaped array for estimation of 2-D wave arrival, we compare it with several other 2-D arrays as shown in Fig. 1. All sensors in Fig. 1 are identical and omnidirectional, and they are located on a uniform grid (so that the CRB can be derived). The arriving waves are assumed to be plane waves. The array output

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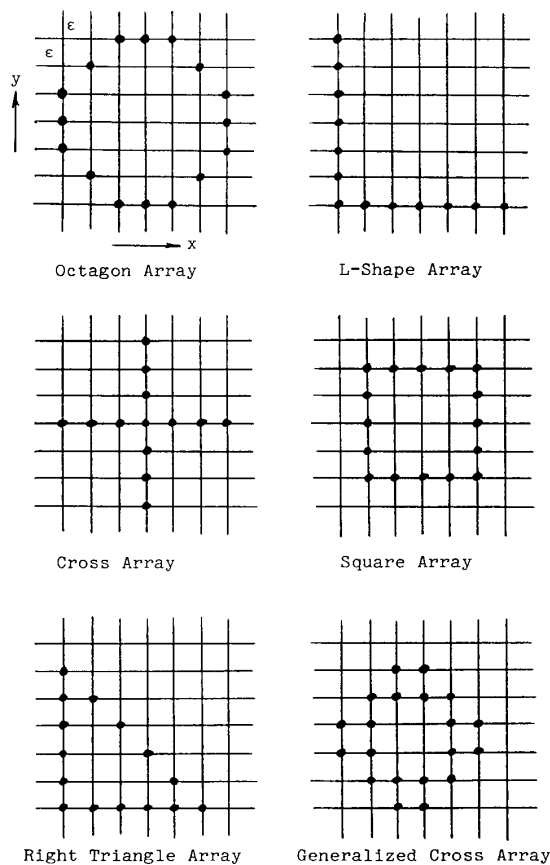


Fig. 1. Six array configurations on uniform  $x - y$  grid.

vector can then be written as

$$y(t) = \sum_{k=1}^D a_k(t) z(\alpha_k, \beta_k) + n(t) \quad (1)$$

with

$$y(t) = [y_1(t), \dots, y_N(t)]^T \quad (2)$$

$$z(\alpha_k, \beta_k) = \begin{bmatrix} \exp [j2\pi(\epsilon/\lambda) \cos(\alpha_k) x_1 + j2\pi(\epsilon/\lambda) \cos(\beta_k) y_1] \\ \dots \\ \exp [j2\pi(\epsilon/\lambda) \cos(\alpha_k) x_N + j2\pi(\epsilon/\lambda) \cos(\beta_k) y_N] \end{bmatrix} \quad (3)$$

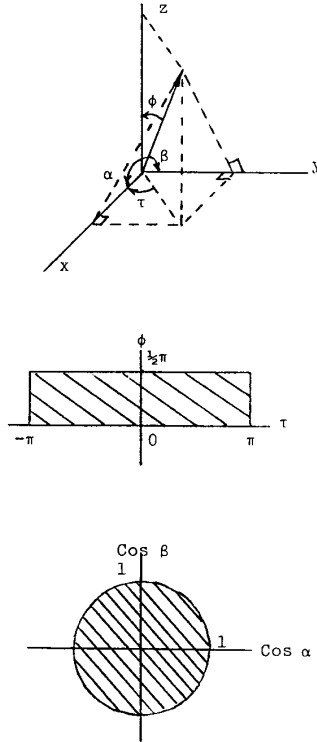


Fig. 2. Relation between two coordinate systems of a 2-D direction.

where  $t$  denotes the  $t$ th snapshot of the total  $T$ ;  $D$  denotes the number of waves;  $a_k(k)$  is the complex amplitude of the  $k$ th wave, the  $N \times 1$  vector  $\mathbf{z}(\dots)$  is called the steering vector;  $\alpha_k$  and  $\beta_k$  are the two angles of the  $k$ th wave with respect to the  $x$  and  $y$  axes as shown in Fig. 2;  $\epsilon$  is the spacing in the  $x$ - $y$  grid;  $\lambda$  is the wavelength;  $(x_i, y_i)$  is the coordinate of the  $i$ th sensor of the total  $N$ ; and  $\mathbf{n}(t)$  is the noise vector.

Computing the CRB [4] involves the inversion of a Fisher information matrix of the dimension equal to the number of unknowns. In order to be able to derive the CRB analytically, we consider the case where  $T = 1$ ,  $D = 1$ , and the noise is the white Gaussian with the variance  $2\sigma^2$ . Then, there are the four unknowns:  $\alpha_1$ ,  $\beta_1$ , the magnitude and the phase of  $a_1(1)$ . Now, computing the CRB's of the estimates of  $\alpha_1$  and  $\beta_1$  for each of the 2-D arrays in Fig. 1 becomes straightforward. Note that the  $4 \times 4$  Fisher information matrix is a sparse matrix which can be inverted analytically. In Table I, the CRB's of  $\cos \alpha_1$  and  $\cos \beta_1$  are listed for each array. The CRB's on  $\cos \alpha_1$  and  $\cos \beta_1$  are equal because of the symmetry of all the arrays considered. The simple expressions given in the table also requires  $N \gg 1$ .

It is seen from this table that the CRB of the L-shaped array is significantly ( $\cong 37\%$ ) smaller than that of the conventional cross array!

It is also seen from the table that the octagon array (which is close to the circular array in structure) has only  $\cong 5\%$  smaller CRB than the L-shaped array.

### III. MAXIMUM LIKELIHOOD ESTIMATION (MLE)

It is known that the CRB can be achieved by the MLE. For the L-shaped array, a computationally efficient ML algorithm is developed in the following way.

TABLE I  
CRB( $\cos \alpha_1$ ) = CRB( $\cos \beta_1$ )

Array Type	CRB Value
Octagon Array	$\frac{57}{\delta N^3}$
L-Shaped Array	$\frac{60}{\delta N^3}$
Cross Array	$\frac{96}{\delta N^3}$
Square Array	$\frac{96}{\delta N^3}$
Right Triangle Array	$\frac{108}{\delta N^3}$
Generalized Cross Array	$\frac{192}{\delta N^3}$

$\delta = 2 \text{SNR}_1 (2\pi\epsilon/\lambda)^2$  and  $\text{SNR}_1 = |a_1|^2 / 2\sigma^2$ .  $2\sigma^2$  is the variance of the white noise. Note that  $\text{CRB}(\alpha_1) = \text{CRB}(\cos \alpha_1) / \sin^2 \alpha_1$  and  $\text{CRB}(\beta_1) = \text{CRB}(\cos \beta_1) / \sin^2 \beta_1$ .

Under the Gaussian assumption of the noise, the ML estimates minimize the cost function:

$$J = \left\| \mathbf{y} - \sum_{k=1}^D A_k \mathbf{z}(\alpha_k, \beta_k) \right\|^2 \quad (4)$$

where  $\|\cdot\|$  denotes the 2-norm and

$$\mathbf{y} = \begin{bmatrix} y(1) \\ \vdots \\ y(T) \end{bmatrix} \quad (5)$$

$$A_k = \begin{bmatrix} a_k(1) I_N \\ \vdots \\ a_k(T) I_N \end{bmatrix}. \quad (6)$$

$I_N$  is the  $N \times N$  identity matrix. The cost function  $J$  is a highly nonlinear function of the unknowns:  $\{a_k(t), \alpha_k$  and  $\beta_k; k = 1, \dots, D$  and  $t = 1, \dots, T\}$ . Based on the structure of the L-shaped array, the following iterative ML algorithm has been developed.

#### A. Outer Loop

Let  $A'_k$  and  $\mathbf{z}'_k$  be the previous estimates of  $A_k$  and  $\mathbf{z}_k$ , where  $\mathbf{z}_k = \mathbf{z}(\alpha_k, \beta_k)$ . Define for  $k = 1, \dots, D$ ,

$$\mathbf{x}_k = A'_k \mathbf{z}'_k + w_k \left( \mathbf{y} - \sum_{k=1}^D A'_k \mathbf{z}'_k \right) \quad (7)$$

where  $w_k, k = 1, \dots, D$ , are nonnegative real values satisfying

$$\sum_{k=1}^D w_k = 1. \quad (8)$$

Then, for each  $k$ , the new estimates of  $A_k$  and  $\mathbf{z}_k$  are obtained by minimizing

$$J_k = \left\| \mathbf{x}_k - A_k \mathbf{z}_k \right\|^2 \quad (9)$$

with respect to  $a_k(t)$ ,  $\alpha_k$ , and  $\beta_k$ .

### B. Inner Loop (minimizing $J_k$ )

Rewrite (9) as

$$J_k = \sum_{t=1}^T \| \mathbf{x}_k(t) - a_k(t) \mathbf{z}_k \|^2 \quad (10)$$

where  $\mathbf{x}_k(t)$  is the  $t$ th subvector of  $\mathbf{x}_k^T = [\mathbf{x}_k(1)^T, \dots, \mathbf{x}_k(T)^T]$ . For any given  $\mathbf{z}_k$ , the optimum estimates of  $a_k(t)$ ,  $t = 1, \dots, T$ , are given by

$$\begin{aligned} a_k(t) &= (\mathbf{z}_k^H \mathbf{z}_k)^{-1} \mathbf{z}_k^H \mathbf{x}_k(t) \\ &= \mathbf{z}_k^H \mathbf{x}_k(t) / N \end{aligned} \quad (11)$$

where the superscript  $H$  denotes the conjugate transpose. Then  $J_k$  becomes

$$J_k = \sum_{t=1}^T \| (I_N - \mathbf{z}_k \mathbf{z}_k^H / N) \mathbf{x}_k(t) \|^2. \quad (12)$$

Now  $J_k$  is a function of the two unknowns  $\alpha_k$  and  $\beta_k$  only. For the  $L$ -shaped array,  $\mathbf{z}_k$  can be written as

$$\mathbf{z}_k = [p_k^M, \dots, p_k, 1, q_k, \dots, q_k^M]^T \quad (13)$$

$$X_k(t) = \begin{bmatrix} x_1 - x_2 & -j(x_1 + x_2) & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ x_M - x_{M+1} & -j(x_M + x_{M+1}) & 0 & \vdots \\ -x_{M+1} + x_{M+2} & 0 & -j(x_{M+1} + x_{M+2}) & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ -x_{2M} + x_{2M+1} & 0 & -j(x_{2M} + x_{2M+1}) & \vdots \end{bmatrix}. \quad (23)$$

where  $M = (N - 1)/2$  and

$$p_k = \exp(j2\pi(\epsilon/\lambda)\cos\alpha_k) \quad (14)$$

$$q_k = \exp(j2\pi(\epsilon/\lambda)\cos\beta_k). \quad (15)$$

$p_k$  and  $q_k$  are also uniquely represented by

$$p_k = (1 + jc_k)/(1 - jc_k) \quad (16)$$

$$q_k = (1 + jd_k)/(1 - jd_k) \quad (17)$$

where  $c_k$  and  $d_k$  are real values.  $(\alpha_k, \beta_k)$  is one-to-one function of  $(c_k, d_k)$  or  $(p_k, q_k)$ . Define the  $(2M + 1)$  by  $2M$  matrix:

$$B_k = \begin{bmatrix} 1 + jc_k \\ -1 + jc_k \\ \vdots \\ 1 + jc_k \\ -1 + jc_k - 1 + jd_k \\ 1 + jd_k \\ \vdots \\ -1 + jd_k \\ 1 + jd_k \end{bmatrix}. \quad (18)$$

It is easy to check that the  $(2M + 1)$  by one vector  $\mathbf{z}_k$  is orthogonal to all  $2M$  columns of  $B_k$ . Therefore, the orthogonal

complement of the span of single vector  $\mathbf{z}_k$  is equal to the span of all the columns of  $B_k$ , i.e.,

$$I_N - (1/N) \mathbf{z}_k \mathbf{z}_k^H = B_k B_k^+ \quad (19)$$

where the superscript plus sign denotes the pseudo-inverse. Since  $B_k$  has independent columns,  $B_k^+ = (B_k^H B_k)^{-1} B_k^H$ . Substituting (19) into (12) yields

$$\begin{aligned} J_k &= \sum_{t=1}^T \| B_k B_k^+ \mathbf{x}_k(t) \|^2 \\ &= \sum_{t=1}^T \mathbf{x}_k(t)^H B_k (B_k^H B_k)^{-1} B_k^H \mathbf{x}_k(t). \end{aligned} \quad (20)$$

It can be verified that

$$\mathbf{x}_k(t)^H B_k = \mathbf{b}_k^T X_k(t) \quad (21)$$

where

$$\mathbf{b}_k = \begin{bmatrix} 1 \\ c_k \\ d_k \end{bmatrix} \quad (22)$$

and  $X_k(t)$  is the  $2M \times 3$  matrix:

In (23),  $x_j$  denotes the  $j$ th element of the vector  $\mathbf{x}_k(t)$ . With (21), (20) becomes

$$J_k = \mathbf{b}_k^T \left[ \sum_{t=1}^T X_k(t) (B_k^H B_k)^{-1} X_k(t)^H \right] \mathbf{b}_k. \quad (24)$$

Now the inner loop can be summarized as: Given the previous estimates  $c_k$  and  $d_k$  (which are the one-to-one function of  $\alpha_k$  and  $\beta_k$ ), the  $3 \times 3$  matrix within the bracket in (24) is computed, and the new estimates of  $c_k$  and  $d_k$  are given by the  $3 \times 1$  eigenvector (according to (22)) corresponding to the smallest eigenvalue of that  $3 \times 3$  matrix. This process is repeated until convergence.

At each iteration of the inner loop, the  $(2M + 1)$  by  $(2M + 1)$  matrix inverse  $(B_k^H B_k)^{-1}$  can be computed very efficiently [7] due to its sparse structure.

The inner loop has been tested to be a stable algorithm (it converged after five iterations in our simulations for one-wave and two-wave cases), which is a consistent property as observed with a similar ML algorithm [2], [3] designed for the ULA's. The outer loop is actually an application of the estimate maximize (EM) approach, which is guaranteed to be stable [5]–[7]. In our simulations, it converged after 10 iterations.

### C. Initial Estimates

Since  $J$  is a highly nonlinear function of the unknowns, good initial estimates are important to make the iterative algorithm converge to the global optimum point. Fortunately, the  $L$ -shaped array consists of two ULA's so that any algorithms [1]–[3] designed for the ULA's can be used to provide the initial estimates.

For multiple waves, the correct pairing between the estimated  $\alpha$  and the estimated  $\beta$  obtained from the ULA's still needs to be done. The optimum pairing is to minimize  $J$  with respect to all possible pairings of  $\alpha$  and  $\beta$ . A more computationally efficient pairing technique follows.

Computing the sample correlation matrix ( $N \times N$ ) of  $y(t)$ :

$$R_y = 1/T \sum_{t=1}^T y(t)y(t)^H$$

$$= Z \left\{ 1/T \sum_{t=1}^T a(t)a(t)^H \right\} Z^H + 1/T \sum_{t=1}^T n(t)n(t)^H \quad (25)$$

where

$$a(t) = [a_1(t), \dots, a_D(t)]^T \quad (26)$$

$$Z = [z_1, \dots, z_D]. \quad (27)$$

Assume both the signal amplitude vector  $a(t)$  and the noise vector  $n(t)$  are ergodic random processes with the covariance matrices:  $R_a$  (of the full rank  $D$ ) and  $R_n = 2\sigma^2 I_N$ , respectively. Then, for large  $T$  (i.e., asymptotically),

$$R_y \cong ZR_a Z^H + 2\sigma^2 I_N. \quad (28)$$

It can be shown that the  $D$  principal eigenvectors of  $R_y$  asymptotically span the same space as the columns of  $Z$ , and the  $N-D$  nonprincipal eigenvectors are asymptotically orthogonal to each of the columns of  $Z$ . Let the  $N$  eigenvectors of  $R_y$  be denoted by  $v_1, \dots, v_D, v_{D+1}, \dots, v_N$ , which are corresponding to the decreasing order of the eigenvalues. Then, the pairing can be obtained by minimizing the following weighted sum of the inner products between the steering vector  $z(\alpha, \beta)$  and the  $N-D$  nonprincipal eigenvectors  $v_m$ ,  $m = D+1, \dots, N$ :

$$J'' = \sum_{m=1}^D w'_m |z^H(\alpha, \beta)v_m|^2 \quad (29)$$

(with respect to  $D$  possible choices of  $\beta_k$ ,  $k = 1, \dots, D$ , for each  $\alpha_k$ ,  $k = 1, \dots, D$ ) where  $w'_m$  are positive weights.  $J''$  is asymptotically zero if the ML estimates of  $\alpha_k$  and  $\beta_k$  are correctly paired.

#### IV. CONCLUSION

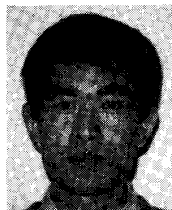
For 2-D wave direction finding, the L-shaped array has higher accuracy potential than the conventional cross array and many other simple structured arrays. Due to the property of the two ULA's in the L-shaped array, the maximum likelihood estimation of the wave directions can be implemented in a computationally efficient way.

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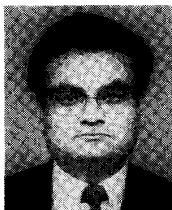
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