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L^2-BOUNDEDNESS OF THE CAUCHY INTEGRAL OPERATOR FOR CONTINUOUS MEASURES

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1. Introduction. Let μ be a continuous (i.e., without atoms) positive Radon measure on the complex plane. The truncated Cauchy integral of a compactly supported function f in $L^p(\mu)$, $1 \le p \le +\infty$, is defined by

$$\mathscr{C}_\varepsilon f(z) = \int_{|\xi - z| > \varepsilon} \frac{f(\xi)}{\xi - z}\, d\mu(\xi), \qquad z \in \mathbb{C},\ \varepsilon > 0.$$

In this paper, we consider the problem of describing in geometric terms those measures μ for which

$$\int |\mathscr{C}_\varepsilon f|^2\, d\mu \le C \int |f|^2\, d\mu, \tag{1}$$

for all (compactly supported) functions $f \in L^2(\mu)$ and some constant C independent of $\varepsilon > 0$. If (1) holds, then we say, following David and Semmes [DS2, pp. 7–8], that the Cauchy integral is bounded on $L^2(\mu)$.

A special instance to which classical methods apply occurs when μ satisfies the doubling condition

$$\mu(2\Delta) \le C\mu(\Delta),$$

for all discs Δ centered at some point of $\operatorname{spt}(\mu)$, where 2Δ is the disc concentric with Δ of double radius. In this case, standard Calderón-Zygmund theory shows that (1) is equivalent to

$$\int \left|\mathscr{C}^* f\right|^2 d\mu \le C \int |f|^2\, d\mu, \tag{2}$$

where

$$\mathscr{C}^* f(z) = \sup_{\varepsilon > 0} |\mathscr{C}_\varepsilon f(z)|.$$

If, moreover, one can find a dense subset of $L^2(\mu)$ for which

$$\mathscr{C} f(z) = \lim_{\varepsilon \to 0} \mathscr{C}_\varepsilon f(z) \tag{3}$$

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exists a.e. (μ) (i.e., almost everywhere with respect to μ), then (2) implies the a.e. (μ) existence of (3), for any $f \in L^2(\mu)$, and

$$\int |\mathscr{C} f|^2 \, d\mu \leq C \int |f|^2 \, d\mu,$$

for any function $f \in L^2(\mu)$ and some constant C.

For a general μ, we do not know if the limit in (3) exists for $f \in L^2(\mu)$ and almost all (μ) $z \in \mathbb{C}$. This is why we emphasize the role of the truncated operators $\mathscr{C}_\varepsilon$.

Proving (1) for particular choices of μ has been a relevant theme in classical analysis in the last thirty years. Calderón's paper [Ca] is devoted to the proof of (1) when μ is the arc length on a Lipschitz graph with small Lipschitz constant. The result for a general Lipschitz graph was obtained by Coifman, McIntosh, and Meyer in 1982 in the celebrated paper [CMM]. The rectifiable curves Γ, for which (1) holds for the arc length measure μ on the curve, were characterized by David [D1] as those satisfying

$$\mu\big(\Delta(z, r)\big) \leq C r, \qquad z \in \Gamma,\ r > 0, \tag{4}$$

where $\Delta(z, r)$ is the closed disc centered at z of radius r. It has been shown in [MMV] that if μ satisfies the Ahlfors-David regularity condition

$$C^{-1} r \leq \mu\big(\Delta(z, r)\big) \leq C r, \qquad z \in E,\ 0 < r < \operatorname{diam}(E),$$

where E is the support of μ, then (1) is equivalent to E being a subset of a rectifiable curve satisfying (4).

A necessary condition for (1) is the *linear growth condition*

$$\mu\big(\Delta(z, r)\big) \leq C_0 r, \qquad z \in \operatorname{spt}(\mu),\ r > 0, \tag{5}$$

as shown, for example, in [D2, p. 56]. To find another relevant necessary condition, we need to introduce a new object. The Menger curvature of three pairwise different points $x, y, z \in \mathbb{C}$ is

$$c(x, y, z) = \frac{1}{R(x, y, z)},$$

where $R(x, y, z)$ is the radius of the circumference passing through x, y, z (with $R(x, y, z) = \infty$ and $c(x, y, z) = 0$, if x, y, z lie on the same line). If two among the points x, y, z coincide, we let $c(x, y, z) = 0$. The relation between the Cauchy kernel and Menger curvature was found by Melnikov in [Me2]. It turns out that a necessary condition for (1) is (see [MV] and [MMV])

$$\int_\Delta \int_\Delta \int_\Delta c(x, y, z)^2 \, d\mu(x)\, d\mu(y)\, d\mu(z) \leq C_1 \mu(\Delta), \tag{6}$$

for all discs Δ. The main result of this paper is that, conversely, (5) and (6) are also sufficient for (1). It is not difficult to realize that (6) can be rewritten as $\mathscr{C}(1) \in$

BMO(μ), when μ is doubling and satisfies (5). Therefore, our result can be understood as a $T(1)$-theorem for the Cauchy kernel with an *underlying measure not necessarily doubling*. In fact, the absence of a doubling condition is the greatest problem we must confront. We overcome the difficulty thanks to the fact that the operators to be estimated have positive kernels. Following an idea of Sawyer, we resort to an appropriate "good λ-inequality" to obtain a preliminary weak form of the L^2-estimate. In a second step, we use an inequality of Melnikov [Me2] relating analytic capacity to Menger curvature to prove the weak (1,1)-estimate for $\mathcal{C}_\varepsilon$, uniform in $\varepsilon > 0$. It is worthwhile to mention that this part of the argument involves complex analysis in an essential way and no real variables proof is known to the author. From the weak (1,1)-estimate, we get the restricted weak-type (2,2) of $\mathcal{C}_\varepsilon$. By interpolation, one obtains the strong-type (p, p), for $1 < p < 2$, and then by duality, one obtains the strong-type (p, p), for $2 < p < \infty$. One more appeal to interpolation finally gives the strong-type (2,2). For other applications of the notion of Menger curvature, see [L] and [Ma2].

We now proceed to introduce some notation and terminology to state a more formal and complete version of our main result. We say that μ satisfies the *local curvature condition* if there is a constant C_1 such that (6) holds for any disc Δ centered at some point of spt(μ).

We say that the Cauchy integral is bounded on $L^p(\mu)$ whenever the operators $\mathcal{C}_\varepsilon$ are bounded on $L^p(\mu)$ uniformly on ε. Let $M(\mathbb{C})$ be the set of all finite complex Radon measures on the plane. If $\nu \in M(\mathbb{C})$, then we set

$$\mathcal{C}_\varepsilon(\nu)(z) = \int_{|\xi - z| > \varepsilon} \frac{1}{\xi - z}\, d\nu(\xi).$$

We say that the Cauchy integral is bounded from $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$, the usual space of weak L^1-functions with respect to μ, whenever the operators $\mathcal{C}_\varepsilon$ are bounded from $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$ uniformly on ε.

We can now state our main result.

THEOREM 1.1. *Let μ be a continuous positive Radon measure on $\mathbb{C}$. Then the following statements are equivalent.*

(1) *μ has linear growth and satisfies the local curvature condition.*
(2) *The Cauchy integral is bounded on $L^2(\mu)$.*
(3) *The Cauchy integral is bounded from $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$.*
(4) *The Cauchy integral is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.*

Notice that if any of the statements (1), (2), (3), or (4) of Theorem 1.1 holds, then the Cauchy integral is bounded on $L^p(\mu)$, for $1 < p < \infty$, by interpolation and duality. Conversely, if there exists $p \in (1, \infty)$ such that the Cauchy integral is bounded on $L^p(\mu)$, then the Cauchy integral is bounded on $L^2(\mu)$ by duality and interpolation.

Using the results of Theorem 1.1, in the final part of the paper we give a geometric characterization of the analytic capacity γ_+, and we show that γ_+ is semiadditive for sets of area zero.

The paper is organized as follows. In Section 2 we define the curvature operator K, and we prove that if μ has linear growth and satisfies the weak local curvature condition, then K is bounded on $L^p(\mu)$, for all $p \in (1, \infty)$. As a consequence, we get that for each μ-measurable subset $A \subset \mathbb{C}$,

$$\int_A \int_A \int_A c(x, y, z)^2 \, d\mu(x) \, d\mu(y) \, d\mu(z) \leq C\mu(A).$$

In Section 3 we explore the relation between the Cauchy integral, analytic capacity, and curvature. In Section 4 we complete the proof of Theorem 1.1. Finally, in Section 5 we study the analytic capacity γ_+.

A constant with a subscript, such as C_0, retains its value throughout the paper, while constants denoted by the letter C may change in different occurrences.

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2. The curvature operator. Throughout the paper, μ is a positive continuous Radon measure on the complex plane. Also, if $A \subset \mathbb{C}$ is μ-measurable, we set

$$c^2(x, y, A) = \int_A c(x, y, z)^2 \, d\mu(z), \qquad x, y \in \mathbb{C},$$

and, if $A, B, C \subset \mathbb{C}$ are μ-measurable, then

$$c^2(x, A, B) = \int_A \int_B c(x, y, z)^2 \, d\mu(y) \, d\mu(z), \qquad x \in \mathbb{C},$$

and

$$c^2(A, B, C) = \int_A \int_B \int_C c(x, y, z)^2 \, d\mu(x) \, d\mu(y) \, d\mu(z).$$

The total curvature of A (with respect to μ) is defined as

$$c^2(A) = \int_A \int_A \int_A c(x, y, z)^2 \, d\mu(x) \, d\mu(y) \, d\mu(z).$$

Also, we define the curvature operator K as

$$K(f)(x) = \int k(x, y) f(y) \, d\mu(y), \qquad x \in \mathbb{C}, \ f \in Ł^1_{\text{loc}}(\mu),$$

where $k(x, y)$ is the kernel

$$k(x, y) = \int c(x, y, z)^2 \, d\mu(z) = c^2(x, y, \mathbb{C}), \qquad x, y \in \mathbb{C}.$$

For a μ-measurable $A \subset \mathbb{C}$, we set

$$K_A(f)(x) = \int c^2(x, y, A) f(y)\, d\mu(y), \qquad x \in \mathbb{C},\ f \in Ł^1_{\mathrm{loc}}(\mu).$$

Thus,

$$K(f) = K_A(f) + K_{\mathbb{C}\setminus A}(f).$$

We say that μ satisfies the *weak local curvature condition* if there are constants $0 < \alpha \le 1$ and C_2 such that for each disc Δ centered at some point of $\mathrm{spt}(\mu)$, there exists a compact subset $S \subset \Delta$ such that

$$\mu(S) \ge \alpha\, \mu(\Delta) \quad \text{and} \quad c^2(S) \le C_2 \mu(S). \tag{7}$$

In this section we prove the following result.

THEOREM 2.1. *Let μ be a positive Radon measure with linear growth that satisfies the weak local curvature condition. Then K is bounded from $L^p(\mu)$ to $L^p(\mu)$, $1 < p < \infty$, and from $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$.*

COROLLARY 2.2. *Let μ be a positive Radon measure with linear growth that satisfies the weak local curvature condition. Then there exists a constant C such that for all μ-measurable sets $A, B \subset \mathbb{C}$,*

$$c^2(A, B, \mathbb{C}) \le C\sqrt{\mu(A)\mu(B)}.$$

In particular,

$$c^2(A) \le C\mu(A).$$

Proof. Since K is of strong-type (2,2),

$$\begin{aligned} c^2(A, B, \mathbb{C}) &= \int_A K(\chi_B)\, d\mu \\ &\le \|\chi_A\|_{L^2(\mu)} \|K(\chi_B)\|_{L^2(\mu)} \\ &\le C\sqrt{\mu(A)\mu(B)}. \end{aligned}$$ □

From Corollary 2.2 it follows that if μ has linear growth, the local and the weak local curvature conditions are equivalent.

Some remarks about Theorem 2.1 are in order. The proof of the L^p-boundedness of the curvature operator is based on a "good λ-inequality." The fact that K is a positive operator seems to be essential to proving the L^p-boundedness of K without assuming that μ is a doubling measure. Recall that in [S1], [S2], and [SW] the boundedness of some *positive* operators in $L^p(\mu)$ is studied without assuming that μ is doubling, too. Our proof is inspired by these papers.

We consider the centered Hardy-Littlewood maximal operator

$$M_\mu(f)(x) = \sup_{r>0} \frac{1}{\mu\big(\Delta(x,r)\big)} \int_{\Delta(x,r)} |f|\, d\mu.$$

As is well known, M_μ is bounded on $L^p(\mu)$ for all $p \in (1, \infty)$ and from $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$. This follows from the usual argument, by the Besicovitch covering lemma (see [Ma1, p. 40]).

We now prove some lemmas for the proof of Theorem 2.1.

LEMMA 2.3. *If μ has linear growth with constant C_0 and $f \in L^1_{\mathrm{loc}}(\mu)$, then for all $x \in \mathbb{C}$ and $d > 0$ we have*

$$\int_{|x-y|\geq d} \frac{1}{|x-y|^2} |f(y)|\, d\mu(y) \leq \frac{4C_0}{d} M_\mu f(x).$$

In particular,

$$\int_{|x-y|\geq d} \frac{1}{|x-y|^2}\, d\mu(y) \leq \frac{4C_0}{d}.$$

The proof of this lemma is straightforward. One has only to integrate on annuli centered at x. See [D1, Lemma 3], for example.

The next lemma shows how the Menger curvature of three points changes as one of these points moves. Before stating the lemma, let us remark that if $x, y, z \in \mathbb{C}$ are three pairwise different points, then elementary geometry shows that

$$c(x, y, z) = \frac{2\,d(x, L_{yz})}{|x-y||x-z|},$$

where $d(x, L_{yz})$ stands for the distance from x to the straight line L_{yz} passing through y, z.

LEMMA 2.4. *Let $x, y, z \in \mathbb{C}$ be three pairwise different points, and let $x' \in \mathbb{C}$ be such that*

$$C^{-1}|x-y| \leq |x'-y| \leq C|x-y|, \tag{8}$$

where $C > 0$ is some constant. Then

$$\left|c(x, y, z) - c\big(x', y, z\big)\right| \leq (4+2C)\frac{\left|x-x'\right|}{|x-y||x-z|}. \tag{9}$$

Proof. Since $x \neq y$, we have $x' \neq y$ by (8). If $x' = z$, then $c(x', y, z) = 0$. In this case, (9) is straightforward:

$$\left|c(x, y, z) - c\big(x', y, z\big)\right| = c(x, y, z) \leq \frac{2}{|x-y|} = 2\frac{\left|x-x'\right|}{|x-y||x-z|}.$$

For $x' \neq y$ and $x' \neq z$, we have

$$\begin{aligned}
\left|c(x,y,z)-c\big(x',y,z\big)\right| &= \left|\frac{2\,d(x,L_{yz})}{|x-y||x-z|}-\frac{2\,d\big(x',L_{yz}\big)}{|x'-y||x'-z|}\right| \\
&= 2\left|\frac{d(x,L_{yz})|x'-y||x'-z|-d(x',L_{yz})|x-y||x-z|}{|x-y||x-z||x'-y||x'-z|}\right| \\
&\leq 2\frac{|d(x,L_{yz})-d(x',L_{yz})|\,|x'-y||x'-z|}{|x-y||x-z||x'-y||x'-z|} \\
&\quad +2\,d\big(x',L_{yz}\big)\left|\frac{|x'-y||x'-z|-|x-y||x-z|}{|x-y||x-z||x'-y||x'-z|}\right| \\
&= A+B. \qquad (10)
\end{aligned}$$

To estimate the term A, notice that $|d(x,L_{yz})-d(x',L_{yz})| \leq |x-x'|$, and so

$$A \leq 2\frac{|x-x'|}{|x-y||x-z|}.$$

We turn now to the term B in (10). We have

$$\begin{aligned}
&\left|\,\big|x'-y\big|\big|x'-z\big|-|x-y||x-z|\,\right| \\
&\qquad = \left|\,(|x'-y|-|x-y|)\,|x'-z|+\big(|x'-z|-|x-z|\big)\,|x-y|\,\right| \\
&\qquad \leq \big|x'-x\big|\big|x'-z\big|+\big|x'-x\big||x-y|.
\end{aligned}$$

Thus, using that $d(x',L_{yz}) \leq |x'-y|$ and $d(x',L_{yz}) \leq |x'-z|$, we obtain

$$\begin{aligned}
B &\leq 2\big|x-x'\big|\left(\frac{d\big(x',L_{yz}\big)\big|x'-z\big|}{|x-y||x-z||x'-y||x'-z|}+\frac{d\big(x',L_{yz}\big)|x-y|}{|x-y||x-z||x'-y||x'-z|}\right) \\
&\leq 2\frac{\big|x-x'\big|}{|x-y||x-z|}+2\frac{\big|x-x'\big|}{|x'-y||x-z|} \\
&\leq (2+2C)\,\frac{\big|x-x'\big|}{|x-y||x-z|}.
\end{aligned}$$

Now, adding the inequalities obtained for A and B, we get (9). □

The following lemma is a kind of "maximum principle" for Menger curvature, which is essential in the proof of Theorem 2.1.

LEMMA 2.5. *Let us suppose that μ has linear growth with constant C_0. Let $A, B \subset \mathbb{C}$ be μ-measurable, and assume that for some β we have*

$$c^2(x,A,B) \leq \beta, \qquad x \in A.$$

Then there exists a constant β' depending on C_0 and β such that

$$c^2(x,A,B) \leq \beta', \qquad x \in \mathbb{C}. \qquad (11)$$

Proof. It is enough to prove the lemma assuming that A is compact. Otherwise, we can consider an increasing sequence of compact sets $A_n \subset A$ such that $\mu(A \setminus \bigcup_{n=1}^{\infty} A_n) = 0$. Then we have $c^2(x, A_n, B) \leq c^2(x, A, B) \leq \beta$, for all $x \in A_n$. Applying the lemma to A_n, it follows that $c^2(x, A_n, B) \leq \beta'$, for all $x \in \mathbb{C}$. Hence, by monotone convergence, we conclude that $c^2(x, A, B) = \lim_{n\to\infty} c^2(x, A_n, B) \leq \beta'$.

Therefore, we assume that A is compact. We only have to prove inequality (11) when $x \notin A$. Let $r > 0$ be the distance from x to A. Then we split $c^2(x, A, B)$ as

$$\begin{aligned} c^2(x, A, B) = {} & c^2\big(x, A \cap \Delta(x, 4r), B \cap \Delta(x, 4r)\big) \\ & + c^2\big(x, A \cap \Delta(x, 4r), B \setminus \Delta(x, 4r)\big) \\ & + c^2\big(x, A \setminus \Delta(x, 4r), B \cap \Delta(x, 4r)\big) \\ & + c^2\big(x, A \setminus \Delta(x, 4r), B \setminus \Delta(x, 4r)\big). \end{aligned} \tag{12}$$

We estimate each term on the right-hand side of (12) separately. For the first, using $c(x, y, z) \leq 2|y - x|^{-1} \leq 4r^{-1}$, we have

$$\begin{aligned} c^2\big(x, A \cap \Delta(x, 4r), B \cap \Delta(x, 4r)\big) & \leq \frac{4}{r^2} \int_{|y-x|\leq 4r} \int_{|z-x|\leq 4r} d\mu(y) d\mu(z) \\ & \leq \frac{4}{r^2} (C_0 4r)^2 \\ & = 64 C_0^2. \end{aligned}$$

For the second term in (12), we use Lemma 2.3:

$$\begin{aligned} c^2\big(x, A \cap \Delta(x, 4r), B \setminus \Delta(x, 4r)\big) & \leq \int_{y \in A \cap \Delta(x,4r)} \int_{|z-x|>4r} \frac{4}{|z-x|^2} \, d\mu(y) \, d\mu(z) \\ & \leq \int_{y \in A \cap \Delta(x,4r)} \frac{C}{r} \, d\mu(y) \\ & \leq C. \end{aligned}$$

The third term is estimated like the second one with z replaced by y.

Finally, we consider the last term in (12). By the definition of r, there exists a point $x' \in A \cap \Delta(x, 2r)$. Then, if $y \in A \setminus \Delta(x, 4r)$ and $z \in B \setminus \Delta(x, 4r)$, we have by Lemma 2.4:

$$c(x, y, z) \leq c\big(x', y, z\big) + C \frac{r}{|y-x||z-x|}.$$

Hence,

$$\begin{aligned} & c^2\big(x, A \setminus \Delta(x, 4r), B \setminus \Delta(x, 4r)\big) \\ & \qquad \leq C \int_{y \in A \setminus \Delta(x,4r)} \int_{z \in B \setminus \Delta(x,4r)} \frac{r^2}{|y-x|^2 |z-x|^2} \, d\mu(y) \, d\mu(z) \\ & \qquad \quad + 2 \int_{y \in A \setminus \Delta(x,4r)} \int_{z \in B \setminus \Delta(x,4r)} c(x', y, z)^2 \, d\mu(y) \, d\mu(z) \end{aligned}$$

$$
\begin{aligned}
&\le Cr^2 \left(\int_{|y-x|>4r} \frac{1}{|y-x|^2}\, d\mu(y) \right) \left(\int_{|z-x|>4r} \frac{1}{|z-x|^2}\, d\mu(z) \right) \\
&\quad + 2c^2\big(x', A, B\big) \\
&\le C + 2\beta,
\end{aligned}
$$

where the last inequality follows from the fact that $x' \in A$. The proof is complete. □

LEMMA 2.6. *Let μ be a positive Radon measure that has linear growth (with constant C_0) and satisfies the weak local curvature condition (7) (with constants $0 < \alpha \le 1$ and C_2). Then there exists some constant β depending on C_0, C_2, and α such that for each disc Δ centered at some point of* $\operatorname{spt}(\mu)$ *there exists a μ-measurable subset $S \subset \Delta$ such that*

$$
\mu(S) \ge \frac{\alpha}{4}\mu(\Delta) \quad \textit{and} \quad c^2(x, S, \Delta) \le \beta, \quad x \in \mathbb{C}.
$$

Proof. Let Δ be a disc centered at some point of $\operatorname{spt}(\mu)$. Because of (7) there exists a subset $S_0 \subset \Delta$ such that

$$
\mu(S_0) \ge \alpha\mu(\Delta) \quad \text{and} \quad c^2(S_0) \le C_2\mu(S_0).
$$

By Chebyshev, there exists a subset $S_1 \subset S_0$ such that

$$
\mu(S_1) \ge \frac{1}{2}\mu(S_0) \ge \frac{\alpha}{2}\mu(\Delta)
$$

and

$$
c^2(x, S_0, S_0) \le 2C_2, \qquad x \in S_1.
$$

Then, for all $x \in S_1$, we have $c^2(x, S_1, S_1) \le 2C_2$. Applying Lemma 2.5, we conclude that there is some constant C_2' such that $c^2(x, S_1, S_1) \le C_2'$, for all $x \in \mathbb{C}$. Therefore,

$$
c^2(S_1, S_1, \Delta) \le C_2'\mu(\Delta) \le \frac{2C_2'}{\alpha}\mu(S_1).
$$

Applying Chebyshev again, we find a subset $S \subset S_1$ such that

$$
\mu(S) \ge \frac{1}{2}\mu(S_1) \ge \frac{\alpha}{4}\mu(\Delta)
$$

and

$$
c^2(x, S_1, \Delta) \le \frac{4C_2'}{\alpha}, \qquad x \in S.
$$

Thus, for all $x \in S$, we have $c^2(x, S, \Delta) \le 4C_2'/\alpha$, and by Lemma 2.5 we get some constant β such that $c^2(x, S, \Delta) \le \beta$, for all $x \in \mathbb{C}$. □

LEMMA 2.7. *Suppose μ has linear growth with constant C_0. Let $Q \subset \mathbb{C}$ be a square and $f \in L^1_{\text{loc}}(\mu)$, $f \geq 0$. Set*

$$f_1 = f\chi_{2Q} \quad \textit{and} \quad f_2 = f - f_1.$$

Let $\varepsilon > 0$ and $R > 2$ be given. Then there exists some constant $C_{R,\varepsilon}$, depending on R, ε, and C_0, such that for all $x, x_0 \in Q$ and for all $\omega \in RQ \setminus 2Q$ we have

$$K(f_2)(x) \leq (1+\varepsilon)K(f)(\omega) + C_{R,\varepsilon} M_\mu(f)(x_0).$$

Proof. We split $K(f_2)(x)$ as

$$\begin{aligned}
K(f_2)(x) &= \int_{y\notin 2Q}\int_{z\in\mathbb{C}} c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z) \\
&= \int_{y\notin 2Q}\int_{z\in 3RQ} c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z) \\
&\quad + \int_{y\notin 2Q}\int_{z\notin 3RQ} c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z).
\end{aligned} \tag{13}$$

To estimate the first integral on the right-hand side of (13), we apply Lemma 2.3:

$$\begin{aligned}
&\int_{y\notin 2Q}\int_{z\in 3RQ} c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z) \\
&\quad \leq C\int_{z\in 3RQ}\left(\int_{|y-x_0|>\left(l(Q)/2\right)} \frac{1}{|y-x_0|^2} f(y)\,d\mu(y)\right)d\mu(z) \\
&\quad \leq C\int_{z\in 3RQ} \frac{1}{l(Q)} M_\mu f(x_0)\,d\mu(z) \\
&\quad \leq C M_\mu f(x_0),
\end{aligned} \tag{14}$$

where $l(Q)$ is the side length of Q.

We split the second integral on the right-hand side of (13) into two parts:

$$\begin{aligned}
&\int_{y\notin 2Q}\int_{z\notin 3RQ} c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z) \\
&\quad = \int_{y\in 3RQ\setminus 2Q}\int_{z\notin 3RQ} c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z) \\
&\quad\quad + \int_{y\notin 3RQ}\int_{z\notin 3RQ} c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z).
\end{aligned} \tag{15}$$

To estimate the first integral on the right-hand side of (15), we apply Lemma 2.3 again:

$$\begin{aligned}
\int_{y\in 3RQ\setminus 2Q}\int_{z\notin 3RQ} & c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z) \\
&\leq C\int_{y\in 3RQ\setminus 2Q} f(y)\left(\int_{|z-x_0|>2l(Q)}\frac{1}{|z-x_0|^2}\,d\mu(z)\right)d\mu(y)\\
&\leq C\int_{y\in 3RQ\setminus 2Q} f(y)\frac{1}{l(Q)}\,d\mu(y)\\
&\leq C M_\mu f(x_0). \qquad (16)
\end{aligned}$$

Finally, we only have to estimate the second integral on the right-hand side of (15). If $y,z\notin 3RQ$, since $x,x_0,\omega\in RQ$, we can apply Lemma 2.4 and obtain

$$|c(x,y,z)-c(\omega,y,z)|\leq C\frac{l(Q)}{|y-x_0||z-x_0|}.$$

Then

$$\begin{aligned}
c(x,y,z)^2 &\leq \left(c(\omega,y,z)+C\frac{l(Q)}{|y-x_0||z-x_0|}\right)^2\\
&\leq (1+\varepsilon)\,c(\omega,y,z)^2 + C\big(1+\varepsilon^{-1}\big)\left(\frac{l(Q)}{|y-x_0||z-x_0|}\right)^2.
\end{aligned}$$

Therefore, applying Lemma 2.3 twice,

$$\begin{aligned}
\int_{y\notin 3RQ}\int_{z\notin 3RQ} & c(x,y,z)^2 f(y)\,d\mu(y)\,d\mu(z)\\
&\leq (1+\varepsilon)\int_{y\notin 3RQ}\int_{z\notin 3RQ} c(\omega,y,z)^2 f(y)\,d\mu(y)\,d\mu(z)\\
&\quad + C\big(1+\varepsilon^{-1}\big)\int_{y\notin 3RQ}\int_{z\notin 3RQ}\frac{l(Q)^2}{|y-x_0|^2|z-x_0|^2} f(y)\,d\mu(y)\,d\mu(z)\\
&\leq (1+\varepsilon)Kf(\omega)+C\big(1+\varepsilon^{-1}\big)l(Q)^2\\
&\quad \times\int_{|y-x_0|>2l(Q)}\frac{1}{|y-x_0|^2}f(y)\,d\mu(y)\int_{|z-x_0|>2l(Q)}\frac{1}{|z-x_0|^2}\,d\mu(z)\\
&\leq (1+\varepsilon)Kf(\omega)+C\big(1+\varepsilon^{-1}\big)M_\mu f(x_0). \qquad (17)
\end{aligned}$$

Adding inequalities (14), (16), and (17), the lemma is proved. □

It is easy to check that if $f\geq 0$, then $K(f)$ is lower-semicontinuous. That is, the set $\Omega_\lambda := \{x\in\mathbb{C} : K(f)(x)>\lambda\}$ is open for each λ. Our proof of Theorem 2.1 uses Whitney's decomposition of this open set. In the next lemma, we state the precise version of the decomposition we need.

LEMMA 2.8. *If $\Omega \subset \mathbb{C}$ is open, $\Omega \neq \mathbb{C}$, then Ω can be decomposed as*

$$\Omega = \bigcup_{k=1}^{\infty} Q_k,$$

where Q_k are dyadic closed squares with disjoint interiors such that for some constants $R > 20$ and $D \geq 1$, the following hold:

(i) $20Q_k \subset \Omega$;

(ii) $RQ_k \cap \Omega^c \neq \emptyset$;

(iii) *for each square Q_k, there are at most D squares Q_j such that $10Q_k \cap 10Q_j \neq \emptyset$.*

Moreover, if μ is a positive Radon measure on $\mathbb{C}$ and $\mu(\Omega) < +\infty$, we can choose some subfamily $\{Q_{s_i}\}_{i\geq 1}$ of $\{Q_k\}_{k\geq 1}$ such that $10Q_{s_i} \cap 10Q_{s_j} = \emptyset$, if $i \neq j$, and

$$\mu\left(\bigcup_{i=1}^{\infty} Q_{s_i}\right) \geq \frac{1}{D}\mu(\Omega).$$

Proof. Whitney's decomposition for closed squares satisfying (i), (ii), and (iii) is a well-known result. See, for example, [St1, pp. 167–169] or [S2]. Let us prove that we can choose some subfamily $\{Q_{s_i}\}_{i\geq 1} \subset \{Q_k\}_{k\geq 1}$ as stated in the lemma. We assume that the squares $\{Q_k\}_{k\geq 1}$ are ordered in such a way that $\mu(Q_m) \geq \mu(Q_n)$, if $m < n$.

We take $Q_{s_1} = Q_1$. By induction, if $Q_{s_1}, \dots, Q_{s_n}$ have been chosen, then we define $Q_{s_{n+1}}$ as the square Q_k such that

$$10Q_k \cap 10Q_{s_1} = \cdots = 10Q_k \cap 10Q_{s_n} = \emptyset \tag{18}$$

and k is minimal with this property. In other words, $Q_{s_{n+1}}$ is the square satisfying (18) of maximal μ-measure.

Observe that, by condition (iii), there are at most $(n-1)D$ squares Q_k such that $10Q_k$ intersects some of the squares $10Q_{s_1}, \dots, 10Q_{s_{n-1}}$. So at least one of the squares $10Q_1, \dots, 10Q_{(n-1)D+1}$ does not intersect any of the squares $10Q_{s_1}, \dots, 10Q_{s_{n-1}}$. Then s_n must satisfy

$$s_n \leq (n-1)D + 1. \tag{19}$$

Now, since $\{\mu(Q_k)\}_{k\geq 1}$ is nonincreasing, by (19) we have

$$\mu\left(\bigcup_{k=(n-1)D+1}^{nD} Q_k\right) \leq D\mu\big(Q_{(n-1)D+1}\big) \leq D\mu\big(Q_{s_n}\big).$$

Therefore,

$$\mu\left(\bigcup_{i=1}^{\infty} Q_i\right) \leq \sum_{n=1}^{\infty} \mu\left(\bigcup_{k=(n-1)D+1}^{nD} Q_k\right) \leq D\sum_{n=1}^{\infty} \mu\big(Q_{s_n}\big).$$

□

Proof of Theorem 2.1. We prove that there exists $0 < \eta < 1$ such that for all $\varepsilon > 0$ there is some $\delta > 0$ for which the following "good λ-inequality" holds:

$$\mu\{x : K(f)(x) > (1+\varepsilon)\lambda, M_\mu f(x) \le \delta\lambda\} \le (1-\eta)\mu\{x : K(f)(x) > \lambda\}, \tag{20}$$

for $f \in L^p(\mu)$, $f \ge 0$. Let $\Omega_\lambda = \{x \in \mathbb{C} : Kf(x) > \lambda\}$. Then Ω_λ is open. Decompose Ω_λ as

$$\Omega_\lambda = \bigcup_{k=1}^{\infty} Q_k,$$

according to Lemma 2.8. Choose a subfamily $\{Q_i\}_{i\in I}$ of $\{Q_k\}_{k\ge1}$ in such a way that $10Q_i \cap 10Q_j = \emptyset$, if $i, j \in I$, $i \ne j$, and

$$\mu\left(\bigcup_{i\in I} Q_i\right) \ge \frac{1}{D}\mu(\Omega_\lambda).$$

We denote by I_1 the set of indices $i \in I$ such that there exists some $x_i \in Q_i \cap \operatorname{spt}\mu$ so that $M_\mu f(x_i) \le \delta\lambda$. Also, we denote $I_2 = I \setminus I_1$. For each $i \in I_1$, we define $\Delta_i = \Delta(x_i, 3l(Q_i))$. Then we have

$$Q_i \subset 3Q_i \subset \Delta_i \subset 10Q_i \subset \Omega_\lambda.$$

On the other hand, since the weak local curvature condition holds, by Lemma 2.6 we know that there is some constant β such that for each disc Δ_i, $i \in I_1$, there exists a μ-measurable subset $S_i \subset \Delta_i$ such that

$$\mu(S_i) \ge \frac{\alpha}{4}\mu(\Delta_i), \qquad c^2(x, S_i, \Delta_i) \le \beta, \quad \text{for all } x \in \mathbb{C}, \tag{21}$$

where β depends on C_2, C_0, and α. For $i \in I_2$, we define $\Delta_i = S_i = Q_i$.

Since $\Delta_i \cap \Delta_j = \emptyset$, if $i, j \in I$, $i \ne j$, we have

$$\begin{aligned}\mu\left(\bigcup_{i\in I} S_i\right) &\ge \frac{\alpha}{4}\sum_{i\in I}\mu(\Delta_i)\\ &\ge \frac{\alpha}{4}\sum_{i\in I}\mu(Q_i)\\ &\ge \frac{\alpha}{4D}\mu(\Omega_\lambda).\end{aligned}$$

Therefore,

$$\mu\left(\Omega_\lambda \setminus \bigcup_{i\in I} S_i\right) \le \left(1 - \frac{\alpha}{4D}\right)\mu(\Omega_\lambda). \tag{22}$$

We prove below that for each $i \in I_1$ we have

$$\mu\big(S_i \cap \big\{x : Kf(x) > (1+\varepsilon)\lambda\big\}\big) \leq \frac{\alpha}{8D}\mu(\Delta_i), \tag{23}$$

if $\delta = \delta(\varepsilon, \alpha)$ is small enough. Then, by (22) and (23),

$$\begin{aligned}
\mu\{Kf > (1+\varepsilon)\lambda, M_\mu f \leq \delta\lambda\} &\leq \mu\left(\Omega_\lambda \setminus \bigcup_{i\in I} S_i\right) \\
&\quad + \mu\left(\left(\bigcup_{i\in I} S_i\right) \cap \{Kf > (1+\varepsilon)\lambda, M_\mu f \leq \delta\lambda\}\right) \\
&\leq \left(1 - \frac{\alpha}{4D}\right)\mu(\Omega_\lambda) + \sum_{i\in I_1} \mu\big(S_i \cap \big\{Kf > (1+\varepsilon)\lambda\big\}\big) \\
&\leq \left(1 - \frac{\alpha}{4D}\right)\mu(\Omega_\lambda) + \frac{\alpha}{8D}\sum_{i\in I_1} \mu(\Delta_i) \\
&\leq \left(1 - \frac{\alpha}{4D} + \frac{\alpha}{8D}\right)\mu(\Omega_\lambda) \\
&= \left(1 - \frac{\alpha}{8D}\right)\mu(\Omega_\lambda).
\end{aligned}$$

Then, taking $\eta = \alpha/(8D)$, (20) follows.

We now prove (23). Observe that, due to (21), for $i \in I_1$, we have

$$K_{\Delta_i}\big(\chi_{S_i}\big)(x) = c^2\big(x, S_i, \Delta_i\big) \leq \beta, \qquad \text{for all } x \in \mathbb{C}.$$

Then, if $i \in I_1$, we have

$$\begin{aligned}
\mu\big\{x \in S_i : K_{\Delta_i}\big(f\chi_{\Delta_i}\big)(x) > (\varepsilon/4)\lambda\big\} &\leq \frac{4}{\varepsilon\lambda}\int_{S_i} K_{\Delta_i}\big(f\chi_{\Delta_i}\big)\,d\mu \\
&= \frac{4}{\varepsilon\lambda}\int f\chi_{\Delta_i} K_{\Delta_i}\big(\chi_{S_i}\big)\,d\mu \\
&\leq \frac{4\beta}{\varepsilon\lambda}\int_{\Delta_i} f\,d\mu \\
&\leq \frac{4\beta}{\varepsilon\lambda}\mu(\Delta_i) M_\mu f(x_i) \\
&\leq \frac{4\beta\delta}{\varepsilon}\mu(\Delta_i) \\
&\leq \frac{\alpha}{8D}\mu(\Delta_i),
\end{aligned} \tag{24}$$

where the last inequality holds provided we choose $\delta \leq (\varepsilon\alpha/32D\beta)$. In fact, we set

$$\delta = \delta(\varepsilon, \alpha) = \min\left\{\frac{\varepsilon}{4C_{R,\varepsilon/4}}, \frac{\varepsilon\alpha}{32D\beta}, \frac{\varepsilon}{4C_3}\right\},$$

where $C_{R,\varepsilon/4}$ is the constant given by Lemma 2.7 and C_3 is some constant, which we define later.

Due to the properties of Whitney's decomposition, there exists a point ω_i in $(RQ_i \setminus 2Q_i) \cap \Omega_\lambda^c$, and if moreover $i \in I_1$, then by Lemma 2.7,

$$\begin{aligned} K\big(f\chi_{\mathbb{C}\setminus 2Q_i}\big)(x) &\le \Big(1+\frac{\varepsilon}{4}\Big) Kf(\omega_i) + C_{R,\varepsilon/4} M_\mu f(x_i) \\ &\le \Big(1+\frac{\varepsilon}{4}\Big)\lambda + C_{R,\varepsilon/4}\delta\lambda \le \Big(1+\frac{\varepsilon}{2}\Big)\lambda. \end{aligned}$$

Consequently, if $x \in Q_i$, $i \in I_1$, and $Kf(x) > (1+\varepsilon)\lambda$, then

$$K\big(f\chi_{2Q_i}\big)(x) > \frac{\varepsilon}{2}\lambda. \tag{25}$$

We prove that

$$K_{\mathbb{C}\setminus\Delta_i}\big(f\chi_{2Q_i}\big)(x) \le \frac{\varepsilon}{4}\lambda, \qquad x \in Q_i,\, i \in I_1. \tag{26}$$

Since $K = K_{\Delta_i} + K_{\mathbb{C}\setminus\Delta_i}$, (25) and (26) give

$$K_{\Delta_i}\big(f\chi_{\Delta_i}\big)(x) \ge K_{\Delta_i}\big(f\chi_{2Q_i}\big)(x) > \frac{\varepsilon}{4}\lambda, \tag{27}$$

provided $x \in Q_i$, $i \in I_1$, and $Kf(x) > (1+\varepsilon)\lambda$. Now (23) is a consequence of (24).

Let us check that (26) holds:

$$\begin{aligned} &K_{\mathbb{C}\setminus\Delta_i}\big(f\chi_{2Q_i}\big)(x) \\ &\quad\le C\int_{y\in 2Q_i} f(y)\left(\int_{z\notin\Delta_i} \frac{1}{|y-z|^2}\,d\mu(z)\right) d\mu(y) \\ &\quad\le C\int_{y\in 2Q_i} f(y)\left(\int_{|y-z|\ge l(Q_i)/2} \frac{1}{|y-z|^2}\,d\mu(z)\right) d\mu(y) \quad \text{(because } 3Q_i \subset \Delta_i\text{)} \\ &\quad\le C\frac{1}{l(Q_i)}\int_{y\in 2Q_i} f(y)\,d\mu(y) \quad \text{(by Lemma 2.3)} \\ &\quad\le C_3 M_\mu f(x_i) \quad \text{(defining } C_3 \text{ appropriately)} \\ &\quad\le \frac{\varepsilon}{4}\lambda \quad \text{(by the choice of } \delta\text{)}. \end{aligned}$$

Hence (20) follows.

It is well known (see [St2, p. 152] or [D2, p. 60]) that from (20), one gets, for $p \in (1,+\infty)$,

$$\|Kf\|_{L^p(\mu)} \le C_p\|M_\mu f\|_{L^p(\mu)} \le C'_p\|f\|_{L^p(\mu)}, \tag{28}$$

provided $f \ge 0$ and

$$\inf(1, K(f)) \in L^p(\mu). \tag{29}$$

Let us see that if f has compact support and is bounded by some constant A, then (29) holds. Suppose that $\operatorname{spt}(f) \subset \Delta(0,R)$. Then $K(f) \leq A \cdot K(\chi_{\Delta(0,R)})$. For $x \in \mathbb{C} \setminus \Delta(0,2R)$, with $d = |x|$, we have

$$\begin{aligned} &K\big(\chi_{\Delta(0,R)}\big)(x) \\ &= c^2\big(x, \Delta(0,R), \mathbb{C}\big) = c^2(x, \Delta(0,R), \Delta(0,2d)\big) + c^2\big(x, \Delta(0,R), \mathbb{C} \setminus \Delta(0,2d)\big). \end{aligned} \tag{30}$$

Now we estimate the first term on the right-hand side:

$$\begin{aligned} c^2\big(x, \Delta(0,R), \Delta(0,2d)\big) &\leq \int_{y\in\Delta(0,R)} \int_{z\in\Delta(0,2d)} \frac{4}{|y-x|^2}\, d\mu(y)\, d\mu(z) \\ &\leq C \int_{y\in\Delta(0,R)} \int_{z\in\Delta(0,2d)} \frac{1}{d^2}\, d\mu(y)\, d\mu(z) \\ &\leq C \frac{1}{d} \int_{y\in\Delta(0,R)} d\mu(y) \\ &= C \frac{1}{d} \int_{y\in\Delta(x,2d)} \chi_{\Delta(0,R)}(y)\, d\mu(y) \\ &\leq C M_\mu \chi_{\Delta(0,R)}(x). \end{aligned}$$

The second term on the right-hand side of (30) is estimated as

$$\begin{aligned} c^2\big(x, \Delta(0,R), \mathbb{C} \setminus \Delta(0,2d)\big) &\leq \int_{y\in\Delta(0,R)} \int_{|z-x|>d} \frac{4}{|z-x|^2}\, d\mu(y)\, d\mu(z) \\ &\leq C \frac{1}{d} \int_{y\in\Delta(0,R)} d\mu(y) \\ &= C \frac{1}{d} \int_{y\in\Delta(x,2d)} \chi_{\Delta(0,R)}(y)\, d\mu(y) \\ &\leq C M_\mu \chi_{\Delta(0,R)}(x). \end{aligned}$$

Thus,

$$\inf\big(1, K(f)\big) \leq \inf\big(1, A \cdot K(\chi_{\Delta(0,R)})\big) \leq \chi_{\Delta(0,2R)} + A \cdot C M_\mu \chi_{\Delta(0,R)},$$

and so $\inf(1, K(f)) \in L^p(\mu)$. Now, (28) holds for $f \geq 0$ bounded with compact support, and a routine argument shows that K is bounded on $L^p(\mu)$.

The boundedness of K from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$ can be proved as follows. One checks easily that (20) also holds for a positive finite measure $\nu \in M(\mathbb{C})$:

$$\mu\big\{x : K(\nu)(x) > (1+\varepsilon)\lambda,\, M_\mu\nu(x) \leq \delta\lambda\big\} \leq (1-\eta)\mu\{x : K(\nu)(x) > \lambda\},$$

where $K(\nu)$ and $M_\mu\nu$ are defined in the obvious way. As before, this inequality yields the desired weak (1,1)-estimate. □

The curvature operator K behaves like a Calderón-Zygmund operator. Moreover, it enjoys the very nice property of being a positive operator. If μ is a doubling measure with linear growth that satisfies the local curvature condition, it can be checked that K is bounded from $H^1_{at}(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to BMO(μ). Then, by interpolation, K is bounded on $L^p(\mu)$, for $p \in (1, \infty)$.

3. Relation of curvature with the Cauchy integral and analytic capacity. The relation between the Cauchy integral and curvature stems from the following formula [Me2], whose proof is a simple calculation (see [MV], for example).

LEMMA 3.1. *Let $x, y, z \in \mathbb{C}$ be three pairwise different points. Then*

$$c(x, y, z)^2 = 2\,\mathrm{Re}\left(\frac{1}{(y-x)\overline{(z-x)}} + \frac{1}{(z-y)\overline{(x-y)}} + \frac{1}{(x-z)\overline{(y-z)}}\right). \tag{31}$$

For any μ-measurable set $A \subset \mathbb{C}$, we set

$$c^2_\varepsilon(A) = \iiint_{\substack{x,y,z \in A \\ |x-y|>\varepsilon \\ |x-z|>\varepsilon \\ |y-z|>\varepsilon}} c(x, y, z)^2\, d\mu(x)\, d\mu(y)\, d\mu(z).$$

The following lemma is a generalization of an identity proved in [MV].

LEMMA 3.2. *Suppose that μ has linear growth, and consider $h : \mathbb{C} \longrightarrow [0, 1]$ to be μ-measurable. Then, for any μ-measurable set $A \subset \mathbb{C}$, we have*

$$\begin{aligned} 2\int |\mathcal{C}_\varepsilon(\chi_A)|^2 h\, d\mu &= \iiint_{\substack{|x-y|>\varepsilon \\ |x-z|>\varepsilon \\ |y-z|>\varepsilon}} c(x, y, z)^2 \chi_A(x)\chi_A(y)h(z)\, d\mu(x)\, d\mu(y)\, d\mu(z) \\ &\quad - 4\,\mathrm{Re}\int_A \mathcal{C}_\varepsilon(\chi_A)\overline{\mathcal{C}_\varepsilon(h)}\, d\mu + O\big(\mu(A)\big) \end{aligned} \tag{32}$$

and

$$c^2_\varepsilon(A) = 6\int_A \big|\mathcal{C}_\varepsilon(\chi_A)\big|^2 d\mu + O\big(\mu(A)\big). \tag{33}$$

Proof. By (31),

$$\iiint_{\substack{|x-y|>\varepsilon \\ |y-z|>\varepsilon \\ |z-x|>\varepsilon}} c(x, y, z)^2 \chi_A(x)\chi_A(y)h(z)\, d\mu(x)\, d\mu(y)\, d\mu(z)$$

$$
\begin{aligned}
&= 2 \iiint_{\substack{|x-y|>\varepsilon \\ |y-z|>\varepsilon \\ |z-x|>\varepsilon}} \chi_A(x)\chi_A(y)h(z) \\
&\quad \times \operatorname{Re}\left(\frac{1}{(y-x)(\overline{z-x})} + \frac{1}{(z-y)(\overline{x-y})} + \frac{1}{(x-z)(\overline{y-z})} \right) d\mu(x)\,d\mu(y)\,d\mu(z) \\
&= I + II + III. \qquad (34)
\end{aligned}
$$

We first estimate the integral I:

$$
\begin{aligned}
I &= 2\operatorname{Re} \iiint_{\substack{|x-y|>\varepsilon \\ |z-x|>\varepsilon}} \chi_A(x)\chi_A(y)h(z) \frac{1}{(y-x)(\overline{z-x})}\, d\mu(x)\,d\mu(y)\,d\mu(z) \\
&\quad - 2\operatorname{Re} \iiint_{\substack{|x-y|>\varepsilon \\ |z-x|>\varepsilon \\ |y-z|\le\varepsilon}} \chi_A(x)\chi_A(y)h(z) \frac{1}{(y-x)(\overline{z-x})}\, d\mu(x)\,d\mu(y)\,d\mu(z) \\
&= I_1 + I_2.
\end{aligned}
$$

We clearly have

$$
\begin{aligned}
I_1 &= 2\operatorname{Re}\left[\int_A \left(\int_{|z-x|>\varepsilon} \frac{\overline{h(z)}}{\overline{z-x}}\, d\mu(z) \right) \left(\int_{|x-y|>\varepsilon} \frac{\chi_A(y)}{y-x}\, d\mu(y) \right) d\mu(x) \right] \\
&= 2\operatorname{Re} \int_A \overline{\mathcal{C}_\varepsilon h(x)} \cdot \mathcal{C}_\varepsilon \chi_A(x)\, d\mu(x). \qquad (35)
\end{aligned}
$$

Now we proceed to estimate the integral I_2. Notice that if $|y-z| \le \varepsilon < |z-x|$, then

$$
\left| \frac{1}{(y-x)(\overline{z-x})} - \frac{1}{|y-x|^2} \right| = \left| \frac{\overline{y-z}}{|y-x|^2(\overline{z-x})} \right| \le \frac{1}{|y-x|^2},
$$

and so

$$
\left| \frac{1}{(y-x)(\overline{z-x})} \right| \le \frac{2}{|y-x|^2}.
$$

Thus,

$$
\begin{aligned}
|I_2| &\le 4 \iiint_{\substack{|x-y|>\varepsilon \\ |z-x|>\varepsilon \\ |y-z|\le\varepsilon}} \chi_A(x)\chi_A(y)h(z) \frac{1}{|y-x|^2}\, d\mu(x)\,d\mu(y)\,d\mu(z) \\
&\le C\varepsilon \int \chi_A(x) \left(\int_{|x-y|>\varepsilon} \frac{1}{|y-x|^2}\, d\mu(y) \right) d\mu(x) \\
&\le C\mu(A). \qquad (36)
\end{aligned}
$$

Interchanging the roles of x and y, it easily follows that

$$
II = I. \qquad (37)
$$

We turn now our attention to III:

$$
\begin{aligned}
III &= 2\,\mathrm{Re} \iiint_{\substack{|y-z|>\varepsilon\\ |z-x|>\varepsilon}} \chi_A(x)\chi_A(y)h(z)\frac{1}{(x-z)\overline{(y-z)}}\,d\mu(x)\,d\mu(y)\,d\mu(z)\\
&\quad -2\,\mathrm{Re} \iiint_{\substack{|y-z|>\varepsilon\\ |z-x|>\varepsilon\\ |x-y|\le\varepsilon}} \chi_A(x)\chi_A(y)h(z)\frac{1}{(x-z)\overline{(y-z)}}\,d\mu(x)\,d\mu(y)\,d\mu(z)\\
&= III_1 + III_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
III_1 &= 2\,\mathrm{Re}\left[\int\left(\int_{|z-x|>\varepsilon}\frac{\chi_A(x)}{x-z}\,d\mu(x)\right)\left(\int_{|y-z|>\varepsilon}\frac{\chi_A(y)}{\overline{(y-z)}}\,d\mu(y)\right)h(z)\,d\mu(z)\right]\\
&= 2\,\mathrm{Re}\int |\mathscr{C}_\varepsilon\chi_A(z)|^2 h(z)\,d\mu(z). \qquad (38)
\end{aligned}
$$

We need an estimate for III_2. If $|x-y|\le\varepsilon<|y-z|$, then

$$
\left|\frac{1}{(x-z)\overline{(y-z)}}-\frac{1}{|x-z|^2}\right| = \left|\frac{\overline{x-y}}{|x-z|^2\overline{(y-z)}}\right| \le \frac{1}{|x-z|^2},
$$

and so

$$
\left|\frac{1}{(x-z)\overline{(y-z)}}\right| \le \frac{2}{|x-z|^2}.
$$

Therefore,

$$
\begin{aligned}
|III_2| &\le 4\iiint_{\substack{|y-z|>\varepsilon\\ |z-x|>\varepsilon\\ |x-y|\le\varepsilon}} \chi_A(x)\chi_A(y)h(z)\frac{1}{|x-z|^2}\,d\mu(x)\,d\mu(y)\,d\mu(z)\\
&\le C\varepsilon\int\chi_A(x)\left(\int_{|z-x|>\varepsilon}\frac{1}{|x-z|^2}\,d\mu(z)\right)d\mu(x)\\
&\le C\mu(A). \qquad (39)
\end{aligned}
$$

Because of (35), (36), (37), (38), and (39), we get

$$
\begin{aligned}
I+II+III &= 4\,\mathrm{Re}\int_A \overline{\mathscr{C}_\varepsilon h(x)}\cdot\mathscr{C}_\varepsilon\chi_A(x)\,d\mu(x)\\
&\quad +2\int|\mathscr{C}_\varepsilon\chi_A(z)|^2 h(z)\,d\mu(z) + O\big(\mu(A)\big).
\end{aligned}
$$

From this equation and (34), we finally obtain (32). Identity (33) follows readily from (32) by taking $h=\chi_A$. □

The analytic capacity γ of a compact set $E \subset \mathbb{C}$ is

$$\gamma(E) = \sup_f \left| f'(\infty) \right|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$ such that $|f| \leq 1$, with the notation $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$.

Melnikov proved an inequality that relates curvature to analytic capacity, which we proceed to describe. Let $E \subset \mathbb{C}$ be compact, and assume that E supports a positive Radon measure μ that has linear growth with constant C_0 and such that

$$c^2(E,\mu) \equiv \int_E \int_E \int_E c(x,y,z)^2 \, d\mu(x)\, d\mu(y)\, d\mu(z) < +\infty.$$

Then Melnikov's inequality [Me2] is

$$\gamma(E) \geq C_4 \frac{\mu(E)^{3/2}}{\left(\mu(E) + c^2(E,\mu)\right)^{1/2}}, \tag{40}$$

where $C_4 > 0$ is some constant depending only on C_0. For our purposes, we need a slightly improved version of (40), which we state as Theorem 3.3. Given a compactly supported measure $\nu \in M(\mathbb{C})$, we denote by $\mathscr{C}(\nu)$ the locally integrable function $(1/z) * \nu$.

THEOREM 3.3. *Let $E \subset \mathbb{C}$ be compact, and let μ be a positive Radon measure supported on E that has linear growth with constant C_0 and such that $c^2(E,\mu) < \infty$. Then there exists a complex finite measure ν supported on E such that $\|\nu\| \leq \mu(E)$, $|\mathscr{C}(\nu)(z)| < 1$ for all $z \in E^c$, $|\mathscr{C}(\nu)(z)| \leq 1$ for almost all (with respect to Lebesgue measure) $z \in \mathbb{C}$, and*

$$\left| \int d\nu \right| \geq C_4 \frac{\mu(E)^{3/2}}{\left(\mu(E) + c^2(E,\mu)\right)^{1/2}},$$

where C_4 is some positive constant depending on C_0.

Proof. Given a positive integer n, set $\delta = 1/n$. In [Me2], Melnikov shows that there exist finitely many discs $\Delta_j^{(n)}$, pairwise disjoint, such that $E_n \equiv \bigcup_j \Delta_j^{(n)} \subset V_\delta(E)$ (where $V_\delta(E)$ is the δ-neighborhood of E), $2\pi \sum_j \operatorname{radius}(\Delta_j^{(n)}) \leq \mu(E)$, and

$$\gamma(E_n) \geq C_4 \frac{\mu(E)^{3/2}}{\left(\mu(E) + c^2(E,\mu)\right)^{1/2}}.$$

Let h_n be the Ahlfors function of E_n. (That is, h_n is analytic on the complement of E_n, $|f(z)| \leq 1$, $z \in \mathbb{C} \setminus E_n$, and $f'(\infty) = \gamma(E_n)$.) Set $\nu_n = h_n(z)\, dz$, where $dz = dz_{\Gamma_n}$, $\Gamma_n = \partial E_n$ (a finite union of disjoint circumferences), and $h_n(z), z \in \Gamma_n$, are the boundary values of h_n. Then $\mathcal{C}(\nu_n)(z) = h_n(z), z \in E_n^c$, and $\mathcal{C}(\nu_n)(z) = 0, z \in \mathring{E}_n$. Hence, $|\mathcal{C}(\nu)(z)| \leq 1$, for almost all (with respect to Lebesgue measure) $z \in \mathbb{C}$. Moreover, $\|\nu_n\| \leq \mu(E)$, for each n.

Passing to a subsequence if necessary, we can assume that $\nu_n \to \nu$ in the weak $*$-topology of $M(\mathbb{C})$, for some $\nu \in M(\mathbb{C})$, and we also assume that $\mathcal{C}(\nu_n) \to h$ in the weak $*$-topology of $L^\infty(\mathbb{C})$ (with respect to Lebesgue measure), for some $h \in L^\infty(\mathbb{C})$. Since $\mathcal{C}(\nu_n)$ converges in the sense of distributions to $\mathcal{C}(\nu)$, we get $\mathcal{C}(\nu) = h$.

We clearly have

$$\left|\int d\nu\right| = \left|\lim_{n\to\infty}\int d\nu_n\right| = \lim_{n\to\infty}\gamma(E_n),$$

and so ν fulfills the required conditions. $\square$

4. Boundedness of the Cauchy integral. We state now some known results, which we use in the proof of Theorem 1.1. The following proposition is basic for the study of the boundedness of the Cauchy integral from $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$.

PROPOSITION 4.1. *Let X be a locally compact Hausdorff space, μ be a positive Radon measure on X, and T be a linear operator bounded from the space $M(X)$ of complex finite Radon measures on X into $\mathcal{C}_0(X)$, the space of continuous functions on X vanishing at ∞. Suppose, furthermore, that T^*, the adjoint operator of T, boundedly sends $M(X)$ into $\mathcal{C}_0(X)$. Then the following statements are equivalent.*

(a) *There exists a constant C such that*

$$\mu\bigl\{x : |T^*\nu(x)| > \lambda\bigr\} \leq C\frac{\|\nu\|}{\lambda}, \qquad \forall\lambda > 0,$$

for all $\nu \in M(X)$.

(b) *There exists a constant C such that, for each Borel set $E \subset X$, we have*

$$\mu(E) \leq 2\sup\left\{\int h\,d\mu : h\ \mu\text{-measurable},\ \operatorname{spt}(h) \subset E, 0 \leq h \leq 1, |T(h\,d\mu)(x)| \leq C, \forall x \in X\right\}.$$

(c) *There exists a constant C such that, for each compact set $E \subset X$, we have*

$$\mu(E) \leq C\sup\left\{\left|\int d\nu\right| : \nu \in M(X), \operatorname{spt}(\nu) \subset E, \|\nu\| \leq \mu(E), |T\nu(x)| \leq 1, \forall x \in X\right\}.$$

Moreover, the least constants in (a), (b), (c), which we denote by C_a, C_b, and C_c, satisfy

$$A^{-1}C_a \leq C_b \leq AC_a, \qquad A^{-1}C_a^{1/3} \leq C_c \leq A(C_a+1),$$

for some constant $A > 0$.

The implication (a) $\Rightarrow$ (b) is proved in [Ch, p. 107]. It is trivial that (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a) can be proved as in [Mu, pp. 78–79] or [Ve2]. We give a sketch of the argument for (c) $\Rightarrow$ (a).

Proof. Suppose that (c) holds. Let ρ^{-1} be the constant that appears in (c). For $\lambda > 0$ and $\theta \in [0, 2\pi)$, let

$$E_{\lambda,\theta} = \left\{x \in X : T^*\nu(x) \in \Delta\left(\lambda e^{i\theta}, \rho\lambda/4\right)\right\}.$$

Covering $\mathbb{C}$ by disks $\Delta(\lambda e^{i\theta}, \rho\lambda/4)$, one can see (as in [Mu, pp. 78–79] or [Ve2]) that (a) follows if, for all $\lambda > 0$ and $\theta \in [0, 2\pi)$,

$$\mu(E_{\lambda,\theta}) \leq C\frac{\|\nu\|}{\lambda}. \tag{41}$$

Let us see that (41) holds. There is a compact set $F_{\lambda,\theta} \subset E_{\lambda,\theta}$ such that $\mu(F_{\lambda,\theta}) \geq \mu(E_{\lambda,\theta})/2$. Let $\eta \in M(X)$ be such that $\operatorname{spt}(\eta) \subset F_{\lambda,\theta}$, $\|\eta\| \leq \mu(F_{\lambda,\theta})$, $|T\eta(x)| \leq 1$ for all $x \in X$, and $\mu(F_{\lambda,\theta}) \leq 2\rho^{-1}\left|\int d\eta\right|$. Then

$$\begin{aligned}
\frac{\rho}{2}\lambda\mu(F_{\lambda,\theta}) &\leq \left|\int \lambda e^{i\theta}\,d\eta\right| \\
&\leq \left|\int T^*(\nu)\,d\eta\right| + \left|\int \left(\lambda e^{i\theta} - T^*(\nu)\right)d\eta\right| \\
&\leq \left|\int T(\eta)\,d\nu\right| + \frac{\rho}{4}\lambda\|\eta\| \\
&\leq \|\nu\| + \frac{\rho}{4}\lambda\mu(F_{\lambda,\theta}).
\end{aligned}$$

Therefore, $\mu(E_{\lambda,\theta}) \leq 2\mu(F_{\lambda,\theta}) \leq 8\|\nu\|/(\rho\lambda)$. □

See also [Ve1] for an interesting application of Proposition 4.1.

Remark 4.2. To apply the lemma, we use a standard technique. We replace $\mathcal{C}_\varepsilon$ by the regularized operator $\widetilde{\mathcal{C}}_\varepsilon$, defined as

$$\widetilde{\mathcal{C}}_\varepsilon\nu(x) = \int r_\varepsilon(x-y)\,d\nu(y),$$

where ν is a complex finite measure and where r_ε is the kernel

$$r_\varepsilon(z) = \begin{cases} \dfrac{1}{z}, & \text{if } |z| > \varepsilon, \\[2mm] \dfrac{\bar{z}}{\varepsilon^2}, & \text{if } |z| \leq \varepsilon. \end{cases}$$

Then $\widetilde{\mathcal{C}}_\varepsilon\nu$ is the convolution of the complex measure ν with the uniformly continuous kernel r_ε, and so $\widetilde{\mathcal{C}}_\varepsilon\nu$ is a continuous function.

Also, we have

$$r_\varepsilon(z) = \frac{1}{z} * \frac{\chi_\varepsilon}{\pi\varepsilon^2},$$

where χ_ε is the characteristic function of $\Delta(0,\varepsilon)$. If ν is a compactly supported Radon measure, we have the identity

$$\widetilde{\mathcal{C}}_\varepsilon \nu = \frac{1}{z} * \frac{\chi_\varepsilon}{\pi\varepsilon^2} * \nu = \frac{\chi_\varepsilon}{\pi\varepsilon^2} * \mathcal{C}\nu.$$

Assume now that $|\mathcal{C}(\nu)| \le A$ a.e. with respect to Lebesgue measure. Since

$$\left\| \frac{\chi_\varepsilon}{\pi\varepsilon^2} \right\|_{L^1(\mathbb{C})} = 1,$$

we obtain $|\widetilde{\mathcal{C}}_\varepsilon(\nu)(z)| \le A$, for all $z \in \mathbb{C}$.

Also, notice that

$$\left|\widetilde{\mathcal{C}}_\varepsilon(\nu)(x) - \mathcal{C}_\varepsilon(\nu)(x)\right| = \frac{1}{\varepsilon^2}\left|\int_{|y-x|<\varepsilon} (\overline{y-x})\, d\nu(y)\right| \le C_0 M_\mu \nu(x). \qquad (42)$$

Hence, if μ has linear growth, $\mathcal{C}_\varepsilon$ is of strong-type (p,p) uniformly in ε if and only if $\widetilde{\mathcal{C}}_\varepsilon$ is of strong-type (p,p) uniformly in ε. Also, $\mathcal{C}_\varepsilon$ is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$ uniformly in ε if and only if the same holds for $\widetilde{\mathcal{C}}_\varepsilon$.

For the proof of Theorem 1.1, the following proposition is also necessary.

PROPOSITION 4.3. *Let μ be a positive continuous Radon measure on $\mathbb{C}$. If the Cauchy integral is bounded on $L^p(\mu)$, for some $p \in (1,\infty)$, or if it is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$, then μ has linear growth.*

The proof of this result can be found in [D2, p. 56]. Let us remark that, in fact, in [D2, p. 56] David states only that if the Cauchy integral operator is of strong-type (p,p), for some $p \in (1,\infty)$, then μ has linear growth. With some minor changes in the proof, one can check that if the Cauchy integral is of weak-type (1,1), then μ has linear growth also.

4.1. Proof of the equivalence between the L^2-boundedness of the Cauchy integral and the linear growth condition plus the local curvature condition of μ. If the Cauchy integral operator is of strong-type (2,2), μ has linear growth by Proposition 4.3 and μ satisfies the local curvature condition by (33) of Lemma 3.2.

Assume now that μ has linear growth and satisfies the local curvature condition. First we show that the Cauchy integral operator is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$. That is, we prove the implication (1) $\Rightarrow$ (3) of Theorem 1.1.

We see that the operator $\widetilde{\mathcal{C}}_\varepsilon^*$, the adjoint of $\widetilde{\mathcal{C}}_\varepsilon$, is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$ uniformly in ε, which is equivalent to the uniform boundedness of $\mathcal{C}_\varepsilon : M(\mathbb{C}) \longrightarrow L^{1,\infty}(\mu)$. To do so, we apply Proposition 4.1 to the operator $\widetilde{\mathcal{C}}_\varepsilon^*$, taking $X = \mathbb{C}$. (Notice that $\widetilde{\mathcal{C}}_\varepsilon$ and $\widetilde{\mathcal{C}}_\varepsilon^*$ are bounded from $M(\mathbb{C})$ into $\mathcal{C}_0(\mathbb{C})$, with the norm depending on ε.) We show that statement (c) of Proposition 4.1 holds; that is, for each compact set $E \subset \mathbb{C}$ and each $\varepsilon > 0$, there exists some complex finite measure $\nu \in M(\mathbb{C})$ such that $\mu(E) \leq C \left|\int d\nu\right|$ (with C independent of ε), $\operatorname{spt}(\nu) \subset E$, $\|\nu\| \leq \mu(E)$, and $|\widetilde{\mathcal{C}}_\varepsilon(\nu)(x)| \leq 1$ for all $x \in \mathbb{C}$.

Since μ has linear growth and satisfies the local curvature condition, the curvature operator is bounded on $L^2(\mu)$. Hence, $c^2(E) \leq C\mu(E)$. If we apply Theorem 3.3 to the compact set E and the measure $\mu_{|E}$, we conclude that there exist a constant $C_4 > 0$ and a complex measure $\nu \in M(\mathbb{C})$ such that $\operatorname{spt}(\nu) \subset E$, $\|\nu\| \leq \mu(E)$, $|\mathcal{C}(\nu)(x)| < 1$ for all $x \in E^c$, $|\mathcal{C}(\nu)| \leq 1$ a.e. with respect to Lebesgue measure, and

$$\left|\int d\nu\right| \geq C_4 \frac{\mu(E)^{3/2}}{\left(\mu(E) + c^2(E,\mu)\right)^{1/2}} \geq C_5 \mu(E),$$

with $C_5 > 0$. By Remark 4.2 we have $|\widetilde{\mathcal{C}}_\varepsilon(\nu)(x)| \leq 1$, for all $x \in \mathbb{C}$. The measure ν has all the required properties, and hence the Cauchy integral is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$.

Now, we prove that the Cauchy integral operator is of restricted weak-type (2,2). That is, for each μ-measurable set $A \subset \mathbb{C}$ and all $\lambda > 0$,

$$\mu\left\{x \in \mathbb{C} : \left|\mathcal{C}_\varepsilon(\chi_A)(x)\right| > \lambda\right\} \leq C \frac{\mu(A)}{\lambda^2},$$

where C is some constant independent of ε.

Set

$$E_\lambda = \left\{x \in \mathbb{C} : \left|\mathcal{C}_\varepsilon \chi_A(x)\right| > \lambda\right\}.$$

Since $\widetilde{\mathcal{C}}_\varepsilon^*$ is bounded from $M(\mathbb{C})$ into $L^{1,\infty}(\mu)$ uniformly in ε, by (b) of Proposition 4.1 there exists a constant C such that for all $\varepsilon > 0$ there is a function $h : \mathbb{C} \longrightarrow [0,1]$ such that $h(x) = 0$, if $x \notin E_\lambda$,

$$\mu(E_\lambda) \leq 2\int_{E_\lambda} h\, d\mu, \quad \text{and} \quad \left|\widetilde{\mathcal{C}}_\varepsilon(h)(x)\right| \leq C, \quad \text{for all } x \in \mathbb{C}.$$

By (42) we get $|\mathcal{C}_\varepsilon(h)(x)| \leq C + C_0$, for all $x \in \mathbb{C}$.

Applying (32) we obtain

$$\mu(E_\lambda) \leq 2\int_{E_\lambda} h\, d\mu \leq \frac{2}{\lambda^2} \int |\mathcal{C}_\varepsilon \chi_A|^2 h\, d\mu$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda^2}\iiint_{\substack{|x-y|>\varepsilon\\|x-z|>\varepsilon\\|y-z|>\varepsilon}} c(x,y,z)^2\chi_A(x)\chi_A(y)h(z)\,d\mu(x)\,d\mu(y)\,d\mu(z)\\
&\quad+\frac{4}{\lambda^2}\left|\int_A \mathcal{C}_\varepsilon(\chi_A)\overline{\mathcal{C}_\varepsilon(h)}\,d\mu\right|+\frac{C}{\lambda^2}\mu(A)\\
&\leq \frac{1}{\lambda^2}\langle K\chi_A,\chi_A\rangle+\frac{C}{\lambda^2}\int_A\left|\mathcal{C}_\varepsilon(\chi_A)\right|d\mu+\frac{C}{\lambda^2}\mu(A)\\
&\leq \frac{C}{\lambda^2}\int_A\left|\mathcal{C}_\varepsilon(\chi_A)\right|d\mu+\frac{C}{\lambda^2}\mu(A).
\end{aligned}
$$

The only task left is to estimate the integral $\int_A |\mathcal{C}_\varepsilon(\chi_A)|\,d\mu$. By (33), since K is of strong-type (2,2), we obtain

$$
\begin{aligned}
\int_A\left|\mathcal{C}_\varepsilon(\chi_A)\right|d\mu &\leq \mu(A)^{1/2}\left(\int_A\left|\mathcal{C}_\varepsilon(\chi_A)\right|^2d\mu\right)^{1/2}\\
&\leq C\mu(A)^{1/2}\left(c^2(A)+\mu(A)\right)^{1/2}\\
&\leq C\mu(A).
\end{aligned}
$$

Therefore, $\mathcal{C}_\varepsilon$ is of restricted weak-type (2,2) uniformly on ε.

Finally, since the Cauchy integral is of weak-type (1,1) and of restricted weak-type (2,2), by interpolation (see [Gu, p. 59] or [StW, p. 197]) we conclude that it is of strong-type (p,p), for $1<p<2$. By duality, it is of strong-type (p,p), for $2<p<\infty$, and again by interpolation it is of strong-type (2,2).

4.2. Proof of the remaining implications in Theorem 1.1. We have proved (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3). On the other hand, it is obvious that (3) $\Rightarrow$ (4). So if we show that (4) $\Rightarrow$ (1), the proof of Theorem 1.1 will be complete.

Suppose that the Cauchy integral operator boundedly sends $L^1(\mu)$ into $L^{1,\infty}(\mu)$. By Proposition 4.3 we know that μ has linear growth. To prove that μ satisfies the local curvature condition, we need the following lemma.

LEMMA 4.4. *Let μ be a positive Radon measure on $\mathbb{C}$. Suppose that $\mathcal{C}_\varepsilon$ is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ uniformly in ε; that is, there exists some constant C such that*

$$\mu\left\{x:\left|\mathcal{C}_\varepsilon f(x)\right|>\lambda\right\}\leq C\frac{\|f\|_{L^1(\mu)}}{\lambda},$$

for all $\lambda>0$, $f\in L^1(\mu)$, and $\varepsilon>0$. Then there exists a constant C' depending on C such that for any compact set $E\subset\mathbb{C}$ there is a μ-measurable function $h:\mathbb{C}\longrightarrow[0,1]$ (independent of ε) such that, for all $\varepsilon>0$,

$$h(x)=0,\qquad \textit{for all } x\notin E, \tag{43}$$

$$\mu(E) \le 8 \int h\, d\mu, \tag{44}$$

$$|\mathcal{C}h(x)| \le C', \quad \textit{for all } x \notin E \textit{ and a.e. } (\mathcal{H}^2)\ x \in \mathbb{C}, \tag{45}$$

and, for all $\varepsilon > 0$,

$$\left|\mathcal{C}_\varepsilon h(x)\right| \le C' + C_0, \quad \textit{for all } x \in \mathbb{C}. \tag{46}$$

Proof. Observe that (46) follows from (45), since for each $\varepsilon > 0$ we have

$$\left|\widetilde{\mathcal{C}}_\varepsilon(h)(x)\right| = \left|\left(\frac{1}{\pi\varepsilon^2}\chi_\varepsilon * \mathcal{C}(h)\right)(x)\right| \le C_6, \quad \text{for all } x \in \mathbb{C},$$

and by (42),

$$|\mathcal{C}_\varepsilon(h)(x)| \le \left\|\widetilde{\mathcal{C}}_\varepsilon(h)\right\|_{L^\infty(\mathbb{C})} + C_0\|h\|_{L^\infty(\mu)} \le C_6 + C_0, \quad \text{for all } x \in C.$$

So, given $E \subset \mathbb{C}$ compact, we must show that there exists a function $h : \mathbb{C} \longrightarrow [0,1]$ satisfying (43), (44), and (45). By the Lebesgue-Radon-Nikodym theorem, we have

$$d\mu_{|E} = g\, d\mathcal{H}^2_{|E} + d\sigma,$$

where $\mathcal{H}^2$ stands for the 2-dimensional Hausdorff measure, $g \in L^1(\mathcal{H}^2_{|E})$, $g \ge 0$, and σ is a positive finite Radon measure that is singular with respect to $\mathcal{H}^2$. So there is a Borel set $E_0 \subset E$ such that $\mathcal{H}^2(E_0) = 0$ and $\sigma(E) = \sigma(E_0)$.

Let us consider the case $\sigma(E_0) \le \mu(E)/2$. Since $\mu(E) = \int_{E\setminus E_0} g\, d\mathcal{H}^2 + \sigma(E_0)$, we have $\int_{E\setminus E_0} g\, d\mathcal{H}^2 \ge \mu(E)/2$. Take $N \in \mathbb{N}$ and $F \subset E \setminus E_0$ compact such that $g(x) \le N$, for all $x \in F$, and $\mu(F) = \int_F g\, d\mathcal{H}^2 \ge \mu(E)/4$.

Since $\widetilde{\mathcal{C}}_\varepsilon$ is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ uniformly in ε, we can apply Proposition 4.1 to the space $X = \operatorname{spt}(\mu)$. We conclude that there exists some function $h_\varepsilon : F \longrightarrow [0,1]$ so that $\mu(F) \le 2\int h_\varepsilon\, d\mu$ and $|\widetilde{\mathcal{C}}_\varepsilon h_\varepsilon(x)| \le C$, for all $x \in \operatorname{spt}(\mu)$. Also, by (42) we have

$$|\mathcal{C}_\varepsilon(h_\varepsilon)(x)| \le \left\|\widetilde{\mathcal{C}}_\varepsilon(h_\varepsilon)\right\|_{L^\infty(\mu)} + C_0\|h_\varepsilon\|_{L^\infty(\mu)} \le C + C_0 = C', \quad \text{for all } x \in F. \tag{47}$$

There is a sequence $\{\varepsilon_k\}_k$ tending to zero such that $(h_{\varepsilon_k})_k$ converges weak $*$ in $L^\infty(\mu)$ to some function $h \in L^\infty(\mu)$. Then it is straightforward to check that

$$\mu(F) \le 2\int h\, d\mu,$$

$$h(x) = 0, \quad \text{for all } x \notin F,$$

and

$$\|h\|_{L^\infty(\mu)} \le 1.$$

Also, we have

$$\lim_{k\to\infty} \mathcal{C}_{\varepsilon_k} h_{\varepsilon_k}(x) = \mathcal{C}h(x), \tag{48}$$

for all $x \in \mathbb{C}$, since

$$\left|\mathcal{C}_{\varepsilon_k} h_{\varepsilon_k}(x) - \mathcal{C}h(x)\right| \le \left|\mathcal{C}_{\varepsilon_k} h_{\varepsilon_k}(x) - \mathcal{C}h_{\varepsilon_k}(x)\right| + \left|\mathcal{C}h_{\varepsilon_k}(x) - \mathcal{C}h(x)\right| = I + II. \tag{49}$$

The term I tends to zero as $k \to \infty$, since

$$I = \left|\int_{|y-x|\le\varepsilon_k} \frac{h_{\varepsilon_k}(y)}{y-x} d\mu(y)\right| \le \int_{|y-x|\le\varepsilon_k} \frac{1}{|y-x|} N d\mathcal{H}^2(y) = 2\pi N\varepsilon_k. \tag{50}$$

Now we consider the term II in (49):

$$II = \left|\int_F \frac{1}{y-x}\big(h_{\varepsilon_k}(y) - h(y)\big)\, d\mu(y)\right|.$$

Notice that, for any fixed $x \in \mathbb{C}$, the function $\chi_F(y)/(y-x)$ belongs to $L^1(\mu)$, as $d\mu_{|F} = g d\mathcal{H}^2$, with g bounded. Therefore, II tends to zero as $k \to \infty$. Hence, (48) holds.

By (47) and (48) we get that $|\mathcal{C}h(x)| \le C'$, for all $x \in F$. It is easily checked that $\mathcal{C}h$ is continuous on $\mathbb{C}$. Therefore, by the maximum principle, $|\mathcal{C}h(x)| \le C'$, for all $x \in \mathbb{C}$. Thus h satisfies (43), (44), and (45).

Suppose now that $\sigma(E_0) > \mu(E)/2$. Let $G \subset E_0$ be compact with $\mu(G) \ge \mu(E)/4$. By Proposition 4.1 there exists some function $h_\varepsilon : G \longrightarrow [0,1]$ so that $\mu(G) \le 2\int h_\varepsilon \, d\mu$ and $|\widetilde{\mathcal{C}}_\varepsilon h_\varepsilon(x)| \le C$, for all $x \in \operatorname{spt}(\mu)$. As above, by (42) we obtain

$$\left|\mathcal{C}_\varepsilon(h_\varepsilon)(x)\right| \le C, \qquad \text{for all } x \in G. \tag{51}$$

We show that there is some constant C_7 such that, for all $w \in \mathbb{C}$ satisfying

$$d(w, G) = 2\varepsilon, \tag{52}$$

we have

$$\left|\mathcal{C}_\varepsilon h_\varepsilon(w)\right| \le C_7. \tag{53}$$

Since $\mathcal{C}_\varepsilon h_\varepsilon(w) = \mathcal{C}h_\varepsilon(w)$ by (52), we get that $|\mathcal{C}h_\varepsilon(x)| \le C_7$, for all $x \in \mathbb{C}$, such that $d(x, G) > 2\varepsilon$, by the maximum principle. Now, we can take a sequence $\{\varepsilon_k\}_k$ tending to zero such that $(h_{\varepsilon_k})_k$ converges weak $*$ in $L^\infty(\mu)$ to some function $h \in L^\infty(\mu)$ satisfying

$$\mu(G) \le 2\int h\, d\mu,$$

$$h(x) = 0, \qquad \text{for all } x \notin G,$$

and

$$\|h\|_{L^\infty(\mu)} \leq 1.$$

Also, for all $x \notin G$, we have

$$\lim_{k\to\infty} \mathcal{C}h_{\varepsilon_k}(x) = \mathcal{C}h(x).$$

As $\mathcal{H}^2(G) = 0$, (43), (44), and (45) hold.

Let us prove (53) for w satisfying $d(w, G) = 2\varepsilon$. Let $x_0 \in G$ be such that $|w - x_0| = 2\varepsilon$. We denote

$$h_{\varepsilon,1} = h_\varepsilon \cdot \chi_{\Delta(x_0,4\varepsilon)}$$

and

$$h_{\varepsilon,2} = h_\varepsilon - h_{\varepsilon,1}.$$

Then, by (51),

$$\begin{aligned}\left|\mathcal{C}_\varepsilon h_\varepsilon(w)\right| &\leq \left|\mathcal{C}_\varepsilon h_\varepsilon(x_0)\right| + \left|\mathcal{C}_\varepsilon h_\varepsilon(w) - \mathcal{C}_\varepsilon h_\varepsilon(x_0)\right| \\ &\leq C + \left|\mathcal{C}_\varepsilon h_{\varepsilon,1}(w)\right| + \left|\mathcal{C}_\varepsilon h_{\varepsilon,1}(x_0)\right| + \left|\mathcal{C}_\varepsilon h_{\varepsilon,2}(w) - \mathcal{C}_\varepsilon h_{\varepsilon,2}(x_0)\right|.\end{aligned}$$

Now, since $|h_\varepsilon| \leq 1$, it is easily checked that $|\mathcal{C}_\varepsilon h_{\varepsilon,1}(w)| \leq C$, $|\mathcal{C}_\varepsilon h_{\varepsilon,1}(x_0)| \leq C$, and $|\mathcal{C}_\varepsilon h_{\varepsilon,2}(w) - \mathcal{C}_\varepsilon h_{\varepsilon,2}(x_0)| \leq C$ for some constant C; (53) follows. □

We can now continue the proof of (4) $\Rightarrow$ (1). By Corollary 2.2 we only have to prove that μ satisfies the weak local curvature condition. Given a disc $\Delta \subset \mathbb{C}$, there exists a function $h : \mathbb{C} \longrightarrow [0, 1]$ (independent of ε) such that

$$\mu(\Delta) \leq 8 \int h\, d\mu,$$

$$h(x) = 0, \qquad \text{for all } x \notin \Delta,$$

and

$$|\mathcal{C}_\varepsilon h(x)| \leq C, \qquad \text{for all } x \in \mathbb{C},$$

with C not depending on the disc Δ. Consider the set

$$S = \left\{ z \in \Delta : h(z) \geq \frac{1}{16} \right\}.$$

Then

$$\begin{aligned}\mu(\Delta) &\leq 8 \left(\int_S h\, d\mu + \int_{\Delta\setminus S} h\, d\mu \right) \\ &\leq 8 \left(\mu(S) + \frac{1}{16} \mu(\Delta \setminus S) \right) \\ &= 8 \left(\mu(\Delta) - \frac{15}{16} \mu(\Delta \setminus S) \right).\end{aligned}$$

Therefore,

$$\mu(\Delta \setminus S) \leq \frac{14}{15}\mu(\Delta),$$

and so

$$\mu(S) \geq \frac{1}{15}\mu(\Delta). \tag{54}$$

By equation (33) applied to the measure $h\,d\mu$, we have

$$c_\varepsilon^2(\Delta, h\,d\mu) = 6\int_\Delta |\mathcal{C}_\varepsilon(h)|^2 h\,d\mu + O\big(\mu(\Delta)\big) \leq C\mu(\Delta),$$

for all $\varepsilon > 0$, and since $h(x) \geq (1/16)$, for $x \in S$, we obtain

$$c_\varepsilon^2(S) \leq 16^3 c_\varepsilon^2(S, h\,d\mu) \leq 16^3 c_\varepsilon^2(\Delta, h\,d\mu) \leq C\mu(\Delta) \leq C\mu(S),$$

for all $\varepsilon > 0$, and consequently $c^2(S) \leq C\mu(S)$. Therefore, by (54), μ satisfies the weak local curvature condition. □

Remark 4.5. After this paper was written, Nazarov, Treil, and Volberg [NTV1] obtained some results that are related to the ones proved here. In particular, they proved that if μ is a positive Radon measure on $\mathbb{C}$ (not doubling, in general) with linear growth and if T is a Calderón-Zygmund operator such that

$$\int |T\chi_Q|^2\,d\mu \leq C\mu(Q),$$

for all squares $Q \subset \mathbb{C}$, then T is bounded on $L^2(\mu)$. They also have shown that the L^2-boundedness of T implies the weak (1,1)-boundedness.

5. A geometric characterization of the analytic capacity γ_+. The *analytic capacity γ_+* (or *capacity γ_+*) of a compact set $E \subset \mathbb{C}$ is defined as

$$\gamma_+(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$, with $|f| \leq 1$ on $\mathbb{C} \setminus E$, which are the *Cauchy transforms* of some positive Radon measure μ supported on E. Obviously,

$$\gamma(E) \geq \gamma_+(E).$$

The analytic capacity γ was first introduced by Ahlfors [Ah] in order to study removable singularities of bounded analytic functions. He showed that a compact set is removable for all bounded analytic functions if and only if it has zero analytic capacity. However, this did not solve the problem of characterizing these sets in a geometric way (this is known as Painlevé's problem), because of the lack of a geometric or metric characterization of analytic capacity. In fact, it is not even known if analytic capacity as a set function is semiadditive; that is, if there is some absolute constant C such that

$$\gamma(E \cup F) \leq C\left(\gamma(E)+\gamma(F)\right),$$

for all compact sets $E, F \subset \mathbb{C}$ (see [Me1], [Su], [Vi], and [VM], for example).

On the other hand, as far as we know, the capacity γ_+ was introduced by Murai [Mu, pp. 71–72]. He introduced this notion only for sets supported on rectifiable curves, and he obtained some estimates involving γ_+ about the weak (1,1)-boundedness of the Cauchy transform on these curves.

Also, until now, no characterization of γ_+ in geometric or metric terms has been known.

In the following theorem, we obtain a more precise version of inequality (40), and we get a geometric and metric characterization of γ_+ for compact sets with area zero.

THEOREM 5.1. *If $E \subset \mathbb{C}$ is compact, then*

$$\gamma_+(E) \geq C \sup \frac{\|\mu\|^{3/2}}{\left(\|\mu\|+c^2(\mu)\right)^{1/2}}, \tag{55}$$

where $C > 0$ is some absolute constant and the supremum is taken over all positive Radon measures μ supported on E such that $\mu(\Delta(x,r)) \leq r$, for all $x \in \mathbb{C}$, $r > 0$. If, moreover, $\mathcal{H}^2(E) = 0$, then

$$\gamma_+(E) \approx \sup \frac{\|\mu\|^{3/2}}{\left(\|\mu\|+c^2(\mu)\right)^{1/2}}, \tag{56}$$

where the supremum is taken as above.

The notation $a \approx b$ in (56) means that there is some positive absolute constant C such that $C^{-1} a \leq b \leq C a$. Using Theorem 5.1, we get the semiadditivity of γ_+.

THEOREM 5.2. *Let $E, F \subset \mathbb{C}$ be compact with $\mathcal{H}^2(E) = \mathcal{H}^2(F) = 0$. Then*

$$\gamma_+(E \cup F) \leq C\left(\gamma_+(E)+\gamma_+(F)\right),$$

where C is some absolute constant.

Proof of Theorem 5.1. First we show that (55) holds. Let μ be a positive finite Radon measure supported on E with linear growth with constant 1 and such that $c^2(\mu) < \infty$. Suppose that $k = c^2(\mu)/\|\mu\| > 1$. Then we set

$$\sigma = \frac{\mu}{k^{1/2}}.$$

Notice that σ has linear growth with constant less than or equal to 1 and

$$c^2(\sigma) = \frac{c^2(\mu)}{k^{3/2}} = \frac{k\|\mu\|}{k^{3/2}} = \|\sigma\|. \tag{57}$$

Furthermore,

$$\frac{\|\sigma\|^{3/2}}{\big(\|\sigma\|+c^2(\sigma)\big)^{1/2}} = \frac{1}{2^{1/2}}\|\sigma\| = \frac{1}{(2k)^{1/2}}\|\mu\|.$$

Thus, by the definition of k,

$$\begin{aligned}\frac{\|\mu\|^{3/2}}{\big(\|\mu\|+c^2(\mu)\big)^{1/2}} &= \frac{1}{(1+k)^{1/2}}\|\mu\| \\ &= \left(\frac{2k}{1+k}\right)^{1/2}\frac{\|\sigma\|^{3/2}}{\big(\|\sigma\|+c^2(\sigma)\big)^{1/2}} \\ &\le 2^{1/2}\frac{\|\sigma\|^{3/2}}{\big(\|\sigma\|+c^2(\sigma)\big)^{1/2}}.\end{aligned}$$

Therefore, taking (57) into account, to prove (55) we can assume that the supremum is taken only over measures μ, supported on E, having linear growth with constant 1 such that $c^2(\mu) \le \|\mu\|$. Let us remark that this fact was already noticed in [Me2].

So, if μ is a positive finite measure supported on E with linear growth with constant 1 and such that

$$c^2(\mu) \le \|\mu\|, \tag{58}$$

we have to show that

$$\gamma_+(E) \ge C_8\|\mu\|, \tag{59}$$

with $C_8 > 0$, since

$$\frac{\|\mu\|^{3/2}}{\big(\|\mu\|+c^2(\mu)\big)^{1/2}} \approx \|\mu\|,$$

because of (58).

By Chebyshev, from (58) we get

$$\mu\big\{x\in\mathbb{C} : c^2_\mu(x) > 2\big\} \le \frac{1}{2}c^2(\mu) \le \frac{1}{2}\|\mu\|.$$

(Here we use the notation $c^2_\mu(x)$ instead of $c^2(x)$.) So if we set

$$\sigma = \mu_{|\{c^2_\mu(x)\le 2\}},$$

then

$$\|\sigma\| \ge \frac{1}{2}\|\mu\|.$$

Also,

$$c^2_\sigma(x) \le 2, \quad \text{for } \sigma\text{-almost all } x\in\mathbb{C}.$$

Notice that σ has linear growth with constant 1 and satisfies

$$c^2(\sigma_{|F}) \leq 2\sigma(F), \tag{60}$$

for any Borel set F. Using Theorem 1.1, we get that the Cauchy transform is bounded on $L^2(\sigma)$ and is bounded also from $L^1(\sigma)$ into $L^{1,\infty}(\sigma)$, with the norm bounded by some absolute constant. Therefore, by Lemma 4.4, there exists some absolute constant C_9 such that for any compact set $F \subset \mathbb{C}$, there is a σ-measurable function $h : F \longrightarrow [0,1]$ such that $\sigma(F) \leq 8 \int h\, d\sigma$ and $|\mathscr{C}(h\, d\sigma)(x)| \leq C_9$, for all $x \notin F$. In particular, if we choose $F = E$, we get

$$\|\sigma\| \leq 8 \int h\, d\sigma,$$

for some σ-measurable function $h : E \longrightarrow [0,1]$, and

$$|\mathscr{C}h(x)| \leq C_9, \quad \text{for all } x \notin E.$$

Thus (59) holds.

For the second part of the theorem, assuming $\mathscr{H}^2(E) = 0$, we only have to show that there exists some constant C such that

$$\gamma_+(E) \leq C \sup \frac{\|\mu\|^{3/2}}{\big(\|\mu\| + c^2(\mu)\big)^{1/2}}. \tag{61}$$

Let ν be some positive measure supported on E such that $|\mathscr{C}\nu(x)| < 1$ for all $x \notin E$ and $\gamma_+(E) \leq 2\|\nu\|$. Let us check that ν has linear growth. Let $\Delta(x,r) \subset \mathbb{C}$ be some closed disc. Recall that if

$$\int_{|z-x|=r} \int \frac{1}{|z-y|}\, d\nu(y)\, d\mathscr{H}^1(z) < \infty, \tag{62}$$

where $\mathscr{H}^1$ stands for the 1-dimensional Hausdorff measure, then

$$\nu\big(\Delta(x,r)\big) = \frac{-1}{2\pi i} \int_{|z-x|=r} \mathscr{C}\nu(z)\, dz. \tag{63}$$

See [Ga, p. 40], for example.

Notice that for all $x \in \mathbb{C}$, (62) holds for a.e. ($\mathscr{H}^1$) $r > 0$ (this follows by Fubini). So we get that for all $x \in \mathbb{C}$, (63) holds for a.e. ($\mathscr{H}^1$) $r > 0$. Since $\mathscr{H}^2(E) = 0$, we also get that for each fixed x, $\mathscr{H}^1(\partial\Delta(x,r) \cap E) = 0$, for $\mathscr{H}^1$-almost all $r > 0$. Therefore, by (63), for all $x \in \mathbb{C}$, we have

$$\nu\big(\Delta(x,r)\big) \leq r, \tag{64}$$

for $\mathscr{H}^1$-almost all $r > 0$, and by approximation this can be extended to all $r > 0$.

Now, by Remark 4.2, $|\widetilde{\mathcal{C}}_\varepsilon \nu(x)| \leq C$, for all $x \in \mathbb{C}$ and for some constant C not depending on ε, so $|\mathcal{C}_\varepsilon \nu(x)| \leq C+1$, for all $x \in \mathbb{C}$ by (42). Using the identity (33), we obtain

$$c^2(\nu) \leq C\|\nu\|.$$

Therefore,

$$\gamma_+(E) \leq 2\|\nu\| \leq C \frac{\|\nu\|^{3/2}}{\big(\|\nu\| + c^2(\nu)\big)^{1/2}},$$

and (61) follows. □

We do not know if the second part of Theorem 5.1 holds for sets with positive area. However, we have the following nonquantitative result.

COROLLARY 5.3. *Let $E \subset \mathbb{C}$ be compact. Then $\gamma_+(E) > 0$ if and only if E supports some positive finite Radon measure μ with linear growth such that $c^2(\mu) < \infty$.*

Proof. If $\mathcal{H}^2(E) > 0$, the result follows, choosing $\mu = \mathcal{H}^2_{|E}$. If $\mathcal{H}^2(E) = 0$, we apply (56) of Theorem 5.1. □

Proof of Theorem 5.2. The semiadditivity of γ_+ for sets of area zero follows from (56) of Theorem 5.1 and the fact that the quantity

$$\sup \frac{\|\mu\|^{3/2}}{\big(\|\mu\| + c^2(\mu)\big)^{1/2}} \tag{65}$$

is semiadditive.

Arguing as in the proof of Theorem 5.1, we know that there is a positive finite Radon measure μ with linear growth with constant 1, μ supported on $E \cup F$, such that

$$\gamma_+(E \cup F) \leq C_{10}\,\|\mu\|$$

and

$$c^2_\mu(x) \leq 2,$$

for μ-almost all $x \in \mathbb{C}$. Then $c^2(\mu_{|E}) \leq 2\mu(E)$ and $c^2(\mu_{|F}) \leq 2\mu(F)$. Thus, by Theorem 5.1, $\gamma_+(E) \geq C_{11}\,\mu(E)$ and $\gamma_+(F) \geq C_{11}\,\mu(F)$, where $C_{11} > 0$ is some absolute constant. Therefore,

$$\gamma_+(E \cup F) \leq C_{10}\big(\mu(E) + \mu(F)\big) \leq C\big(\gamma_+(E) + \gamma_+(F)\big). \qquad \square$$

With some minor changes in the proof of Theorem 5.2, one can check that, in fact, γ_+ is *countably* semiadditive on compact sets with area zero.

Remark 5.4. Let us define the capacity $\tilde{\gamma}_+$. Given a compact $E \subset \mathbb{C}$, we set

$$\tilde{\gamma}_+(E) = \sup \|\mu\|,$$

where the supremum is taken over all positive Radon measures μ supported on E such that $|\mathcal{C}\mu| \leq 1$ a.e. ($\mathcal{H}^2$) in $\mathbb{C}$. Obviously, if $\mathcal{H}^2(E) = 0$, then

$$\tilde{\gamma}_+(E) = \gamma_+(E).$$

However, we do not know if $\tilde{\gamma}_+(E) = \gamma_+(E)$ or $\tilde{\gamma}_+(E) \approx \gamma_+(E)$ holds for compact sets E with positive area. Arguing as in Theorem 5.1, it is easily seen that

$$\tilde{\gamma}_+(E) \approx \sup \frac{\|\mu\|^{3/2}}{\big(\|\mu\| + c^2(\mu)\big)^{1/2}},$$

with the supremum taken as in Theorem 5.1, for any compact set $E \subset \mathbb{C}$. Also, as in Theorem 5.2,

$$\tilde{\gamma}_+(E \cup F) \leq C\big(\tilde{\gamma}_+(E) + \tilde{\gamma}_+(F)\big),$$

for all compact sets $E, F \subset \mathbb{C}$. So the notion of $\tilde{\gamma}_+$ seems to be more natural than the notion of γ_+.

On the other hand, observe that if we showed that

$$\gamma(E) \approx \gamma_+(E), \tag{66}$$

for all sets E with $\mathcal{H}^2(E) = 0$, then by Theorem 5.2 we could obtain easily that analytic capacity is a semiadditive function on compact sets. Of course, proving (66) seems difficult. In fact, (66) clearly implies the conjecture of Melnikov stating that $\gamma(E) > 0$ if and only if E supports some positive finite measure with linear growth and finite curvature.

Remark 5.5. Nazarov, Treil, and Volberg [NTV3] have shown that

$$\gamma_+(E) > 0 \quad \text{if and only if} \quad \gamma_c(E) > 0,$$

where $\gamma_c(E)$ stands for

$$\gamma_c(E) = \sup |f'(\infty)|,$$

with the supremum taken over all functions f analytic on $\mathbb{C} \setminus E$, with $|f| \leq 1$, which are the Cauchy transforms of some complex measure. However, from their estimates, one cannot derive that $\gamma_+(E) \approx \gamma_c(E)$. (It is not difficult to see that this would imply (66).)

Nazarov, Treil, and Volberg informed the author that they know how to obtain Theorem 5.1 with different arguments, using the $T(b)$ obtained in [NTV2].

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