L^2 ESTIMATES AND EXISTENCE THEOREMS FOR THE $\bar{\partial}$ OPERATOR

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Contents

Preface

The theory of analytic function of several complex variables, as presented for example in the Cartan seminars [7], consists in a reduction to the theory of analytic functions of one complex variable. First one only studies functions in polycylinders (products of open sets in the different coordinate planes). The extension of the results to more general domains is then achieved by embedding them as submanifolds of polycylinders in spaces of high dimension. The success of this procedure depends of course on the invariance of

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the notion of analytic function under analytic mappings, so similar techniques do not seem applicable to many overdetermined systems of differential equations other than the Cauchy-Riemann equations for analytic function of several complex variables. It is therefore of interest to give a different treatment of these equations which is more suitable for extension to general overdetermined systems.

Such a technique was suggested by Garabedian and Spencer [11]. The execution of their ideas caused considerable difficulties, however, and it was not until 1958 that Morrey [22] found a general method for proving the L^2 estimates required in this approach. His method was extended and simplified by Kohn [14] and Ash [2]. In [15] Kohn has also announced some results on boundary regularity which are required in this context and were still missing in Morrey's fundamental work. The proofs of these results have recently appeared in [15 a] and have later on been simplified by Kohn and Nirenberg jointly, and also by Morrey.

The aim of this paper is to simplify and develop this work. The most important simplification is that we bypass the difficult questions of boundary regularity discussed by Kohn [15] and use instead only fairly elementary results on "identity of weak and strong extensions of differential operators". These can be proved with the methods of Friedrichs [10] and are essentially well known before in a different context (see Lax-Phillips [16]). Further, we characterize the open sets for which estimates of the Morrey-Kohn type are valid. This leads to new proofs of results obtained by Andreotti and Grauert [1] with sheaf theoretic methods; our results are essentially the restriction of theirs to the sheaf of germs of analytic functions. To prove global existence theorems and approximation theorems of the Runge type, we introduce L^2 estimates which involve densities depending on a parameter. This technique has its origin in the Carleman method for proving uniqueness theorems for solutions of a partial differential equation, which we have combined with the ideas of Morrey and Kohn. Part of our results have been obtained with similar methods by Andreotti and Vesentini in a manuscript to appear in Publ. Inst. Hautes Études.

The plan of the paper is as follows. In Chapter I we present the facts from functional analysis and the theory of first order differential operators which we need. Chapter II is devoted to the study of function theory in pseudo-convex domains in $\mathbf{C}^n$. The basic a priori estimates are then easy to prove, and they lead to very precise existence and approximation theorems for the $\bar\partial$ operator in such domains. The results obtained can be used to construct analytic functions satisfying growth conditions, which does not seem as easy to do with the classical methods. (See however Ehrenpreis [9] and Malgrange [19].) We give a few applications here. For further applications of results of this type we refer to the papers just quoted.

In Chapter III we consider function theory in open subsets of a complex manifold. We then aim at maximum generality rather than precision in the results as in Chapter II. The estimates discussed are of the same types as in Chapter II, but in Chapter III we determine almost completely when they are valid. As we have already mentioned, this leads to results of Andreotti and Grauert [1], due in part to Ehrenpreis [8]. In a final section we also show that the L^2 methods developed here give in a very simple way results on the boundary behavior of the Bergman kernel function extending those given by Bergman [3] for domains of holomorphy in $\mathbf{C}^2$.

Apart from the results involving precise bounds, this paper does not give any new existence theorems for functions of several comples variables. However, we believe that it is justified by the methods of proof.

I. Functional analysis and first order differential operators

1.1. *Basic facts from functional analysis*

In this section we shall collect some classical facts on operators in Hilbert space in a form which is suitable for the following applications.

Let H_1 and H_2 be two Hilbert spaces and let

$$T: H_1 \to H_2$$

be a linear, closed, densely defined operator. Then $T^*: H_2 \to H_1$ has the same properties, and $T^{**} = T$. (See e.g. Nagy [23], p. 29.) By definition of the adjoint operator, the orthogonal complement of the range R_T of T is the null space N_{T^*} of T^*, which implies that the orthogonal complement of N_{T^*} is the closure $[R_T]$ of R_T. When R_T is closed we therefore have a good description of R_T in terms of N_{T^*}.

THEOREM 1.1.1. *The following conditions on T are equivalent:*

(a) *R_T is closed.*

(b) *There is a constant C such that*

$$\|f\|_1 \leqslant C \|Tf\|_2, \quad f \in D_T \cap [R_{T^*}]. \tag{1.1.1}$$

(c) *R_{T^*} is closed.*

(d) *There is a constant C such that*

$$\|g\|_2 \leqslant C \|T^* g\|_1, \quad g \in D_{T^*} \cap [R_T]. \tag{1.1.2}$$

The best constants in (1.1.1) *and in* (1.1.2) *are the same.*

Proof. Assume that (a) holds. Since the orthogonal complement of $[R_{T^*}]$ is equal to N_T, the restriction of T to $D_T \cap [R_{T^*}]$ is a closed, one to one, linear mapping onto the closed subspace R_T of H_2. Hence the inverse is continuous by the closed graph theorem, which proves (b). Conversely, (b) obviously implies (a). In view of the symmetry between T and T^*, it is now clear that (c) and (d) are also equivalent, and it suffices to prove that (b) implies (d). From (b) we obtain

$$|(g, Tf)_2| = |(T^*g, f)_1| \leqslant \|T^*g\|_1\|f\|_1 \leqslant C\|T^*g\|_1\|Tf\|_2; \quad g \in D_{T^*},\ f \in D_T \cap [R_{T^*}].$$

Hence $|(g,h)_2| \leqslant C\|T^*g\|_1\|h\|_2$, $g \in D_{T^*}$, $h \in R_T$, which implies (d).

In the usual applications of Theorem 1.1.1 to existence theorems for differential operators T, the range R_T is expected to have at most finite codimension, and this makes (1.1.2) much easier to study than (1.1.1). In the applications to overdetermined systems of differential operators, on the other hand, one can only hope that R_T shall consist of all elements in H_2 satisfying certain compatibility conditions given by the vanishing of some differential operators—and perhaps a finite number of additional linear equations. To put this in an abstract form we assume given another Hilbert space H_3 and a closed densely defined linear operator $S: H_2 \to H_3$ such that

$$ST = 0. \tag{1.1.3}$$

Then the range of T is of course included in the null space of S.

THEOREM 1.1.2. *A necessary and sufficient condition for R_T and R_S both to be closed is that*

$$\|g\|_2^2 \leqslant C^2(\|T^*g\|_1^2 + \|Sg\|_3^2); \quad g \in D_{T^*} \cap D_S,\ g \perp N = N_{T^*} \cap N_S = N_S \ominus [R_T]. \tag{1.1.4}$$

Proof. First note that
$$H_2 = [R_T] \oplus N \oplus [R_{S^*}]. \tag{1.1.5}$$

In fact, (1.1.3) implies that R_T and R_{S^*} are orthogonal, and the intersection of the orthogonal complements of these spaces is N. Now S vanishes on $[R_T]$, and T^* vanishes on $[R_{S^*}]$ since $T^*S^* = 0$. By (1.1.2) R_T is closed if and only if the inequality (1.1.4) is valid when $g \in D_{T^*} \cap [R_T]$. Similarly, by (1.1.1) with T replaced by S, R_S is closed if and only if the inequality (1.1.4) is valid when $g \in D_S \cap [R_{S^*}]$. Since every g occurring in (1.1.4) can be split into two such orthogonal components, the theorem follows.

Note that the dimension of N is equal to the codimension of $[R_T]$ in N_S so that in the applications there is hope that N shall be finite dimensional. It is the fact that (1.1.4) is expected to hold essentially for all g such that the right-hand side is defined which makes it easier to study than (1.1.1) or (1.1.2). Sufficient conditions for (1.1.4) can be obtained by compactness arguments:

THEOREM 1.1.3. *Assume that from every sequence $g_k \in D_{T^*} \cap D_S$ with $\|g_k\|_2$ bounded and $T^*g_k \to 0$ in H_1, $Sg_k \to 0$ in H_3, one can select a strongly convergent subsequence. Then* (1.1.4) *holds and N is finite dimensional.*

Proof. By hypothesis the unit sphere in N is compact, so N has to be finite dimensional. Now if (1.1.4) were not valid, we could choose a sequence $g_k \perp N$ such that $\|g_k\|_2 = 1$ and $T^*g_k \to 0$ in H_1, $Sg_k \to 0$ in H_3. Let g be a strong limit of the sequence g_k, which exists by hypothesis. Then $\|g\|_2 = 1$ and g is orthogonal to N although $T^*g = Sg = 0$, so that $g \in N$. This contradiction proves (1.1.4).

In the applications we shall also encounter modified forms of (1.1.4):

THEOREM 1.1.4. *Let A be a closed, densely defined, linear operator in H_2, and let F be a closed subspace of H_2 which contains R_T. Assume that*

$$\|Af\|_2^2 \leqslant \|T^*f\|_1^2 + \|Sf\|_3^2; \quad f \in D_{T^*} \cap D_S \cap F, \tag{1.1.6}$$

which in particular shall mean that $f \in D_{T^*} \cap D_S \cap F$ implies $f \in D_A$. Then we have $R_{A^*} \cap N_S \cap F \subset R_T$; if $g = A^*h$, $h \in D_{A^*}$, and $g \in N_S \cap F$, we can find $u \in D_T$ so that $Tu = g$ and $\|u\|_1 \leqq \|h\|_2$. Furthermore, if $v \in R_{T^*}$, we can choose $f \in D_A \cap D_{T^*}$ so that $T^*f = v$ and $\|Af\|_2 \leqslant \|v\|_1$.

Proof. With g and h as in the theorem we have to find $u \in H_1$ so that $\|u\|_1 \leqslant \|h\|_2$ and $Tu = g$, that is,

$$(u, T^*f)_1 = (g, f)_2, \quad f \in D_{T^*}.$$

By the Hahn-Banach theorem this is equivalent to proving the inequality

$$|(g, f)_2| \leqslant \|h\|_2 \|T^*f\|_1, \quad f \in D_{T^*}. \tag{1.1.7}$$

First note that if $f \perp N_S \cap F$, we have $T^*f = 0$ because $R_T \subset N_S \cap F$. Since $g \in N_S \cap F$, it is therefore enough to prove (1.1.7) when $f \in N_S \cap F$ and $f \in D_{T^*}$. But then we obtain from (1.1.6) that $\|Af\|_2 \leqslant \|T^*f\|_1$, which gives

$$|(g, f)_2| = |(A^*h, f)_2| = |(h, Af)_2| \leqslant \|h\|_2 \|Af\|_2 \leqslant \|h\|_2 \|T^*f\|_1.$$

This proves (1.1.7) and the first part of the theorem. To prove the second part we note that the range of T^* is equal to the range of its restriction to the orthogonal complement of N_{T^*}, that is $[R_T]$, which is contained in $N_S \cap F$. Hence one can find $f \in N_S \cap F \cap D_{T^*}$ so that $T^* f = v$. But then it follows from (1.1.6) that $f \in D_A$ and that $\|Af\|_2 \leqslant \|v\|_1$. The proof is complete.

1.2. *Identity of weak and strong extensions of first order differential operators*

In our applications of the results proved in section 1.1, the operators T and S will be first order systems of differential operators. The a priori estimates discussed in section 1.1 will first be obtained only for smooth elements in $D_{T^*} \cap D_S$, and to prove them in general it will be necessary to show that such elements are dense in $D_{T^*} \cap D_S$ for the graph norm. This follows essentially from known results (Friedrichs [10], Lax-Phillips [16]) but we shall sum up what is required here.

Let μ be a positive measure with compact support in $\mathbf{R}^N$ and $\mu(1) = 1$. Define μ_ε by

$$\int u(x) d\mu_\varepsilon(x) = \int u(\varepsilon x) d\mu(x)$$

when u is continuous and has compact support. Then we have $\mu_\varepsilon(1) = 1$, so if $v \in L^2$ it follows that

$$\| \mu_\varepsilon \ast v \|_{L^2} \leqslant \| v \|_{L^2}.$$

Since $\mu_\varepsilon \ast v \to v$ uniformly if v is a continuous function with compact support and since such functions are dense in L^2, it follows that $\mu_\varepsilon \ast v \to v$ in L^2 when $\varepsilon \to 0$ for every $v \in L^2$.

A much more subtle fact concerning the regularization by convolutions is given by Friedrichs' lemma (Friedrichs [10]; see also Hörmander [13]).

LEMMA 1.2.1. *Let μ be a positive measure with compact support in $\mathbf{R}^N$ such that $\mu(1) = 1$ and $D_i \mu = \partial \mu / \partial x_i$ is a measure for a certain i $(1 \leqslant i \leqslant N)$. If $v \in L^2(\mathbf{R}^N)$ has compact support and a is a Lipschitz continuous function in a neighborhood of the support of v, it follows that*

$$a(D_i v \ast \mu_\varepsilon) - (a D_i v) \ast \mu_\varepsilon \to 0 \text{ in } L^2 \text{ when } \varepsilon \to 0.$$

Note that the product of a Lipschitz continuous function and a first order derivative of an L^2 function is well defined in the sense of distribution theory.

Proof. If M is a Lipschitz constant for a and if m_i is the total variation of $|y| D_i \mu$, the arguments of Friedrichs [10] give (see [13], p. 393)

$$\| a(D_i v \ast \mu_\varepsilon) - (a D_i v) \ast \mu_\varepsilon \|_{L^2} \leqslant M(1 + m_i) \| v \|_{L^2}. \tag{1.2.1}$$

Since the left-hand side of (1.2.1) tends to 0 when $\varepsilon \to 0$ if $v \in C_0^1$, which is a dense set in L^2, the assertion follows.

LEMMA 1.2.2. *Let $u_1, \ldots, u_J$ be L^2 functions with compact support in an open set $U \subset R^N$, let a_{ij} ($i=1, \ldots, N$; $j=1, \ldots, J$) be Lipschitz continuous in U and assume that for each i and j either a_{ij} is a constant or $D_i\mu$ is a measure. Then*

$$\left\| \sum_{i=1}^{N} \sum_{j=1}^{J} a_{ij} D_i (u_j \ast \mu_\varepsilon) - \left(\sum_{i=1}^{N} \sum_{j=1}^{J} a_{ij} D_i u_j \right) \ast \mu_\varepsilon \right\|_{L^2} \to 0, \ \varepsilon \to 0, \tag{1.2.2}$$

and $D_i(u_j \ast \mu_\varepsilon) \in L^2$ for all i such that $D_i\mu$ is a measure.

Proof. Since multiplication by a_{ij} and convolution with μ_ε commute if a_{ij} is a constant, the lemma is an immediate consequence of Lemma 1.2.1.

We shall now consider a system of differential equations

$$\sum_{i=1}^{N} \sum_{j=1}^{J} a_{ij}^k D_i u_j + \sum_{j=1}^{J} b_j^k u_j = f_k, \quad k = 1, \ldots, K,$$

which we write in the form

$$Au + Bu = f. \tag{1.2.3}$$

As norm on u we take $\| u \|_{L^2} = (\sum_1^J \| u_j \|_{L^2}^2)^{\frac{1}{2}}$ and similarly for f.

PROPOSITION 1.2.3. *Let U be an open set in $\mathbf{R}^N$, let $\varphi \in C^1(U)$ be real valued, and assume that* grad $\varphi \neq 0$ *when $\varphi = 0$. Set $U^{\pm} = \{x; x \in U, \varphi(x) \gtrless 0\}$. Suppose we have a solution of* (1.2.3) *in the interior of U^-, such that the components of u and of f are in $L^2(U^-)$ and vanish outside a compact subset of U^-. The coefficients of A are assumed to be Lipschitz continuous and those of B bounded measurable in U. Then there is a sequence $u^\nu \in C^\infty(U^-)$, vanishing outside a fixed compact subset of U^-, such that*

$$\| u^\nu - u \|_{L^2(U^-)} \to 0, \quad \| Au^\nu + Bu^\nu - f \|_{L^2(U^-)} \to 0 \quad \text{when} \quad \nu \to \infty.$$

If the Cauchy data of u on the surface $\varphi = 0$ with respect to the system (1.2.3) *vanish in the sense that $Au + Bu = f$ in U if u and f are defined as 0 in $U \cap \complement U^-$, one can choose u^ν with support in the interior of U^-.*

Proof. First assume that there is an open convex set Γ with $0 \in \bar{\Gamma}$ such that

$$\varphi(x) = 0, \quad x \in \operatorname{supp} u \Rightarrow x \pm y \in U^{\pm}, \quad y \in \Gamma. \tag{1.2.4}$$

Extend u and f to be 0 in U outside U^-. Then

$$Au + Bu = f + g, \tag{1.2.3$'$}$$

where the support of g lies in $\{x; x \in \operatorname{supp} u, \varphi(x)=0\}$; the hypothesis in the latter part of the theorem is that $g=0$. Now choose $\mu \in C_0^\infty(\Gamma)$, which implies that $\mu_\varepsilon \in C_0^\infty(\Gamma)$, $0<\varepsilon<1$. Then $u_j * \mu_\varepsilon \in C_0^\infty(U)$ and by Lemma 1.2.2

$$A(u*\mu_\varepsilon)+B(u*\mu_\varepsilon)-f*\mu_\varepsilon-g*\mu_\varepsilon \to 0 \quad \text{in} \quad L^2(U) \quad \text{when} \quad \varepsilon \to 0.$$

But $g*\mu_\varepsilon=0$ in U^- in view of (1.2.4), and $\|f*\mu_\varepsilon-f\|_{L^2}\to 0$ when $\varepsilon\to 0$ so $u^\nu=u*\mu_{1/\nu}$ has the required properties. To prove the last statement we choose ε between -1 and 0. Then the support of $u*\mu_\varepsilon$ lies in the interior of U^- if ε is small enough, again by (1.2.4), and since $g=0$ by hypothesis now, we have $\|A(u*\mu_\varepsilon)+B(u*\mu_\varepsilon)-f\|_{L^2(U)}\to 0$.

In general there is no convex set Γ with the required properties, but for every point $x\in\operatorname{supp} u$ one can choose a set Γ which can be used in a neighborhood of x. By using a partition of unity we can therefore decompose u into a sum of a finite number of terms such that the hypotheses in the first part of the proof are fulfilled for each term. This completes the proof.

In the next proposition we shall consider solutions of a system of differential equations (1.2.3) satisfying Cauchy boundary conditions only with respect to some of the equations. Thus let $K^0 \leqq K$ (the number of equations in (1.2.3)), set $f^0=(f_1, \dots, f_{K^0})$ and write the first K^0 equations (1.2.3) in the form

$$A^0u+B^0u=f^0. \tag{1.2.5}$$

If $\varphi\in C^1$ we set
$$A(\operatorname{grad}\varphi)=\left(\sum_{i=1}^N a_{ij}^k \frac{\partial\varphi}{\partial x_i}\right)_{\substack{k=1,\dots,K\\ j=1,\dots,J}},$$

and define the matrix A^0 similarly with K replaced by K^0.

PROPOSITION 1.2.4. *Let U be an open set in $\mathbf{R}^N$, let $\varphi\in C^{r+1}(U)$ be real valued, $r\geqslant 1$, and assume that $\operatorname{grad}\varphi\neq 0$ when $\varphi=0$. Set $U^-=\{x; x\in U, \varphi(x)\leqslant 0\}$. Suppose we have a solution of (1.2.3) in the interior U_0^- of U^-, such that the components of u and of f are in $L^2(U_0^-)$ and vanish outside a compact subset of U^-. We assume that the coefficients of A are in $C^r(U)$, that those of B are bounded measurable in U, and that the matrices $A(\operatorname{grad}\varphi)$ and $A^0(\operatorname{grad}\varphi)$ have constant rank in a neighborhood of $\{x; x\in U, \varphi(x)=0\}$. In addition assume that the Cauchy data of u with respect to the operator A^0 on the surface $\varphi=0$ vanish in the sense that (1.2.5) is valid in U if u and f^0 are defined as 0 outside U_0^-. Then there is a sequence u^ν with components in $C^r(U^-)$, vanishing outside a fixed compact subset of U^-, such that*

$$\|u^\nu-u\|_{L^2(U_0^-)}\to 0, \quad \|Au^\nu+Bu^\nu-f\|_{L^2(U_0^-)}\to 0, \quad \nu\to\infty,$$

and the Cauchy data of u^ν with respect to the operator A^0 vanish, that is, $A^0(\operatorname{grad}\varphi)u^\nu=0$ when $\varphi=0$.

Proof. First assume that $\varphi(x)=x_N$ and that the coefficients a^k_{Nj} in $A(\operatorname{grad}\varphi)$ all vanish except when $j=k=1, ..., r_0$, the rank of $A^0(\operatorname{grad}\varphi)$, and when $J+1-j=K+1-k=1, ..., r-r_0$, where r is the rank of the whole matrix $A(\operatorname{grad}\varphi)$; these coefficients are assumed to be equal to 1. Define u and f as 0 in U outside U_0^-; the equations (1.2.5) are then fulfilled in the whole of U. Now choose μ as a measure with support in the plane $x_N=0$ with a C^∞ density. Since $D_i\mu$ is then a measure for every $i \neq N$, the hypotheses of Lemma 1.2.2 are fulfilled. Hence the components of $u^\varepsilon = u * \mu^\varepsilon$ and all their derivatives with respect to other variables than x_N are in L^2 and we have

$$Au^\varepsilon + Bu^\varepsilon - f \to 0 \text{ in } L^2(U_0^-) \text{ when } \varepsilon \to 0; \quad A^0u^\varepsilon + B^0u^\varepsilon - f^0 \to 0 \text{ in } L^2(U) \text{ when } \varepsilon \to 0. \tag{1.2.6}$$

This proves that u^ε has Cauchy data 0 with respect to the equations (1.2.5). Also note that (1.2.6) proves that $\partial u_j^\varepsilon/\partial x^N \in L^2(U)$ if $j \leq r_0$ and that $\partial u_j^\varepsilon/\partial x^N \in L^2(U_0^-)$ if $j > J+r_0-r$. These are the only x_N derivatives occurring in the operator A.

Now choose positive measures μ^+ and μ^- with supports in the half spaces $\{x; x_N>0\}$ and $\{x; x_N<0\}$ respectively, with total mass 1 and density in C_0^∞. We set with $\delta>0$

$$u_j^{\varepsilon\delta} = u_j^\varepsilon * \mu_\delta^-, \quad j=1, \ldots, K_0; \qquad u_j^{\varepsilon\delta} = u_j^\varepsilon * \mu_\delta^+, \quad j=K_0+1, \cdots, K.$$

Then $u_j^{\varepsilon\delta} \in C_0^\infty(U)$ for small ε and δ, and the support is contained in the interior of U^- when $j \leqslant K_0$. When $\delta \to 0$ we have $D_i u_j^{\varepsilon\delta} \to D_i u_j^\varepsilon$ in $L^2(U)$ if $i<N$ or if $i=N$ and $j \leqslant r_0$. In addition, $D_N u_j^{\varepsilon\delta} \to D_N u_j^\varepsilon$ in $L^2(U_0^-)$ if $j>J+r_0-r$. If we define u^ν as $u^{\varepsilon\delta}$ with first ε and then δ chosen sufficiently small, we can therefore achieve that

$$\| u^\nu - u \|_{L^2(U^-)} + \| Au^\nu + Bu^\nu - f \|_{L^2(U^-)} < \frac{1}{\nu}.$$

This completes the proof in the special case.

In general it suffices to prove that every point in U where $\varphi=0$ has a neighborhood where a suitable change of dependent and independent variables leads to the situation just considered. Indeed, when we have proved that, a partition of unity can be used to split u into a finite sum consisting of one term with support in the interior of U^-, to which we can apply Proposition 1.2.3, and otherwise only terms which can be approximated in view of the first part of the proof.

Thus take a point $x_0 \in U$ with $\varphi(x_0)=0$. By the implicit function theorem there is a C^{r+1} change of variables in a neighborhood of x_0 such that $\varphi(x)$ is one of the new coordinates. This substitution preserves the regularity properties of the coefficients required in the theorem and also keeps the class of C^r functions invariant. We may therefore without any restriction assume that $\varphi(x)=x_N$. By hypothesis, the matrix

$$(a^k_{Nj})^{k=1,\dots,K_0}_{j=1,\dots,J}$$

has constant rank equal to r_0 in a neighborhood of x_0. We may assume that the matrix with $j, k=1, \dots, r_0$ is non-singular at x_0 and therefore in a neighborhood of x_0. In this neighborhood we can then introduce

$$u'_k = \sum_{j=1}^{J} a^k_{Nj} u_j, \quad k=1, \dots, r_0; \qquad u'_k = u_k, \ r_0 < k \leqslant J,$$

as new dependent variables. Since the coefficients of this transformation and its inverse are in C^r, the regularity hypotheses in the theorem will be fulfilled by the new system. The equations (1.2.3) now assume the form

$$\sum_1^N \sum_1^J a'^k_{ij} D_i u'_j + \sum_1^J b'^k_j u'_j = f_k, \quad k=1, \dots, K,$$

with Cauchy boundary conditions for the first K_0 equations; we have $a'^k_{Nj} = \delta_{jk}$ for $k=1,\dots, r_0$; $j=1, \dots, J$, and $a'^k_{Nj}=0$, $j>r_0$, $k \leqslant K_0$ since the rank of the matrix a'^k_{Nj}, $k=1, \dots, K_0$, $j=1, \dots, J$, is r_0 everywhere. By subtracting linear combinations of the first r_0 equations from the others we may attain that $a'^k_{Nj}=0$ when $j \leqslant r_0$ for every $k > r_0$.

The first K_0 equations have now obtained the desired form. Further, the matrix a'^k_{Nj} with $j > r_0$ and $k > K_0$ must now have constant rank equal to $r - r_0$. Introducing suitable linear combinations of $u'_{r_0+1}, \dots, u'_J$ as new dependent variables in the same way as above and forming linear combinations of the equations with $k > K_0$, we obviously obtain a system of differential equations of the special form considered in the beginning of the proof. The linear change of dependent variables as well as its inverse has C^r coefficients. This completes the proof.

II. Function theory in pseudo-convex domains in $\mathbf{C}^n$

2.1. *Notations and estimates*

We shall denote the real coordinates in $\mathbf{C}^n$ by x_j, $1 \leqslant j \leqslant 2n$, and the complex coordinates by $z_j = x_{2j-1} + i x_{2j}$, $j=1, \dots, n$. A differential form f is said to be of type (p, q) if it can be written in the form

$$f = \sum_{|I|=p,\ |J|=q}' f_{I,J}\, dz^I \wedge d\bar{z}^J,$$

where $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are multi-indices, that is, sequences of indices between 1 and n. The notation $\sum'$ means that the summation only extends over strictly increasing multi-indices, and we have written

$$dz^I \wedge d\bar{z}^J = dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

The coefficients $f_{I,J}$ may be distributions in an open set, and are supposed to be defined for arbitrary I and J so that they are antisymmetric both in the indices of I and in those of J. We set $\partial/\partial\bar{z}_k = (\partial/\partial x_{2k-1} + i\partial/\partial x_{2k})/2$ and

$$\bar{\partial} f = \sum_{I,J}{}' \sum_k \frac{\partial f_{I,J}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz^I \wedge d\bar{z}^J. \tag{2.1.1}$$

The form $\bar{\partial} f$ is then of type $(p, q+1)$ and

$$\bar{\partial}\bar{\partial} f = 0. \tag{2.1.2}$$

If $\mathfrak{F}$ is a space of distributions we denote by $\mathfrak{F}_{(p,q)}$ the space of forms of type (p, q) with coefficients belonging to $\mathfrak{F}$. In particular we shall use this notation with $\mathfrak{F} = C^k(\Omega)$, where Ω is an open set in $\mathbf{C}^n$, or with $\mathfrak{F} = C^k(\bar{\Omega})$, the space of restrictions to Ω of functions which $\in C^k$ in the whole space. We shall also use the space $\dot{C}^k(\Omega)$ consisting of elements in $C^k(\bar{\Omega})$ vanishing outside a large sphere. If φ is a measurable function in Ω, locally bounded from above, we denote by $L^2(\Omega, \varphi)$ the space of functions in Ω which are square integrable with respect to the density $e^{-\varphi}$; the norm in $L^2_{(p,q)}(\Omega, \varphi)$ is defined by

$$\| f \|_\varphi^2 = \int | f(z) |^2 e^{-\varphi} dV, \quad f \in L^2_{(p,q)}(\Omega, \varphi), \tag{2.1.3}$$

where dV is the Lebesgue measure and

$$| f(z) |^2 = \langle f(z), f(z) \rangle = \sum{}' | f_{I,J}(z) |^2. \tag{2.1.4}$$

Finally, we write $L^2(\Omega, \mathrm{loc})$ for the space of functions which are square integrable on all compact subsets of Ω.

It is clear that $L^2_{(p,q)}(\Omega, \varphi)$ is a Hilbert space. If p and q are fixed with $q > 0$ we denote by T the maximal (weak) differential operator from $L^2_{(p,q-1)}(\Omega, \varphi)$ into $L^2_{(p,q)}(\Omega, \varphi)$ defined by $\bar{\partial}$; thus a form $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ is in D_T if and only if $\bar{\partial} u$, defined in the sense of distribution theory, belongs to $L^2_{(p,q)}(\Omega, \varphi)$. It is clear that T is closed and densely defined if φ is continuous. Similarly, $\bar{\partial}$ defines a closed and densely defined operator S from $L^2_{(p,q)}(\Omega, \varphi)$ to $L^2_{(p,q+1)}(\Omega, \varphi)$. By (2.1.2) we have

$$ST = 0, \tag{2.1.5}$$

which makes the results of section 1.1 applicable provided that we can prove the required estimates. To do so, we first need the following

PROPOSITION 2.1.1. *$\dot{C}^1_{(p,q)}(\bar\Omega)\cap D_{T^*}$ is dense in $D_{T^*}\cap D_S$ in the graph norm $f\to(\|f\|^2+\|T^*f\|^2+\|Sf\|^2)^{\frac12}$ if the boundary $\partial\Omega$ of Ω is of class C^2 and $\varphi\in C^1(\bar\Omega)$. Further $\dot{C}^1_{(p,q-1)}(\bar\Omega)$ is dense in D_T in the graph norm $f\to(\|f\|^2+\|Tf\|^2)^{\frac12}$.*

Proof. First note that if $\chi\in\dot{C}^\infty(\bar\Omega)$ and $f\in D_S$, then $\chi f\in D_S$ and

$$\|S(\chi f)-\chi Sf\|_\varphi\leqslant C\sup|\operatorname{grad}\chi|\;\|f\|_\varphi.$$

A similar result holds for T. From the fact that

$$|(\chi f,\,Tu)_\varphi-(f,\,T(\bar\chi u))_\varphi|\leqslant C\sup|\operatorname{grad}\chi|\;\|f\|_\varphi\|u\|_\varphi,\quad u\in D_T,$$

we also conclude that if $f\in D_{T^*}$ then $\chi f\in D_{T^*}$ and

$$\|T^*(\chi f)-\chi T^*f\|_\varphi\leqslant C\sup|\operatorname{grad}\chi|\;\|f\|_\varphi,\quad f\in D_{T^*}.$$

Now let $\chi\in C_0^\infty(\mathbf{C}^n)$ satisfy the condition $\chi(0)=1$ and set $\chi^\varepsilon(z)=\chi(\varepsilon z)$. If $f\in D_{T^*}\cap D_S$ it follows that $\chi^\varepsilon f\in D_{T^*}\cap D_S$ and that $\chi^\varepsilon f\to f$, $S(\chi^\varepsilon f)\to Sf$, $T^*(\chi^\varepsilon f)\to T^*f$ in the appropriate L^2 spaces when $\varepsilon\to 0$. To prove the theorem we therefore only have to approximate elements f in $D_{T^*}\cap D_S$ which vanish outside a large sphere. If we note that T^* is a differential operator with constant coefficients in the first order terms and continuous coefficients otherwise and that elements in D_{T^*} satisfy the Cauchy boundary conditions in the weak sense, the result then follows from Proposition 1.2.4. That the hypotheses of Proposition 1.2.4 are fulfilled is obvious in view of the unitary invariance of the $\bar\partial$-operator. The last statement follows in the same way from Proposition 1.2.3.

In what follows we assume throughout that the boundary $\partial\Omega$ of Ω is in C^2, and we denote by ϱ a real valued function in $C^2(\bar\Omega)$, which vanishes on $\partial\Omega$, is negative in Ω and satisfies the condition $|\operatorname{grad}\varrho|=1$ on $\partial\Omega$. These conditions imply that $\operatorname{grad}\varrho$ is the exterior unit normal on $\partial\Omega$, so Green's formula may be written in the following form when $v,w\in\dot{C}^1(\bar\Omega)$.

$$\int_\Omega\frac{\partial v}{\partial x_j}\bar w e^{-\varphi}dV=-\int_\Omega v\overline{\left(\frac{\partial w}{\partial x_j}-w\frac{\partial\varphi}{\partial x_j}\right)}e^{-\varphi}dV+\int_{\partial\Omega}\frac{\partial\varrho}{\partial x_j}v\bar w e^{-\varphi}dS,$$

where dS is the Euclidean surface element on $\partial\Omega$. Writing

$$\delta_j=\frac{\partial w}{\partial z_j}-w\frac{\partial\varphi}{\partial z_j}=e^\varphi\frac{\partial(we^{-\varphi})}{\partial z_j}\tag{2.1.6}$$

we obtain
$$\int_\Omega\frac{\partial v}{\partial\bar z_j}\bar w e^{-\varphi}dV=-\int_\Omega v\overline{\delta_j w}e^{-\varphi}dV+\int_{\partial\Omega}\frac{\partial\varrho}{\partial\bar z_j}v\bar w e^{-\varphi}dS.\tag{2.1.7}$$

For later reference we note that when $\varphi \in C^2$ we have the commutation relations

$$\left(\delta_k \frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial \bar{z}_j} \delta_k\right) w = w \frac{\partial^2 \varphi}{\partial z_k \partial \bar{z}_j}, \quad w \in C^2, \tag{2.1.8}$$

which imply the identities

$$\int_\Omega \delta_j v \overline{\delta_k w} e^{-\varphi} dV - \int_\Omega \frac{\partial v}{\partial \bar{z}_k} \overline{\frac{\partial w}{\partial \bar{z}_j}} e^{-\varphi} dV$$

$$= \int_\Omega v \bar{w} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} e^{-\varphi} dV + \int_{\partial\Omega} \frac{\partial \varrho}{\partial z_j} v \overline{\delta_k w} e^{-\varphi} dS$$

$$- \int_{\partial\Omega} \frac{\partial \varrho}{\partial \bar{z}_k} v \overline{\frac{\partial w}{\partial \bar{z}_j}} e^{-\varphi} dS; \quad v, w \in \dot{C}^1(\overline{\Omega}). \tag{2.1.8$'$}$$

In fact, (2.1.8)$'$ is an immediate consequence of (2.1.7) and (2.1.8) if $w \in \dot{C}^2(\overline{\Omega})$ and follows when $w \in \dot{C}^1(\overline{\Omega})$ since $\dot{C}^2(\overline{\Omega})$ is a dense subset.

We shall now describe explicitly the space $\dot{C}^1_{(p,q)}(\overline{\Omega}) \cap D_{T^*}$ occurring in Proposition 2.1.1. To do so, we form

$$(\bar\partial u, f)_\varphi = \int_\Omega \langle \bar\partial u, f \rangle e^{-\varphi} dV,$$

where $f \in \dot{C}^1_{(p,q)}(\overline{\Omega})$ and $u \in \dot{C}^1_{(p,q-1)}(\overline{\Omega})$. We shall move the differentiations from u to f. Writing $u = \sum' u_{I,K} dz^I \wedge d\bar{z}_K$, where $|I| = p$ and $|K| = q-1$, we have

$$\bar\partial u = (-1)^p \textstyle\sum'_{I,K} \sum_j \partial u_{I,K}/\partial \bar{z}_j dz^I \wedge d\bar{z}_j \wedge d\bar{z}^K,$$

which gives in view of (2.1.7)

$$(\bar\partial u, f) = (-1)^p \int_\Omega \sum_{I,K}{}' \sum_j \frac{\partial u_{I,K}}{\partial \bar{z}_j} \overline{f_{I,jK}} e^{-\varphi} dV$$

$$= (-1)^{p-1} \int_\Omega \sum_{I,K}{}' \sum_j u_{I,K} \overline{\delta_j f_{I,jK}} e^{-\varphi} dV + (-1)^p \int_{\partial\Omega} \sum_{I,K}{}' u_{I,K} \overline{\sum_j f_{I,jK} \frac{\partial \varrho}{\partial z_j}} e^{-\varphi} dS.$$

Since $\dot{C}^1_{(p,q-1)}$ is dense in D_T for the graph norm by Proposition 2.1.1, we conclude that an element $f \in \dot{C}^1_{(p,q)}(\overline{\Omega})$ belongs to D_{T^*} if and only if

$$\sum_{j=1}^n f_{I,jK} \frac{\partial \varrho}{\partial z_j} = 0 \quad \text{on } \partial\Omega \text{ for all } I \text{ and } K, \tag{2.1.9}$$

and then we have
$$T^* f = (-1)^{p-1} \sum_{I,k}{}' \sum_{j=1}^n \delta_j f_{I,jK} dz^I \wedge d\bar{z}^K. \tag{2.1.10}$$

If $f \in \dot{C}^1_{(p,q)}(\bar{\Omega}) \cap D_{T^*}$ we obtain from (2.1.10) and (2.1.1)

$$\| T^* f \|_\varphi^2 + \| Sf \|_\varphi^2 = \sum_{I,K}{}' \sum_{j,k} \int_\Omega \delta_j f_{I,jK} \overline{\delta_k f_{I,kK}} \, e^{-\varphi} dV + \sum_{I,J,L}{}' \sum_{j,l} \int_\Omega \frac{\partial f_{I,J}}{\partial \bar{z}_j} \overline{\frac{\partial f_{I,L}}{\partial \bar{z}_l}} \, \varepsilon^{jJ}_{lL} \, e^{-\varphi} dV, \tag{2.1.11}$$

where $\varepsilon^{jJ}_{lL} = 0$ unless $j \notin J$, $l \notin L$ and $\{j\} \cup J = \{l\} \cup L$, in which case ε^{jJ}_{lL} is the sign of the permutation $\binom{jJ}{lL}$. We shall rearrange the terms in the last sum. First consider the terms with $j = l$. Then $J = L$ and $j \notin J$ unless $\varepsilon^{jJ}_{lL} = 0$, so the sum of these terms is

$$\sum_{I,J}{}' \sum_{j \notin J} \int \left| \frac{\partial f_{I,J}}{\partial \bar{z}_j} \right|^2 e^{-\varphi} dV.$$

Next consider the terms with $j \neq l$. Then we have $l \in J$ and $j \in J$ if $\varepsilon^{jJ}_{lL} \neq 0$, and deletion of l from J or j from L gives the same multi-index K. Since

$$\varepsilon^{jJ}_{lL} = \varepsilon^{jJ}_{jlK} \, \varepsilon^{jlK}_{ljK} \, \varepsilon^{ljK}_{lL} = -\varepsilon^{J}_{lK} \, \varepsilon^{jK}_{L},$$

the sum of the terms in question is

$$-\sum_{I,K}{}' \sum_{j \neq l} \int_\Omega \frac{\partial f_{I,lK}}{\partial \bar{z}_j} \overline{\frac{\partial f_{I,jK}}{\partial \bar{z}_l}} e^{-\varphi} dV.$$

We can therefore rewrite (2.1.11) in the form

$$\| T^* f \|_\varphi^2 + \| Sf \|_\varphi^2 = \sum_{I,K}{}' \sum_{j,k} \int_\Omega \delta_j f_{I,jK} \overline{\delta_k f_{I,kK}} e^{-\varphi} dV$$
$$- \sum_{I,K}{}' \sum_{j,k} \int_\Omega \frac{\partial f_{I,jK}}{\partial \bar{z}_k} \overline{\frac{\partial f_{I,kK}}{\partial \bar{z}_j}} e^{-\varphi} dV + \sum_{I,J}{}' \sum_{j} \int_\Omega \left| \frac{\partial f_{I,J}}{\partial \bar{z}_j} \right|^2 e^{-\varphi} dV. \tag{2.1.12}$$

So far we have only reorganized the terms in (2.1.11). However, we shall now integrate by parts, moving all differentiations to the right. Using (2.1.8)′ and the boundary condition (2.1.9) we obtain

$$\| T^* f \|_\varphi^2 + \| Sf \|_\varphi^2 = \sum_{I,K}{}' \sum_{j,k} \int_\Omega f_{I,jK} \overline{f_{I,kK}} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} e^{-\varphi} dV$$
$$+ \sum_{I,J}{}' \sum_{j} \int_\Omega \left| \frac{\partial f_{I,J}}{\partial \bar{z}_j} \right|^2 e^{-\varphi} dV - \sum_{I,K}{}' \sum_{j,k} \int_{\partial\Omega} f_{I,jK} \frac{\partial \varrho}{\partial \bar{z}_k} \overline{\frac{\partial f_{I,kK}}{\partial \bar{z}_j}} e^{-\varphi} dS. \tag{2.1.13}$$

Now the function $\sum_k f_{I,kK}\partial\varrho/\partial z_k$ vanishes on $\partial\Omega$, so its gradiend is there proportional to grad ϱ. This means that for every boundary point there is a constant λ so that

$$\sum_k \left(\frac{\partial f_{I,kK}}{\partial \bar{z}_j}\frac{\partial \varrho}{\partial z_k} + f_{I,kK}\frac{\partial^2 \varrho}{\partial \bar{z}_j \partial z_k}\right) = \lambda \frac{\partial \varrho}{\partial \bar{z}_j}, \quad j=1, \ldots, n.$$

If we multiply by $\overline{f_{I,jK}}$ and add, we obtain in view of (2.1.9)

$$\sum_{j,k}\left(\overline{f_{I,jK}}\frac{\partial f_{I,kK}}{\partial \bar{z}_j}\frac{\partial \varrho}{\partial z_k} + \overline{f_{I,jK}} f_{I,kK}\frac{\partial^2 \varrho}{\partial \bar{z}_j \partial z_k}\right) = 0, \text{ on } \partial\Omega,$$

and using this equation in the last sum of (2.1.13) we have proved

PROPOSITION 2.1.2. *The following identy is valid when* $f \in \dot{C}^1_{(p,q)}(\bar{\Omega}) \cap D_{T^*}$

$$\|T^*f\|_\varphi^2 + \|Sf\|_\varphi^2 = \sum_{I,K}' \sum_{j,k} \int_\Omega f_{I,jK}\overline{f_{I,kK}} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} e^{-\varphi} dV$$
$$+ \sum_{I,J}' \sum_j \int_\Omega \left|\frac{\partial f_{I,J}}{\partial \bar{z}_j}\right|^2 e^{-\varphi} dV + \sum_{I,K}' \sum_{j,k} \int_{\partial\Omega} f_{I,jK}\overline{f_{I,kK}} \frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k} e^{-\varphi} dS. \quad (2.1.14)$$

The proof of this result has entirely followed the ideas of Morrey [22], Kohn [14] and Ash [2], the only difference being the introduction of the weight function $e^{-\varphi}$. However, we shall now see that the first sum on the right of (2.1.14), which is caused by the weight function $e^{-\varphi}$, is extremely useful in proving estimates, and makes it possible to simplify and extend the work just quoted which is based on the surface integral in (2.1.14). First we recall a definition.

Definition 2.1.3. The boundary $\partial\Omega$ of Ω is said to be *pseudo-convex* if at every point on $\partial\Omega$

$$\sum_{j,k=1}^n t_j \bar{t}_k \frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_k} \geqslant 0 \quad \text{if} \quad \sum_1^n t_j \frac{\partial \varrho}{\partial z_j} = 0. \quad (2.1.15)$$

Here $(t_1, \ldots, t_n)$ is a vector with complex components. If the hermitian form is strictly positive for all such $t \neq 0$, the boundary is called *strictly pseudo-convex*. Note that these definitions are independent of the choice of the function ϱ.

If $\partial\Omega$ is pseudo-convex, it follows from (2.1.9) that the last sum in (2.1.14) is non-negative, so we obtain

THEOREM 2.1.4. If $\partial\Omega$ *is pseudo-convex, we have when* $f \in \dot{C}^1_{(p,q)}(\bar{\Omega}) \cap D_{T^*}$

$$\int_\Omega \sum_{I,K}{}' \sum_{j,k} f_{I,jK} \overline{f_{I,kK}} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} e^{-\varphi} dV \leq \| T^* f \|_\varphi^2 + \| Sf \|_\varphi^2. \tag{2.1.16}$$

Remark. In the passage from (2.1.14) to (2.1.16) we have entirely neglected the terms in the second sum on the right-hand side of (2.1.14). We shall see in Chapter III that using the full force of these terms one can relax the hypotheses on $\partial\Omega$ very much.

To obtain a useful estimate from (2.1.16) we must of course choose φ so that the hermitian form

$$\sum_j \sum_k t_j \bar{t}_k \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \tag{2.1.17}$$

is positive definite at every point in Ω, that is, we have to choose the function φ strictly plurisubharmonic. (See e.g. Lelong [17].)

2.2. *Existence theorems*

Combination of Proposition 2.1.1 and Thorem 2.1.4 with the first part of Theorem 1.1.4 (with $F = H_2$) gives the following result:

THEOREM 2.2.1. *Let* Ω *be an open set in* $\mathbf{C}^n$ *with a* C^2 *pseudo-convex boundary. Let* $\varphi \in C^2(\bar{\Omega})$ *be strictly plurisubharmonic in* Ω *and let* $e^\varkappa$ *where* $\varkappa \in C(\Omega)$ *be the lowest eigenvalue of the matrix* $(\partial^2\varphi/\partial z_j \partial \bar{z}_k)$. *For every* $f \in L^2_{(p,q)}(\Omega, \varphi)$, $q > 0$, *such that* $\bar{\partial} f = 0$ *and*

$$\int_\Omega |f|^2 e^{-(\varphi+\varkappa)} dV < \infty$$

we can then find a form $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ *such that* $\bar{\partial} u = f$ *and*

$$q \int_\Omega |u|^2 e^{-\varphi} dV \leq \int_\Omega |f|^2 e^{-(\varphi+\varkappa)} dV. \tag{2.2.1}$$

We now wish to remove the hypotheses concerning the smoothness of $\partial\Omega$ and of φ in Theorem 2.2.1, which is quite easy because we have the estimate (2.2.1). First recall that in general a function φ with values in $[-\infty, +\infty)$ is called plurisubharmonic if it is semi-continuous from above and locally integrable, and the sum

$$\sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k,$$

defined in the sense of distribution theory, is a positive measure for arbitrary complex numbers t_j. In particular, $\partial^2\varphi/\partial z_j\partial\bar{z}_k$ is then a measure for all j and all k. We shall say that $e^{\varkappa}$ where $\varkappa\in C(\Omega)$ is a lower bound for the plurisubharmonicity of φ if the difference

$$\sum_{j,k=1}^{n}\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k}t_j\bar{t}_k-e^{\varkappa}\sum_{1}^{n}|t_j|^2$$

is a positive measure for arbitrary complex numbers t_j. We also have to extend Definition 2.1.3 so that not only domains with smooth boundaries are allowed:

Definition 2.2.2. An open set $\Omega\subset\mathbf{C}^n$ is called *pseudo-convex* if there exists a plurisubharmonic function σ in Ω such that $\Omega_M=\{z; z\in\Omega, \sigma(z)<M\}$ is relatively compact in Ω for every real number M.

It is a well-known and elementary fact that if $\partial\Omega\in C^2$ then $\partial\Omega$ is pseudo-convex in the sense of Definition 2.1.3 if and only if Ω is pseudo-convex in the sense of Definition 2.2.2. (Cf. Bremermann [5], Oka [26, 27].) If d is the distance to $\complement\Omega$ and Ω is pseudo-convex, then $\sigma(z)=|z|^2-\log d(z)$, is a continuous plurisubharmonic function satisfying the requirements in the definition.

THEOREM 2.2.1′. *Let Ω be a pseudo-convex open set in $\mathbf{C}^n$, let φ be plurisubharmonic in Ω and let $e^{\varkappa}$ where $\varkappa\in C(\Omega)$ be a lower bound for the plurisbuharmonicity of φ. For every $f\in L^2_{(p,q)}(\Omega,\mathrm{loc})$, $q>0$, such that $\bar\partial f=0$ and*

$$\int_{\Omega}|f|^2e^{-(\varphi+\varkappa)}dV<\infty,$$

one can then find a form $u\in L^2_{(p,q-1)}(\Omega,\varphi)$ such that $\bar\partial u=f$ and

$$q\int_{\Omega}|u|^2e^{-\varphi}dV\leqslant\int_{\Omega}|f|^2e^{-(\varphi+\varkappa)}dV. \tag{2.2.2}$$

Proof. We shall first solve the equation $\bar\partial u=f$ in a relatively compact open subset ω of Ω. Choose M so that $\sup_{\bar\omega}\sigma<M$, where σ is the function in Definition 2.2.2, and let $\delta>0$ be a lower bound for the distance from Ω_M to $\partial\Omega$. With a function $\chi\in C_0^\infty(\mathbf{C}^n)$, such that $\chi\geqslant 0$, $\int\chi dV=1$, $\chi(z)$ depends only on $|z|$ and vanishes when $|z|>1$, we put for $0<\varepsilon<\delta$ and $z\in\Omega_M$

$$\varphi_\varepsilon(z)=\int\varphi(z-\varepsilon z')\chi(z')dV(z'). \tag{2.2.3}$$

Then $\varphi_\varepsilon\in C^\infty(\Omega_M)$, φ_ε is plurisubharmonic and $\varphi_\varepsilon\searrow\varphi$ when $\varepsilon\searrow 0$. If we define $\varkappa_\varepsilon$ so that

$$e^{\varkappa_\varepsilon(z)} = \int e^{\varkappa(z-\varepsilon z')}\chi(z')dV(z'),$$

then $e^{\varkappa_\varepsilon}$ is a lower bound for the plurisubharmonicity of φ_ε and $\varkappa_\varepsilon \to \varkappa$ uniformly in Ω_M when $\varepsilon \to 0$.

Next define σ_ε by substituting σ for φ in (2.2.3). Then σ_ε is plurisubharmonic in Ω_M when $0<\varepsilon<\delta$. If $\sup_{\bar\omega}\sigma<m<M$ we have $\sigma_\varepsilon(z)<m<M$ for every $z\in\bar\omega$ if ε is small, and $\sigma_\varepsilon(z)<M$, $z\in\bar\Omega_M$ implies $z\in\Omega_M$. By a theorem of Morse [21], the set of all $t\in(m, M)$ such that there is a critical point for σ_ε with $\sigma_\varepsilon(z)=t$ is a set of measure 0. (Since $\sigma_\varepsilon\in C^\infty$ the result we need is in fact quite elementary.) For fixed small ε we can therefore choose t with $m<t<M$ so that

$$\Omega' = \{z;\, z\in\Omega_M,\, \sigma_\varepsilon(z)<t\}$$

has a C^∞ boundary. The boundary is then pseudo-convex in the sense of Definition 2.1.3. Application of Theorem 2.2.1 with Ω replaced by Ω' and φ replaced by φ_ε now shows that if f satisfies the hypotheses of Theorem 2.2.1′ we can find a form $u_\varepsilon\in L^2_{(p,q-1)}(\Omega', \varphi_\varepsilon)$ such that $\bar\partial u_\varepsilon = f$ in Ω' and

$$q\int_{\Omega'} |u_\varepsilon|^2 e^{-\varphi_\varepsilon} dV \leqslant \int_{\Omega'} |f|^2 e^{-(\varphi+\varkappa_\varepsilon)} dV.$$

Here we have used that $\varphi_\varepsilon \geqslant \varphi$. Since $\Omega' \supset \omega$ and φ_ε is uniformly bounded from above in ω we can find a weak limit u of u_ε in $L^2_{(p,q-1)}(\omega, 0)$ when $\varepsilon\to 0$. It is clear that $\bar\partial u = f$ in ω and since

$$\int_\omega |u|^2 e^{-\varphi_\delta} dV \leqslant \overline{\lim_{\varepsilon\to 0}} \int_\omega |u_\varepsilon|^2 e^{-\varphi_\delta} dV$$

for every $\delta>0$, we obtain

$$q\int_\omega |u|^2 e^{-\varphi} dV \leqslant \int_{\Omega_M} |f|^2 e^{-(\varphi+\varkappa)} dV.$$

Now let ω_ν be an increasing sequence of relatively compact open subsets of Ω with union equal to Ω. We have already proved that for every ν there is a solution of the equation $\bar\partial u = f$ in ω_ν such that the estimate (2.2.2) holds if the integration in the left-hand side is restricted to ω_ν. Taking again a weak limit when $\nu\to\infty$, we have proved the theorem.

We shall now give some consequences of Theorem 2.2.1′.

THEOREM 2.2.3. *Let Ω be a bounded pseudo-convex open set in $\mathbf{C}^n$, let $\delta = \sup_{z, z' \in \Omega} |z - z'|$ be the diameter of Ω, and let φ be a plurisubharmonic function in Ω. For every $f \in L^2_{(p,q)}(\Omega, \varphi)$, $q > 0$, with $\bar\partial f = 0$, one can then find $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ such that $\bar\partial u = f$ and*

$$q \int_\Omega |u|^2 e^{-\varphi} dV \leqslant e\delta^2 \int_\Omega |f|^2 e^{-\varphi} dV. \tag{2.2.4}$$

Proof. We may assume that $0 \in \Omega$, which implies that $|z| \leqslant \delta$ when $z \in \Omega$. With a positive constant a we now replace φ by $\varphi(z) + a|z|^2 = \varphi(z) + a(z_1\bar z_1 + \dots + z_n \bar z_n)$ in Theorem 2.2.1′. Then we can choose $e^\varkappa = a$, and Theorem 2.2.1′ gives that there exists a solution u of the equation $\bar\partial u = f$ such that

$$q \int_\Omega |u|^2 e^{-\varphi} dV \leqslant e^{a\delta^2} a^{-1} \int_\Omega |f|^2 e^{-\varphi} dV.$$

If we choose $a = \delta^{-2}$, the right-hand side attains its minimum with respect to a and the theorem is proved.

THEOREM 2.2.4. *If Ω is pseudo-convex, $f \in L^2_{(p,q)}(\Omega, \mathrm{loc})$, $q > 0$, and f satisfies the integrability condition $\bar\partial f = 0$, there exists a form $u \in L^2_{(p,q-1)}(\Omega, \mathrm{loc})$ such that $\bar\partial u = f$.*

Proof. It follows immediately from Definition 2.2.2 that we can find an increasing function χ of a real variable, vanishing for negative arguments, such that $f \in L^2_{(p,q)}(\Omega, \chi(\sigma))$. Since every such function has a convex increasing majorant, we may assume χ convex and increasing. But then $\chi(\sigma)$ is plurisubharmonic so it follows from Theorem 2.2.1′ with $\varphi(z) = \chi(\sigma(z)) + |z|^2$ that there is a form $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ such that $\bar\partial u = f$. This proves the theorem.

THEOREM 2.2.5 (*Cartan–Oka–Serre*). *If O is the sheaf of germs of holomorphic functions in Ω, we have $H^q(\Omega, O) = 0$, $q > 0$, for every pseudo-convex Ω.*

Proof. This follows immediately from Theorem 2.2.4 by the Dolbeault isomorphism, where of course we use the fine sheaf of germs of L^2 forms instead of the sheaf of germs of infinitely differentiable forms, which does not change the sheaf of germs of forms of type (0, 0) for which $\bar\partial u = 0$. See e.g. Malgrange [18].

We recall that Theorem 2.2.5 implies that the first Cousin problem in Ω can be solved and that the second Cousin problem is solvable when it is possible topologically. (See Cartan [7].) From Theorem 2.2.5 it is also easy to deduce that a pseudo-convex domain is a domain of holomorphy (see e.g. Bers [4] p. 74), so that these classes of domains are identical (the Levi problem). However, we shall give a different proof of this fact in the next section.

2.3. *Approximation theorems*

In this section we shall study the properties of the operator T^* which follow from Theorem 2.1.4. This leads to approximation theorems for the solutions of the equation $\bar\partial u=0$.

In the following theorem we use that $L^2_{(p,q-1)}(\Omega,-\varphi)$ and $L^2_{(p,q-1)}(\Omega,\varphi)$ are antiduals of each other with respect to the sesquilinear form

$$\langle u, v\rangle=\int_\Omega \sum_{I,K}{}' u_{I,K}\overline{v_{I,K}}\,dV;\quad u\in L^2_{(p,q-1)}(\Omega,-\varphi),\ v\in L^2_{(p,q-1)}(\Omega,\varphi).$$

PROPOSITION 2.3.1. *Let Ω be an open set in $\mathbf{C}^n$ with a C^2 pseudo-convex boundary. Let $\varphi\in C^2(\bar\Omega)$ be strictly plurisubharmonic in Ω, and let u be a form in $L^2_{(p,q-1)}(\Omega,-\varphi)$ such that $\langle u,v\rangle=0$ for every solution $v\in L^2_{(p,q-1)}(\Omega,\varphi)$ of the equation $\bar\partial v=0$. Then there exists a form $f\in L^2_{(p,q)}(\Omega,\text{loc})$ such that*

$$\vartheta f=(-1)^{p-1}\sum_{I,K}{}'\sum_j\frac{\partial f_{I,jK}}{\partial z_j}dz^I\wedge d\bar z^K=u, \tag{2.3.1}$$

where the first equality is a definition of ϑ, and

$$\int_\Omega\sum_{I,K}{}'\sum_{j,k}f_{I,jK}\overline{f_{I,kK}}\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}e^{\varphi}dV\leqslant\int_\Omega|u|^2e^{\varphi}dV. \tag{2.3.2}$$

Proof. If we put $U=ue^{\varphi}$, the hypotheses concerning u mean that $U\in L^2_{(p,q-1)}(\Omega,\varphi)$ and that $(U,v)_\varphi=0$ for every $v\in L^2_{(p,q-1)}(\Omega,\varphi)$ with $\bar\partial v=0$. With the notations used in section 2.1 this implies that U is in the closure of R_{T^*}. First assume that U belongs to R_{T^*}. Choose $F\in L^2_{(p,q)}(\Omega,\varphi)$ so that $T^*F=U$ and F is orthogonal to the null space of T^*. Then $SF=0$ so from Proposition 2.1.1 and Theorem 2.1.4 it follows that

$$\int_\Omega\sum_{I,K}{}'\sum_{j,k}F_{I,jK}\overline{F_{I,kK}}\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}e^{-\varphi}dV\leqslant\int_\Omega|U|^2e^{-\varphi}dV. \tag{2.3.3}$$

The equation $T^*F=U$ implies that $e^{\varphi}\vartheta(Fe^{-\varphi})=U$. If U is only in the closure of R_{T^*}, we take a sequence $U^\nu\to U$ in $L^2_{(p,q-1)}(\Omega,\varphi)$, with $U^\nu\in R_{T^*}$, and determine corresponding F^ν such that (2.3.3) holds with $F=F^\nu$, $U=U^\nu$, and $e^{\varphi}\vartheta(F^\nu e^{-\varphi})=U^\nu$. By (2.3.3) we can extract a subsequence which converges weakly in L^2 on all compact subsets of Ω, and for the limit F we have (2.3.3) and $e^{\varphi}\vartheta(Fe^{-\varphi})=U$. If we set $f=Fe^{-\varphi}$, the proposition is proved.

Remark. It would of course have been possible to show that f satisfies the boundary condition (2.1.9) in a weak sense. We shall not need this fact below but it could be used to give somewhat more precise theorems.

PROPOSITION 2.3.2. *Let the hypotheses on Ω and on φ in Proposition 2.3.1 be fulfilled and let $\psi \in C^2(\bar\Omega)$ be another strictly plurisubharmonic function. Let $u \in L^2_{(p,q-1)}(\Omega, -\varphi)$, let $u=0$ where $\psi>0$ and assume that $\langle u, v\rangle =0$ for every v such that $\bar\partial v=0$ and*

$$v \in L^2_{(p,q-1)}(\Omega, \varphi+\lambda\psi^+)$$

for some $\lambda>0$; here $\psi^+=\sup(\psi, 0)$. Then there is a form f satisfying (2.3.1) *and* (2.3.2) *which vanishes where $\psi>0$.*

Proof. Let $\chi \in C^2(\mathbf{R})$ be a convex function such that $\chi(t)=0$ when $t<0$ and $0<\chi'(t)\leqslant 1$ when $t>0$. With a positive parameter λ we set $\varphi_\lambda=\varphi+\lambda\chi(\psi)$. Then we have $\varphi\leqslant\varphi_\lambda\leqslant\varphi+\lambda\psi^+$, and using the convexity of χ we obtain

$$\sum_{j,k}\frac{\partial^2\varphi_\lambda}{\partial z_j\partial\bar z_k}t_j\bar t_k \geqslant \sum_{j,k}\frac{\partial^2\varphi}{\partial z_j\partial\bar z_k}t_j\bar t_k+\lambda\sum_{j,k}\chi'(\psi)\frac{\partial^2\psi}{\partial z_j\partial\bar z_k}t_j\bar t_k. \tag{2.3.4}$$

Now apply Proposition 2.3.1 with φ replaced by φ_λ. Since $\varphi\leqslant\varphi_\lambda$ with equality in the support of u, it follows that for every λ one can find $f=f^\lambda$ such that (2.3.1) and (2.3.2) hold, and in addition

$$\lambda\int_\Omega \sum_{I,K}{}' \sum_{j,k} f^\lambda_{I,jK}\overline{f^\lambda_{I,kK}}\,\chi'(\psi)\frac{\partial^2\psi}{\partial z_j\partial\bar z_k}e^\varphi dV \leqslant \int_\Omega |u|^2e^\varphi dV.$$

Hence $f^\lambda\to 0$ on every compact subset of $\{z; z\in\Omega, \psi(z)>0\}$ when $\lambda\to+\infty$. Since f^λ satisfies (2.3.2) for every λ we can find a weak limit f of f^λ when $\lambda\to+\infty$, and f also satisfies (2.3.1) and (2.3.2). When $\psi>0$ we have $f=0$ so this proves the theorem.

We shall now derive an approximation theorem from Proposition 2.3.2. It is then convenient to use the following terminology.

Definition 2.3.3. A compact subset K of an open set $\Omega\subset\mathbf{C}^n$ is called *pseudo-convex with respect to* Ω if for every $z\in\Omega\cap\complement K$ there is a plurisubharmonic function ψ in Ω such that $\psi(z)>0$ but $\psi<0$ in K.

LEMMA 2.3.4. *Let K be a compact set which is pseudo-convex with respect to a pseudo-convex open set $\Omega\supset K$, and let ω be an open neighborhood of K. Then there exists a continuous plurisubharmonic function ψ in Ω such that $\psi<0$ in K but $\psi>0$ in $\Omega\cap\complement\omega$; moreover, ψ can be chosen so that $\{z; z\in\Omega, \psi(z)<M\}$ is relatively compact in Ω for every M.*

Proof. Let σ be a continuous function satisfying the requirements in Definition 2.2.2. Adding a constant to σ, if necessary, we may assume that $\sigma<0$ in K. Set

$$K'=\{z; z\in\Omega, \sigma(z)\leqslant 2\} \text{ and } L=\{z; z\in\Omega\cap\complement\omega, \sigma(z)\leqslant 0\};$$

these sets are both compact. For every $z \in L$ we can choose a function ψ which is plurisubharmonic in Ω, so that $\psi(z) > 0$ and $\psi < 0$ in K. Forming a regularization of ψ as in the proof of Theorem 2.2.1′ we obtain a continuous plurisubharmonic function ψ', defined in a neighborhood of K', such that $\psi' < 0$ in K and $\psi' > 0$ in a neighborhood of z. Since L is compact we conclude, using the Borel–Lebesgue lemma and the fact that the supremum of a finite family of plurisubharmonic functions is plurisubharmonic, that there is a continuous plurisubharmonic function ψ_1 in a neighborhood of K', such that $\psi_1 > 0$ in a neighborhood of L and $\psi_1 < 0$ in K. Let C be the maximum of ψ_1 in K', and set for $z \in \Omega$

$$\psi(z) = \sup(\psi_1(z), C\sigma(z)) \quad \text{if} \quad \sigma(z) < 2; \quad \text{and} \quad \psi(z) = C\sigma(z) \quad \text{if} \quad \sigma(z) > 1.$$

The two definitions agree when $1 < \sigma(z) < 2$, so ψ is a continuous plurisubharmonic function in Ω. It is obvious that ψ has all the required properties.

THEOREM 2.3.5. *Let Ω be an open pseudo-convex set in $\mathbf{C}^n$ and let K be a compact subset of Ω which is pseudo-convex with respect to Ω. Let $u \in L^2_{(p,q-1)}(K, 0)$ and let $\bar\partial u = 0$ on K in the strong sense that $\int_K \langle u, \vartheta f\rangle dV = 0$ for every $f \in L^2_{(p,q)}(\Omega, 0)$ such that $f = 0$ outside K and $\vartheta f \in L^2_{(p,q-1)}(\Omega, 0)$. Then one can approximate u arbitrarily closely in $L^2_{(p,q-1)}(K, 0)$ by forms $u' \in L^2_{(p,q-1)}(\Omega, \text{loc})$ such that $\bar\partial u' = 0$.*

Remarks. (1) Note that the assumption on u is satisfied if $u \in L^2_{(p,q-1)}$ and $\bar\partial u = 0$ in a neighborhood of K, for then we have $\langle u, \vartheta f\rangle = \langle \bar\partial u, f\rangle = 0$. If K is the closure of an open set with C^1 boundary and $\bar\partial u = 0$ only in this open set the assumption is also fulfilled in view of Proposition 1.2.3.

(2) Since Ω is pseudo-convex it follows from Definition 2.2.2 that Ω is the union of an increasing sequence of compact subsets which are pseudo-convex with respect to Ω.

Before proving the theorem we note that Theorem 2.3.5 implies the Oka-Weil approximation theorem.

COROLLARY 2.3.6. *Let Ω be a pseudo-convex open set in $\mathbf{C}^n$ and let K be a compact subset which is pseudo-convex with respect to Ω. If u is a function which is analytic in a neighborhood of K, it is possible to approximate u arbitrarily closely in the maximum norm over K by functions which are analytic in Ω.*

Proof. Let u be analytic in the open set $\omega \supset K$ and choose ψ according to Lemma 2.3.4. For sufficiently small $\varepsilon > 0$ the set $K_\varepsilon = \{z; \psi(z) \leqslant -\varepsilon\}$ is then a compact subset of ω which is a neighborhood of K, and K_ε is obviously pseudo-convex with respect to Ω. Theorem 2.3.5 with $p = q = 0$ shows that there is a sequence $u^j \in L^2(\Omega, \text{loc})$ with $\bar\partial u^j = 0$

such that $u^j \to u$ in $L^2(K_\varepsilon)$. But every u^j is by Weyl's lemma an analytic function in Ω (after correction on a null set), and $u^j \to u$ uniformly on compact subsets of the interior of K_ε. This proves the corollary.

Proof of Theorem 2.3.5. Let Ω' be a relatively compact open subset of Ω with a C^2 pseudo-convex boundary, and let $K \subset \Omega'$. We shall first prove that u can be approximated by solutions $u' \in L^2_{(p,\, q-1)}(\Omega', 0)$ of the equation $\bar\partial u' = 0$. Let therefore U be an arbitrary form in $L^2_{(p,\, q-1)}(\Omega', 0)$ such that $U = 0$ outside K and $\langle U, u' \rangle = 0$ for every such u'. If we can prove that $U = \vartheta f$ for some $f \in L^2_{(p,\, q)}(\Omega', 0)$ vanishing outside K, it will follow that $\langle U, u \rangle = \langle \vartheta f, u \rangle = 0$, and Hahn–Banach's theorem then gives that u can be approximated by the forms u' in question.

To prove the existence of f we let K' be a compact neighborhood of K contained in Ω'. Taking a regularization of the function ψ in Lemma 2.3.4 and adding a small multiple of $|z|^2$ we can construct a strictly plurisubharmonic function $\psi \in C^\infty(\Omega')$ such that $\psi < 0$ in K but $\psi > 0$ outside K'. We can therefore apply Proposition 2.3.2, with $\varphi(z) = |z|^2$, for example, and Ω replaced by Ω'. It follows that there is a form $f \in L^2_{(p,\, q)}(\Omega', 0)$ with $\vartheta f = U$, such that $f = 0$ outside K' and

$$q \int_{\Omega'} |f|^2 e^{|z|^2} dV \leqslant \int_{\Omega'} |U|^2 e^{|z|^2} dV.$$

We now take a decreasing sequence of compact neighborhoods K^j of K, all contained in Ω', and for each of them we choose a form f^j with $\vartheta f^j = U$ and $f^j = 0$ outside K^j so that the estimate just given holds for each f^j. If f is a weak limit of f^j when $j \to \infty$, we obtain $\vartheta f = U$ and $f = 0$ outside K.

To complete the proof we have to approximate a general pseudo-convex Ω by subsets of the type just discussed. Let σ be a continuous plurisubharmonic function in Ω such that $\Omega_M = \{z;\, z \in \Omega,\, \sigma(z) < M\}$ is relatively compact in Ω for every M. We may assume that $K \subset \Omega_1 \subset \bar\Omega_1 \subset \Omega_2 \subset \bar\Omega_2 \subset \ldots$. The construction used in the proof of Theorem 2.2.1′ shows that there exist pseudo-convex open sets ω_j with C^2 boundaries such that

$$\bar\Omega_j \subset \omega_j \subset \Omega_{j+1}, \quad j = 1, 2, \ldots.$$

For every $\varepsilon > 0$ there is a sequence of forms $u^j \in L^2_{(p,\, q-1)}(\omega_j, \mathrm{loc})$ such that $\bar\partial u^j = 0$ and

$$\int_K |u^1 - u|^2 dV < \tfrac{1}{4}\varepsilon^2, \quad \int_{\Omega_j} |u^{j+1} - u^j|^2 dV < \varepsilon^2 4^{-j-1}, \quad j = 1, 2, \ldots.$$

In fact, the compact subset $\{z;\, \sigma(z) \leqslant j\}$ of ω_j contains Ω_j and since it is pseudo-convex with respect to ω_{j+1} the existence of u^{j+1} follows from the first part of the proof if u^j is already selected. The estimates just given imply that $u' = \lim_{j\to\infty} u^j$ exists in $L^2_{(p,q-1)}(\Omega, \text{loc})$. Since $\bar\partial u' = 0$ and $\int_K |u-u'|^2 dV < \varepsilon^2$, the proof is complete.

We shall now give the solution of the Levi problem in the case considered here.

Definition 2.3.7. A compact subset K of an open set Ω in $\mathbf{C}^n$ is called *holomorph-convex with respect to* Ω if for every $z \in \Omega \cap \complement K$ there is an analytic function ψ in Ω such that $|\psi(z)| > 1$ but $|\psi| < 1$ in K.

Since $\log |\psi|$ is plurisubharmonic, comparison with Definition 2.3.3 shows that holomorph-convexity implies pseudo-convexity. Conversely, we have

THEOREM 2.3.8. *If Ω is a pseudo-convex open set in $\mathbf{C}^n$ and K is a compact subset which is pseudo-convex with respect to Ω, then K is holomorph-convex with respect to Ω.*

Before the proof we note that the theorem implies the following essentially equivalent result of Oka [26], [27], Bremermann [6] and Norguet [25].

COROLLARY 2.3.9. *An open set $\Omega \subset \mathbf{C}^n$ is pseudo-convex if and only if it is a domain of holomorphy.*

Proof. It is an elementary fact that every domain of holomorphy is pseudo-convex (see Bremermann [5]). Conversely, if Ω is pseudo-convex and σ is a continuous plurisubharmonic function satisfying the conditions in Definition 2.2.2, then $\{z; \sigma(z) \leqslant M\}$ is pseudo-convex, hence holomorph-convex, with respect to Ω for every M. Since this set contains an arbitrary compact subset of Ω when M is sufficiently large, it follows from a classical theorem of Cartan–Thullen that Ω is a domain of holomorphy (see [7]).

Proof of Theorem 2.3.8. It is sufficient to prove that if $0 \in \Omega \cap \complement K$ there is a function $u \in A(\Omega)$ such that $\sup_K |u| < |u(0)|$. By Lemma 2.3.4 there is a continuous plurisubharmonic function σ in Ω such that $\sigma < 0$ in K, $\sigma(0) = 0$ and

$$\Omega_c = \{z; z \in \Omega, \sigma(z) < c\} \subset\subset \Omega \quad \text{for all} \quad c \in \mathbf{R}.$$

By means of a regularization we can, as in the proof of Theorem 2.2.1′, approximate σ by a strictly plurisubharmonic C^∞ function φ in Ω_1 so closely that $\varphi < 0$ in K, $\varphi(0) = 0$, but $\varphi > 0$ outside a compact subset of Ω_1. Taylor's formula gives

$$\varphi(z) = \operatorname{Re} w(z) + \sum_{j,k=1}^{n} \frac{\partial^2 \varphi(0)}{\partial z_j\, \partial \bar z_k} z_j \bar z_k + o(|z|^2),$$

where w is an analytic second degree polynomial vanishing at 0. Since the hermitian form is positive definite, we conclude that there is a neighborhood ω_0 of 0 such that $\varphi(z) > \operatorname{Re} w(z)$

if $0 \neq z \in \omega_0$. We may assume that $\omega_0 \cap K = \emptyset$. If ω_1 and ω_2 are neighborhoods of 0 such that $\omega_2 \subset\subset \omega_1 \subset\subset \omega_0$, it follows that we can choose $\varepsilon > 0$ and $\delta > 0$ so that

$$\operatorname{Re} w(z) < -\varepsilon \quad \text{if} \quad z \in \omega_1 \cap \complement\omega_2 \quad \text{and} \quad \varphi(z) < \delta. \tag{2.3.6}$$

Let $\Omega' = \{z; z \in \Omega_1, \varphi(z) < \delta\}$. This is a pseudo-convex open set since $(1-\sigma)^{-1} + (\delta - \varphi)^{-1}$ is plurisubharmonic in Ω'. With a positive parameter t and a function $\chi \in C_0^\infty(\omega_1)$ which is equal to 1 in ω_2 we now set

$$u_t = \chi e^{tw} - v_t$$

where v_t shall be chosen so that $u_t \in A(\Omega')$, that is, so that

$$\bar\partial v_t = \bar\partial \chi e^{tw}. \tag{2.3.7}$$

Since $\operatorname{Re} w < -\varepsilon$ if $z \in \Omega' \cap \operatorname{supp} \bar\partial\chi$, in view of (2.3.6), the L^2 norm of the right-hand side of (2.3.7) is $O(e^{-t\varepsilon})$. Hence it follows from Theorem 2.2.3 that one can find a solution v_t of (2.3.7) such that

$$\int_{\Omega'} |v_t|^2 dV \leqslant C e^{-2t\varepsilon}.$$

Since v_t is analytic in a neighborhood of K and of 0, it follows that $v_t(z) = O(e^{-\varepsilon t})$ uniformly when $z \in K \cup \{0\}$ and $t \to \infty$. For large t we obtain

$$\sup_K |u_t| < |u_t(0)|.$$

Now $\{z; z \in \Omega_1, \varphi(z) \leqslant 0\}$ is a compact pseudo-convex set relative Ω_1, and it is contained in Ω', so Theorem 2.3.5 shows that u_t can be approximated uniformly on this set by functions in $A(\Omega_1)$. Theorem 2.3.5 also implies that functions in $A(\Omega_1)$ can be approximated uniformly on $\{z; z \in \Omega, \sigma(z) \leqslant 0\}$ by functions in $A(\Omega)$. Hence one can find $u \in A(\Omega)$ so close to u_t that

$$\sup_K |u| < |u(0)|.$$

The proof is complete.

2.4. *Cohomology with bounds*

Theorem 2.2.5 was obtained as a consequence of Theorem 2.2.4 which is a considerably weakened form of Theorem 2.2.1'. We shall now give an example of the analogous results which follows by direct application of Theorem 2.2.1' and the usual proof of the Dolbeault isomorphism. In order to obtain shorter statements we only consider the case $\Omega = \mathbb{C}^n$

which has a particular interest in certain applications. (See Ehrenpreis [9], Malgrange [19].) However, there is no difficulty in proving corresponding results for an arbitrary pseudo-convex open set.

Let $\{\Omega_\nu\}$, $\nu=1,2,\ldots$, be a fixed covering of $\mathbf{C}^n$ by open subsets, that is, $\bigcup_1^\infty \Omega_\nu = \mathbf{C}^n$. If s is an integer $\geqslant 0$ we denote by $C^s(Z_{(p,q)}(\{\Omega_\nu\},\varphi))$ the set of all alternating cochains $c=\{c_\alpha\}$ where $\alpha=(\alpha_0,\ldots,\alpha_s)$ is an $(s+1)$-tuple of positive integers, $c_\alpha \in L^2_{(p,q)}(\Omega_\alpha)$, $\Omega_\alpha = \Omega_{\alpha_0}\cap\ldots\cap\Omega_{\alpha_s}$, $\bar\partial c_\alpha = 0$ and

$$\|c\|_\varphi^2 = \sum_\alpha{}' \int_{\Omega_\alpha} |c_\alpha|^2 e^{-\varphi}\,dV < \infty.$$

As usual we define the coboundary operator δ from $C^s(Z_{(p,q)}(\{\Omega_\nu\},\varphi))$ to $C^{s+1}(Z_{(p,q)}(\{\Omega_\nu\},\varphi))$ by

$$(\delta c)_{\alpha_0,\ldots,\alpha_{s+1}} = \sum_{j=0}^{s+1}(-1)^j c_{\alpha_0,\ldots,\hat\alpha_j,\ldots,\alpha_{s+1}}$$

where $\hat\alpha_j$ means that the index α_j shall be deleted.

THEOREM 2.4.1. *Assume that the covering* $\{\Omega_\nu\}$ *has the following three properties:*

(i) *All* Ω_ν *are pseudo-convex and the diameter of* Ω_ν *is bounded by a constant independent of* ν.

(ii) *There is an integer* N *such that more than* N *different sets* Ω_ν *always have an empty intersection.*

(iii) *There exists a partition of unity* $\chi_\nu \in C_0^\infty(\Omega_\nu)$ *such that* $\sum \chi_\nu = 1$, $\chi_\nu \geqslant 0$ *and* $\sum_\nu |\bar\partial \chi_\nu| \leqslant$ constant.

Let φ *be a plurisubharmonic function and let* $\varkappa$ *be a continuous function* $\leqslant 0$ *such that* $e^\varkappa$ *is a lower bound for the plurisubharmonicity of* φ. *For every* $c \in C^s(Z_{(p,q)}(\{\Omega_\nu\},\varphi+\varkappa))$ *with* $\delta c = 0$, $s \geqslant 1$ *one can then find a cochain* $c' \in C^{s-1}(Z_{(p,q)}(\{\Omega_\nu\},\varphi))$ *such that* $\delta c' = c$ *and* $\|c'\|_\varphi \leqslant K\|c\|_{\varphi+\varkappa}$, *where the constant* K *does not depend on* c.

We are of course mainly interested in the case $p=q=0$ but the general statement is needed in the proof.

Proof. If we set

$$b_\alpha = \sum_j \chi_j c_{j,\alpha}, \quad |\alpha| = s,$$

with the product $\chi_j c_{j,\alpha}$ defined as 0 outside the support of χ_j, we obtain a form $b_\alpha \in L^2_{(p,q)}(\Omega_\alpha)$, and for the $(s-1)$ cochain thus defined we have $\delta b = c$. In fact, using the assumption that $\delta c = 0$ we find

$$(\delta b)_{\alpha_0,\ldots,\alpha_s} = \sum_j \sum_{k=0}^{s}(-1)^k \chi_j c_{j,\alpha_0,\ldots\hat\alpha_k\ldots,\alpha_s} = \sum_j \chi_j c_{\alpha_0,\ldots,\alpha_s} = c_{\alpha_0,\ldots,\alpha_s}.$$

Cauchy's inequality and the equation $\sum \chi_j = 1$ give

$$\int_{\Omega_\alpha} |b_\alpha|^2 e^{-\varphi-\varkappa} dV \leqslant \sum_j \int_{\Omega_\alpha} \chi_j |c_{j,\alpha}|^2 e^{-\varphi-\varkappa} dV.$$

If we sum over all increasing α and use the fact that the square of the norm of a fixed component $c_{\alpha_0,\dots,\alpha_s}$ occurs with a coefficient $\leqslant 1$, we obtain

$$\sum_{|\alpha|=s-1}' \int_{\Omega_\alpha} |b_\alpha|^2 e^{-\varphi-\varkappa} dV \leqslant \sum_{|\alpha|=s}' \int_{\Omega_\alpha} |c_\alpha|^2 e^{-\varphi-\varkappa} dV. \tag{2.4.2}$$

Obviously, we do not necessarily have $\bar\partial b_\alpha = 0$. However, since $\bar\partial c = 0$ and $c = \delta b$, we know that $\delta\bar\partial b = 0$, where

$$\bar\partial b_\alpha = \sum_j \bar\partial \chi_j \wedge c_{j,\alpha}.$$

Using (ii) and (iii) we obtain with a constant C

$$\sum_{|\alpha|=s}' \int_{\Omega_\alpha} |\bar\partial b_\alpha|^2 e^{-\varphi-\varkappa} dV \leqslant C^2 \| c \|^2_{\varphi+\varkappa}. \tag{2.4.3}$$

First assume that $s = 1$. Then the fact that $\delta\bar\partial b = 0$ means that $\bar\partial b$ defines a global form of type $(p, q+1)$, and for this form, which we denote by f, we have the estimate

$$\int |f|^2 e^{-\varphi-\varkappa} dV \leqslant C^2 \| c \|^2_{\varphi+\varkappa}. \tag{2.4.3$'$}$$

Since $\bar\partial f = 0$ it follows from Theorem 2.2.1′ that there is a form $u \in L^2_{(p,q)}(\mathbf{C}^n, \varphi)$ such that $\bar\partial u = f$ and

$$\int |u|^2 e^{-\varphi} dV \leqslant \int |f|^2 e^{-\varphi-\varkappa} dV \leqslant C^2 \| c \|^2_{\varphi+\varkappa}. \tag{2.4.4}$$

By condition (ii) this implies that

$$\sum_\nu \int_{\Omega_\nu} |u|^2 e^{-\varphi} dV \leqslant NC^2 \| c \|^2_{\varphi+\varkappa}. \tag{2.4.5}$$

Now we have $\bar\partial b = f = \bar\partial u$. If we put $c'_\nu = b_\nu - u$ in Ω_ν, we therefore obtain a cochain in $C^0(Z_{(p,q)}(\Omega, \varphi))$ such that $\delta c' = c$, and from the estimates (2.4.2) and (2.4.5) it follows that $\|c'\|_\varphi \leqslant K \|c\|_{\varphi+\varkappa}$, if we recall that $\varkappa \leqslant 0$.

Next we consider the case $s > 1$. In doing so we first note that finite intersections of the sets Ω_ν are also pseudo-convex. In fact, this is an immediate consequence of Definition

2.2.2, since the sum of a finite number of plurisubharmonic functions is plurisubharmonic. We may of course assume that the theorem has already been proved for smaller values of s. Now $\bar\partial b \in C^{s-1}(Z_{(p,q+1)}(\{\Omega_\nu\}, \varphi+\varkappa))$ and $\delta\bar\partial b = \bar\partial c = 0$, so it follows from the inductive hypothesis that there is a cochain $b' \in C^{s-2}(Z_{(p,q+1)}(\{\Omega_\nu\}, \varphi))$ such that $\delta b' = \bar\partial b$, and there is an estimate for b' of the form

$$\|b'\|_\varphi \leqslant K\|\bar\partial b\|_{\varphi+\varkappa}. \tag{2.4.6}$$

Since $\bar\partial b'_\alpha = 0$ and Ω_α is pseudo-convex, Theorem 2.2.3 implies in view of the hypothesis (i) that we can choose a form $b''_\alpha \in L^2_{(p,q)}(\Omega_\alpha)$ such that $b'_\alpha = \bar\partial b''_\alpha$ and

$$\int_{\Omega_\alpha} |b''_\alpha|^2 e^{-\varphi} dV \leqslant K \int_{\Omega_\alpha} |b'_\alpha|^2 e^{-\varphi} dV, \tag{2.4.7}$$

where K is a constant. But then we have $\bar\partial(b-\delta b'') = \bar\partial b - \delta b' = 0$, so that with $c' = b - \delta b''$ we obtain $\bar\partial c' = 0$ and $\delta c' = \delta b = c$. Furthermore, the estimates (2.4.2), (2.4.3), (2.4.6) and (2.4.7) give the estimate $\|c'\|_\varphi \leqslant K\|c\|_{\varphi+\varkappa}$ for some constant K, which completes the proof.

2.5. *Some applications*

The purpose of this section is only to give some examples of constructions of analytic functions which can be based on the results of sections 2.2 and 2.3. For this reason we do not state the results in as general a form as possible.

THEOREM 2.5.1. *Let φ be a plurisubharmonic function in $\mathbf{C}^n$ such that for some constant C*

$$|\varphi(z+z') - \varphi(z)| \leqslant C, \quad |z'| \leqq 1. \tag{2.5.1}$$

Let Σ be a complex linear subspace of $\mathbf{C}^n$ of codimension k and denote the Lebesgue measure in Σ by $d\sigma$. Then, for every analytic function u in Σ such that

$$\int_\Sigma |u|^2 e^{-\varphi} d\sigma < \infty, \tag{2.5.2}$$

there exists an analytic function U in $\mathbf{C}^n$ such that $U = u$ in Σ and

$$\int_{\mathbf{C}^n} |U|^2 e^{-\varphi}(1+|z|^2)^{-3k} dV \leqslant K \int_\Sigma |u|^2 e^{-\varphi} d\sigma \tag{2.5.3}$$

where K is a constant independent of u.

Proof. First note that $\log(1+|z|^2)$ is strictly plurisubharmonic, for

$$\sum_{j,k=1}^{n} t_j \bar{t}_k \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log(1+|z|^2) = (1+|z|^2)^{-2}(|t|^2(1+|z|^2) - |\langle t,z\rangle|^2) \geqslant (1+|z|^2)^{-2}|t|^2 \tag{2.5.4}$$

by Cauchy's inequality. It is therefore enough to prove the theorem when Σ is a hyperplane and iterate this special result k times. We may of course assume that Σ is the hyperplane $z_n = 0$. Then u is an analytic function of $z' = (z_1, \ldots, z_{n-1})$ which we may regard as an analytic function in $\mathbf{C}^n$ which is independent of z_n. By (2.5.1) we have

$$\int_{|z_n|<1} |u|^2 e^{-\varphi} dV \leqslant \pi e^C \int_\Sigma |u|^2 e^{-\varphi} d\sigma. \tag{2.5.5}$$

Let $\psi \in C_0^\infty(\mathbf{C})$ be equal to 1 in the disc with radius $\frac{1}{2}$ and center at 0 and let $\psi = 0$ outside the unit disc. Writing

$$U(z) = \psi(z_n)u(z') - z_n v(z),$$

we have $U(z) = u(z')$ when $z_n = 0$ so it only remains to show that v can be chosen so that $\bar\partial U = 0$ and U has the required bound. The equation $\bar\partial U = 0$ is equivalent to

$$\bar\partial v = z_n^{-1} u(z') \bar\partial \psi(z_n) = z_n^{-1} u(z') \frac{\partial \psi}{\partial \bar{z}_n} d\bar{z}_n = f. \tag{2.5.6}$$

It is clear that $\bar\partial f = 0$, for $\partial\psi/\partial\bar{z}_n = 0$ when $|z_n| < \frac{1}{2}$, and if C_1 is an upper bound for $|\partial\psi/\partial\bar{z}_n|$, we have by (2.5.5)

$$\int_{\mathbf{C}^n} |f|^2 e^{-\varphi} dV \leqslant (2C_1)^2 \int_{|z_n|<1} |u|^2 e^{-\varphi} dV \leqslant \pi e^C (2C_1)^2 \int_\Sigma |u|^2 e^{-\varphi} d\sigma.$$

We now apply Theorem 2.2.1′ with $\Omega = \mathbf{C}^n$ and with φ replaced by $\varphi + 2\log(1+|z|^2)$. By (2.5.4) and the plurisubharmonicity of φ we can choose $e^\varkappa = 2(1+|z|^2)^{-2}$ and conclude that (2.5.6) has a solution v such that

$$2\int_{\mathbf{C}^n} |v|^2 e^{-\varphi}(1+|z|^2)^{-2} dV \leqslant \int_{\mathbf{C}^n} |f|^2 e^{-\varphi} dV. \tag{2.5.7}$$

From (2.5.5) and (2.5.7), the estimate (2.5.3) with $k = 1$ follows immediately.

Theorem 2.5.1 can be used to prove an extension to several variables of a theorem of Pólya [28], which was given by Martineau [20] and is also included in the fundamental principle of Ehrenpreis [9]. Let us first recall som basic definitions. By $\mathcal{A}$ we denote the set of all entire analytic functions in $\mathbf{C}^n$. This is a Fréchet space with the topology of uniform convergence on all compact sets. The dual space is denoted by $\mathcal{A}'$. If $\mu \in \mathcal{A}'$, the Laplace transform is defined by $\tilde{\mu}(\zeta) = \mu_z(\exp[z,\zeta])$ where $[z,\zeta] = z_1\zeta_1 + \ldots + z_n\zeta_n$. It is obvious that $\tilde{\mu}$ is an entire function of exponential type, and $\tilde{\mu}$ determines μ uniquely since linear combinations of exponential functions are dense in $\mathcal{A}$. Conversely, every entire function of exponential type is the Laplace transform of an element $\mu \in \mathcal{A}'$. (This is also a consequence of the proof of Theorem 2.5.2.)

Let K be a compact set. We shall say that μ is carried by K if for every neighborhood ω of K there is a constant C such that

$$|\mu(f)| \leqslant C \sup_{\omega} |f|, \quad f \in \mathcal{A}. \tag{2.5.8}$$

Set

$$H_K(\zeta) = \sup_{z \in K} \operatorname{Re}[z, \zeta]. \tag{2.5.9}$$

THEOREM 2.5.2 (*Pólya–Ehrenpreis–Martineau*). *A functional $\mu \in \mathcal{A}'$ is carried by the convex compact set K if and only if for every $\delta > 0$ there is a constant C_δ such that*

$$|\tilde{\mu}(\zeta)| \leqslant C_\delta \exp(H_k(\zeta) + \delta|\zeta|), \quad \zeta \in \mathbf{C}^n. \tag{2.5.10}$$

Proof. The necessity is an obvious consequence of (2.5.8) and (2.5.9). In proving the sufficiency we wish to construct for every $\varepsilon > 0$ a Schwartz distribution ν with support in the set K_ε of points with distance $\leqslant \varepsilon$ from K, so that $\nu(f) = \mu(f)$, $f \in \mathcal{A}$, This will prove the theorem, for every distribution with compact support defines an analytic functional carried by the support of the distribution. In fact, the derivatives of an analytic function in a compact set can be estimated by the maximum of the function in a neighborhood of that set. Let $\hat{\nu}$ be the Fourier–Laplace transform of ν, which is an analytic function of $2n$ complex variables $\theta_1, \ldots, \theta_{2n}$ defined by

$$\hat{\nu}(\theta_1, \ldots, \theta_{2n}) = \nu(\exp(-ix_1\theta_1 - \ldots - ix_{2n}\theta_{2n})).$$

The analytic functional μ is defined by ν if and only if

$$\tilde{\mu}(\zeta_1, \ldots, \zeta_n) = \nu(\exp((x_1 + ix_2)\zeta_1 + \ldots + (x_{2n-1} + ix_{2n})\zeta_n)).$$

(Recall that the real and complex coordinate in $\mathbf{C}^n$ are related by $z_k = x_{2k-1} + ix_{2k}$.) Thus we must have

$$\hat{\nu}(i\zeta_1, -\zeta_1, i\zeta_2, -\zeta_2, \ldots, i\zeta_n, -\zeta_n) = \tilde{\mu}(\zeta_1, \ldots, \zeta_n), \quad \zeta \in \mathbf{C}^n. \tag{2.5.11}$$

That ν has its support in K_ε means by the Paley–Wiener theorem that for some constants C and N

$$|\hat{\nu}(\theta_1, \ldots, \theta_{2n})| \leqslant C(1 + |\theta_1| + \ldots + |\theta_{2n}|)^N e^{\varphi(\theta)} \tag{2.5.12}$$

where
$$\varphi(\theta) = \sup_{x \in K_\varepsilon} (x_1 \operatorname{Im} \theta_1 + \ldots + x_{2n} \operatorname{Im} \theta_{2n}). \tag{2.5.13}$$

It is therefore sufficient to construct an entire analytic function satisfying (2.5.11) and (2.5.12).

Let u denote the function

$$(i\zeta_1, -\zeta_1, \ldots, i\zeta_n, -\zeta_n) \to \hat{\mu}(\zeta_1, \ldots, \zeta_n)$$

which is defined and analytic in a subspace Σ of $\mathbf{C}^{2n}$ of codimension n. If

$$\theta = (i\zeta_1, -\zeta_1, \ldots, i\zeta_n, -\zeta_n),$$

we obtain from (2.5.13)

$$\varphi(\theta) = \sup_{z \in K_\varepsilon} (x_1 \operatorname{Re} \zeta_1 - x_2 \operatorname{Im} \zeta_1 + \ldots) = \sup_{z \in K_\varepsilon} \operatorname{Re}[z, \zeta] = H_K(\zeta) + \varepsilon |\zeta|.$$

Hence (2.5.10) implies that

$$|u(\theta)| \leqslant C_\delta \exp(\varphi(\theta) + (\delta - \varepsilon)|\zeta|), \quad \delta > 0,$$

and choosing $\delta < \varepsilon$ we obtain
$$\int_\Sigma |u|^2 e^{-2\varphi} d\sigma < \infty. \tag{2.5.14}$$

Since φ is convex and therefore plurisubharmonic, it follows from Theorem 2.5.1 that there is an entire analytic function U in $\mathbf{C}^{2n}$ such that $U = u$ in Σ and

$$\int_{\mathbf{C}^{2n}} |U(\theta)|^2 e^{-2\varphi(\theta)} (1 + |\theta|^2)^{-3n} dV < \infty. \tag{2.5.15}$$

By Cauchy's integral formula this implies that $|U(\theta)| e^{-\varphi(\theta)} (1 + |\theta|)^{-3n}$ is bounded. Hence (2.5.12) and (2.5.11) are fulfilled by $\hat{\nu} = U$, which completes the proof.

Next we shall give an application of Proposition 2.3.2.

THEOREM 2.5.3. *Let Φ be a plurisubharmonic function $\in C^2(\mathbf{C}^n)$. Then the set of entire functions v such that*

$$\int |v|^2 e^{-\Phi} (1 + |z|^2)^{-N} dV < \infty, \tag{2.5.16}$$

for some integer N, contains functions not identically zero. In fact, it is a dense subset of the space $\mathcal{A}$ of all entire functions with the topology of uniform convergence on compact sets.

Proof. An equivalent topology in $\mathcal{A}$ is defined by L^2 convergence on all compact sets, and every element $\mu \in \mathcal{A}'$ can therefore be extended to a continuous linear form on L^2_{loc}, that is, there is a function $u \in L^2$ with compact support such that $\mu(v) = \langle v, u \rangle$, $v \in \mathcal{A}$. If μ is orthogonal to all entire functions v satisfying (2.5.16) for some N, we claim that there exist functions $f_j \in L^2$ such that

$$u = \sum_1^n \frac{\partial f_j}{\partial z_j} \tag{2.5.17}$$

and all f_j have compact support. In fact, let $u(z) = 0$ when $|z| > R$, and apply Proposition 2.3.2 with $\varphi(z) = \Phi(z) + \log(1 + |z|^2)$ and $\psi(z) = \log((1 + |z|^2)/(1 + R^2))$. The hypotheses of Proposition 2.3.2 are then fulfilled in view of (2.5.4). Hence (2.5.17) is valid for suitable f_j with compact support. But (2.5.17) implies that

$$\mu(v) = \int v \bar{u} \, dV = - \sum_1^n \int \frac{\partial v}{\partial \bar{z}_j} f_j \, dV = 0; \quad v \in \mathcal{A},$$

so the theorem follows from the Hahn-Banach theorem.

III. Function theory on manifolds

3.1. *Preliminaries*

Let M be a complex analytic paracompact manifold of complex dimension n. The decomposition of differential forms into forms of type (p, q), the definition of the $\bar{\partial}$ operator and the definition of plurisubharmonic functions which we have introduced in Chapter II for domains in $\mathbf{C}^n$ can immediately be extended to forms and functions on the manifold M, for all these definitions are invariant for analytic changes of coordinates.

In order to study the operator $\bar{\partial}$ with the Hilbert space techniques of section 1.1, we must introduce hermitian norms on differential forms on M. We therefore choose a hermitian metric on M, that is, a Riemannian metric which in any analytic coordinate system with coordinates $z_1, \ldots, z_n$ has the form

$$\sum_{j,k=1}^n h_{jk} \, dz_j \, \overline{dz_k},$$

where h_{jk} is a positive definite hermitian matrix with C^∞ coefficients. The existence of such a hermitian structure is trivial locally, and is immediately proved in the large by means of a partition of unity. We keep the hermitian structure on M fixed in all that follows. The element of volume defined by the structure we denote by dV and the element of area on a smooth hypersurface we denote by dS. (For definitions see also Weil [29].)

If f is a form of type (1, 0) and $f=\sum_1^n f_j dz_j$ in a local coordinate system, we set

$$\langle f,f\rangle=\sum \bar{h}^{jk} f_j \bar{f}_k$$

where (h^{jk}) is the inverse of (h_{jk}). This has invariant meaning, for

$$\langle f,f\rangle=\sup_{dz}\frac{|\sum f_j dz_j|^2}{\sum h_{jk} dz_j d\bar{z}_k}.$$

By the Gram–Schmidt orthogonalization process every point in M has a neighborhood U where there are n forms $\omega^1, \ldots, \omega^n$ of type (1,0) with C^∞ coefficients such that $\langle\omega^j, \omega^k\rangle=\delta_{jk}$; $j, k=1, \ldots, n$. If we set $f=\sum f'_j \omega^j$, it follows that

$$\langle f,f\rangle=\sum_1^n |f'_j|^2.$$

More generally, a differential form f of type (p, q) can be written in a unique way in the form

$$f=\sum_{|I|=p,\,|J|=q}{}' f_{I,J}\,\omega^I\wedge\bar{\omega}^J$$

(for notations see also section 2.1) where $f_{I,J}$ are antisymmetric in I and in J,

$$\omega^I=\omega^{i_1}\wedge\ldots\wedge\omega^{i_p} \text{ and } \bar{\omega}^J=\bar{\omega}^{j_1}\wedge\ldots\wedge\bar{\omega}^{j_q}.$$

We can define $\langle f,f\rangle$ by $\qquad \langle f,f\rangle=|f|^2=\sum' |f_{I,J}|^2,$

for this definition is independent of the choice of orthonormal basis $\omega^1, \ldots, \omega^n$.

Let Ω be an open subset of M and φ a continuous function in Ω. We then define $L^2_{(p,q)}(\Omega,\varphi)$ as the space of all measurable forms f in Ω of type (p, q), that is, forms with measurable components in any local coordinate system, such that

$$\|f\|^2_\varphi=\int_\Omega |f|^2 e^{-\varphi} dV<\infty,$$

forms which are equal almost everywhere being identified. If $q\geqslant 1$, the operator $\bar\partial$ defines in the weak sense a closed densely defined operator

$$T: L^2_{(p,q-1)}(\Omega,\varphi)\to L^2_{(p,q)}(\Omega,\varphi)$$

and another $\qquad S: L^2_{(p,q)}(\Omega,\varphi)\to L^2_{(p,q+1)}(\Omega,\varphi).$

If Ω is relatively compact in M and has a C^3 boundary, which we assume from now on, and if $\varphi\in C^1(\bar\Omega)$, it follows from Proposition 1.2.4 by application of a partition of unity that $C^2_{(p,q)}\cap D_{T^*}$ is dense in $D_{T^*}\cap D_S$ for the graph norm.

If $u \in C^1$ and the forms $\omega^1, \ldots, \omega^n$ are a local basis for forms of type $(1, 0)$ in an open analytic coordinate patch U, we set

$$du = \sum_1^n \frac{\partial u}{\partial \omega^i} \omega^i + \sum_1^n \frac{\partial u}{\partial \bar{\omega}^i} \bar{\omega}^i$$

as definition of the first order linear differential operators $\partial/\partial\omega^i$ and $\partial/\partial\bar{\omega}^i$ in U. Then we have

$$\bar{\partial} u = \sum_1^n \frac{\partial u}{\partial \bar{\omega}^j} \bar{\omega}^j,$$

and if $f = \sum' f_{I,J}\, \omega^I \wedge \bar{\omega}^J$ it follows that

$$\bar{\partial} f = \sum_{I,J}{}' \sum \frac{\partial f_{I,J}}{\partial \bar{\omega}^j} \bar{\omega}^j \wedge \omega^I \wedge \bar{\omega}^J + \ldots,$$

where the dots indicate terms in which no $f_{I,J}$ is differentiated; they occur because $\bar{\partial}\omega^i$ and $\bar{\partial}\bar{\omega}^j$ need not be 0. If the sum is denoted by Af we obviously have $|\bar{\partial} f - Af| \leqslant C|f|$, where C is independent of f.

Now let $f \in C^2_{(p,q)}(\bar{\Omega}) \cap D_{T^*}$ and let $f = 0$ outside a compact subset of $U \cap \bar{\Omega}$. Then we have for $u \in C^1_{(p,q-1)}(\bar{\Omega})$

$$\int_{U\cap\Omega} \langle T^* f, u\rangle e^{-\varphi} dV = \int_{U\cap\Omega} \langle f, \bar{\partial} u\rangle e^{-\varphi} dV = (-1)^p \sum_{I,K}{}' \sum_j \int_{U\cap\Omega} f_{I,jK} \overline{\frac{\partial u_{I,K}}{\partial \bar{\omega}^j}} e^{-\varphi} dV + \ldots \tag{3.1.1}$$

where dots indicate terms where no derivatives occur. We shall integrate by parts in (3.1.1). First note that with the notation

$$\delta_j w = e^{\varphi} \frac{\partial (w e^{-\varphi})}{\partial \omega^j}$$

Green's formula assumes the form

$$\int_{U\cap\Omega} \frac{\partial v}{\partial \bar{\omega}^j} \bar{w} e^{-\varphi} dV = -\int_{U\cap\Omega} v \overline{\delta_j w} e^{-\varphi} dV + \int_{U\cap\Omega} \sigma_j v \bar{w} e^{-\varphi} dV + \int_{U\cap\partial\Omega} \frac{\partial \varrho}{\partial \bar{\omega}^j} v \bar{w} e^{-\varphi} dS$$

where ϱ denotes the distance to $\partial\Omega$ with respect to the hermitian metric, defined to be >0 in $\complement\bar{\Omega}$ and <0 in Ω, and where σ_j is in $C^\infty(U)$. Integrating by parts in (3.1.1) we conclude that f satisfies the boundary condition

$$\sum f_{I,jK} \frac{\partial \varrho}{\partial \omega^j} = 0 \text{ on } U \cap \partial\Omega, \tag{3.1.2}$$

and that $$T^*f = (-1)^{p-1} \sum \delta_j f_{I,jK}\, \omega^I \wedge \bar{\omega}^K + \ldots = Bf + \ldots, \tag{3.1.3}$$

where the dots indicate terms where no $f_{I,jK}$ is differentiated and which do not involve φ. Hence $|T^*f - Bf| \leqslant C|f|$, and we obtain with another constant C independent of f and φ

$$\big|\, \|Af\|_\varphi^2 + \|Bf\|_\varphi^2 - \|Sf\|_\varphi^2 - \|T^*f\|_\varphi^2 \big| \leqslant C\|f\|_\varphi (\|Sf\|_\varphi + \|T^*f\|_\varphi + \|f\|_\varphi). \tag{3.1.4}$$

The arguments which led from (2.1.11) to (2.1.12) apply without change and give

$$\|Af\|_\varphi^2 + \|Bf\|_\varphi^2 = \sum_{I,K}{}' \sum_{j,k} \int_{U\cap\Omega} \delta_j f_{I,jK} \overline{\delta_k f_{I,kK}}\, e^{-\varphi} dV$$

$$- \sum_{I,K}{}' \sum_{j,k} \int_{U\cap\Omega} \frac{\partial f_{I,jK}}{\partial \bar{\omega}^k} \overline{\frac{\partial f_{I,kK}}{\partial \bar{\omega}^j}}\, e^{-\varphi} dV + \sum_{I,J}{}' \sum_{j} \int_{U\cap\Omega} \left| \frac{\partial f_{I,J}}{\partial \bar{\omega}^j} \right|^2 e^{-\varphi} dV. \tag{3.1.5}$$

Before repeating the integration by parts performed next in section 2.1 we must study the commutators of the operators $\partial/\partial\bar{\omega}^j$ and δ_k.

Thus let w be a smooth function and consider

$$\bar{\partial}\partial w = \bar{\partial} \sum_1^n \frac{\partial w}{\partial \omega^k} \omega^k = \sum_{j,k=1}^n \frac{\partial^2 w}{\partial \bar{\omega}^j \partial \omega^k} \bar{\omega}^j \wedge \omega^k + \sum_1^n \frac{\partial w}{\partial \omega^i} \bar{\partial} \omega^i.$$

Since $\bar{\partial}\omega^i$ is a form of type $(1, 1)$ we may write

$$\bar{\partial}\omega^i = \sum_{j,k=1}^n c_{jk}^i \bar{\omega}^j \wedge \omega^k \tag{3.1.6}$$

which gives $$\bar{\partial}\partial w = \sum_{j,k} \left(\frac{\partial^2 w}{\partial \bar{\omega}^j \partial \omega^k} + \sum_i c_{jk}^i \frac{\partial w}{\partial \omega^i} \right) \bar{\omega}^j \wedge \omega^k.$$

If we replace w by $\bar{w}$ and take complex conjugates of all terms, we also obtain

$$\partial\bar{\partial} w = \sum_{j,k} \left(\frac{\partial^2 w}{\partial \omega^j \partial \bar{\omega}^k} + \sum_i \bar{c}_{jk}^i \frac{\partial w}{\partial \bar{\omega}^i} \right) \omega^j \wedge \bar{\omega}^k.$$

The identity $\bar{\partial}\partial w = -\partial\bar{\partial} w$ therefore implies

$$w_{kj} = \frac{\partial^2 w}{\partial \bar{\omega}^j \partial \omega^k} + \sum_i c_{jk}^i \frac{\partial w}{\partial \omega^i} = \frac{\partial^2 w}{\partial \omega^k \partial \bar{\omega}^j} + \sum_i \bar{c}_{kj}^i \frac{\partial w}{\partial \bar{\omega}^i}. \tag{3.1.7}$$

where the left-hand equality is a definition. Note that with this notation we have

$$\partial\bar{\partial} w = \sum w_{jk}\, \omega^j \wedge \bar{\omega}^k.$$

A function $\varphi \in C^2$ is therefore plurisubharmonic if the form $\sum \varphi_{jk} f_j \bar{f}_k$ is positive definite, and the Levi form of $\partial\Omega$ is $\sum \varrho_{jk} f_j \bar{f}_k$.

From (3.1.7) it follows that

$$\left(\delta_k \frac{\partial w}{\partial \bar{\omega}^j} - \frac{\partial \delta_k w}{\partial \bar{\omega}^j}\right) = \frac{\partial^2 \varphi}{\partial \bar{\omega}^j \partial \omega^k} w + \sum_i c^i_{jk} \frac{\partial w}{\partial \omega^i} - \sum_i \bar{c}^i_{kj} \frac{\partial w}{\partial \bar{\omega}^i},$$

or if we use the definition of δ_i and (3.1.7) again, with w replaced by φ,

$$\left(\delta_k \frac{\partial w}{\partial \bar{\omega}^j} - \frac{\partial \delta_k w}{\partial \bar{\omega}^j}\right) = \varphi_{kj} w + \sum_i c^i_{jk} \delta_i w - \sum_i \bar{c}^i_{kj} \frac{\partial w}{\partial \bar{\omega}^i}. \tag{3.1.8}$$

Using Green's formula and (3.1.8) we now integrate by parts in (3.1.5), which gives

$$\|Af\|^2_\varphi + \|Bf\|^2_\varphi = \sum_{I,J}' \sum_j \int_{U\cap\Omega} \left|\frac{\partial f_{I,J}}{\partial \bar{\omega}^j}\right|^2 e^{-\varphi} dV$$

$$+ \sum_{I,K}' \sum_{j,k} \int_{U\cap\Omega} \varphi_{jk} f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dV + t_1 + t_2 + t_3 + t_4 \tag{3.1.9}$$

where
$$t_1 = \sum_{I,K}' \sum_{j,k} \int_{U\cap\partial\Omega} \left(f_{I,jK} \frac{\partial \varrho}{\partial \omega^j} \overline{\delta_k f_{I,kK}} - f_{I,jK} \frac{\partial \varrho}{\partial \bar{\omega}^k} \overline{\frac{\partial f_{I,kK}}{\partial \bar{\omega}^j}}\right) e^{-\varphi} dS,$$

$$t_2 = \sum_{I,K}' \sum_{j,k} \int_{U\cap\Omega} \left(f_{I,jK} \bar{\sigma}_j \overline{\delta_k f_{I,kK}} - f_{I,jK} \sigma_k \overline{\frac{\partial f_{I,kK}}{\partial \bar{\omega}^j}}\right) e^{-\varphi} dV,$$

$$t_3 = \sum_{I,K}' \sum_{i,j,k} \int_{U\cap\Omega} f_{I,jK} \bar{c}^i_{jk} \overline{\delta_i f_{I,kK}} e^{-\varphi} dV,$$

$$t_4 = -\sum_{I,K}' \sum_{i,j,k} \int_{U\cap\Omega} f_{I,jK} c^i_{kj} \overline{\frac{\partial f_{I,kK}}{\partial \bar{\omega}^i}} e^{-\varphi} dV.$$

The first term in the definition of t_1 vanishes in view of the boundary condition (3.1.2), and arguing exactly as in section 2.1 when we passed from (2.1.13) to (2.1.14), we thus obtain

$$t_1 = \sum_{I,K}' \sum_{j,k} \int_{U\cap\partial\Omega} \frac{\partial^2 \varrho}{\partial \omega^j \partial \bar{\omega}^k} f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dS. \tag{3.1.10}$$

When studying the other terms we use the notation

$$|||f|||^2_\varphi = \sum_{I,J}' \sum_k \int_{U\cap\Omega} \left|\frac{\partial f_{I,J}}{\partial \bar{\omega}^k}\right|^2 e^{-\varphi} dV + \|f\|^2_\varphi.$$

This gives immediately
$$|t_4| \leq C |||f|||_\varphi \|f\|_\varphi, \tag{3.1.11}$$

where the constant like all the following ones is independent of both f and φ. If we integrate by parts in t_3 we find that $t_3 = t_3' + t_3''$ where

$$t_3' = \sum_{I,K}' \sum_{i,j,k} \int_{U \cap \partial\Omega} f_{I,jK} \bar{c}^i_{jk} \frac{\partial \varrho}{\partial \bar{\omega}^i} \overline{f_{I,kK}} e^{-\varphi} dS, \tag{3.1.12}$$

$$|t_3''| \leqslant C |||f|||_\varphi \|f\|_\varphi. \tag{3.1.13}$$

Combination of (3.1.10) and (3.1.12) gives in view of (3.1.7)

$$t_1 + t_3' = \sum_{I,K}' \sum_{j,k} \int_{U \cap \partial\Omega} \varrho_{jk} f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dS. \tag{3.1.14}$$

To estimate t_2 finally we note that if we integrate by parts in the terms containing δ_k it follows from (3.1.2) that there will be no boundary terms, so we obtain

$$|t_2| \leqslant C |||f|||_\varphi \|f\|_\varphi. \tag{3.1.15}$$

Summing up (3.1.4), (3.1.9)–(3.1.15), we have proved

PROPOSITION 3.1.1. *For forms $f \in C^2_{(p,q)}(\bar{\Omega}) \cap D_{T^*}$ vanishing outside a fixed compact subset of a coordinate patch U in M we have, if $\varphi \in C^2(\bar{\Omega})$ and $\partial\Omega \in C^3$,*

$$| \|T^*f\|^2_\varphi + \|Sf\|^2_\varphi - Q_1(f,f) - Q_2(f,f) - Q_3(f,f)| \leqslant C \|f\|_\varphi (\|T^*f\|_\varphi + \|Sf\|_\varphi + |||f|||_\varphi), \tag{3.1.16}$$

where

$$Q_1(f,f) = \sum_{I,J}' \sum_j \int_{U \cap \Omega} \left| \frac{\partial f_{I,J}}{\partial \bar{\omega}^j} \right|^2 e^{-\varphi} dV, \tag{3.1.17}$$

$$Q_2(f,f) = \sum_{I,K}' \sum_{j,k} \int_{U \cap \Omega} \varphi_{jk} f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dV, \tag{3.1.18}$$

$$Q_3(f,f) = \sum_{I,K}' \sum_{j,k} \int_{U \cap \partial\Omega} \varrho_{jk} f_{I,jK} \overline{f_{I,kK}} e^{-\varphi} dS. \tag{3.1.19}$$

Note that Q_2 and Q_3 are independent of the choice of $\omega^1, \ldots, \omega^n$.

So far we have essentially followed Ash [2] and Kohn [15]. In the next two sections we shall use Proposition 3.1.1 to give a rather complete study of the estimates in which we are interested. When doing so we note that (3.1.16) implies that for every $\varepsilon > 0$ there is a constant C_ε such that

$$(1-\varepsilon) Q_1(f,f) + Q_2(f,f) + Q_3(f,f) \leqslant (1+\varepsilon)(\|T^*f\|^2_\varphi + \|Sf\|^2_\varphi) + C_\varepsilon \|f\|^2_\varphi, \tag{3.1.20}$$

$$(1-\varepsilon)(\|T^*f\|^2_\varphi + \|Sf\|^2_\varphi) \leqslant (1+\varepsilon) Q_1(f,f) + Q_2(f,f) + Q_3(f,f) + C_\varepsilon \|f\|^2_\varphi. \tag{3.1.21}$$

In the proof of estimates we shall later on make another integration by parts, using the next two propositions.

PROPOSITION 3.1.2. *Let U be a coordinate patch $\subset \Omega$ and let $\varphi \in C^2(U)$. If $w \in C^2(U)$ and w vanishes outside a fixed compact subset of U, we have*

$$\left| \int_U \frac{\partial w}{\partial \bar{\omega}^k} \overline{\frac{\partial w}{\partial \bar{\omega}^j}} e^{-\varphi} dV - \int_U \delta_j w \overline{\delta_k w} e^{-\varphi} dV + \int_U \varphi_{jk} |w|^2 e^{-\varphi} dV \right| \leqslant C \| w \|_\varphi ||| w |||_\varphi, \quad (3.1.22)$$

where C is independent of w and of φ.

Proof. If we multiply the complex conjugate of (3.1.8) by w and integrate by parts, we find that the estimate to prove is equivalent to

$$\left| \int_U w \sigma_k \overline{\frac{\partial w}{\partial \bar{\omega}^j}} e^{-\varphi} dV - \int_U w \bar{\sigma}_j \overline{\delta_k w} e^{-\varphi} dV - \sum_i \int_U \bar{c}^i_{jk} w \overline{\delta_i w} e^{-\varphi} dV + \sum_i \int_U c^i_{kj} w \overline{\frac{\partial w}{\partial \bar{\omega}^i}} e^{-\varphi} dV \right|$$
$$\leqslant C \| w \|_\varphi ||| w |||_\varphi.$$

But this follows immediately if we integrate by parts in the integrals containing the differential operator δ.

If U intersects $\partial\Omega$ we have to modify Theorem 3.1.2 since integration by parts will give rise to certain boundary terms. Following Kohn [15] we can then since $\partial\partial\varrho = 0$ choose the forms ω^j so that $\omega^n = 2\partial\varrho$, which implies that $\partial\varrho/\partial\omega^j = 0$ when $j < n$. The forms $\omega^1, \ldots, \omega^n$ can of course not be chosen with C^∞ coefficients as we have assumed until now, but if $\partial\Omega \in C^3$ the forms $\omega^1, \ldots, \omega^n$ can be chosen with C^2 coefficients which implies that $c^i_{jk} \in C^1$.

PROPOSITION 3.1.3. *If $w \in C^2(U \cap \bar{\Omega})$, if w vanishes outside a fixed compact subset of $U \cap \bar{\Omega}$, and if $\varphi \in C^2(U \cap \bar{\Omega})$, we have if both j and k are $< n$*

$$\left| \int_{U \cap \Omega} \frac{\partial w}{\partial \bar{\omega}^k} \overline{\frac{\partial w}{\partial \bar{\omega}^j}} e^{-\varphi} dV - \int_{U \cap \Omega} \delta_j w \overline{\delta_k w} e^{-\varphi} dV + \int_{U \cap \Omega} \varphi_{jk} |w|^2 e^{-\varphi} dV + \int_{U \cap \partial\Omega} \varrho_{jk} |w|^2 e^{-\varphi} dS \right|$$
$$\leqslant C \| w \|_\varphi ||| w |||_\varphi, \quad (3.1.23)$$

where C is independent of w and φ.

Proof. Since $\partial\varrho/\partial\omega^j = 0$ on $\partial\Omega$ when $j \neq n$, it follows from (3.1.8) that

$$\left| \int_{U \cap \Omega} \frac{\partial w}{\partial \bar{\omega}^k} \overline{\frac{\partial w}{\partial \bar{\omega}^j}} e^{-\varphi} dV - \int_{U \cap \Omega} \delta_j w \overline{\delta_k w} e^{-\varphi} dV + \int_{U \cap \Omega} \varphi_{jk} |w|^2 e^{-\varphi} dV + \int_{U \cap \partial\Omega} \bar{c}^n_{jk} |w|^2 \frac{\partial \varrho}{\partial \bar{\omega}^n} e^{-\varphi} dS \right|$$
$$\leqslant C \| w \|_\varphi ||| w |||_\varphi.$$

If we apply (3.1.7) with j and k interchanged and with w replaced by ϱ, it follows that $\varrho_{jk} = \bar{c}^n_{jk}\, \partial\varrho/\partial\bar{\omega}^n$ which proves (3.1.23).

Remark. Note that in proving Propositions 3.1.2–3.1.3 we have only used that M has an integrable almost complex structure. Hence the estimates in the following paragraphs also hold under that hypotheses and can be used to prove the theorem of Newlander–Nirenberg [24] (see Kohn [14]).

3.2. *Estimates for fixed norms*

In this section we shall study estimates of the type discussed by Morrey [22], Kohn [14] and Ash [2]. Let U be an open coordinate patch in M such that $U \cap \partial\Omega$ is in C^3, and let φ be a fixed function in $C^2(\bar{\Omega} \cap U)$. If $z \in U \cap \partial\Omega$ we denote by $\lambda_1(z), \ldots, \lambda_{n-1}(z)$ the eigenvalues of the Levi form

$$\sum_{j,k=1}^{n} \varrho_{jk} f_j \bar{f}_k$$

with respect to the form $\langle f, f\rangle = \sum |f_j|^2$ in the plane $\sum \partial\varrho/\partial\omega^j f_j = 0$. The eigenvalues are of course independent of the choice of the forms ω^j. For real λ we set $\lambda^+ = \max(\lambda, 0)$, $\lambda^- = \max(-\lambda, 0)$, so that $\lambda = \lambda^+ - \lambda^-$ and $|\lambda| = \lambda^+ + \lambda^-$.

THEOREM 3.2.1. *Assume that there are constants K and K' such that*

$$\int_{U \cap \partial\Omega} |f|^2 e^{-\varphi} dS \leqslant K(\|T^* f\|^2_\varphi + \|Sf\|^2_\varphi) + K' \|f\|^2_\varphi \tag{3.2.1}$$

for all $f \in C^2_{(p,q)}(\bar{\Omega}) \cap D_{T^*}$ vanishing outside a compact subset of $\bar{\Omega} \cap U$. Then we have for every point on $U \cap \partial\Omega$ and every multi-index J formed with q different of the indices $1, \ldots, n-1$

$$1 \leqslant K\left(\sum_1^{n-1} \lambda_j^- + \sum_{j \in J} \lambda_j\right) = K\left(\sum_{j \in J} \lambda_j^+ + \sum_{j \notin J} \lambda_j^-\right). \tag{3.2.2}$$

Proof. We can choose the local coordinates z_j in U so that $z = 0$ at the point on $U \cap \partial\Omega$ where we wish to prove (3.2.2), and $\omega^j = dz_j$ at $z = 0$. Shrinking U if necessary we may assume that $U \cap \bar{\Omega}$ is defined by an inequality of the form

$$\operatorname{Im} z_n = x_{2n} \geqslant \varrho_1(x_1, \ldots, x_{2n-1}), \quad z \in U,$$

where $\varrho_1 \in C^2$ and vanishes to the second order at 0. (To shorten notations we do not distinguish here between a point in U and the corresponding point in $\mathbf{C}^n$.) Since the shortest distance to $\partial\Omega$ will in the first approximation be attained in the direction of the x_{2n} axis, when z is near 0, and since the hermitian metric is $\sum |dz_i|^2$ when $z=0$, we have $\varrho(z)=(x_{2n}-\varrho_1(x_1, ..., x_{2n-1}))(1+O(|z|))$.

Hence
$$\frac{\partial^2\varrho(0)}{\partial z_j\partial\bar{z}_k}=\frac{\partial^2\varrho_1(0)}{\partial z_j\partial\bar{z}_k}; \quad j,k=1, \ldots, n-1.$$

By a unitary transformation of the variables $z_1, \ldots, z_{n-1}$ we may achieve that

$$\sum_{j,k=1}^{n-1}\frac{\partial^2\varrho(0)}{\partial z_j\partial\bar{z}_k}z_j\bar{z}_k=\sum_1^{n-1}\lambda_j|z_j|^2.$$

Then it follows from Taylor's formula that

$$\varrho_1(z)=\sum_1^{n-1}\lambda_j|z_j|^2+\operatorname{Re}A(z')+O(|z_n||z'|+|z_n|^2)+o(|z|^2),$$

where $z'=(z_1, ..., z_{n-1})$ and A is a homogeneous analytic second degree polynomial.

Let J be a multi-index of length q formed with the indices $1, ..., n-1$, and let I be an arbitrary multi-index of length p. Then we can choose a form $f\in C^2_{(p,q)}(U)$ such that $f=dz^I\wedge d\bar{z}^J$ at 0 and f satisfies the boundary conditions (3.1.2). In fact, if $p=0$ and $q=1$, $J=\{j\}$, we can choose

$$f=2i\left(\frac{\partial\varrho}{\partial z_n}dz_j-\frac{\partial\varrho}{\partial z_j}dz_n\right).$$

For general p and q we need only take the exterior product of dz^I with the forms constructed above when j runs through J.

With $\psi\in C_0^\infty(\mathbf{C}^n)$ and a positive parameter τ we now set

$$f^\tau(z)=f(z)\psi(\tau z)\exp(i\tau^2z_n).$$

Since the last factor is analytic, we have

$$\frac{\partial f^\tau(z)}{\partial\bar{\omega}^j}=\left(f(z)\frac{\partial\psi(\tau z)}{\partial\bar{\omega}^j}+O(1)\psi(\tau z)\right)\exp(i\tau^2z_n).$$

Note that the first term on the right-hand side will involve a factor τ when calculated. If we introduce as new variables τ^2x_{2n} and τx_j for $j<2n$, we easily obtain

$$\lim_{\tau\to+\infty}Q_1(f^\tau,f^\tau)\,\tau^{2n-1}\to\int|\Psi(x',0)|^2dx'\int_{\varrho_2(x')}^\infty e^{-2x_{2n}}dx_{2n},$$

where
$$\Psi'^2=\sum_1^n\left|\frac{\partial\psi}{\partial\bar{z}^i}\right|^2, \quad x'=(x_1, \ldots, x_{2n-1});$$

dx' denotes the Lebesgue measure, and ϱ_2 is the second order part of the Taylor expansion of ϱ_1. Hence

$$2\tau^{2n-1}Q_1(f^\tau,f^\tau)\to\int|\Psi(x',0)|^2e^{-2\varrho_2(x')}dx',\quad \tau\to+\infty.$$

By the same substitution it follows that

$$Q_2(f^\tau,f^\tau)=O(\tau^{-2n-1}),\quad (f^\tau,f^\tau)_\varphi=O(\tau^{-2n-1}),\quad \tau\to+\infty,$$

$$\tau^{2n-1}Q_3(f^\tau,f^\tau)\to\sum_{j\in J}\lambda_j\int|\psi(x',0)|^2e^{-2\varrho_2(x')}dx',\quad \tau\to+\infty,$$

$$\tau^{2n-1}\int_{\partial\Omega}|f^\tau|^2e^{-\varphi}dS\to\int|\psi(x',0)|^2e^{-2\varrho_2(x')}dx',\quad \tau\to+\infty.$$

If (3.2.1) holds we thus conclude, using (3.1.21), that

$$\begin{aligned}(1-\varepsilon)&\int|\psi(x',0)|^2e^{-2\varrho_2(x')}dx'\\&\leqslant K\left\{(1+\varepsilon)\tfrac{1}{2}\int|\Psi(x',0)|^2e^{-2\varrho_2(x')}dx'+\sum_{j\in J}\lambda_j\int|\psi(x',0)|^2e^{-2\varrho_2(x')}dx'\right\}\end{aligned}\tag{3.2.3}$$

for every $\varepsilon>0$, and therefore when $\varepsilon=0$. Now choose $\psi=\psi_1\psi_2$ where $\psi_1\in C_0^\infty(\mathbf{C}^{n-1})$ is a function of $x''=(x_1,\ldots,x_{2n-2})$ and $\psi_2\in C_0^\infty(\mathbf{C})$ is a function of (x_{2n-1},x_{2n}) such that $\partial\psi_2/\partial\bar z_n=0$ when $x_{2n}=0$. This equation does not in any way restrict the values of ψ_2 when $x_{2n}=0$, so (3.2.3) implies

$$\left(1-K\sum_{j\in J}\lambda_j\right)\int|\psi_1(x'')|^2e^{-2\varrho_2(x'',0)}dx''\leqslant\frac{K}{2}\sum_1^{n-1}\int\left|\frac{\partial\psi_1}{\partial\bar z_j}\right|^2e^{-2\varrho_2(x'',0)}dx''.\tag{3.2.4}$$

Now recall that $\varrho_2(x'',0)=L(z')+\operatorname{Re}A(z')$ where A is an analytic polynomial and

$$L(z')=\sum_1^{n-1}\lambda_j|z_j|^2.$$

If in (3.2.4) we replace ψ_1 by ψ_1e^A and note that multiplication by the analytic function e^A commutes with $\partial/\partial\bar z_j$, we get

$$\left(1-K\sum_{j\in J}\lambda_j\right)\int|\psi_1|^2e^{-2L}dx''\leqslant\frac{K}{2}\sum_1^{n-1}\int\left|\frac{\partial\psi_1}{\partial\bar z_j}\right|^2e^{-2L}dx''.\tag{3.2.5}$$

From (3.2.5) the inequality (3.2.2) follows easily by a slight extension of Lemma 8.1.2 in Hörmander [12] which we shall now prowe.

LEMMA 3.2.2. *Let $L=\sum_1^\nu \lambda_j |z_j|^2$ and assume that*

$$2c\int |\chi|^2 e^{-2L}dx \leqslant \int \sum_1^\nu \left|\frac{\partial \chi}{\partial \bar{z}_j}\right|^2 e^{-2L}dx, \quad \chi \in C_0^\infty(\mathbf{C}^\nu), \tag{3.2.6}$$

where dx is the Lebesgue measure in $\mathbf{C}^\nu$. Then it follows that

$$c \leqslant \sum_1^\nu \lambda_j^-, \tag{3.2.7}$$

which conversely implies (3.2.6).

Proof. First assume that $\lambda_j > 0$ for every j. With $\chi \in C_0^\infty$, $\chi(0)=1$, we set $\chi^\varepsilon(z)=\chi(\varepsilon z)$. Then the right-hand side of (3.2.6) with χ replaced by χ^ε is $O(\varepsilon^2)$ but $\int |\chi^\varepsilon|^2 e^{-2L}dx \to \int e^{-2L}dx \neq 0$ when $\varepsilon \to 0$, which proves that $c \leqslant 0$. More generally, if $\lambda_j \geqslant 0$ for all j, with equality for exactly μ values of j, the right-hand side of (3.2.6) is $O(\varepsilon^{2-\mu})$ whereas the integral on the left is only $O(\varepsilon^{-\mu})$, which again proves that $c \leqslant 0$. To study the general case we note that with $\delta_j = e^{2L}\partial(e^{-2L}\chi)/\partial z_j = \partial\chi/\partial z_j - 2\partial L/\partial z_j \chi$ we have (see 2.1.8))

$$\delta_j \frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial \bar{z}_j}\delta_j = 2\frac{\partial^2 L}{\partial z_j \partial \bar{z}_j} = 2\lambda_j.$$

Hence an integration by parts gives

$$\int \left|\frac{\partial \chi}{\partial \bar{z}_j}\right|^2 e^{-2L}dx = \int |\delta_j \chi|^2 e^{-2L}dx - 2\lambda_j \int |\chi|^2 e^{-2L}dx.$$

If we put $\quad L' = \sum_1^\nu |\lambda_j||z_j|^2 \quad$ and $\quad \chi'(z) = \chi(w_1, \dots, w_\nu)\exp\left(2\sum_1^\nu \lambda_j^- |z_j|^2\right),$

where $w_j = z_j$ when $\lambda_j \geqslant 0$ and $w_j = \bar{z}_j$ when $\lambda_j < 0$, the inequality (3.2.6) now reduces to

$$2\left(c - \sum_1^\nu \lambda_j^-\right)\int |\chi'|^2 e^{-2L'}dx \leqslant \int \sum_1^\nu \left|\frac{\partial \chi'}{\partial \bar{z}_j}\right|^2 e^{-2L'}dx, \quad \chi \in C_0^\infty(\mathbf{C}^\nu).$$

By the first part of the proof, this implies (3.2.7). The converse is obvious.

COROLLARY 3.2.3. *Assume that for some constant C*

$$\int_{\partial\Omega} |f|^2 e^{-\varphi} dS \leqslant C(\|T^*f\|_\varphi^2 + \|Sf\|_\varphi^2 + \|f\|_\varphi^2), \quad f \in C^2_{(p,q)}(\bar{\Omega}) \cap D_{T^*}. \tag{3.2.8}$$

Then the Levi form has either at least $n-q$ positive eigenvalues or else at least $q+1$ negative eigenvalues at every point on $\partial\Omega$.

Proof. Assume that at a boundary point there are at most q eigenvalues <0 and at most $n-1-q$ eigenvalues >0. Then we can find a multi-index J of length q formed with indices $\leqslant n-1$, such that $\lambda_j \leqslant 0$ when $j \in J$ and $\lambda_j \geqslant 0$ when $j \notin J$. But then (3.2.2) cannot hold for any K.

On the other hand, we note that if the Levi form has at least $n-q$ positive eigenvalues and $|J|=q$, we can find $j\in J$ such that $\lambda_j>0$, and if the Levi forms has at least $q+1$ negative eigenvalues we can find $j\notin J$ with $\lambda_j<0$. In both cases (3.2.2) is therefore valid for some K at every point on $\partial\Omega$. For reasons of continuity this implies that (3.2.2) is valid on the whole of $\partial\Omega$ for some K, if the conclusion of Corollary 3.2.3 holds.

THEOREM 3.2.4. *Assume that* (3.2.2) *is valid for the eigenvalues of the Levi form at a point* $z_0 \in \partial\Omega$, *and let* $\varepsilon>0$. *Then there exists a neighborhood* U *of* z_0 *such that* (3.2.1) *holds with* K *replaced by* $K(1+\varepsilon)$ *and a suitable constant* K' *for all* $f \in C^2_{(p,q)}(\bar\Omega) \cap D_{T^*}$ *vanishing outside a compact subset of* $\bar\Omega \cap U$.

Proof. We choose coordinates z so that the coordinates of z_0 are 0, and choose the forms ω^j so that $\omega^n = 2\partial\varrho$. By a unitary transformation of $\omega^1, \ldots, \omega^{n-1}$ we can achieve that the Levi form $\sum_1^{n-1}\varrho_{jk}(0)f_j\bar f_k$ assumes the form $\sum_1^{n-1}\lambda_j|f_j|^2$. Let $\lambda_j<0$, $j=1,\ldots,\mu$; $\lambda_j\geqslant 0$, $j>\mu$. By Proposition 3.1.3 we have for fixed φ with a constant C

$$\int_{U\cap\partial\Omega} -\varrho_{jj}|w|^2e^{-\varphi}dS \leqslant \int_{U\cap\Omega}\left|\frac{\partial w}{\partial\bar\omega^j}\right|^2 e^{-\varphi}dV + C\|w\|_\varphi |||w|||_\varphi.$$

Taking $w=f_{I,J}$, adding and using (3.1.17) we obtain if $0<\delta<1$

$$\sum_{I,J}{}' \int_{U\cap\partial\Omega}\sum_1^\mu -\varrho_{jj}|f_{I,J}|^2e^{-\varphi}dS \leqslant Q_1(f,f) + C\|f\|_\varphi |||f|||_\varphi \leqslant (1-\delta)^{-1}Q_1(f,f) + C_\delta\|f\|^2_\varphi.$$

In view of (3.1.20) with ε replaced by δ this implies

$$\int_{U\cap\partial\Omega}\left((1-\delta)^2\sum_1^\mu -\varrho_{jj}\sum_{I,J}{}'|f_{I,J}|^2 + \sum_{I,K}{}'\sum_{j,k}\varrho_{jk}f_{I,jK}\overline{f_{I,kK}}\right)e^{-\varphi}dS$$
$$\leqslant (1+\delta)(\|T^*f\|^2_\varphi + \|Sf\|^2_\varphi) + C_\delta\|f\|^2_\varphi.$$

At $z=0$ the quadratic form in the integrand is

$$\sum_{I,J}{}'|f_{I,J}|^2\left((1-\delta)^2\sum_1^{n-1}\lambda_j^- + \sum_{j\in J}\lambda_j\right) > (1+\delta)\sum_{I,J}{}'\frac{|f_{I,J}|^2}{K(1+\varepsilon)}, \quad f\neq 0,$$

if δ is small enough. Here J varies over multi-indices of length q not containing the index n, for $f_{I,J}=0$ on $\partial\Omega$ if $n\in J$ in view of (3.1.2). If U is chosen sufficiently small it follows for reasons of continuity that

$$\int_{U \cap \partial\Omega} |f|^2 e^{-\varphi} dS \leq K(1+\varepsilon)(\|T^*f\|_\varphi^2 + \|Sf\|_\varphi^2) + C_\varepsilon \|f\|_\varphi^2,$$

which proves the theorem.

By a carefully applied partition of unity we could prove a global version of Theorem 3.2.4. (See Hörmander [12], remark on p. 198.) However, we only give a form where we neglect the size of the constants.

THEOREM 3.2.5. *Let Ω be relatively compact and have a C^3 boundary, the Levi form of which has either at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues at every boundary point. For a fixed φ, the estimate* (3.2.8) *is then valid for some constant C.*

Proof. This follows immediately from Theorem 3.2.4 by application of a partition of unity.

In Kohn [14] it is shown that an estimate of the form (3.2.8), combined with the results of section 1.2 and the theory of elliptic systems of differential operators, implies that the unit ball of $D_{T^*} \cap D_S$ (with respect to the graph norm) is relatively compact in $L^2_{(p,q)}(\Omega, \varphi)$. The hypotheses of Theorem 1.1.3 are therefore fulfilled. However, we shall proceed in a different way here, making essential use of the weight function φ. Thus we shall in the next section consider the dependence of our estimates on φ, which will also yield other results such as approximation theorems.

3.3. *Domination estimates*

In this section we shall prove estimates which allow us to extend the arguments of section 2.3 to open sets satisfying the conditions to which we were led in section 3.2. Thus we are interested in studying estimates where the weight function φ is replaced by a convex increasing function χ of φ and the dependence of the estimates on χ is examined carefully. First we consider the case of linear functions χ. The notations used are the same as in the two preceding paragraphs, and we assume throughout that $\varphi \in C^3(\overline{\Omega})$ and that $\partial\Omega \in C^3$.

THEOREM 3.3.1. *Let $z_0 \in \Omega$. In order that there shall exist a neighborhood $U \subset \Omega$ of z_0 and constants C, τ_0 such that*

$$\tau \|f\|_{\tau\varphi}^2 \leq C(\|T^*f\|_{\tau\varphi}^2 + \|Sf\|_{\tau\varphi}^2), \quad \tau > \tau_0, \tag{3.3.1}$$

for all $f \in C^2_{(p,q)}(\overline{\Omega})$ with compact support in U, it is necessary and sufficient that the hermitian form

$$\sum_{j,k=1}^{n} \varphi_{jk}(z_0)\, t_j \bar{t}_k \tag{3.3.2}$$

has either at least $q+1$ *negative or at least* $n-q+1$ *positive eigenvalues.*

Here T^* denotes the adjoint of the operator $T=\bar{\partial}$ with respect to the norms $\| \ \|_{\tau\varphi}$, so the coefficients of this differential operator depend on τ.

Proof. a) *Necessity.* If (3.3.1) holds, we obtain from (3.1.21) with φ replaced by $\tau\varphi$ and $\varepsilon=\frac{1}{2}$, for example,(1)

$$\tau\int |f|^2 e^{-\tau\varphi}\, dV \leqslant C'\left\{\sum_{I,J}{}' \sum_{j} \int \left|\frac{\partial f_{I,J}}{\partial\bar{\omega}^j}\right|^2 e^{-\tau\varphi} dV + \tau \sum_{I,K}{}' \sum_{j,k} \int \varphi_{jk} f_{I,jK} \overline{f_{I,kK}}\, e^{-\tau\varphi} dV\right\} \tag{3.3.3}$$

when τ is large enough and $f\in C^2_{(p,q)}(\Omega)$ has compact support in U. We may assume that U is contained in one coordinate patch and choose the local coordinates and the forms ω^j so that the coordinates of z_0 are 0 and $\omega^j=dz^j$ at 0. By a unitary transformation we may achieve that

$$\sum_{j,k=1}^{n} \frac{\partial^2\varphi(0)}{\partial z_j \partial\bar{z}_k} t_j \bar{t}_k = 2\sum_{1}^{n} \lambda_j |t_j|^2 = 2L(t).$$

Write
$$\varphi(z) = 2\{\operatorname{Re}\,(\langle z, N\rangle + A(z)) + L(z)\} + o(|z|^2),$$

where A is an analytic second degree polynomial. With a fixed $f\in C^2_{(p,q)}(\Omega)$ with support in U and $\psi\in C_0^\infty(C^n)$ we now set

$$f^\tau(z) = \tau^{n/2}\psi(z\sqrt{\tau})\exp(\tau(\langle z, N\rangle + A(z)))f.$$

When $\tau\to+\infty$ it follows from (3.3.3) applied to f^τ that

$$|f(0)|^2 \int |\psi|^2 e^{-2L} dx \leqslant C'\left\{|f(0)|^2 \int \sum_{1}^{n} \left|\frac{\partial\psi}{\partial\bar{z}_j}\right|^2 e^{-2L} dx + 2\sum_{I,K}{}' \sum_{j} \lambda_j |f_{I,jK}(0)|^2 \int |\psi|^2 e^{-2L} dx\right\},$$

where dx is the Lebesgue measure in $\mathbf{C}^n$. If for fixed J with $|J|=q$ and fixed I with $|I|=p$ we choose $f=dz^I\wedge d\bar{z}^J$ at the origin, we obtain

$$\left(1-2C'\sum_{j\in J}\lambda_j\right)\int |\psi|^2 e^{-2L} dx \leqslant C'\int \sum_{1}^{n} \left|\frac{\partial\psi}{\partial\bar{z}_j}\right|^2 e^{-2L} dx.$$

By Lemma 3.2.2 this implies

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(1) By using arbitrarily small numbers ε we could determine the infimum of the constants C that can be used in (3.3.1) as we did in a similar context in section 3.2.

$$1-2C'\sum_{j\in J}\lambda_j\leqslant 2C'\sum_1^n\lambda_j^-,$$

that is,
$$1\leqslant 2C'\left(\sum_{j\in J}\lambda_j^+ + \sum_{j\notin J}\lambda_j^-\right).$$

If this holds for all J with $|J|=q$, it follows as in the proof of Corollary 3.2.3 that there are either $q+1$ negative or $n+1-q$ positive eigenvalues λ_j.

b) *Sufficiency*. Again we choose the coordinates so that

$$\sum_{j,k=1}^n\varphi_{jk}(0)t_j\bar t_k=2\sum_1^n\lambda_j|t_j|^2.$$

Let $\lambda_j<0$ for $j=1, \dots, \mu$ and $\lambda_j\geqslant 0$ for $j>\mu$. By (3.1.20) we have when $f\in C^2_{(p,q)}(\Omega)$ and f has support in a fixed coordinate patch U with $0\in U\subset\Omega$

$$(1-2\varepsilon)\sum_{I,J}\sum_j\int_\Omega\left|\frac{\partial f_{I,J}}{\partial\bar\omega^j}\right|^2 e^{-\tau\varphi}dV+\varepsilon|||f|||^2_{\tau\varphi}+\tau\sum_{I,K}{}'\sum_{j,k}\int_\Omega\varphi_{jk}f_{I,jK}\overline{f_{I,kK}}\,e^{-\tau\varphi}dV$$
$$\leqslant(1+\varepsilon)(\|T^*f\|^2_{\tau\varphi}+\|Sf\|^2_{\tau\varphi})+C'_\varepsilon\|f\|^2_{\tau\varphi}.$$

In those terms in the first sum where $j\leqslant\mu$ we now use (3.1.22) and obtain

$$\tau\int Q_\varepsilon(z,f,f)\,e^{-\tau\varphi}dV+\varepsilon|||f|||^2_{\tau\varphi}\leqslant(1+\varepsilon)(\|T^*f\|^2_{\tau\varphi}+\|Sf\|^2_{\tau\varphi})+C'_\varepsilon\|f\|^2_{\tau\varphi}+C\|f\|_{\tau\varphi}|||f|||_{\tau\varphi},\tag{3.3.4}$$

where we have used the notation

$$Q_\varepsilon(z,f,f)=(1-2\varepsilon)\sum_{I,J}{}'\sum_{j=1}^\mu-\varphi_{jj}|f_{I,J}|^2+\sum_{I,K}{}'\sum_{j,k}\varphi_{jk}f_{I,jK}\overline{f_{I,kK}}.\tag{3.3.5}$$

If we estimate the last term in (3.3.4) by $\varepsilon|||f|||^2_{\tau\varphi}+C^2\varepsilon^{-1}\|f\|^2_{\tau\varphi}$, it follows that

$$\tau\int Q_\varepsilon(z,f,f)\,e^{-\tau\varphi}dV\leqslant(1+\varepsilon)(\|T^*f\|^2_{\tau\varphi}+\|Sf\|^2_{\tau\varphi})+C_\varepsilon\|f\|^2_{\tau\varphi}.\tag{3.3.6}$$

Now we have
$$Q_0(0,f,f)=\sum_{I,J}{}'\left(\sum_1^n\lambda_j^-+\sum_{j\in J}\lambda_j\right)|f_{I,J}|^2$$

which is a positive definite hermitian form since by hypothesis either $\lambda_j>0$ for some $j\in J$ or $\lambda_j<0$ for some $j\notin J$. Hence $Q_\varepsilon(z,f,f)$ is uniformly positive definite if ε is sufficiently small and z belongs to a sufficiently small neighborhood U of 0. From (3.3.6) it follows therefore that

$$\tau \int |f|^2 e^{-\tau\varphi} dV \leqslant C(\|T^*f\|_{\tau\varphi}^2 + \|Sf\|_{\tau\varphi}^2) + C'\|f\|_{\tau\varphi}^2, \tag{3.3.7}$$

when the support of f belongs to U and ε is fixed. When $\tau > 2C'$, the estimate (3.3.1) follows.

At the boundary and for non-linear functions $\chi(\varphi)$ of φ instead of $\tau\varphi$ our results are not quite complete but still adequate for the applications.

Definition 3.3.2. We shall say that a real valued function $\varphi \in C^2$ satisfies the condition A_q at a point z_0 if $\operatorname{grad}\varphi(z_0) \neq 0$ and

$$\lambda_1 + \ldots + \lambda_q + \sum_1^{n-1} \mu_j^- > 0, \tag{3.3.8}$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ are the eigenvalues of the quadratic form (3.3.2) and $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_{n-1}$ are the eigenvalues of the same quadratic form restricted to the plane

$$\sum_1^n t_j \varphi_j = 0.$$

We note that the minimum–maximum principle for the eigenvalues gives

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \mu_{n-1} \leqslant \lambda_n,$$

so the condition (3.3.8) implies

$$\mu_1 + \ldots + \mu_q + \sum_1^{n-1} \mu_j^- > 0 \tag{3.3.9}$$

if $q < n$. Conversely, if (3.3.9) holds, a slight modification of φ will yield (3.3.8):

LEMMA 3.3.3. *Assume that* $\operatorname{grad}\varphi(z_0) \neq 0$ *and that* (3.3.9) *is valid or that* $q = n$. *Then* $e^{\tau\varphi}$ *satisfies the condition* A_q *at* z_0 *if* τ *is sufficiently large.*

Proof. We may assume that $\varphi(z_0) = 0$. It is then clear that

$$(e^{\tau\varphi})_{jk} = \tau\varphi_{jk} + \tau^2 \varphi_j \bar{\varphi}_k$$

at z_0. Thus we have to prove that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the quadratic form

$$\sum \varphi_{jk} t_j \bar{t}_k + \tau \left| \sum_1^n t_j \varphi_j \right|^2 \tag{3.3.10}$$

for large τ satisfy (3.3.8), where μ_j are independent of τ. The case $q = n$ is trivial so we assume that $q < n$. We have to show that the trace of the restriction of the form (3.3.10) to any q-dimensional subspace of $\mathbf{C}^n$ is $> -\sum_1^{n-1} \mu_j^- + \varepsilon$ for large τ and some $\varepsilon > 0$. Suppose that this were not true. Since the set of all q-dimensional subspaces form a compact space and the form (3.3.10) increases with τ, it follows that for every $\varepsilon>0$ one can find a fixed q-dimensional subspace such that the trace of the restriction of the form (3.3.10) to this space is $\leqslant -\sum_1^{n-1}\mu_j^- +\varepsilon$ for every τ. But then the subspace must lie in the plane $\sum t_j\varphi_j=0$ and we have a contradiction with (3.3.9) if ε is small enough.

We recall that (3.3.9) means that at least $n-q$ of the eigenvalues μ_j are >0 or that at least $q+1$ of them are >0.

Definition 3.3.4. We shall say that a real valued function $\varphi\in C^2$ satisfies the condition a_q at a point z_0 if $\operatorname{grad}\varphi(z_0)\neq 0$ and if $q<n$ the form (3.3.2) restricted to the plane $\sum_1^n t_j\varphi_j=0$ has at least $q+1$ negative or at least $n-q$ positive eigenvalues.

Note that the condition a_q is independent of the choice of hermitian metric in M (which is not true for A_q) and that it only depends on the surface $\{z;\varphi(z)=\varphi(z_0)\}$ and the side of this surface on which $\varphi(z)>\varphi(z_0)$. (Cf. Hörmander [12], p. 203.) The condition a_q may therefore be considered as a condition on a piece of oriented C^2 surface. In particular we shall say that $\partial\Omega$ satisfies condition a_q if the function ϱ introduced in section 3.1 satisfies this condition.

THEOREM 3.3.5. *Let $z_0\in\bar\Omega$ and let φ satisfy condition A_q at z_0. If $z_0\in\partial\Omega$ we also assume that φ is constant on $\partial\Omega$ and $<\varphi(z_0)$ in Ω. Then there is a neighborhood U of z_0 and a constant C such that for all convex increasing functions $\chi\in C^2(\mathbf{R}^1)$ we have*

$$\int\chi'(\varphi)\,|f|^2e^{-\chi(\varphi)}dV\leqslant C(\|T^*f\|^2_{\chi(\varphi)}+\|Sf\|^2_{\chi(\varphi)}+\|f\|^2_{\chi(\varphi)})\tag{3.3.11}$$

for all $f\in C^2_{(p,q)}(\bar\Omega)\cap D_{T^*}$ with support in $U\cap\bar\Omega$.

Proof. We start with the estimate (3.1.20) with φ replaced by $\chi(\varphi)$. Noting that

$$\sum_{j,k}(\chi(\varphi))_{jk}f_{I,jK}\overline{f_{I,kK}}=\chi'(\varphi)\sum_{j,k}\varphi_{jk}f_{I,jK}\overline{f_{I,kK}}+\chi''(\varphi)\Big|\sum_j\varphi_jf_{I,jK}\Big|^2,$$

where the last term is $\geqslant 0$, we obtain if $0<\varepsilon<\frac12$

$$(1-2\varepsilon)\sum_{I,J}{}'\sum_j\int_{U\cap\Omega}\left|\frac{\partial f_{I,J}}{\partial\bar\omega^j}\right|^2e^{-\chi(\varphi)}dV+\varepsilon|||f|||^2_{\chi(\varphi)}+\sum_{I,K}{}'\sum_{j,k}\int_{U\cap\Omega}\chi'(\varphi)\,\varphi_{jk}f_{I,jK}\overline{f_{I,kK}}\,e^{-\chi(\varphi)}dV$$
$$+\sum_{I,K}{}'\sum_{j,k}\int_{U\cap\partial\Omega}\varrho_{jk}f_{I,jK}\overline{f_{I,kK}}\,e^{-\chi(\varphi)}dS\leqslant(1+\varepsilon)(\|T^*f\|^2_{\chi(\varphi)}+\|Sf\|^2_{\chi(\varphi)})+C_\varepsilon\|f\|^2_{\chi(\varphi)}.\tag{3.3.12}$$

We choose the basis ω^j for forms of type $(1,0)$ so that ω^n is proportional to $\partial\varphi$, hence

$\varphi_j = \partial\varphi/\partial\omega^j = 0$ if $j<n$. Since grad φ and grad ϱ are proportional on $\partial\Omega$, this means that Proposition 3.1.3 is applicable. By a unitary transformation of the forms ω^j, $j<n$, we can achieve that

$$\sum_{j,k=1}^{n-1} \varphi_{jk}(z_0)\, t_j \bar{t}_k = \sum_1^{n-1} \mu_j |t_j|^2.$$

Let $\mu_j<0$ when $j=1, \ldots, \sigma$ and $\mu_j \geqslant 0$ when $j>\sigma$. We apply Proposition 3.1.3 to the terms in the first sum in (3.3.12) where $j\leqslant\sigma$. This gives

$$\int_{U\cap\Omega} \chi'(\varphi)\, Q_\varepsilon(z,f,f)\, e^{-\chi(\varphi)}dV + \int_{U\cap\partial\Omega} R_\varepsilon(z,f,f)\, e^{-\chi(\varphi)}dS + \varepsilon\, |||f|||^2_{\chi(\varphi)}$$
$$\leqslant (1+\varepsilon)\left(\|T^*f\|_{\chi(\varphi)} + \|Sf\|^2_{\chi(\varphi)}\right) + C_\varepsilon \|f\|^2_{\chi(\varphi)} + C\,\|f\|_{\chi(\varphi)}\, |||f|||_{\chi(\varphi)}, \tag{3.3.13}$$

where we have used the notations

$$Q_\varepsilon(z,f,f) = -(1-2\varepsilon) \sum_{I,J}{}' \sum_{j=1}^{\sigma} \varphi_{jj} |f_{I,J}|^2 + \sum_{I,K}{}' \sum_{j,k} \varphi_{jk} f_{I,jK} \overline{f_{I,kK}}, \tag{3.3.14}$$

$$R_\varepsilon(z,f,f) = -(1-2\varepsilon) \sum_{I,J}{}' \sum_{j=1}^{\sigma} \varrho_{jj} |f_{I,J}|^2 + \sum_{I,K}{}' \sum_{j,k} \varrho_{jk} f_{I,jK} \overline{f_{I,kK}}. \tag{3.3.15}$$

(In the computation it is important that $\varphi_j=0$ when $j<n$.)

The hermitian form $Q_0(z_0,f,f)$ is positive definite. In fact, if $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_n$ are the eigenvalues of the form (3.3.2), we have

$$(\lambda_1+\ldots+\lambda_q)|f|^2 \leqslant \sum_{I,K}{}' \sum_{j,k} \varphi_{jk} f_{I,jK} \overline{f_{I,kK}}$$

as is immediately seen if a frame is used where the form (3.3.2) is diagonalized. Hence it follows from (3.3.8) that $Q_0(z_0,f,f)$ is positive definite. If U is a sufficiently small neighborhood of z_0 and ε is given a fixed but sufficiently small value, it follows that $|f|^2 \leqslant CQ_\varepsilon(z,f,f)$ for some constant C when $z\in U$. This implies that $R_\varepsilon(z,f,f)\geqslant 0$ when $z\in U\cap\partial\Omega$, for f satisfies the boundary condition $f_{I,J}=0$ when $n\in J$, and the fact that φ is constant on the boundary implies that $\varphi_{jk}=c\varrho_{jk}$, for some $c>0$ if $j,k<n$. Theorem 3.3.5 now follows from (3.3.13), even with the L^2 norm of f over $U\cap\partial\Omega$ with respect to the density $e^{-\chi(\varphi)}dS$ added on the left-hand side of (3.3.11).

3.4. *Existence and approximation theorems*

The first existence theorem which we shall prove could also have been obtained from Theorem 3.2.5 and Proposition 1.2.4 by the arguments of Kohn [14]. However, we prefer to use the estimates involving weight functions in all the existence proofs.

THEOREM 3.4.1. *Let Ω be relatively compact in M and have a C^3 boundary $\partial\Omega$, satisfying the condition a_q. Then R_T is closed and has finite codimension in N_S.*

We recall that T is the weak maximal operator from $L^2_{(p,q-1)}(\Omega, \varphi)$ to $L^2_{(p,q)}(\Omega, \varphi)$ defined by $\bar\partial$, and that S is the weak maximal $\bar\partial$ operator from $L^2_{(p,q)}(\Omega, \varphi)$ to $L^2_{(p,q+1)}(\Omega, \varphi)$. Here φ is any function $\in C(\overline{\Omega})$.

Proof. The assertion is obviously independent of the choice of the function $\varphi \in C(\overline{\Omega})$, since changing φ only means introducing equivalent norms in the three Hilbert spaces concerned. We choose $\varphi \in C^3(\overline{\Omega})$ so that φ is near $\partial\Omega$ of the form $e^{\lambda\varrho}$ with λ so large that according to Lemma 3.3.3 the condition A_q is satisfied by φ at every point on $\partial\Omega$. It suffices to prove the assertion of the theorem with φ replaced by some multiple of φ. From Theorem 3.3.5 it follows that every point on $\partial\Omega$ has a neighborhood U such that

$$\tau \int_{U \cap \Omega} |f|^2 e^{-\tau\varphi} dV \leqslant C(\|T^* f\|^2_{\tau\varphi} + \|Sf\|^2_{\tau\varphi} + \|f\|^2_{\tau\varphi})$$

for large τ and all $f \in C^2_{(p,q)}(\overline{\Omega}) \cap D_{T^*}$ with support in $U \cap \overline{\Omega}$. We can cover $\partial\Omega$ by a finite number of such neighborhoods U_ν and choose $\psi_\nu \in C_0^\infty(U_\nu)$ so that $\Sigma\psi_\nu = 1$ in Ω outside a compact subset K. In view of the obvious estimates

$$\|T^*(\psi_\nu f)\|_{\tau\varphi} \leqslant \|T^* f\|_{\tau\varphi} + C\|f\|_{\tau\varphi},$$

where C is independent of f and of τ (see also the proof of Proposition 2.1.1) we obtain with another C

$$\tau \int_{\complement K} |f|^2 e^{-\tau\varphi} dV \leqslant C(\|T^* f\|^2_{\tau\varphi} + \|Sf\|^2_{\tau\varphi} + \|f\|^2_{\tau\varphi}), \quad f \in C^2_{(p,q)}(\overline{\Omega}) \cap D_{T^*}.$$

If we choose τ so that $\tau > 2C$, it follows that

$$\int_{\complement K} |f|^2 e^{-\tau\varphi} dV \leqslant \|T^* f\|^2_{\tau\varphi} + \|Sf\|^2_{\tau\varphi} + \int_K |f|^2 e^{-\tau\varphi} dV, \quad f \in C^2_{(p,q)}(\overline{\Omega}) \cap D_{T^*}. \tag{3.4.1}$$

Since Proposition 1.2.4 implies that $C^2_{(p,q)}(\overline{\Omega}) \cap D_{T^*}$ is dense in $D_S \cap D_{T^*}$ for the graph norm, the estimate is valid for all $f \in D_S \cap D_{T^*}$.

To prove the theorem it suffices to show that (3.4.1) implies that the hypotheses of Theorem 1.1.3 are fulfilled. Thus let $f_j \in D_S \cap D_{T^*}$ ($j = 1, 2, \ldots$) and let $T^* f_j \to 0$, $Sf_j \to 0$ in $L^2_{(p,q\mp1)}(\Omega, \tau\varphi)$ respectively. If the sequence f_j converges in $L^2_{(p,q)}(K, \tau\varphi)$ we conclude that f_j converges in $L^2_{(p,q)}(\Omega, \tau\varphi)$ by applying (3.4.1) to $f = f_j - f_k$ and letting $j, k \to \infty$. (Note that τ is a fixed number such that (3.4.1) is valid.) Therefore it only remains to prove the following simple lemma, where we write ψ instead of $\tau\varphi$.

LEMMA 3.4.2. *Let $\psi \in C^2$ and set*

$$B = \{f;\, f \in D_S \cap D_{T^*},\, \|f\|_\psi^2 + \|Sf\|_\psi^2 + \|T^*f\|_\psi^2 \leqslant 1\}.$$

Then B is relatively compact in $L^2_{(p,q)}(\Omega, \mathrm{loc})$.

Proof. Let U be a coordinate patch $\subset \Omega$ and let $\chi \in C_0^\infty(U)$. Writing $g = \chi f$, we have for some constant C and all $f \in B$

$$\|g\|_\psi^2 + \|Sg\|_\psi^2 + \|T^*g\|_\psi^2 \leqslant C.$$

From (3.1.20) we therefore obtain with another C

$$\sum_{I,J}{}' \sum_j \int \left| \frac{\partial g_{I,J}}{\partial \bar{\omega}^j} \right|^2 e^{-\psi} dV \leqslant C.$$

With still another C it follows that

$$\sum_{I,J}{}' \int \left(|g_{I,J}|^2 + \sum_k \left| \frac{\partial g_{I,J}}{\partial \bar{z}^k} \right|^2 \right) dx \leqslant C,$$

where z^k are the local coordinates, and taking Fourier transforms we obtain

$$\sum_{I,J}{}' \int (|g_{I,J}|^2 + |\operatorname{grad} g_{I,J}|^2/4)\, dx \leqslant C.$$

The set of all such g with support in a fixed compact set is compact in L^2 by Rellich's lemma (see e.g. Hörmander [12], Theorem 2.2.3).

From every sequence of elements in B we can thus select a subsequence which is L^2 convergent in a neighborhood of any given point in Ω. Using the Borel–Lebesgue lemma and taking a diagonal sequence we can find a subsequence converging in $L^2_{(p,q)}$ on any compact subset of Ω. This completes the proof of Lemma 3.4.2 and therefore the proof of Theorem 3.4.1.

Definition 3.4.3. If Ω is relatively compact in the manifold M and φ is continuous in $\bar{\Omega}$, we denote the quotient space N_S/R_T by $\bar{H}_{(p,q)}(\Omega)$. (We recall that

$$N_S = \{f;\, f \in L^2_{(p,q)}(\Omega, \varphi),\, \bar\partial f = 0\},$$

and that R_T is the range of the weak maximal $\bar\partial$ operator from $L^2_{(p,q-1)}(\Omega, \varphi)$ to $L^2_{(p,q)}(\Omega, \varphi)$. This quotient space is of course independent of φ.) We also denote by $H_{(p,q)}(\Omega)$ the quotient space of $\{f;\, f \in L^2_{(p,q)}(\Omega, \mathrm{loc}),\, \bar\partial f = 0\}$ with respect to

$$L^2_{(p,q)}(\Omega, \mathrm{loc}) \cap \{\bar\partial f;\, f \in L_{(p,q-1)}(\Omega, \mathrm{loc})\};$$

here Ω may be any paracompact complex analytic manifold.

By the Dolbeault theorem (see the proof of Corollary 2.2.5) there is a natural isomorphism between the space $H_{(p,q)}(\Omega)$ and the qth cohomology group of Ω with values in the sheaf of germs of holomorphic p-forms.

When the hypotheses of Theorem 3.4.1 are fulfilled, we know that $\bar{H}_{(p,q)}(\Omega)$ is a finite dimensional vector space. For a given φ there is a natural isomorphism

$$\bar{H}_{(p,q)}(\Omega) \approx N_S \ominus R_T = N_S \cap N_{T^*} = N_{(p,q)}(\Omega, \varphi)$$

where the last equality is a definition. If $\Omega' \supset \Omega$ we obtain by taking restrictions a canonical homomorphism $\bar{H}_{(p,q)}(\Omega') \to \bar{H}_{(p,q)}(\Omega)$. Our next purpose is to give conditions which guarantee that this homomorphism is surjective or injective. At the same time we shall obtain an approximation theorem for solutions of the equation $\bar{\partial} u = 0$. The proof depends on the estimates contained in the following two propositions.

PROPOSITION 3.4.4. *Let Ω be relatively compact with $\partial\Omega \in C^3$, let $\varphi \in C^3(\bar{\Omega})$ be equal to 0 on $\partial\Omega$ and <0 in Ω, and assume that φ satisfies condition A_q in $\{z; z \in \bar{\Omega}, \varphi(z) \geq c\}$ for some $c<0$. Then there is a compact subset K of $\Omega_c = \{z; z \in \Omega, \varphi(z) < c\}$ and a constant C such that for all convex increasing functions $\chi \in C^2(\mathbf{R}^1)$*

$$\int_{\complement K} \chi'(\varphi)\,|f|^2 e^{-\chi(\varphi)} dV \leq C(\|T^* f\|^2_{\chi(\varphi)} + \|Sf\|^2_{\chi(\varphi)} + \|f\|^2_{\chi(\varphi)}), \quad f \in C^2_{(p,q)}(\bar{\Omega}) \cap D_{T^*}. \quad (3.4.2)$$

Proof. In virtue of Theorem 3.3.5 we can find a neighborhood U of any point in $\bar{\Omega} \cap \complement\Omega_c$ such that (3.4.2) holds when f has support in $U \cap \bar{\Omega}$. If we use a partition of unity as in the proof of Theorem 3.4.1, the estimate (3.4.2) follows immediately from the fact that it holds locally. The details may be left to the reader.

We shall now derive from (3.4.2) a more useful estimate. To do so we choose a sequence of convex increasing functions $\chi_k \in C^2$ such that

$$\chi_k(\tau) = \gamma\tau \quad \text{when} \quad \tau < c, \quad \chi_k'(\tau) \nearrow +\infty \quad \text{when} \quad k \to \infty \text{ and } \tau > c. \quad (3.4.3)$$

Here γ is a constant larger than the constant C in (3.4.2). We set $\varphi_k = \chi_k(\varphi)$.

PROPOSITION 3.4.5. *Under the hypotheses of Proposition 3.4.4 and with the notation just introduced, one can find a constant C' such that for large k*

$$\|f\|_{\varphi_k} \leq C'(\|T^* f\|_{\varphi_k} + \|Sf\|_{\varphi_k}) \quad \text{if} \quad f \in D_{T^*} \cap D_S$$

$$\text{and} \quad \int_{\Omega_c} \langle f, g\rangle e^{-\gamma\varphi} dV = 0 \quad \text{for all} \quad g \in N_{(p,q)}(\Omega_c, \gamma\varphi). \quad (3.4.4)$$

Proof. Assume that the assertion is false. For any C' we can then find arbitrarily large values of k for which (3.4.4) does not hold. Passing if necessary to a subsequence of the given sequence we may assume that (3.4.4) is not valid with $C'=k$. In view of Proposition 1.2.4 we can then choose $f_k \in C^2_{(p,q)}(\bar{\Omega}) \cap D_{T^*}$ (a space which is independent of k) such that

$$\|f_k\|_{\varphi_k}=1,\ \|T^* f_k\|_{\varphi_k}+\|Sf_k\|_{\varphi_k}<1/k,\quad \int_{\Omega_c}\langle f_k, g\rangle e^{-\gamma\varphi}dV=0 \quad \text{for all} \quad g\in N_{(p,q)}(\Omega_c,\gamma\varphi). \tag{3.4.5}$$

Since $\chi_k' \geqslant \gamma$ we obtain from (3.4.2) that

$$\int_{\complement K}|f_k|^2 e^{-\varphi_k}dV \leqslant C\gamma^{-1}(1+k^{-1}),$$

which implies

$$\int_K |f_k|^2 e^{-\gamma\varphi}dV \geqslant 1-C\gamma^{-1}(1+k^{-1}). \tag{3.4.6}$$

No subsequence of the sequence f_k can therefore converge to 0 in L^2 norm over K.

From the first part of (3.4.5) and the fact that $\varphi_k=\gamma\varphi$ in Ω_c it follows that a subsequence of the sequence f_k is weakly convergent in $L^2_{(p,q)}(\Omega_c,\gamma\varphi)$ to a limit f. Changing notations if necessary, we may assume that the whole sequence converges. Now take $\psi\in C_0^\infty(\Omega_c)$ so that $\psi=1$ on K. From (3.4.5) it follows that for some C_1

$$\|T^*(\psi f_k)\|_{\gamma\varphi}+\|S(\psi f_k)\|_{\gamma\varphi}+\|\psi f_k\|_{\gamma\varphi}\leqslant C_1$$

for all k. Hence the weakly convergent sequence ψf_k is strongly compact in $L^2_{(p,q)}(\Omega_c,\gamma\varphi)$ by Lemma 3.4.2, so it must in fact converge strongly. In view of (3.4.6) it follows therefore that $f\neq 0$, while (3.4.5) implies that $Sf=0$ in Ω_c. We shall prove that $f\in D_{T^*}$ and that $T^*f=0$ in Ω_c, that is, $f\in N_{(p,q)}(\Omega_c,\gamma\varphi)$. Since it follows from (3.4.5) that f is orthogonal to every element in $N_{(p,q)}(\Omega_c,\gamma\varphi)$, this will yield a contradiction and prove the proposition.

Thus introduce $f_k e^{-\varphi_k}=g_k$. We have

$$\int_\Omega |g_k|^2 e^{\varphi_k}dV=\|f_k\|^2_{\varphi_k}=1,$$

so g_k converges weakly in $L^2_{(p,q)}(\Omega,-\gamma\varphi)$ to a limit g which is $fe^{-\gamma\varphi}$ in Ω_c and 0 outside Ω_c. Further, we have $e^{-\varphi_k}T^*f_k=\vartheta g_k$ where ϑ is the differential operator adjoint to $\bar\partial$, defined by

$$\int\langle\vartheta g, u\rangle\, dV=\int\langle g,\bar\partial u\rangle\, dV, \tag{3.4.7}$$

for all $u\in C^1_{(p,q)}(\Omega)$ with compact support. Thus ϑ does not depend on k. Since $\varphi_k\geqslant\gamma\varphi$ and

$$\int |\vartheta g_k|^2 e^{\varphi_k} dV = \| T^* f_k \|_{\varphi_k}^2 \to 0, \quad k \to \infty,$$

it follows that $\vartheta g_k \to 0$ in $L^2_{(p,q)}(\Omega, -\gamma\varphi)$. Hence we obtain by applying (3.4.7) to g_k and letting $k \to \infty$

$$\int_{\Omega_c} \langle g, \bar\partial u \rangle \, dV = 0, \quad u \in C^1_{(p,q)}(\bar\Omega_c),$$

for every $u \in C^1_{(p,q)}(\bar\Omega_c)$ can be extended to an element of $C^1_{(p,q)}(\Omega)$ with compact support. If we recall that $g = f e^{-\gamma\varphi}$ in Ω_c, we obtain

$$\int_{\Omega_c} \langle f, \bar\partial u \rangle \, e^{-\gamma\varphi} dV = 0, \quad u \in C^1_{(p,q)}(\bar\Omega_c)$$

and since $C^1_{(p,q)}(\bar\Omega_c)$ is dense in the domain of T in Ω_c (with respect to the graph norm), it follows that $T^* f = 0$. The proof is complete.

THEOREM 3.4.6. *Let Ω be relatively compact with $\partial\Omega \in C^3$, let $\varphi \in C^3(\bar\Omega)$ be equal to 0 on $\partial\Omega$ and <0 in Ω, and assume that φ satisfies condition a_q in $\bar\Omega$ outside $\Omega_c = \{z; z \in \Omega, \varphi(z) < c\}$ for some $c < 0$. If $f \in L^2_{(p,q)}(\Omega, \varphi)$ and $\bar\partial f = 0$, the equation $\bar\partial u = f$ has a solution $u \in L^2_{(p,q-1)}(\Omega, \varphi)$ if in Ω_c it has a solution $\in L^2_{(p,q-1)}(\Omega_c, \varphi)$. In other words, the restriction homomorphism $\bar H_{(p,q)}(\Omega) \to \bar H_{(p,q)}(\Omega_c)$ is injective.*

Proof. If we replace φ by $e^{\lambda\varphi} - 1$ with a sufficiently large λ, we can by Lemma 3.3.3 achieve that φ satisfies A_q outside Ω_c, and this makes Proposition 3.4.5 applicable. Choose a fixed k so that (3.4.4) holds. Let F be the set of all $f \in L^2_{(p,q)}(\Omega, \varphi_k)$ such that $\bar\partial f = 0$ and the equation $\bar\partial u = f$ has a solution $u \in L^2_{(p,q)}(\Omega_c, \gamma\varphi)$ on Ω_c. Since $\int_{\Omega_c} \langle f, g \rangle e^{-\lambda\varphi} dV = 0$ for every $g \in N_{(p,q)}(\Omega_c, \gamma\varphi)$ if $f \in F$, the estimate (3.4.4) shows that we may apply Theorem 1.1.4 with A equal to a multiple of the identity. This proves the theorem.

THEOREM 3.4.7. *Let the hypotheses of Theorem 3.4.6 be fulfilled. For every $u \in L^2_{(p,q-1)}(\Omega_c, \varphi)$ satisfying the equation $\bar\partial u = 0$ in Ω_c and for every $\varepsilon > 0$ one can find $u_1 \in L^2_{(p,q-1)}(\Omega, \varphi)$ satisfying the equation $\bar\partial u_1 = 0$ in Ω, so that $\int_{\Omega_c} |u - u_1|^2 e^{-\varphi} dV < \varepsilon$.*

Proof. As in the proof of Theorem 3.4.6 we may assume that φ satisfies A_q outside Ω_c, which makes Proposition 3.4.5 applicable. Let $v \in L^2_{(p,q-1)}(\Omega_c, -\varphi)$ be orthogonal to the restriction to Ω_c of every $u_1 \in L^2_{(p,q-1)}(\Omega, \varphi)$ satisfying the equation $\bar\partial u_1 = 0$ in Ω. If we define v to be 0 in Ω outside Ω_c, this means that

$$\int_{\Omega} \langle u_1, v \rangle \, dV = 0$$

for all u_1 such that $Tu_1=0$. Hence ve^{φ_k} belongs to R_{T^*} for R_{T^*} is closed by Theorem 3.4.1 and Theorem 1.1.1. By Theorem 1.1.4 and the estimate (3.4.4) we can therefore find $f_k\in D_{T^*}$ so that $T^*f_k=ve^{\varphi_k}$ and $\|f_k\|_{\varphi_k}\leqslant C'\|T^*f_k\|_{\varphi_k}=C'\|v\|_{-\gamma\varphi}$. Now set $g_k=f_ke^{-\varphi_k}$. With the notation ϑ used in the proof of Theorem 3.4.5 we have $\vartheta g_k=v$ for every k, and the estimate just given for f_k implies

$$\int_\Omega |g_k|^2 e^{\varphi_k} dV \leqslant C'^2 \int_{\Omega_c} |v|^2 e^{\gamma\varphi} dV. \tag{3.4.8}$$

Since $\varphi_k\geqslant\gamma\varphi$, we can choose a weak limit g of the sequence g_k in $L^2_{(p,q)}(\Omega,-\gamma\varphi)$, and from (3.4.8) it follows that $g=0$ outside Ω_c. From the equations $\vartheta g_k=v$, we obtain $\vartheta g=v$. Hence

$$\int_{\Omega_c}\langle v,u\rangle\, dV=\int_{\Omega_c}\langle g,\bar\partial u\rangle\, dV$$

for every $u\in C^1_{(p,q)}(\overline{\Omega}_c)$, and therefore for every u in the domain of the maximal differential operator defined by $\bar\partial$ in $L^2(\Omega_c,\varphi)$ (Proposition 1.2.3). This proves that $\int_{\Omega_c}\langle v,u\rangle dV=0$ for every $u\in L^2_{(p,q-1)}(\Omega_c,\varphi)$ satisfying the equation $\bar\partial u=0$, so Theorem 3.4.7 follows from the Hahn–Banach theorem.

THEOREM 3.4.8. *Let Ω be relatively compact with $\partial\Omega\in C^3$, let $\varphi\in C^3(\overline{\Omega})$ be equal to 0 on $\partial\Omega$ and <0 in Ω, and assume that φ satisfies the conditions a_q and a_{q+1} in $\overline{\Omega}$ outside $\Omega_c=\{z;\, z\in\Omega,\, \varphi(z)<c\}$ for some $c<0$. Then the restriction homomorphism $\overline{H}_{(p,q)}(\Omega)\to\overline{H}_{(p,q)}(\Omega_c)$ is an isomorphism.*

Proof. By Theorem 3.4.6 the homomorphism is injective. To prove that it is surjective we shall use Theorem 3.4.7. Choose an orthonormal basis $g_1,\dots,g_\nu$ for $N_{(p,q)}(\Omega_c,\varphi)$. Since $\bar\partial g_j=0$, we can for every $\varepsilon>0$ find $G_j\in L^2_{(p,q)}(\Omega,\varphi)$ such that $\bar\partial G_j=0$ and

$$\left|\int_{\Omega_c}\langle (g_j-G_j),g_k\rangle\, e^{-\varphi} dV\right|<\varepsilon.$$

This follows from Theorem 3.4.7 since φ satisfies a_{q+1}. Writing

$$a_{jk}=\int_{\Omega_c}\langle G_j,g_k\rangle\, e^{-\varphi} dV,$$

we have $|a_{jk}-\delta_{jk}|<\varepsilon$, so the matrix (a_{jk}) has an inverse (A_{jk}) if ε is sufficiently small. If we set $G'_j=\Sigma A_{jl}G_l$, it follows that

$$\int_{\Omega_c}\langle (g_j-G'_j),g_k\rangle\, e^{-\varphi} dV=\delta_{jk}-\sum_l A_{jl}a_{lk}=0$$

for all j and k. Hence the restriction of G_j' to Ω_c has the same image as g_j in $\bar{H}_{(p,q)}(\Omega_c)$, which proves that the homomorphism in the theorem is surjective.

We shall now study $H_{(p,q)}(\Omega)$. In doing so it is convenient to assume that $\varphi \to +\infty$ at the "boundary" of Ω; this can be achieved if the hypotheses of the preceding theorems are fulfilled, for the validity of condition a_q is not affected if φ is replaced by an increasing function of φ.

THEOREM 3.4.9. *Let Ω be a complex analytic manifold of complex dimension n and let φ be a C^3 function in Ω such that the open sets $\Omega_c = \{z; z \in \Omega, \varphi(z) < c\}$ are relatively compact in Ω for every real number c. Further assume that φ satisfies condition a_q in the complement of Ω_c for some c. Then the restriction homomorphism $H_{(p,q)}(\Omega) \to \bar{H}_{(p,q)}(\Omega_c)$ is injective for the same c, which implies that $H_{(p,q)}(\Omega)$ has finite dimension. Further, every $u \in L^2_{(p,q-1)}(\Omega_c, \varphi)$ such that $\bar\partial u = 0$ can be approximated arbitrarily closely in the norm of $L^2_{(p,q-1)}(\Omega_c, \varphi)$ by the restrictions to Ω_c of forms $u_1 \in L^2_{(p,q-1)}(\Omega, \text{loc})$ such that $\bar\partial u_1 = 0$. If φ satisfies both conditions a_q and a_{q+1} outside Ω_c, the homomorphism $H_{(p,q)}(\Omega) \to \bar{H}_{(p,q)}(\Omega_c)$ is an isomorphism.*

Proof. As observed in paragraph 3.1, we can introduce a hermitian metric in Ω. If $d > c$, the preceding theorems are then applicable with Ω replaced by Ω_d. Now let $f \in L^2_{(p,q)}(\Omega, \text{loc})$, assume that $\bar\partial f = 0$ and that the equation $\bar\partial u = f$ has a solution $u \in L^2_{(p,q-1)}(\Omega_c, \varphi)$. For every integer $j > 0$ we can then find $u_j \in L^2_{(p,q-1)}(\Omega_{c+j}, \varphi)$ such that $\bar\partial u_j = f$ in Ω_{c+j} and

$$\int_{\Omega_{c+j}} |u_{j+1} - u_j|^2 e^{-\varphi} dV \leqslant 2^{-j}. \tag{3.4.9}$$

In fact, assume that $u_1, \ldots, u_j$ have already been chosen. To construct u_{j+1} we first note that in virtue of Theorem 3.4.6 we can find $u \in L^2_{(p,q-1)}(\Omega_{c+j+1}, \varphi)$ such that $\bar\partial u = f$ in Ω_{c+j+1}. This implies that $\bar\partial(u - u_j) = 0$ in Ω_{c+j}, so by Theorem 3.4.7 there exists a form $v \in L^2_{(p,q-1)}(\Omega_{c+j+1}, \varphi)$ such that $\bar\partial v = 0$ and (3.4.9) is valid for $u_{j+1} = u - v$. Since $\bar\partial u_{j+1} = \bar\partial u = f$ in Ω_{c+j+1}, this proves the statement. From (3.4.9) it follows that $u = \lim_{j\to\infty} u_j$ exists in $L^2_{(p,q-1)}(\Omega, \text{loc})$, and it is obvious that $\bar\partial u = f$. This proves that the homomorphism $H_{(p,q)}(\Omega) \to \bar{H}_{(p,q)}(\Omega_c)$ is injective, and since $\bar{H}_{(p,q)}(\Omega_c)$ is finite dimensional by Theorem 3.4.1, we conclude that $H_{(p,q)}(\Omega)$ has finite dimension. The approximation theorem follows immediately by iterated use of Theorem 3.4.7. We leave the details to the reader in order not to repeat the arguments already used in the proof of Theorem 2.3.5. The final statement now follows by repetition of the proof of Theorem 3.4.8, so we omit these details too.

Remark. The part of condition a_q, which requires that grad$\varphi \neq 0$, is unnecessarily restrictive and could be removed by applying the theorem of Morse [21] as in Chapter II, provided that $\varphi \in C^{2n}$. We can also give another supplementary result:

THEOREM 3.4.10. *Let Ω be a complex manifold of complex dimension n, and let φ be a C^2 function in Ω such that the open sets $\Omega_c = \{z;\, z \in \Omega,\, \varphi(z) < c\}$ are relatively compact in Ω for every real c. Further assume that φ satisfies condition a_q outside Ω_{c_0} for some c_0 and that the form* (3.3.2) *has either at least $q+1$ negative or $n-q+1$ positive eigenvalues for every $z_0 \in \Omega$. Then it follows that $H_{(p,q)}(\Omega) = 0$.*

Proof. We obtain $\bar{H}_{(p,q)}(\Omega_c) = 0$ for every $c > c_0$ by using Theorem 3.3.1 and Theorem 3.3.5. As weight function we then use $\chi(\varphi)$ where χ is convex on $\mathbf{R}$ and linear with a large slope on $(-\infty, c_0)$. In view of Theorem 3.4.9, this proves the theorem.

3.5. *Behavior of the Bergman kernel function at the boundary*

Let Ω be a paracompact open subset of a complex hermitian manifold M of dimension n, and let $\varphi \in C^0(\bar{\Omega})$. The set of analytic functions in Ω with

$$\|u\|_\varphi^2 = \left(\int_\Omega |u|^2 e^{-\varphi} dV\right)^{\frac{1}{2}} < \infty$$

forms a Hilbert space with this norm. If $u_1, u_2, \ldots$ is a complete orthonormal system in this space, the Bergman kernel function of Ω (with respect to the weight function $e^{-\varphi}$) is defined by

$$K(z;\Omega,\varphi) = K(z) = \sum_1^\infty |u_j(z)|^2 = \sup |\sum_1^\infty a_j u_j(z)|^2 / \sum_1^\infty |a_j|^2 = \sup |u(z)|^2 / \|u\|_\varphi^2, \quad z \in \Omega, \qquad (3.5.1)$$

where the supremum is taken over all $u \neq 0$ in the Hilbert space. The last form of the definition shows that K is independent of the choice of the orthonormal system.

Bergman [3] has studied the behavior of the function $K(z;\Omega,\varphi)$ at the boundary of Ω when Ω is a bounded domain of holomorphy in $\mathbf{C}^2$ and $\varphi = 0$. (It is also well known that the kernel function is regular at a point on the boundary where some eigenvalue of the Levi form is negative.) We shall here extend the results of Bergman as follows:

THEOREM 3.5.1. *Assume that the weak maximal operator $\bar{\partial}: L^2_{(0,0)}(\Omega,\varphi) \to L^2_{(0,1)}(\Omega,\varphi)$ has a closed range, and let z_0 be a point on $\partial\Omega$ such that $\partial\Omega \in C^2$ in a neighborhood of z_0. Further we assume that $\partial\Omega$ is strictly pseudo-convex at z_0, that is, that the Levi form $\Sigma \varrho_{jk}(z_0) t_j \bar{t}_k$ is positive definite in the plane $\Sigma\, \varrho_j(z_0) t_j = 0$. ($\varrho$ is the distance to $\partial\Omega$; for other notations see sections* 3.1 *and* 3.2.) *Let $k(z_0)$ be the product of the $n-1$ eigenvalues of this form. Then*

$$|\varrho(z)|^{n+1} K(z;\Omega,\varphi) \to k(z_0)\, e^{\varphi(z_0)} \frac{n!}{4\pi^n}, \quad z \to z_0. \qquad (3.5.2)$$

The hypothesis that the range of $\bar\partial$ be closed is always fulfilled if Ω is a domain of holomorphy in $\mathbf{C}^n$ and $\varphi(z)=|z|^2$, for example (see Theorem 2.2.1′); or if Ω is a bounded domain of holomorphy in $\mathbf{C}^n$ and φ is any function in $C(\bar\Omega)$; or if $\partial\Omega \in C^3$ and the Levi form of $\partial\Omega$ has everywhere either $n-1$ positive or 2 negative eigenvalues (Theorem 3.4.1). Thus Theorem 3.5.1 implies Grauert's theorem that Ω is holomorph-convex if the boundary is strictly pseudo-convex.

The main step in the proof is to show that the statement (3.5.2) can be localized. First note that if $\Omega' \subset \Omega$ it is a trivial consequence of (3.5.1) that

$$K(z;\Omega',\varphi) \geqslant K(z;\Omega,\varphi), \quad z \in \Omega'. \tag{3.5.3}$$

On the other hand, we shall prove

LEMMA 3.5.2. *Let the range of the weak maximal $\bar\partial$ operator T from $L^2_{(0,0)}(\Omega,\varphi)$ to $L^2_{(0,1)}(\Omega,\varphi)$ be closed. Let $z_0 \in \partial\Omega$ and assume that for some neighborhood U of z_0 there is an analytic function u_0 in $\Omega'=\Omega\cap U$ such that $|u_0|\leqslant 1$ in Ω', $|u_0(z)|\to 1$ when $z\to z_0$, and $|u_0(z)|$ has an upper bound <1 in $\Omega'\cap \complement U_0$ for some neighborhood U_0 of z_0 with compact closure contained in U. Then it follows that*

$$K(z;\Omega,\varphi)/K(z;\Omega',\varphi)\to 1, \quad z\to z_0. \tag{3.5.4}$$

Proof. Let $\chi \in C_0^\infty(U)$ be equal to 1 in U_0, and let $0\leqslant\chi\leqslant 1$ everywhere. If $u' \in L^2_{(0,0)}(\Omega',\varphi)$ and u' is analytic in Ω', we set with an integer ν to be determined later

$$u=\chi u' u_0^\nu - v.$$

The product $\chi u' u_0^\nu$ shall be defined as 0 where $\chi=0$, and $v\in L^2_{(0,0)}(\Omega,\varphi)$ shall be chosen so that $\bar\partial u=0$ in Ω, that is,

$$\bar\partial v=(\bar\partial\chi)u'u_0^\nu.$$

Since R_T is closed by hypothesis, it follows from Theorem 1.1.1 that this equation, besides the obvious solution $v=\chi u' u_0^\nu$, has a solution v with

$$\int_\Omega |v|^2 e^{-\varphi} dV \leqslant C\int_{\Omega'\cap\complement U_0} |u'u_0^\nu|^2 e^{-\varphi} dV. \tag{3.5.5}$$

If ε is any positive number, we obtain

$$\int_{\Omega'} |u-u'u_0^\nu|^2 e^{-\varphi} dV \leqslant 2(C+1)\int_{\Omega'\cap\complement U_0} |u'u_0^\nu|^2 e^{-\varphi} dV \leqslant \varepsilon^2 \int_{\Omega'} |u'|^2 e^{-\varphi} dV, \tag{3.5.6}$$

provided that ν is chosen so large that $|u_0|^{2\nu}<\varepsilon^2/2(C+1)$ in $\Omega'\cap\complement U_0$. From the definition of the kernel function in Ω' and (3.5.6) it follows that

$$|u(z)-u'(z)\,u_0(z)^\nu|^2 \leqslant \varepsilon^2 K(z;\Omega',\varphi)\int_{\Omega'}|u'|^2e^{-\varphi}dV, \quad z\in\Omega'.$$

Hence
$$|u(z)| \geqslant |u'(z)|\,|u_0(z)|^\nu - \varepsilon\left(K(z;\Omega',\varphi)\int_{\Omega'}|u'|^2e^{-\varphi}dV\right)^{\frac{1}{2}}, \quad z\in\Omega'.$$

Since the supremum in the definition (3.5.1) of the kernel function is obviously attained, we can for every $z\in\Omega'$ choose $u'\not\equiv 0$ so that

$$|u'(z)|^2 = K(z;\Omega',\varphi)\int_{\Omega'}|u'|^2e^{-\varphi}dV.$$

For the corresponding function u we then obtain the estimate

$$|u(z)|^2 \geqslant K(z;\Omega',\varphi)\,(|u_0(z)|^\nu-\varepsilon)^2\int_{\Omega'}|u'|^2e^{-\varphi}dV, \tag{3.5.7}$$

when $z\in\Omega'$ and $|u_0(z)|^\nu>\varepsilon$. By the triangle inequality and (3.5.5) we have

$$\int_\Omega |u|^2e^{-\varphi}dV \leqslant (1+\varepsilon)^2\int_{\Omega'}|u'|^2e^{-\varphi}dV.$$

Combined with (3.5.7) this estimate implies

$$K(z;\Omega,\varphi)\geqslant K(z;\Omega',\varphi)\,(|u_0(z)|^\nu-\varepsilon)^2(1+\varepsilon)^{-2} \quad \text{if} \quad z\in\Omega', \quad |u_0(z)|^\nu>\varepsilon.$$

Hence
$$\varliminf_{z\to z_0} K(z;\Omega,\varphi)/K(z;\Omega',\varphi)\geqslant(1-\varepsilon)^2(1+\varepsilon)^{-2},$$

and since ε is arbitrary, this proves the lemma if we recall (3.5.3).

Note that the proof is very close to that of Theorem 2.3.8.

Using Lemma 3.5.2 and the monotonicity (3.5.3) we can reduce the proof of Theorem 3.5.1 to the study of some special domain Ω, for which the kernel function is easy to compute.

LEMMA 3.5.3. *Let Ω_0 be the ellipsoid in $\mathbf{C}^n$ defined by*

$$\Omega_0=\{z;\, z\in\mathbf{C}^n,\, a_1|z_1|^2+\ldots+a_n|z_n|^2<a_0\},$$

where $a_0, a_1, \ldots, a_n$ are positive numbers, and let the element of volume in the definition of the kernel function be the Lebesgue measure. Then

$$K(z;\Omega_0,0)=n!\pi^{-n}a_0\ldots a_n(a_0-a_1|z_1|^2-\ldots-a_n|z_n|^2)^{-n-1}.$$

Proof. We may assume that $a_0=1$ and, after a linear change of variables, that $a_1=\ldots=a_n=1$. In view of the unitary invariance of K it is no restriction to let $z=(0,\ldots,0,\zeta)$.

If u is a square integrable function in $A(\Omega_0)$, then a unitary transformation B of the variables $z_1, \ldots, z_{n-1}$ leaves Ω_0, $u(0, \ldots, 0, \zeta)$ and $\int |u|^2 dV$ invariant. If we form

$$u_1(z) = \int u(Bz', z_n)\, dB,$$

where dB is the normalized Haar measure on the unitary group, we therefore obtain a function $u_1 \in A(\Omega_0)$ such that

$$u_1(0, \ldots, 0, \zeta) = u(0, \ldots, 0, \zeta),$$

and by Minkowski's inequality we have

$$\int_{\Omega_0} |u_1|^2 dV \leqslant \int_{\Omega_0} |u|^2 dV.$$

But u_1 is invariant for unitary transformations of $z_1, \ldots, z_{n-1}$ and must therefore be a function of z_n only. In determining the supremum in (3.5.1) we may thus assume that u is an analytic function of z_n when $|z_n| < 1$. Put

$$u(z) = \sum_0^\infty c_j z_n^j.$$

Since the volume of the unit ball in R^{2n-2} is $\pi^{n-1}/(n-1)!$, we obtain

$$\int_{\Omega_0} |u|^2 dV = \pi^{n-1}/(n-1)! \int_0^{2\pi} \int_0^1 |u(re^{i\theta})|^2 r(1-r^2)^{n-1} dr d\theta$$

$$= \pi^{n-1}/(n-1)! \sum_0^\infty |c_j|^2 2\pi \int_0^1 r^{2j+1}(1-r^2)^{n-1} dr$$

$$= \pi^n \sum_0^\infty |c_j|^2 j!/(j+n)!.$$

By Cauchy–Schwarz' inequality it follows that

$$|u(0, \ldots, 0, \zeta)|^2 \leqslant \pi^{-n} \sum_0^\infty |\zeta|^{2j} (j+n)!/j! \int_{\Omega_0} |u|^2 dV$$

where equality is attained for some u. Since the sum of the series is $n!(1-|\zeta|^2)^{-n-1}$, the lemma is proved.

In the proof of Theorem 3.5.1 it is convenient to apply Lemma 3.5.3 in a slightly different form:

LEMMA 3.5.4. *Let a_{jk} $(j, k=1, \ldots, n)$ be a positive definite hermitian symmetric matrix, and set*

$$\Omega_0 = \left\{z; \operatorname{Im} z_n > \sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k\right\}.$$

Then $$K(z; \Omega_0, 0) = \varkappa n!\, 4^{-1} \pi^{-n} \left(\operatorname{Im} z_n - \sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k\right)^{-n-1}$$

where $\varkappa = \det(a_{jk})_{j,k=1}^{n-1}$.

Proof. By a unitary transformation of the variables $z_1, \ldots, z_{n-1}$ we can reduce the matrix $(a_{jk})_{j,k=1}^{n-1}$ to diagonal form, and the statement of the theorem then remains invariant. Assuming this reduction already made, we can introduce $z_j + z_n a_{nj}/a_{jj}$, $j=1, \ldots, n-1$, and z_n as new variables. The determinant of this transformation is equal to 1, so again the statement is invariant. Hence we may assume that the whole matrix (a_{jk}) has diagonal form. If we write $\operatorname{Im} z_n - a_{nn}|z_n|^2 = 1/4a_{nn} - a_{nn}|z_n - i/2a_{nn}|^2$, the lemma now follows from Lemma 3.5.2.

Proof of Theorem 3.5.1. As in Lemma 3.3.3 we can find a real valued function $\psi \in C^2$ which is strictly plurisubharmonic in a neighborhood of z_0 where Ω is defined by the equation $\psi < 0$, and $\operatorname{grad} \psi$ is the exterior unit normal on $\partial\Omega$. We choose local coordinates at z_0 so that the coordinates of z_0 are all 0 and the differentials dz_j are orthonormal at z_0. This implies that the Riemannian element of integration has density 1 with respect to the Lebesgue measure in the coordinate space. Further we choose the coordinates so that $\psi(z) + \operatorname{Im} z_n = O(|z|^2)$ at z_0. By Taylor's formula, Ω is therefore defined in a neighborhood of z_0 by an inequality of the form

$$\operatorname{Im} z_n > \sum_{j,k=1}^{n} \frac{\partial^2 \psi(0)}{\partial z_j\, \partial \bar{z}_k} z_j \bar{z}_k + \operatorname{Re} A(z) + o(|z|^2),$$

where A is an analytic, homogeneous, second degree polynomial. If we replace the coordinate z_n by $z_n - iA(z)$, the differential at z_0 is not changed, so we may assume without restriction that $A=0$ from the beginning. Put $a_{jk} = \partial^2\psi(0)/\partial z_j \partial \bar{z}_k$, which is a hermitian symmetric, positive definite matrix.

With an arbitrary $\varepsilon > 0$ we set

$$\Omega_\varepsilon = \left\{z; \operatorname{Im} z_n > \sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k + \varepsilon|z|^2\right\}.$$

Then $$\Omega_\varepsilon^\delta = \Omega_\varepsilon \cap \{z; |z| < \delta\}$$

is contained in Ω if δ is sufficiently small. (We do not distinguish between a point in Ω near z_0 and the point in $\mathbf{C}^n$ defined by its coordinates.) For small δ the product of $e^{-\varphi}$ and the density of the Riemannian element of volume with respect to the Lebesgue measure is larger than $\exp(-\varphi(z_0)-\varepsilon)$ in this set. Using the monotonicity (3.5.3) we therefore obtain

$$K(z;\Omega,\varphi)\leqslant e^{\varphi(z_0)+\varepsilon}K(z;\Omega_\varepsilon^\delta,0).$$

If we let $z\to 0$ so that $\operatorname{Im} z_n/|z|$ has a positive lower bound, it follows from Lemma 3.5.2 and from Lemma 3.5.4, applied to $\Omega_\varepsilon^\delta$ and Ω_ε, that

$$\overline{\lim}\,(\operatorname{Im} z_n)^{n+1}K(z;\Omega,\varphi)\leqslant e^{\varphi(z_0)+\varepsilon}\,\overline{\lim}\,(\operatorname{Im} z_n)^{n+1}K(z;\Omega_\varepsilon^\delta,0)$$
$$=e^{\varphi(z_0)+\varepsilon}\,\overline{\lim}\,(\operatorname{Im} z_n)^{n+1}K(z;\Omega_\varepsilon,0)=n!\,4^{-1}\pi^{-n}e^{\varphi(z_0)+\varepsilon}\det(a_{jk}+\varepsilon\delta_{jk})_{j,k=1}^{n-1}.$$

Since ε is arbitrary, this proves with the notations used in Theorem 3.5.1 that

$$\overline{\lim_{z\to z_0}}\,|\varrho(z)|^{n+1}K(z;\Omega,\varphi)\leqslant k(z_0)\,e^{\varphi(z_0)}n!/(4\pi^n), \tag{3.5.9}$$

if $z\to z_0$ while z remains inside a small cone in the coordinate space around the normal of $\partial\Omega$ at z_0. But a moment's reflection shows that this result is valid uniformly in z_0, so it remains true for arbitrary approach to z_0.

So far we have not used the hypothesis in Theorem 3.5.1 that the range of the $\bar\partial$ operator be closed. However, this is of course a vital assumption when we wish to estimate K from below, for without it we would not even know that there are non-trivial analytic functions in Ω.

Let ε be >0 but smaller than the smallest eigenvalue of the matrix (a_{jk}). For sufficiently small δ we have

$$\Omega^\delta=\{z;\,z\in\Omega,\,|z|<\delta\}\subset\Omega_{-\varepsilon}.$$

Hence Lemma 3.5.2 can be applied with $U=\{z;\,|z|<\delta\}$ and $u_0(z)=e^{iz_n}$. From Lemma 3.5.2 and the monotonicity (3.5.3) we then obtain if δ is sufficiently small

$$\underline{\lim}\,(\operatorname{Im} z_n)^{n+1}K(z;\Omega,\varphi)=\underline{\lim}\,(\operatorname{Im} z_n)^{n+1}K(z;\Omega^\delta,\varphi)\geqslant e^{\varphi(z_0)-\varepsilon}\,\underline{\lim}\,(\operatorname{Im} z_n)^{n+1}K(z;\Omega_{-\varepsilon},0),$$

when $z\to z_0$ and remains in a small cone around the normal of $\partial\Omega$. Arguing exactly as in the proof of (3.5.9) we conclude that

$$\underline{\lim_{z\to z_0}}\,|\varrho(z)|^{n+1}K(z;\Omega,\varphi)\geqslant k(z_0)\,e^{\varphi(z_0)}n!/(4\pi^n). \tag{3.5.10}$$

The repetition of the details of this argument may be left to the reader.

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