L^2 estimates for Fourier integral operators with complex phase

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1. Introduction

If X and Y are two C^∞ manifolds and if $C\subset(T^*(X)\diagdown 0)\times(T^*(Y)\diagdown 0)$ is the graph of a homogeneous canonical transformation from $T^*(Y)\diagdown 0$ to $T^*(X)\diagdown 0$ then every $A\in I^0(X\times Y, C', \Omega(X\times Y)^{1/2})$ is L^2 continuous in the sense that it defines a continuous map from $L^2_{\text{comp}}(Y, \Omega(Y)^{1/2})$ to $L^2_{\text{loc}}(X, \Omega(X)^{1/2})$ (Hörmander [3, Section 4.2]). The conclusion is not true for any other real canonical relations C. Here $\Omega(X)$, say, is the density bundle on X; we shall omit the half density bundles frequently in the notation. The symbols will be tacitly assumed to be of type 1, 0 unless otherwise stated.

Melin and Sjöstrand [7] have also proved that every $A\in I^0(X\times Y, C')$ is L^2 continuous if C is the graph of a positive complex canonical transformation in their sense. The proof was based on an extension of the theory of Fourier integral operators to amplitudes of type 1/2, 1/2. This extended calculus shows that A^*A must be a pseudo-differential operator with symbol in $S^0_{1/2,1/2}$ for such A. Now A^*A can also be computed by the calculus of Fourier integral operators with complex phase and $S^0_{1,0}$ symbols, so (locally at least) we have with $b\in S^0_{1,0}$

$$A^*Au(y) = (2\pi)^{-n}\int e^{i\varphi(y,\eta)}b(y,\eta)\hat{u}(\eta)\,d\eta. \tag{1.1}$$

Comparison of the two results shows that if $\varphi(y,\eta)=\langle y,\eta\rangle+\psi(y,\eta)$ then $e^{i\psi}$ must be in $S^0_{1/2,1/2}$, which is equivalent to

$$|\partial\psi(y,\eta)/\partial y|^2/|\eta|+|\partial\psi(y,\eta)/\partial\eta|^2|\eta| \leqq C\,\mathrm{Im}\,\psi(y,\eta). \tag{1.2}$$

In Section 3 we shall prove that (1.2) also follows from a simple and direct geometrical argument. The L^2 continuity of A is then a consequence of the standard calculus and the Calderón—Vaillancourt theorem [1].

However, there exist positive complex canonical relations C such that all $A\in I^0(X\times Y, C')$ are L^2 continuous but C is not of graph type. In fact, an important example occurs in the theory of operators of principal type and was studied by Duistermaat—Sjöstrand [2, Lemma 4.2]. We shall prove here that every $A\in I^0(X\times Y, C')$ is L^2 continuous if and only if at every real point of C the projections of real tangents of C to $T(T^*(X))$ and to $T(T^*(Y))$ are both injective. More precisely, we shall then for $A\in I^0(X\times Y, C')$ determine necessary and sufficient conditions for an estimate of the form

$$(1.3) \qquad \|Au\| \leqq (M+\varepsilon)\|u\|+C_{\varepsilon,K}\|u\|_{(-1)}, \quad u\in C_0^\infty(K, \Omega(Y)^{1/2}), \ \varepsilon>0,$$

to be valid. Here K is a compact subset of Y, $\|u\|$ is the L^2 norm, and $\|u\|_{(-1)}$ is a semi-norm in the Sobolev space $H_{(-1)}$ of derivatives of L^2 half densities. Standard arguments show that for every real point $\gamma=(x_0, \xi_0, y_0, \eta_0)\in C$ this estimate implies another of the form

$$(1.4) \qquad \|A_0 u\| \leqq M\|u\|, \quad u\in C_0^\infty(\mathbf{R}^{n_Y}), \quad n_Y = \dim Y.$$

Here A_0 is a localization of A at infinity in the direction γ; it is a Fourier integral operator defined by a quadratic form as phase function and a constant amplitude. The associated canonical transformation is a linear positive complex canonical relation $C_\gamma\subset T^*(\mathbf{C}^{n_X})\times T^*(\mathbf{C}^{n_Y})$. Estimates of the form (1.4) are invariant under real linear symplectic transformations in $T^*(\mathbf{C}^{n_X})$ or $T^*(\mathbf{C}^{n_Y})$. The corresponding equivalence classes are determined in Section 4, and we show that A_0 is continuous if and only if

$$C_\gamma\cap(T^*(\mathbf{R}^{n_X})\times\{0\}) = \{0\}, \quad C_\gamma\cap(\{0\}\times T^*(\mathbf{R}^{n_Y})) = \{0\},$$

and then we can also compute the norm. The symbol of A_0 defines a positive complex half density (or real 1/4 density) on C_γ. We can therefore consider $\|A_0\|$ as a norm of half densities on C_γ. The definition is symplectically invariant and depends continuously on C_γ in a natural sense. It can be described geometrically by noting that C_γ is the direct sum of a canonical graph in $T^*(\mathbf{C}^{n'})\times T^*(\mathbf{C}^{n'})$ and strictly positive (negative) Lagrangians in $T^*(\mathbf{C}^{n''_X})\subset T^*(\mathbf{C}^{n_X})$ and in $T^*(\mathbf{C}^{n''_Y})\subset T^*(\mathbf{C}^{n_Y})$ respectively. In these one has natural densities defined by lifting the symplectic form to the graph by one of the projections and by the natural positive definite metrics in the strictly positive (negtive) Lagrangian planes. This density allows one to identify a half density in C_γ with a scalar, and the norm of the half density is this scalar multiplied by a product of eigenvalues associated with the canonical graph.

The interpretation of A^*A as a pseudo-differential operator remains valid under the preceding conditions on C. This makes it possible to use standard localization arguments for pseudo-differential operators to study estimates of the form (1.3) for A^*A. In this way we show that (1.3) is valid if and only if for the real part C_K of C over any compact set $K\subset X\times Y$ we have

$$\varlimsup_{C_K\ni\gamma\to\infty}\|\sigma_A(\gamma)\|\leqq M \tag{1.5}$$

where σ_A is the symbol half density normed as described above. (For the detailed definition see Section 4.) The result is of course well known for Fourier integral operators with real phase.

In this paper we shall rely on the techniques of almost analytic continuation developed in [6], [7] and we keep the notation used there. For unexplained notation the reader should consult these papers. However, it is possible to avoid almost analytic continuation by introducing Lagrangian ideals of complex valued functions defined on the real cotangent bundle. We shall develop this point of view elsewhere.

2. The localized estimates

Let X and Y be C^∞ manifolds and C a C^∞ complex positive homogeneous canonical relation $\subset(T^*(X)\setminus 0)\times(T^*(Y)\setminus 0)^{\sim}$ which is closed in $T^*(X\times Y)\setminus 0$. Let $A\in I^0(X\times Y, C'; \Omega(X\times Y)^{1/2})$ be an associated half density of order 0 in $X\times Y$, and assume that A is properly supported and satisfies (1.3). Then $\varphi A\psi$ has the same property if $\varphi\in C_0^\infty(X)$, $\psi\in C_0^\infty(Y)$ and $|\varphi|\leqq 1$, $|\psi|\leqq 1$, both considered as multiplication operators. We can choose φ and ψ with small support and equal to 1 near given points x_0 and y_0. In proving necessary conditions we may therefore assume that $X\subset\mathbf{R}^{n_X}$, $Y\subset\mathbf{R}^{n_Y}$, and that supp A is close to $(x_0, y_0)=(0,0)$. (We identify operators and kernels throughout.) Replacing φ and ψ by pseudo-differential operators we may also assume that $WF'(A)$ is in a small conic neighborhood of the ray through a given point $(x_0, \xi_0, y_0, \eta_0)\in C$. If the coordinates in X and Y are conveniently chosen we have modulo C^∞ an oscillatory integral representation

$$A(x, y) = (2\pi)^{-3n/4}\iint e^{i(\langle x,\xi\rangle-\langle y,\eta\rangle-H(\xi,\eta))}a(\xi,\eta)\,d\xi\,d\eta \tag{2.1}$$

where $n=\dim(X\times Y)$ and $a\in S^{-n/4}$ is supported in a small conic neighborhood of (ξ_0, η_0) where $\operatorname{Im} H\leqq 0$. Choose $u\in C_0^\infty(\mathbf{R}^{n_Y})$, $v\in C_0^\infty(\mathbf{R}^{n_X})$ and set

$$u_t(y) = e^{it^2\langle y,\eta_0\rangle}u(ty)t^{n_Y/2},\quad v_t(x) = e^{it^2\langle x,\xi_0\rangle}v(tx)t^{n_X/2}.$$

Since $\|u_t\|=\|u\|$, $\|v_t\|=\|v\|$ and $u_t\to 0$ weakly in L^2, hence strongly in $H_{(-1)}$, as $t\to\infty$, we obtain if (1.3) is valid that

$$\varlimsup_{t\to\infty}|(Au_t, v_t)|\leqq M\|u\|\,\|v\|.$$

A direct computation gives

$$\hat{u}_t(\eta) = t^{-n_Y/2}\hat{u}((\eta - t^2\eta_0)/t), \quad \hat{v}_t(\xi) = t^{-n_X/2}\hat{v}((\xi - t^2\xi_0)/t),$$

$$(Au_t, v_t) = (2\pi)^{-3n/4} t^{n/2} \iint e^{-iH(t^2\xi_0 + t\xi,\, t^2\eta_0 + t\eta)} \hat{u}(\eta)\overline{\hat{v}(\xi)} a(t^2\xi_0 + t\xi,\, t^2\eta_0 + t\eta)\, d\xi\, d\eta.$$

The exponential is bounded by 1 in absolute value, and since $0 = x_0 = H'_\xi(\xi_0, \eta_0)$, $0 = y_0 = -H'_\eta(\xi_0, \eta_0)$, we obtain

$$t^2 H(\xi_0 + \xi/t,\, \eta_0 + \eta/t) \to Q(\xi, \eta), \quad t \to \infty,$$

where Q is the quadratic part in the Taylor expansion of H at (ξ_0, η_0). That $H(\xi_0, \eta_0) = 0$ follows from the homogeneity of H. When $|\xi| < t|\xi_0|/2$, $|\eta| < t|\eta_0|/2$ we have by the mean value theorem

$$t^{n/2}|a(t^2\xi_0 + t\xi,\, t^2\eta_0 + t\eta) - a(t^2\xi_0,\, t^2\eta_0)| < Ct^{n/2} t^{2(-n/4-1)} t(|\xi| + |\eta|)$$
$$= Ct^{-1}(|\xi| + |\eta|),$$

for $a \in S^{-n/4}$. When $|\xi| > t|\xi_0|/2$ or $|\eta| > t|\eta_0|/2$ we estimate a by a constant and t by $C(|\xi| + |\eta|)$ and obtain since $\hat{u}\hat{v}$ is rapidly decreasing

$$|(Au_t, v_t) - t^{n/2} a(t^2\xi_0,\, t^2\eta_0)(A_0 u, v)| \to 0 \tag{2.2}$$

where A_0 is defined by the oscillatory integral

$$A_0(x, y) = (2\pi)^{-3n/4} \iint e^{i(\langle x, \xi\rangle - \langle y, \eta\rangle - Q(\xi, \eta))}\, d\xi\, d\eta. \tag{2.3}$$

Hence (1.3) implies

$$\overline{\lim_{t\to\infty}}\, t^{n/2} |a(t^2\xi_0,\, t^2\eta_0)|\, \|A_0\| \leqq M. \tag{2.4}$$

If we choose A so that the upper limit is not 0 it follows in particular that $\|A_0\| < \infty$ if A is L^2 continuous.

Lemma 2.1. *If the operator* A_0 *defined by* (2.3) *is* L^2 *continuous, then the corresponding linear canonical relation*

$$\{(\partial Q/\partial\xi,\, \xi;\, -\partial Q/\partial\eta,\, \eta)\} \subset T^*(\mathbf{C}^{n_X}) \times T^*(\mathbf{C}^{n_Y}) \tag{2.5}$$

intersects $T^*(\mathbf{R}^{n_X}) \times \{0\}$ *and* $\{0\} \times T^*(\mathbf{R}^{n_Y})$ *only at* 0.

Proof. Assume that $(\tilde{x}, \tilde{\xi}; 0, 0)$ is real and belongs to the canonical relation (2.5). Then it is symplectically orthogonal, that is,

$$\langle \partial Q/\partial\xi, \tilde{\xi}\rangle - \langle \tilde{x}, \xi\rangle = 0 \quad \text{for all} \quad (\xi, \eta).$$

This implies that with $\varphi(x, \xi, y, \eta) = \langle x, \xi\rangle - \langle y, \eta\rangle - Q(\xi, \eta)$

$$(\langle D_x, \tilde{x}\rangle - \langle x, \tilde{\xi}\rangle) A_0(x, y) = (2\pi)^{-3n/4} \iint (\langle \xi, \tilde{x}\rangle - \langle x, \tilde{\xi}\rangle) e^{i\varphi}\, d\xi\, d\eta$$
$$= (2\pi)^{-3n/4} \iint \langle -D_\xi, \tilde{\xi}\rangle e^{i\varphi}\, d\xi\, d\eta = 0.$$

Choose $u\in C_0^\infty(\mathbf{R}^{n_Y})$ with $A_0u\neq 0$. Since $A_0u\in L^2$ and $(\langle D_x, \tilde{x}\rangle-\langle x, \tilde{\xi}\rangle)(A_0u)=0$ we obtain $\tilde{\xi}=0$ if $\tilde{x}=0$, for A_0u cannot be supported by a hyperplane. If $\tilde{x}\neq 0$ and ψ is a real valued solution of the equation $\langle\partial\psi/\partial x, \tilde{x}\rangle=\langle x, \tilde{\xi}\rangle$ then $e^{-i\psi}A_0u$ is constant in the direction $\tilde{x}$ which contradicts that $0\neq A_0u\in L^2$. Hence $\tilde{x}=\tilde{\xi}=0$. Taking the adjoint of A_0 we conclude that (2.5) cannot have a real element of the form $(0, 0; y, \eta)$ either.

The canonical relation (2.5) is the complex tangent plane $T_\gamma(C)$ of C at γ in the sense of Melin—Sjöstrand [7]. We have therefore proved

Theorem 2.2. *If every $A\in I^0(X\times Y, C'; \Omega(X\times Y)^{1/2})$ is L^2 continuous, then the real tangent plane $T_\gamma^{\mathbf{R}}(C)$ of C at any real $\gamma=(x_0, \xi_0, y_0, \eta_0)$ has injective projections into $T_{x_0, \xi_0}(T^*(X))$ and $T_{y_0, \eta_0}(T^*(Y))$.*

The estimate (2.4) also contains quantitative information. We shall return to it in Section 4 after developing the linear algebra required to give (2.4) the invariant form (1.5). In Section 5 we shall then prove that conversely (1.5) implies (1.3).

3. A^*A as a pseudo-differential operator and L^2 continuity

In this section we shall prove a converse of Theorem 2.2. To do so we must make some preliminary remarks on symplectic linear algebra. Let S_1 and S_2 be real symplectic finite dimensional vector spaces with complexifications $S_{1\mathbf{C}}$ and $S_{2\mathbf{C}}$. A linear canonical relation from S_2 to S_1 is a complex linear subspace $C\subset S_{1\mathbf{C}}\oplus S_{2\mathbf{C}}$ which is Lagrangian with respect to the difference $\sigma_1-\sigma_2$ of the complexified symplectic forms σ_1 and σ_2 in S_1 and in S_2, lifted to $S_1\oplus S_2$. It is called positive if it is positive with respect to this form, that is,

$$i^{-1}(\sigma_1(X, \bar{X})-\sigma_2(Y, \bar{Y}))\geqq 0 \quad \text{when} \quad (X, Y)\in C. \tag{3.1}$$

Lemma 3.1. *If C is positive, $(X, Y)\in C$ and $i^{-1}(\sigma_1(X, \bar{X})-\sigma_2(Y, \bar{Y}))=0$ then $(\bar{X}, \bar{Y})\in C$, that is, the radical of the hermitian form in* (3.1) *is generated by the real elements in C.*

Proof. By the Cauchy—Schwarz inequality

$$i^{-1}(\sigma_1(X', \bar{X})-\sigma_2(Y', \bar{Y})) = 0 \quad \text{for all} \quad (X', Y')\in C$$

so $(\bar{X}, \bar{Y})\in C$ since C is Lagrangian.

Note that this argument is valid for any positive Lagrangian plane.

Lemma 3.2. *Let $C_1\subset S_{1\mathbf{C}}\oplus S_{2\mathbf{C}}$ and $C_2\subset S_{2\mathbf{C}}\oplus S_{3\mathbf{C}}$ be linear positive canonical relations which are transversal in the sense that*

$$\{Y\in S_{2\mathbf{C}};\ (0, Y)\in C_1,\ (Y, 0)\in C_2\} = \{0\}.$$

Then $C_1 \circ C_2 = \{(X, Z) \in S_{1C} \oplus S_{3C};\ (X, Y) \in C_1$ *and* $(Y, Z) \in C_2$ *for some* $Y \in S_{2C}\}$ *is a positive canonical relation. If* $(X, Z) \in C_1 \circ C_2$ *is real, then* Y *is real here.*

Proof. If $(X', Y') \in C_1$ and $(Y', Z') \in C_2$ then

$$\sigma_1(X, X') - \sigma_3(Z, Z') = \sigma_1(X, X') - \sigma_2(Y, Y') + \sigma_2(Y, Y') - \sigma_3(Z, Z') = 0$$

so $C_1 \circ C_2$ is isotropic. The intersection of $C_1 \oplus C_2$ with $\{(X, Y, Y, Z)\} \subset S_{1C} \oplus S_{2C} \oplus S_{2C} \oplus S_{3C}$ has codimension at most $\dim S_{2C}$ in $C_1 \oplus C_2$ so the dimension is at least $(\dim S_{1C} + \dim S_{3C})/2$. The dimension of $C_1 \circ C_2$ is the same so $C_1 \circ C_2$ is Lagrangian. With the notation in the definition of $C_1 \circ C_2$ we have

$$i^{-1}(\sigma(X, \bar{X}) - \sigma(Z, \bar{Z})) = i^{-1}(\sigma(X, \bar{X}) - \sigma(Y, \bar{Y})) + i^{-1}(\sigma(Y, \bar{Y}) - \sigma(Z, \bar{Z}))$$

and both terms on the right hand side are non-negative, so $C_1 \circ C_2$ is positive. If the left hand side vanishes then both terms in the right hand side do, so $(\bar{X}, \bar{Y}) \in C_1$ and $(\bar{Y}, \bar{Z}) \in C_2$ by Lemma 3.1. If X and Z are real, then $(0, \operatorname{Im} Y) \in C_1$ and $(\operatorname{Im} Y, 0) \in C_2$ so $\operatorname{Im} Y = 0$ by assumption. The proof is complete.

Lemma 3.3. *Assume that the projections of* $C_R = C \cap (S_1 \oplus S_2)$ *to* S_1 *and to* S_2 *are both injective. Then there is a unique decomposition* $S_1 = S_{11} \oplus S_{12}$, $S_2 = S_{21} \oplus S_{22}$ *in symplectically orthogonal symplectic subspaces such that* $C = C_1 \oplus \lambda_1 \oplus \lambda_2$ *where* C_1 *is the graph of a symplectic isomorphism* $S_{21C} \to S_{11C}$ *and* λ_1 *resp.* λ_2 *are strictly positive (negative) Lagrangian planes in* S_{12C} *and in* S_{22C} *respectively.*

Proof. It is clear that

$$\lambda_1 = \{X \in S_{1C};\ (X, 0) \in C\}$$

is isotropic. If $X \in \lambda_1$ and $i^{-1}\sigma_1(X, \bar{X}) = 0$ it follows from Lemma 3.1 that $(\bar{X}, 0) \in C$ so $(\operatorname{Re} X, 0)$ and $(\operatorname{Im} X, 0)$ are in C. Hence $X = 0$ by assumption. Since $\sigma_1(\operatorname{Im} X, \operatorname{Re} X) = \sigma_1(X, \bar{X})/2i \neq 0$ if $0 \neq X \in \lambda_1$ it follows that

$$S_{12} = \{\operatorname{Re} X;\ X \in \lambda_1\}$$

is a symplectic subspace of S_1 of dimension $2 \dim_C \lambda_1$. We have $\lambda_1 \subset S_{12C}$ and $2 \dim_C \lambda_1 = \dim_C S_{12C}$ so λ_1 is Lagrangian and strictly positive in S_{12C}. In the same way we find that

$$\lambda_2 = \{Y \in S_{2C};\ (0, Y) \in C\}$$

is a strictly negative Lagrangian subspace of the complexification of the symplectic space S_{22} formed by its real parts. Define S_{11} and S_{21} as the symplectically orthogonal complements of S_{12} and S_{22}. The component in S_{12C} (or S_{22C}) of any element in C is symplectically orthogonal to λ_1 (resp. λ_2) so it is in λ_1 (resp. λ_2). Hence $C = C_1 \oplus \lambda_1 \oplus \lambda_2$ where C_1 is a canonical relation $\subset S_{11C} \oplus S_{21C}$ which has injective projections to S_{11C} and S_{21C}. But this means that C_1 is the graph of a symplectic isomorphism and the lemma is proved.

Lemma 3.4. *If* C *satisfies the hypotheses in Lemma* 3.3 *then*

$$\bar{C}^{-1} = \{(\bar{Y}, \bar{X});\, (X, Y) \in C\} \subset S_{2C} \oplus S_{1C}$$

is a positive canonical relation transversal to C, *so* $\bar{C}^{-1} \circ C \subset S_{2C} \oplus S_{2C}$ *is a positive canonical relation. If* $(Y_1, Y_2) \in \bar{C}^{-1} \circ C$ *and* $i^{-1}(\sigma(Y_1, \bar{Y}_1) - \sigma(Y_2, \bar{Y}_2)) = 0$ *then* $Y_1 = Y_2$.

Proof. With the notation in Lemma 3.3 we have $\lambda_1 \cap \bar{\lambda}_1 = \{0\}$ since λ_1 is strictly positive and $\bar{\lambda}_1$ is strictly negative. This proves the transversality; the positivity of $\bar{C}^{-1}$ is obvious. By Lemma 3.1 it suffices to prove the last statement when Y_1 and Y_2 are real. Then we have $(Y_1, X) \in \bar{C}^{-1}$, $(X, Y_2) \in C$ for some real X, by Lemma 3.2, so $Y_2 - Y_1 \in \lambda_2$, hence $Y_2 = Y_1$ since λ_2 has no real elements.

The preceding lemmas are essentially the infinitesimal version of the proof of the following converse of Theorem 2.2.

Theorem 3.5. *Let* X *and* Y *be* C^∞ *manifolds and* C *a* C^∞ *complex positive homogeneous canonical relation* $\subset (T^*(X)\setminus 0) \times (T^*(Y)\setminus 0)^{\sim}$ *which is closed in* $T^*(X\times Y)\setminus 0$. *Assume that for every real* $\gamma = (x_0, \xi_0, y_0, \eta_0) \in C$ *the projections of the real tangent plane* $T_\gamma(C)_{\mathbf{R}}$ *of* C *into* $T_{x_0, \xi_0}(T^*(X))$ *and* $T_{y_0, \eta_0}(T^*(Y))$ *are injective. Then every* $A \in I^0(X\times Y, C';\, \Omega(X\times Y)^{1/2})$ *is continuous from* $L^2_{\text{comp}}(Y, \Omega(Y)^{1/2})$ *to* $L^2_{\text{loc}}(X, \Omega(X)^{1/2})$.

Proof. We can localize by multiplying A to the left and right by partitions of unity in X or Y. More generally, we can microlocalize by using pseudo-differential partitions of unity, so A may be assumed to have support near a point (x_0, y_0) while $WF'(A)$ is contained in a small conic neighborhood of the ray through $\gamma = (x_0, \xi_0, y_0, \eta_0)$.

If $\varphi(x, y, \theta)$ is a regular phase function of positive type at (x_0, y_0, θ_0) defining C at γ (see Melin—Sjöstrand [6, Def. 3.5]) we can write A modulo C^∞ in the form

$$A(x, y) = (2\pi)^{-(n+2N)/4} \int e^{i\varphi(x, y, \theta)} a(x, y, \theta)\, d\theta$$

where $n = \dim X\times Y$, $\theta \in \mathbf{R}^N$, and $a \in S^{(n-2N)/4}(\mathbf{R}^n \times \mathbf{R}^N)$ has support in a small conic neighborhood of (x_0, y_0, θ_0),

$$\varphi'_\theta(x_0, y_0, \theta_0) = 0, \quad \varphi'_x(x_0, y_0, \theta_0) = \xi_0, \quad -\varphi'_y(x_0, y_0, \theta_0) = \eta_0.$$

The kernel of the adjoint operator

$$A^*(y, x) = \overline{A(x, y)} = (2\pi)^{-(n+2N)/4} \int e^{-i\overline{\varphi(x, y, \theta)}}\, \overline{a(x, y, \theta)}\, d\theta$$

is in $I^0(Y\times X, (\bar{C}^{-1})')$ where $\bar{C}^{-1}$ is defined by the phase function $-\overline{\varphi(x, y, \theta)}$, so it is obtained from C by taking complex conjugates and interchanging X and Y. If we apply Lemma 3.4 to the tangent plane $T_\gamma(C)$ it follows that the composition $\bar{C}^{-1}\circ C$ satisfies the transversality condition in [6] so $A^*A\in I^0(Y\times Y,(\bar{C}^{-1}\circ C)')$, the support of A^*A is close to (y_0, y_0) and $WF'(A)$ is close to $(y_0, \eta_0, y_0, \eta_0)$. The composition $\bar{C}^{-1}\circ C$ is defined by the regular phase function of positive type

$$\varphi(x, y, \theta)-\overline{\varphi(x, z, \tau)} \tag{3.2}$$

where z denotes the variable in the copy of Y to the left. We claim that there is another defining phase function of the form $\tilde{\varphi}(z, \eta)-\langle y, \eta\rangle$. For the proof we recall some basic facts on Fourier integral operators.

Let $\psi(x, y, \theta)$ be a regular phase function of positive type at (x_0, y_0, θ_0) defining a canonical relation C_ψ from $T^*(Y)$ to $T^*(X)$. Thus $\xi_0=\psi_x'(x_0, y_0, \theta_0)$ and $\eta_0=-\psi_y'(x_0, y_0,\theta_0)$ are assumed real and different from 0. Assume also that the critical point of $\psi(x, y, \theta)+\langle y, \eta\rangle$ as a function of y, θ at (y_0, θ_0), when $x=x_0$, $\eta=\eta_0$, is non-degenerate. Let $\tilde{\psi}(x, \eta)$ be the critical value of an almost analytic extension of $\psi(x, y, \theta)+\langle y, \eta\rangle$, that is, the value when $\partial\psi/\partial y+\eta=0$, $\partial\psi/\partial\theta=0$. Then $\tilde{\psi}(x, \eta)-\langle y, \eta\rangle$ is a regular phase function of positive type at (x_0, y_0,η_0) which also defines C_ψ. In fact, the almost analytic continuation of C_ψ is the set of $(x, \partial\psi/\partial x, y, -\partial\psi/\partial y)$ with $\partial\psi/\partial\theta=0$. On the other hand, $\tilde{\psi}$ is of positive type by Lemma 2.1 in [6] and $\tilde{\psi}(x, \eta)-\langle y, \eta\rangle$ defines the canonical relation

$$\{(x, \partial\tilde{\psi}/\partial x, \partial\tilde{\psi}/\partial\eta, \eta)\}. \tag{3.3}$$

Here $\partial\tilde{\psi}(x, \eta)/\partial x\equiv\partial\psi(x, y, \theta)/\partial x$ and $\partial\tilde{\psi}(x, \eta)/\partial\eta\equiv y$ when $\partial\psi(x, y, \theta)/\partial\theta=0$, $\partial\psi(x, y, \theta)/\partial y+\eta=0$ which proves that the canonical relation (3.3) is C_ψ. Note that the non-degeneracy of $\psi(x, y, \theta)+\langle y, \eta\rangle$ means that at $(x_0, y_0, \theta_0, \eta_0)$

$$d_{y,\theta}(\partial\psi/\partial y+\eta) = 0, \quad d_{y,\theta}(\partial\psi/\partial\theta) = 0$$

must imply $dy=d\theta=0$, or equivalently, that $(0, d\xi, dy, 0)$ belongs to the complex tangent plane of C_ψ at $(x_0, \xi_0, y_0, \eta_0)$ only when $d\xi=dy=0$.

Let us now apply this to $\bar{C}^{-1}\circ C$ with the coordinates in $T^*(Y)\times T^*(Y)$ denoted by (z, ζ, y, η). The intersection of the complex tangent plane at (y_0,η_0,y_0,η_0) with the plane $dz=d\eta=0$ is equal to $\{0\}$, for the intersection is in the diagonal by Lemma 3.4 since the plane $dz=d\eta=0$ is Lagrangian and its own conjugate. Hence $\bar{C}^{-1}\circ C$ is also defined by the phase function $\tilde{\varphi}(z, \eta)-\langle y, \eta\rangle$ if $\tilde{\varphi}(z, \eta)$ is the value of the almost analytic extension of

$$\Phi(z, \eta, x, y, \theta, \tau) = \varphi(x, y, \theta)-\bar{\varphi}(x, z, \tau)+\langle y, \eta\rangle \tag{3.4}$$

at the "critical point" defined by

$$\partial\varphi(x, y, \theta)/\partial x-\partial\bar{\varphi}(x, z, \tau)/\partial x = 0, \quad \partial\varphi(x, y, \theta)/\partial y+\eta = 0, \tag{3.5}$$

$$\partial\varphi(x, y, \theta)/\partial\theta = 0, \quad \partial\bar{\varphi}(x, z, \tau)/\partial\tau = 0.$$

By [6, Lemma 2.1] we have for real (z, η) near (y_0, η_0), if $x=X(z,\eta), \dots, \tau=T(z,\eta)$ at this critical point,

$$|\operatorname{Im}(X(z,\eta),\ Y(z,\eta),\ \Theta(z,\eta),\ T(z,\eta))|^2 \leqq C \operatorname{Im} \tilde{\varphi}(z,\eta). \tag{3.6}$$

The derivatives $\partial\Phi/\partial\bar{x}, \dots, \partial\Phi/\partial\bar{\tau}$ all vanish for real arguments since we have an almost analytic extension, so it follows from (3.6) and Taylor's formula that if $X_r = \operatorname{Re} X, \dots, T_r = \operatorname{Re} T$ then

$$\operatorname{Im} \Phi(z, \eta, X_r(z,\eta), \dots, T_r(z,\eta)) \leqq C_1 \operatorname{Im} \tilde{\varphi}(z,\eta). \tag{3.7}$$

For any non-negative C^2 function F we have $|F'|^2 \leqq CF$ locally. If we write $\varphi = \varphi_1 + i\varphi_2$ it follows from (3.7) that in a neighborhood of (y_0, η_0)

$$\begin{aligned} &|\varphi_2'(X_r(z,\eta),\ Y_r(z,\eta),\ \Theta_r(z,\eta))| + |\varphi_2'(X_r(z,\eta),\ z,\ T_r(z,\eta))| \\ &\qquad \leqq C_2(\operatorname{Im} \tilde{\varphi}(z,\eta))^{1/2}. \end{aligned} \tag{3.8}$$

With the notation

$$f(x,y,\theta) = (\partial\varphi_1(x,y,\theta)/\partial(x,\theta),\ \partial\varphi_2(x,y,\theta)/\partial(x,y,\theta))$$

we have in a neighborhood of (y_0, η_0)

$$\begin{aligned} &|f(X_r(z,\eta),\ Y_r(z,\eta),\ \Theta_r(z,\eta)) - f(X_r(z,\eta),\ z,\ T_r(z,\eta))| \\ &\qquad \leqq C_3(\operatorname{Im} \tilde{\varphi}(z,\eta))^{1/2}. \end{aligned} \tag{3.9}$$

For the components of f involving φ_2 this follows from (3.8). By (3.5), (3.6) and Taylor's formula we have

$$\begin{aligned} &|\varphi_x'(X_r, Y_r, \Theta_r) - \tilde{\varphi}_x'(X_r, z, T_r)| + |\varphi_\theta'(X_r, Y_r, \Theta_r) - \tilde{\varphi}_\tau'(X_r, z, T_r)| \\ &\qquad \leqq C_4(\operatorname{Im} \tilde{\varphi}(z,\eta))^{1/2} \end{aligned}$$

which gives the estimate (3.9) of the components of f involving φ_1 since (3.8) estimates the derivatives of φ_2.

The Jacobian matrix $\partial f(x,y,\theta)/\partial(y,\theta)$ is injective at (x_0, y_0, θ_0). Indeed, if $dx=0$ and $df=0$ then $d\varphi_\theta'=0$ so $(dx, d\xi, dy, d\eta) \in T_\gamma(C)$ if $\xi = \varphi_x'$ and $\eta = -\varphi_y'$. If dy and $d\theta$ are real then $d\eta$ is also real since f contains $\partial\varphi_2/\partial y$. Thus $(dx, d\xi, dy, d\eta) = (0, 0, dy, d\eta) = 0$ by our hypothesis on C, which proves the injectivity since f is real. Hence Taylor's formula gives in a neighborhood of (x_0, y_0, θ_0) that

$$|y-z| + |\theta-\tau| \leqq C|f(x,y,\theta) - f(x,z,\tau)|.$$

If we combine this estimate with (3.9) we have proved that

$$|Y_r(z,\eta) - z| + |\Theta_r(z,\eta) - T_r(z,\eta)| \leqq C_5(\operatorname{Im} \tilde{\varphi}(z,\eta))^{1/2}. \tag{3.10}$$

When computing the derivatives of $\psi(z,\eta) = \tilde{\varphi}(z,\eta) - \langle z, \eta \rangle$ we shall use that the derivatives of Φ with respect to x, y, θ, τ vanish at the critical point while the $\tilde{x}, \tilde{y}, \tilde{\theta}, \tilde{\tau}$ derivatives are $O((\operatorname{Im}\psi(z,\eta))^{1/2})$ by (3.6). Since

$$\partial\Phi/\partial z-\eta = -\partial\bar{\varphi}(x,z,\tau)/\partial z-\eta, \quad \partial\Phi/\partial\eta-z = y-z$$

and $\eta=-\varphi_y'(x,y,\theta)$ at the critical point, it follows that

$$|\partial\tilde{\varphi}/\partial z-\eta|+|\partial\tilde{\varphi}/\partial\eta-z| \leqq |\bar{\varphi}_z'(X,z,T)-\varphi_y'(X,Y,\Theta)|+|Y-z|+C_6(\operatorname{Im}\psi)^{1/2}.$$

After another application of Taylor's formula and (3.6), (3.10) we obtain

$$|\partial\psi(z,\eta)/\partial z|^2+|\partial\psi(z,\eta)/\partial\eta|^2 \leqq C\operatorname{Im}\psi(z,\eta) \tag{3.11}$$

in a neighborhood of (y_0,η_0). Taking the almost analytic extensions homogeneous we obtain a homogeneous function ψ and conclude that

$$|\partial\psi(z,\eta)/\partial z|^2/|\eta|+|\partial\psi(z,\eta)/\partial\eta|^2|\eta| \leqq C\operatorname{Im}\psi(z,\eta) \tag{3.12}$$

in a conic neighborhood of (y_0,η_0). This is the inequality (1.2) of the introduction.

The important consequence of (3.12) already referred to in the introduction is that in a conic neighborhood of (y_0,η_0)

$$|D_y^\alpha D_\eta^\beta e^{i\psi(y,\eta)}| \leqq C_{\alpha\beta}|\eta|^{(|\alpha|-|\beta|)/2}e^{-\operatorname{Im}\psi(y,\eta)/2}, \quad |\eta|>1. \tag{3.13}$$

In particular, $e^{i\psi}\in S^0_{1/2,1/2}$. To prove (3.13) we observe that $D_y^\alpha D_\eta^\beta e^{i\psi}$ is a linear combination of terms of the form

$$(D_y^{\alpha_1}D_\eta^{\beta_1}\psi)\dots(D_y^{\alpha_k}D_\eta^{\beta_k}\psi)e^{i\psi}.$$

When $|\alpha_j+\beta_j|\geqq 2$ we just use the estimate

$$|D_y^{\alpha_j}D_\eta^{\beta_j}\psi| \leqq C_{\alpha\beta}|\eta|^{1-|\beta_j|} \leqq C_{\alpha\beta}|\eta|^{(|\alpha_j+\beta_j|-2|\beta_j|)/2}, \quad |\eta|>1,$$

and when $|\alpha_j+\beta_j|=1$ we use that by (3.12)

$$|D_y^{\alpha_j}D_\eta^{\beta_j}\psi| \leqq C|\eta|^{(|\alpha_j|-|\beta_j|)/2}(\operatorname{Im}\psi)^{1/2}.$$

Since $(\operatorname{Im}\psi)^N e^{-(\operatorname{Im}\psi)/2}$ is bounded for every N, the estimate (3.13) follows. (Conversely, it is easy to see that (3.13) with $|\alpha|+|\beta|=1$ implies (3.12).)

Summing up, we have now proved that A^*A is a pseudo-differential operator of order 0 and type $1/2, 1/2$. By the Calderón—Vaillancourt theorem [1] this implies that A^*A is L^2 continuous, which completes the proof of Theorem 3.5.

The preceding proof only shows that A^*A is a pseudo-differential operator when $A\in I^0(X\times Y, C')$ and $WF'(A)$ is in a small conic neighborhood of a given point. In general there may for real $(x,\xi)\in T^*(X)\setminus 0$ exist several distinct $(y,\eta)\in T^*(Y)\setminus 0$ with $(x,\xi,y,\eta)\in C$ although the local results on $\bar{C}^{-1}\circ C$ proved above show that the set of such (y,η) is discrete. However, we have

Theorem 3.6. *If in addition to the hypotheses of Theorem* 3.5 *we assume that the map* $C_{\mathbf{R}}\to T^*(X)\setminus 0$ *is injective and that* A *is properly supported, then* A^*A *is a pseudo-differential operator of order* 0 *and type* $1/2, 1/2$.

Proof. Since $WF'(A)\subset C_{\mathbf{R}}$ the hypothesis implies that $WF'(A^*A)$ is contained in the diagonal of $T^*(Y)\setminus 0$. In fact, if $(x, \xi, y, \eta)\in WF'(A)$ and $(y', \eta', x, \xi)\in WF'(A^*)$ then $(y, \eta)=(y', \eta')$ by assumption. To determine A^*A at a point $(y_0, \eta_0, y_0, \eta_0)$ on the diagonal, we introduce

$$\{(x, \xi)\in T^*(X)\setminus 0;\ (x, \xi, y_0, \eta_0)\in WF'(A)\}. \tag{3.14}$$

Since the local composition $C\circ\bar{C}^{-1}$ only has real points in the diagonal, the set (3.14) is discrete so it can only have finitely many points $(x_j, \xi_j), j=1, \dots, J$. Hence we can write

$$A = \sum_1^J A_j+R$$

where $(x, \xi, y_0, \eta_0)\notin WF'(R)$, for all (x, ξ), and $WF'(A_j)$ is in such a small conic neighborhood of $(x_j, \xi_j, y_0, \eta_0)$ that $A_j^*A_k\in C^\infty$ if $j\neq k$ and $A_j{}^*A_j$ is a pseudo-differential operator of order 0 and type 1/2, 1/2. Since

$$(y_0, \eta_0, y_0, \eta_0)\notin WF'(A^*A-\textstyle\sum_1^J A_j^*A_j)$$

the proof is complete.

Note that if the projection $C\to T^*(Y)$ is also injective then the proof of Theorem 3.6 simplifies since (3.14) consists of a single point (x_0, ξ_0). Thus A^*A is microlocally equal to $A_1{}^*A_1$ at $(y_0, \eta_0, y_0, \eta_0)$ if A is microlocally equal to A_1 at $(x_0, \xi_0, y_0, \eta_0)$.

4. Normal form for positive linear canonical relations

The study of the estimate (1.3) requires a rather complete description of positive linear canonical relations. This will be the subject of the present section. As in Section 3 we let S_1 and S_2 be real symplectic vector spaces with complexifications $S_{1\mathbf{C}}$ and $S_{2\mathbf{C}}$. Denoting by C a positive linear canonical relation satisfying the conditions in Lemma 3.3, we wish to introduce coordinates in the various spaces defined there so that C obtains a simple form. It is well known that one can introduce symplectic coordinates x, ξ in S_{12} such that λ_1 is defined by the equation $\xi=ix$ with the corresponding complex coordinates in $S_{12\mathbf{C}}$. Similarly λ_2 can be defined by $\eta=-iy$. When we now pass to the choice of appropriate coordinates in S_{11} and S_{21} we shall simplify the notation by assuming temporarily that λ_1 and λ_2 do not occur, thus that

$$C=\{(JY, Y),\ Y\in S_{2\mathbf{C}}\}$$

where J is a symplectic isomorphism $S_{2\mathbf{C}}\to S_{1\mathbf{C}}$. We want to find $(X, Y)\in C$

so that $(\mu X, Y)\in \bar{C}$ for some $\mu\in \mathbf{C}$, or in terms of J

$$X = JY, \quad \mu X = \bar{J}Y, \quad \text{that is,} \quad \bar{\mu}\bar{X} = J\bar{Y}, \tag{4.1}$$

for then the space spanned by $\operatorname{Re} Y, \operatorname{Im} Y$ is sent by J in a simple manner to that spanned by $\operatorname{Re} X, \operatorname{Im} X$. Eliminating X we write (4.1) in the form

$$\mu TY = Y \tag{4.2}$$

where $T=\bar{J}^{-1}J$. Thus μ^{-1} must be an eigenvalue of T with eigenvector Y unless $Y=0$. Since (4.1) can be written

$$\bar{X} = \bar{J}\bar{Y}, \quad \bar{\mu}\bar{X} = J\bar{Y}$$

we obtain $T\bar{Y}=\bar{\mu}\bar{Y}$ so $\bar{\mu}$ must also be an eigenvalue with eigenvector $\bar{Y}$. Since $\bar{J}$ is symplectic we have

$$i^{-1}\big(\sigma_1(X, \bar{X})-\sigma_2(Y, \bar{Y})\big) = i^{-1}\big(\sigma_1(JY, \bar{J}\bar{Y})-\sigma_2(Y, \bar{Y})\big)$$
$$= i^{-1}\big(\sigma_2(TY, \bar{Y})-\sigma_2(Y, \bar{Y})\big) = i^{-1}(\mu^{-1}-1)\sigma_2(Y, \bar{Y}) = 2(\mu^{-1}-1)\sigma_2(\operatorname{Im} Y, \operatorname{Re} Y).$$

If $\sigma_2(Y, \bar{Y})=0$ it follows from Lemma 3.1 that $(\bar{X}, \bar{Y})\in C$ so $\mu=1$ then. If $\mu\neq 1$ we conclude that μ is real and that

$$(\mu^{-1}-1)\sigma_2(\operatorname{Im} Y, \operatorname{Re} Y) > 0, \quad \sigma_1(\operatorname{Im} X, \operatorname{Re} X) = \mu^{-1}\sigma_2(\operatorname{Im} Y, \operatorname{Re} Y).$$

Since $J\bar{Y}=\mu\bar{X}$ we may replace μ by $1/\mu$ and Y by $\bar{Y}$ here which gives

$$(\mu-1)\sigma_2(\operatorname{Im} Y, \ \operatorname{Re} Y) < 0,$$

hence $\mu>0$.

If $\mu<1$ we can normalize Y so that $\sigma_2(\operatorname{Im} Y, \operatorname{Re} Y)=1$, hence $\sigma_1(\operatorname{Im} X, \operatorname{Re} X)= \mu^{-1}=e^{2\tau}$, $\tau>0$. Then $\operatorname{Re} Y, \operatorname{Im} Y$ resp. $e^{-\tau}\operatorname{Re} X, e^{-\tau}\operatorname{Im} X$ are symplectic bases in two dimensional subspaces S_2' and S_1' of S_2 and S_1 such that $J\, S_{2\mathbf{C}}'\subset S_{1\mathbf{C}}'$. With the corresponding coordinates y, η and x, ξ we have $J(1, i)=e^{\tau}(1, i)$, $J(1, -i)=e^{-\tau}(1, -i)$ so J has the matrix

$$R(\tau) = \begin{pmatrix} \cosh\tau & -i\sinh\tau \\ i\sinh\tau & \cosh\tau \end{pmatrix}. \tag{4.3}$$

As in the proof of Lemma 3.3 we find that the canonical relation C is the direct sum of the graph of $R(\tau)$ and the graph of a canonical transformation between the complexifications of the symplectically orthogonal spaces of S_2' in S_2 and S_1' in S_1. We can repeat this procedure and split off two dimensional spaces until we reach a situation where T only has the eigenvalue 1, that is, $T-I$ is nilpotent, which we now assume.

Write $T=T_1+iT_2$ with T_j real. Since $T=\bar{J}^{-1}J$ we have $T\bar{T}=I$, that is,

$$T_1^2+T_2^2 = I, \quad T_1T_2 = T_2T_1. \tag{4.4}$$

If $T=I$ then J is real. We choose a symplectic basis in S_2 and map it by the real symplectic map J to a symplectic basis in S_1. With these bases J becomes the identity. Assume now that $T-I$ is not 0. We can then choose Y so that $(T-I)Y\neq 0$ but $(T-I)^2Y=0$. The relations (4.4) give

$$(T-I)^2 = T_1^2-T_2^2+2iT_1T_2+I-2T = 2T(T_1-I)$$

and since T is invertible it follows that $(T_1-I)Y=0$, hence $T_2^2Y=0$ by (4.4). The real or imaginary part of Y must therefore have the same properties as assumed for Y so we may assume that Y is real. The positivity of J gives then

$$0 \leqq i^{-1}(\sigma_1(JY, \bar{J}\bar{Y})-\sigma_2(Y,\bar{Y})) = i^{-1}\sigma_2(TY, Y) = \sigma_2(T_2Y, Y).$$

If this is 0 then Lemma 3.1 shows that $(\bar{J}Y, Y)\in C$ so $\bar{J}Y=JY$ and $TY=Y$. This is against our hypothesis so $\sigma_2(T_2Y, Y)>0$. We can normalize so that $\sigma_2(T_2Y, Y)=1$. Now

$$\bar{J}^{-1}JY = (I+iT_2)Y, \quad \bar{J}^{-1}JT_2Y = T_2Y$$

so $JT_2Y=\bar{J}T_2Y$ is real and $JY-\bar{J}Y=i\bar{J}T_2Y$. For $X=\operatorname{Re} JY$ we have

$$\sigma_1(X, JT_2Y) = (\sigma_1(JY, JT_2Y)+\sigma_1(\bar{J}Y, \bar{J}T_2Y))/2 = \sigma_2(Y, T_2Y) = -1$$

so we can take Y, T_2Y resp. X, JT_2Y as elements in symplectic bases in two dimensional subspaces of S_2 and of S_1. Since

$$JY = X+i\bar{J}T_2Y/2, \quad J(T_2Y) = JT_2Y$$

it follows that J reduces to a map in the complexification which sends $(1,0)$ to $(1, i/2)$ and $(0,1)$ to $(0,1)$. Hence J has the matrix $E(1/2)$, if

$$E(a) = \begin{pmatrix} 1 & 0 \\ ia & 1 \end{pmatrix}. \tag{4.5}$$

(These are symplectically equivalent when $a>0$ so the parameter a is superfluous. However, it will be useful later on.) Summing up, we have now proved:

Theorem 4.1. *Let S_1 and S_2 be real symplectic finite dimensional vector spaces and C a positive linear canonical relation $\subset S_{1\mathbf{C}}\oplus S_{2\mathbf{C}}$ such that the projections of $C_{\mathbf{R}}$ to S_1 and to S_2 are injective. Then there exist symplectic coordinates $x_j, \xi_j, j\leqq n_1$ in S_1 and $y_j, \eta_j, j\leqq n_2$ in S_2, integers $\nu_0\leqq\nu\leqq\min(n_1, n_2)$ and numbers $\tau_j\geqq 0, j\leqq\nu_0$, such that C is defined by*

$$(x_j, \xi_j) = R(\tau_j)(y_j, \eta_j), \quad j\leqq \nu_0; \quad (x_j, \xi_j) = E(1)(y_j, \eta_j), \tag{4.6}$$
$$\nu_0 < j \leqq \nu; \quad \xi_j = ix_j, \quad \nu < j \leqq n_1; \quad \eta_j = -iy_j, \quad \nu < j \leqq n_2.$$

Conversely we shall see from the following discussion of $R(\tau)$ and $E(a)$ that the hypothesis follows from the conclusion. First we observe that $R(\tau)$ is rotation by the angle $i\tau$, so $\tau \to R(\tau)$ is a one parameter group of symplectic maps, $R(\tau'+\tau)=R(\tau')R(\tau)$, which implies $R'(\tau)=R'(0)R(\tau)$. With σ denoting the standard symplectic form in $T^*(\mathbf{C})$ we have

$$\sigma(X, R'(0)Y) = -i(xy+\xi\eta); \quad X, Y \in T^*(\mathbf{C}) \tag{4.7}$$

so $R'(0)$ is the Hamilton map of the quadratic form $-i(x^2+\xi^2)$ and

$$i^{-1}(\sigma(R'(0)Y, \bar{Y})+\sigma(Y, \overline{R'(0)Y})) = 2(|y|^2+|\eta|^2); \quad Y=(y, \eta).$$

This implies that

$$i^{-1}(\sigma(R(\tau)Y, \overline{R(\tau)Y})-\sigma(Y, \bar{Y})) \geqq 0, Y \in T^*(\mathbf{C}), \quad \tau \geqq 0,$$

for the left hand side vanishes when $\tau=0$ and the derivative with respect to τ is $2|R(\tau)Y|^2$. Thus $R(\tau)$ is positive. It is also clear that $E(a)$ is a one parameter group of symplectic maps, and we have

$$\sigma(X, E'(0)Y) = -ixy,$$

$$i^{-1}(\sigma(E'(0)Y, \bar{Y})+\sigma(Y, \overline{E'(0)Y})) = 2|y|^2, \quad Y=(y, \eta),$$

which proves that $E(a)$ is positive when $a>0$. The maps $R(\tau)$ and $E(a)$ can also be regarded as the exponentials of $-i$ times the Hamilton maps of the quadratic forms $\tau(x^2+\xi^2)$ and ax^2 respectively.

We shall now discuss Fourier integral operators associated with the various canonical relations in Theorem 4.1, and also study the corresponding half densities. By a half density in a complex vector space we shall mean a number associated to each system of complex linear coordinates w, such that $a|dw|^{1/2}$ is invariant, that is, a is replaced by $a'=a|Dw/Dw'|^{1/2}$ if new complex coordinates w' are introduced. It is convenient to keep the transformation law in mind by using the notation $a|dw|^{1/2}$ for the half density. We shall only consider absolute values of symbols of Fourier integral distributions in order to avoid lengthy discussions of powers of the imaginary unit.

The polynomial $Q_\tau(x, \eta)$ is a generating function of $R(\tau)$ if

$$R(\tau)(\partial Q_\tau/\partial\eta, \eta) = (x, \partial Q_\tau/\partial x); \quad x, \eta \in \mathbf{C};$$

that is,

$$\partial Q_\tau/\partial\eta \cosh\tau - i\eta\sinh\tau = x, \quad i\partial Q_\tau/\partial\eta \sinh\tau+\eta\cosh\tau = \partial Q_\tau/\partial x.$$

The solution of these equations is

$$Q_\tau(x, \eta) = (x\eta+i(x^2+\eta^2)(\sinh\tau)/2)/\cosh\tau, \tag{4.8}$$

where we observe that $\operatorname{Im} Q_\tau \geqq 0$ which verifies the positivity again. Set

$$A_\tau u(x) = (2\pi)^{-1}a(\tau)\int e^{iQ_\tau(x,\eta)}\hat{u}(\eta)\, d\eta, \quad u \in C_0^\infty(\mathbf{R}). \tag{4.9}$$

We choose $a(\tau)$ so that the symbol $a(\tau)|dx\,d\eta|^{1/2}$ agrees with the symplectic half density $|dx\,d\xi|^{1/2}$, that is,

$$a(\tau)\big(D(x,\eta)/D(x,\xi)\big)^{1/2} = a(\tau)(\partial^2 Q_\tau/\partial x\,\partial\eta)^{-1/2} = 1.$$

Thus we set

$$A_\tau u(x) = (2\pi)^{-1}(\cosh\tau)^{-1/2}\int e^{iQ_\tau(x,\eta)}\hat{u}(\eta)\,d\eta, \quad u\in C_0^\infty. \tag{4.9$'$}$$

The infinitesimal version of the calculus of Fourier integral operators with complex phase shows that $A_{\tau+\tau'}$ can only differ from $A_\tau A_{\tau'}$ by a power of i, for $R(\tau+\tau')=R(\tau)R(\tau')$. Since a simple calculation shows that the kernel of A_τ is positive, we must have $A_{\tau+\tau'}=A_\tau A_{\tau'}$. By the formula

$$A_\tau u(x\cosh\tau) = (2\pi)^{-1}(\cosh\tau)^{-1/2}\int e^{-x^2(\sinh 2\tau)/4}e^{ix\eta-\eta^2(\tanh\tau)/2}\hat{u}(\eta)\,d\eta$$

it is clear that the norm of A_τ in L^2 is at most 1. When $\tau\to 0$ we obtain

$$\big(A_\tau u(x)-u(x)\big)/\tau \to -(2\pi)^{-1}\int e^{ix\eta}(x^2+\eta^2)\hat{u}(\eta)\,d\eta/2 = -(x^2+D^2)u/2,$$

if $u\in C_0^\infty$, so the infinitesimal generator of the semigroup A_τ, $\tau\geqq 0$, is the harmonic oscillator divided by -2. Thus

$$A_\tau = \exp\big(-\tau(x^2+D^2)/2\big) \tag{4.10}$$

which is also easily verified directly and well known. For the L^2 norm we obtain

$$\|A_\tau\| = e^{-\tau/2}, \tag{4.11}$$

for the lowest eigenvalue of the harmonic oscillator is 1, the eigenfunction being $e^{-x^2/2}$.

Similarly the generating function P_a of $E(a)$ is defined by

$$\partial P_a(x,\eta)/\partial\eta = x, \quad \partial P_a(x,\eta)/\partial x = iax+\eta,$$

so we have

$$P_a(x,\eta) = x\eta+iax^2/2. \tag{4.12}$$

The corresponding semigroup of operators is

$$E_a u(x) = (2\pi)^{-1}\int e^{ix\eta-ax^2/2}\hat{u}(\eta)\,d\eta = e^{-ax^2/2}u(x), \quad u\in L^2, \quad a\geqq 0. \tag{4.13}$$

The norm of E_a is of course 1.

For the canonical relation

$$\{(x, ix;\, y, -iy)\}\subset T^*(\mathbf{C}^{n_1})\times T^*(\mathbf{C}^{n_2})$$

a defining phase function with no parameters is $i(|x|^2+|y|^2)/2$. The corresponding operator with kernel

$$(2\pi)^{-(n_1+n_2)/4}e^{-(|x|^2+|y|^2)/2}$$

has symbol $|dx\,dy|^{1/2}$, and the norm is

$$(2\pi)^{-(n_1+n_2)/4}\left(\iint e^{-|x|^2-|y|^2}\,dx\,dy\right)^{1/2} = 2^{-(n_1+n_2)/4}.$$

It is natural to compare $|dx dy|^{1/2}$ with the half density defined by the Hermitian metric $i^{-1}(\sigma((x, ix), \overline{(x, ix)})-\sigma((y, -iy), \overline{(y, -iy)}))=2(|x|^2+|y|^2)$ in the two Lagrangian planes. This is

$$|d(x\sqrt{2})\,d(y\sqrt{2})|^{1/2} = 2^{(n_1+n_2)/4}|dx\,dy|^{1/2},$$

so the principal symbol is this half density multiplied by the norm. To sum up the preceding results we make a definition.

Definition 4.2. *If C is a positive linear canonical relation satisfying the hypotheses in Theorem 4.1, then d_C will denote the half density*

$$d_C = \left|\Pi_1^{\nu}\, dx_j\, d\xi_j e^{\tau_j}\, \Pi_{\nu+1}^{n_1} 2^{1/2}\, dx_j\, \Pi_{\nu+1}^{n_2} 2^{1/2}\, dy_j\right|^{1/2}. \tag{4.14}$$

Equivalently, with the decomposition in Lemma 3.3, one can describe d_C as follows. The canonical relation C_1 corresponds to a map J such that $\bar{J}^{-1}J$ has positive, pairwise reciprocal eigenvalues. If the eigenvalues larger than 1 are denoted by $\mu_1, \mu_2, \ldots$ then d_C is the product of $\Pi\mu_j^{1/4}$, the half density defined by the symplectic form in S_{11}, lifted to C_1, and the half densities in λ_1 and λ_2 defined by the positive hermitian metrics $i^{-1}\sigma_1(X, \overline{X})$ and $i\sigma_2(Y, \overline{Y})$. In particular this shows that Definition 4.2 is independent of the choice of coordinates in S_1 and in S_2. The definition has been chosen so that we have

Theorem 4.3. *Let $C\subset T^*(\mathbf{C}^{n_1})\times T^*(\mathbf{C}^{n_2})$ be a positive linear canonical relation with no real element having only one projection equal to 0, let $Q(x, y, \theta)$ be a quadratic form which is a positive non-degenerate phase function defining C, and set for some constant a*

$$A(x, y) = a\int e^{iQ(x,y,\theta)}\,d\theta. \tag{4.15}$$

Then A is the kernel of an operator which is L^2 bounded, and the absolute value of the symbol is the half density $\|A\|\,d_C$ on C.

Proof. The statement merely rephrases Definition 4.2 and the calculations preceding it if C has already the form in Theorem 4.1. Otherwise we can find real linear symplectic maps χ_1 and χ_2 in $T^*(\mathbf{R}^{n_1})$ and $T^*(\mathbf{R}^{n_2})$ such that if C_1 and C_2 are the complex extensions of their graphs, the composition $C_1\circ C\circ C_2$ has the form in Theorem 4.1. With C_1 and C_2 are associated unitary operators U_1 and U_2 (see e.g. Hörmander [4, Th. 4.3]). The product U_1AU_2 is associated with $C_1\circ C\circ C_2$ and the symbol is that of A transported from C to $C_1\circ C\circ C_2$ by the identification given by the composition. This is just the infinitesimal form of the composition rules for Fourier integral operators when all except one of them corresponds to a canonical transformation. The norm of U_1AU_2 is equal to $\|A\|$ so the absolute value of the symbol is $d_{C_1 \circ C \circ C_2}\|A\|$, which proves the theorem. Note that it contains a converse of Lemma 2.1.

We shall now study what happens when C varies. To do so we shall first look at the general form of the eigenvalue problem studied above when C is a graph. Recall that the problem was to find $(X, Y)\in C$ with $(\mu X, Y)\in \bar{C}$. First of all we homogenize to the problem of satisfying the conditions

$$(X, Y)\in C, \quad (\mu' X, \mu'' Y)\in \bar{C} \quad \text{and} \quad (X, Y) \neq 0, \quad (\mu', \mu'') \neq 0.$$

Let $n=\dim C$ and let $T\colon \mathbf{C}^n\to C$ be a linear parametrization. The problem is then to find $u, v\in \mathbf{C}^n$ such that

$$\big(\mu'(Tu)_1, \quad \mu''(Tu)_2\big)+\bar{T}v = 0.$$

Here $(Tu)_j$ is the component of Tu in $S_{j\mathbf{C}}$. These $2n$ equations for $(u, v)\in \mathbf{C}^{2n}$ have a solution $\neq 0$ if and only if the determinant $D_C(\mu', \mu'')$ of the system vanishes. Apart from a factor the determinant is independent of the parametrization, for any other is of the form TT' where $T'\in GL(n, \mathbf{C})$. It is clear that D_C is homogeneous of degree n in (μ', μ''). To compute D_C we may assume that C is of the form in Theorem 4.1 and parametrize by $y_j, \eta_j, j\leqq \nu$, and $x_j, j>\nu; y_j, j>\nu$. Thus for example $x_{\nu+1}=u_{2\nu+1}$, $\xi_{\nu+1}=iu_{2\nu+1}$ in the parametrization of C and $x_{\nu+1}=v_{2\nu+1}$, $\xi_{\nu+1}=-iv_{2\nu+1}$ in the parametrization of $\bar{C}$, so the equations

$$\mu' u_{2\nu+1}+v_{2\nu+1} = 0, \quad \mu' i u_{2\nu+1}-iv_{2\nu+1} = 0$$

and no others contain $u_{2\nu+1}$ and $v_{2\nu+1}$. This gives a factor $-2i\mu'$ in the determinant. We can argue in the same way for each of the variables x_j and y_j with $j>\nu$, and for the others we have a graph, so

$$D_C(\mu', \mu'') = K\prod_1^\nu\big((\mu'-e^{-\tau_j}\mu'')(e^{-\tau_j}\mu'-\mu'')\big)\mu'^{n_1-\nu}\mu''^{n_2-\nu}.$$

Here K is a constant $\neq 0$.

The polynomial $D_C(\mu', \mu'')$ varies continuously with C so the τ_j also vary continuously apart from the fact that some τ_j may tend to ∞ giving rise to an additional factor $\mu'\mu''$. We must therefore examine what happens to d_C if say $\tau_1\to\infty$ in (4.6). The graph of $R(\tau_1)$ is defined by

$$x_1 = y_1 \cosh \tau_1 - i\eta_1 \sinh \tau_1, \quad \xi_1 = iy_1 \sinh \tau_1 + \eta_1 \cosh \tau_1$$

or equivalently

$$y_1 = x_1 \cosh \tau_1 + i\xi_1 \sinh \tau_1, \quad \eta_1 = -ix_1 \sinh \tau_1 + \xi_1 \cosh \tau_1$$

so the graph converges to the plane defined by

$$\xi_1 = ix_1, \; \eta_1 = -iy_1.$$

The half density in the graph is

$$|dx_1\, d\xi_1 e^{\tau_1}|^{1/2} = |dx_1\, d\eta_1 e^{\tau_1}/\cosh \tau_1|^{1/2} \to 2^{1/2}\, |dx_1\, dy_1|^{1/2}$$

so the limit is the product of the densities in the two limiting Lagrangian planes. This is the basic reason for the continuity of d_C but before proving it we have to give a precise definition.

The canonical relations $C \subset S_{1\mathbf{C}} \oplus S_{2\mathbf{C}}$ form a compact analytic manifold $\mathscr{C}$, the positive canonical relations form a closed subset $\mathscr{C}_+$, and those which satisfy the hypothesis in Theorem 4.1 form an open subset $\mathscr{C}_+^0$ of $\mathscr{C}_+$. The half densities form a complex line bundle $\Omega^{1/2}$ over $\mathscr{C}$. In fact, if $w_1, \ldots, w_n$ are linear foims in $S_{1\mathbf{C}} \oplus S_{2\mathbf{C}}$ which restrict to a coordinate systems in some $C_0 \in \mathscr{C}$, then this is true for all $C \in \mathscr{C}$ in a neighborhood U of C_0. The map

$$U \times \mathbf{C} \ni (C, z) \to z\, |dw_1 \ldots dw_n|_C^{1/2}$$

gives a local trivialization of $\Omega^{1/2}$ which clearly makes $\Omega^{1/2}$ into a line bundle, with real valued transition functions. The continuity of d_C can now be stated as follows:

Theorem 4.4. *The map* $C \to d_C$ *of Definition* 4.2 *is a continuous section of* $\Omega^{1/2}$ *over* $\mathscr{C}_+^0$ *which tends to* 0 *at the boundary of* $\mathscr{C}_+^0$ *in* $\mathscr{C}_+$.

Proof. Let $C_\nu \in \mathscr{C}_+^0$ be a sequence with $C_\nu \to C_0 \in \mathscr{C}_+^0$ as $\nu \to \infty$. It is sufficient to prove that d_{C_ν} has a subsequence converging to d_{C_0}. In doing so we may assume that the theorem is already known for lower dimensions.

a) Assume first that there is a non-zero element $(X_\nu, 0) \in C_\nu$ for every ν. After suitable normalization and passage to a subsequence we may assume that $\lim X_\nu = X_0$ exists and is not 0. Then $\sigma_1(X_0, \overline{X}_0) \neq 0$ so $\sigma_1(\operatorname{Im} X_\nu, \operatorname{Re} X_\nu) \to \sigma_1(\operatorname{Im} X_0, \operatorname{Re} X_0) \neq 0$ as $\nu \to \infty$. We may even assume that

$$\sigma_1(\operatorname{Im} X_\nu, \operatorname{Re} X_\nu) = \sigma_1(\operatorname{Im} X_0, \operatorname{Re} X_0) \neq 0.$$

Then there exists a sequence of complectic maps χ_ν in S_1 converging to the identity as $\nu \to \infty$ such that $\chi_\nu X_\nu = X_0$. If we compose C_ν to the left with the complexification of the graph of χ_ν we obtain new $C'_\nu \in \mathscr{C}_+$ converging to C_0 as $\nu \to \infty$ such that $(X_0, 0) \in C'_\nu$ for all ν. But then C'_ν and C_0 are direct sums of the Lagrangian plane generated by X_0 in the space spanned by X_0 and $\overline{X}_0$ and canonical relations in fewer variables, so $d_{C'_\nu} \to d_{C_0}$, hence $d_{C_\nu} \to d_{C_0}$.

b) The same argument is applicable if there is a non-zero element $(0, Y_\nu) \in C_\nu$ for every ν.

c) Now assume that each C_ν is the graph of a canonical transformation and that some of the $\tau_{j\nu}$ in Theorem 4.1 tends to ∞. Then we can find $0 \neq (X_\nu, Y_\nu) \in C_\nu$ and $\mu_\nu \to \infty$ such that $(\mu_\nu X_\nu, Y_\nu) \in \overline{C}_\nu$. We can normalize so that $(X_\nu, Y_\nu) \to (X_0, Y_0) \neq 0$

as $\nu\to\infty$. Since $(X_0, 0)\in\bar{C}_0$ and $(X_0, Y_0)\in C_0$ we must have $\sigma_1(X_0, \bar{X}_0)=0$ so $(X_0, 0)\in C_0$ by Lemma 3.1. But this implies $X_0=0$, hence $\sigma_2(Y_0, \bar{Y}_0)\neq 0$. It is no restriction to assume that

$$\sigma_2(Y_\nu, \bar{Y}_\nu) = \sigma_2(Y_0, \bar{Y}_0).$$

Since $(\mu_\nu \bar{X}_\nu, \bar{Y}_\nu)\in C_\nu$ and $(\bar{X}_\nu, \bar{Y}_\nu)\in\bar{C}_\nu$ we obtain in the same way that there is a sequence a_ν such that $a_\nu\bar{Y}_\nu\to 0$ and $a_\nu\mu_\nu\bar{X}_\nu=X'_\nu\to X'_0\neq 0$, $0\neq\sigma_1(X'_0, \bar{X}'_0)$. We may of course assume that

$$\sigma_1(X'_\nu, \bar{X}'_\nu) = \sigma_1(X'_0, \bar{X}'_0).$$

We can then compose C_ν left and right with symplectic maps $\chi_{1\nu}$ and $\chi_{2\nu}$ in S_1 and S_2, extended to S_{1C} and S_{2C}, chosen so that $\chi_{2\nu}Y_0=Y_\nu$, $\chi_{1\nu}X'_\nu=X'_0$ and $\chi_{j\nu}$ tends to the identity when $\nu\to\infty$, $j=1, 2$. We then obtain canonical relations splitting into the sum of the graph of an operator $R((\log\mu_\nu)/2)$ from the space spanned by $Y_0, \bar{Y}_0$ to that spanned by $X'_0, \bar{X}'_0$, and a convergent sequence of canonical relations in the symplectically orthogonal spaces. Hence the continuity follows as in case a) from the inductive hypothesis if we also recall the special case discussed before the statement of the theorem.

d) It remains to consider the case where each C_ν is the graph of a canonical transformation J_ν and the eigenvalues of $\bar{J}_\nu^{-1}J_\nu$ remain bounded as $\nu\to\infty$. Then it follows from the general discussion of the eigenvalue problem above that C_0 is also the graph of a canonical transformation J_0. Since $\tau_{j\nu}\to\tau_j$ as $\nu\to\infty$, if $\tau_{j\nu}$ are ordered increasingly, and since the symplectic half density in C_ν converges to that in C_0, it follows that $d_{C_\nu}\to d_C$.

On the other hand, if $C_\nu\in\mathscr{C}_+^0$ and $C_\nu\to C_0\notin\mathscr{C}_+^0$ it follows from Theorem 4.3 that $d_{C_\nu}\to 0$. In fact, we can take operators A_ν as in Theorem 4.3 associated to C_ν such that $A_\nu\to A_0\neq 0$ when $\nu\to\infty$. Then the absolute value of the principal symbol of A_ν has a limit $\neq 0$ but $\|A_\nu\|\to\infty$ since $A_\nu\to A_0$ as operator in $\mathscr{S}$ and $\|A_0\|=\infty$ by Lemma 2.1 since A_0 is associated to C_0. This completes the proof.

The definition of d_C depends on the symplectic forms σ_1 and σ_2. If they are replaced by $t\sigma_1$ and $t\sigma_2$, $t>0$, then d_C is multiplied by $t^{n/4}$ where $n=\dim_{\mathbf{C}}C$. This follows at once from Definition 4.2, for the numbers τ_j do not change while $dx_j d\xi_j$ is replaced by $t\,dx_j d\xi_j$ when $j\leqq\nu$ whereas dx_j and dy_j are multiplied by $t^{1/2}$ when $j>\nu$.

If d is any half density in $C\in\mathscr{C}_+^0$ we define $\|d\|=|d/d_C|$, that is, we introduce a norm such that the norm of d_C is 1. With this notation we shall now give (2.3) an invariant form, so let C now as in Section 2 be a positive homogeneous canonical relation $\subset(T^*(X)\setminus 0)\times(T^*(Y)\setminus 0)^{\sim}$. The symbol of the kernel A in (2.1) in the local coordinates used has absolute value $|a(\xi, \eta)|\;|d\xi\,d\eta|^{1/2}$. We want to calculate the norm at the real point parametrized by $(t^2\xi_0, t^2\eta_0)$. To do so we note that the quotient q between the half density $|d\xi\, d\eta|^{1/2}$ in the tangent space of C at a real point $\gamma=(\partial H/\partial\xi, \xi; -\partial H/\partial\eta, \eta)$ in C and $d_{T_\gamma(C)}$ is homogeneous of degree $n/2-n/4=n/4$ in the sense that $M_t^*q=t^{n/4}q$ if M_t denotes multiplication by t in the fibers of $T^*(X\times Y)$. This follows from the fact that $M_t^*\sigma=t\sigma$ and the remarks above on how d_C depends on the symplectic form. Hence

$$M_t^*(|a(\xi,\eta)|\,|d\xi\, d\eta|^{1/2}/d_C) = |a(t\xi, t\eta)|t^{n/4}|d\xi\, d\eta|^{1/2}/d_C.$$

When $(\xi,\eta)=(\xi_0,\eta_0)$ we have $|d\xi\, d\eta|^{1/2}=\|A_0\|\, d_C$ by Theorem 4.3, if A_0 is defined by (2.3), so (2.4) means that

$$\overline{\lim_{t\to\infty}}\, \|\sigma_A(t^2\xi_0, t^2\eta_0)\| \leqq M \tag{4.16}$$

if σ_A is the symbol half density of A. It is invariantly defined modulo lower order symbols and factors i, so (4.16) is coordinate free. The proof shows that (4.16) is locally uniform in (ξ_0,η_0) which is also a consequence of Theorem 4.4 and the fact that $a(\xi,\eta)\in S^{-n/4}$. Hence we have proved that (1.5) follows from (1.3).

The relation (2.2) means that the difference between $t^{n/2}a(t^2\xi_0, t^2\eta_0)A_0$ and the pullback of $e^{it^2(\langle y,\eta_0\rangle-\langle x,\xi_0\rangle)}A(x,y)$ as a half density by the map $(x,y)\to(x/t, y/t)$ tends to 0 in $\mathscr{D}'$ as $t\to\infty$. By the description of the principal symbol of a product in [7, Section 7] it follows that the difference between $|t^{n/2}a(t^2\xi_0, t^2\eta_0)|^2A_0^*A_0$ and the pullback of $e^{it^2\langle y-z,\eta_0\rangle}A^*A(z,y)$ also tends to 0, thus

$$(A^*Au_t, u_t)-c(t)(A_0^*A_0u, u) \to 0, \quad u\in C_0^\infty(\mathbf{R}^{n_Y}),$$

where $c(t)=|t^{n/2}a(t^2\xi_0, t^2\eta_0)|^2$. If (2.4) is valid then

$$\overline{\lim_{t\to\infty}}\, c(t)\|A_0^*A_0\| \leqq M^2, \tag{2.4$'$}$$

which means that the analogue of (1.5) is valid for A^*A with M replaced by M^2.

5. Precise L^2 estimates

In this section we shall prove the equivalence of (1.3) and (1.5) stated in the introduction, assuming the condition on C in Theorem 2.2 and a related global condition:

Theorem 5.1. *Let X and Y be C^∞ manifolds and C a C^∞ complex positive homogeneous canonical relation $\subset(T^*(X)\setminus 0)\times(T^*(Y)\setminus 0)^\sim$ which is closed in $T^*(X\times Y)\setminus 0$, and assume that*

(i) *the maps $C_\mathbf{R}\to T^*(X)\setminus 0$ and $C_\mathbf{R}\to T^*(Y)\setminus 0$ from the real subset $C_\mathbf{R}$ are injective,*

(ii) *for every* $\gamma=(x_0, \xi_0, y_0, \eta_0)\in C_{\mathbf{R}}$ *the maps from* $T_\gamma(C)_{\mathbf{R}}$ *to the tangent spaces of* $T^*(X)$ *and* $T^*(Y)$ *at* (x_0, ξ_0) *and* (y_0, η_0) *are injective.*

Let $A\in I^0(X\times Y, C'; \Omega(X\times Y)^{1/2})$ *be properly supported. Then* (1.3) *is valid with some* $C_{\varepsilon, K}$ *for every compact set* $K\subset Y$ *and every* $\varepsilon>0$, *if and only if* (1.5) *is valid.*

Proof. That (1.3) implies (1.5) was proved in Section 2 combined with the reformulation of (2.4) given at the end of Section 4. It follows from Theorem 3.6 that A^*A is a pseudo-differential operator, of order 0 and type 1/2, 1/2. The theorem will be proved if we show that for every (y_0, η_0) with $y_0\in K$ and $\eta_0\neq 0$ there is a pseudo-differential operator $\Psi(y, D)$ with principal symbol $\psi(y, \eta)$ equal to 1 in a conic neighborhood of (y_0, η_0) such that

$$\|A\Psi(y, D)u\|^2 \leqq (M+\varepsilon)^2\|u\|^2+C_{\varepsilon, K}\|u\|^2_{(-1)}.$$

In fact, we can then choose pseudo-differential operators Φ_j of order 0 with principal symbol φ_j supported in the set where the principal symbol ψ_j of some such Ψ_j is equal to 1, and $\Sigma|\varphi_j|^2=1$ in a neighborhood of K. Since

$$\|A\Psi_j(y, D)\Phi_j(y, D)u\|^2 \leqq (M+\varepsilon)^2\|\Phi_j(y, D)u\|^2+C'_{\varepsilon, K}\|u\|^2_{(-1)}$$

and

$$\Phi_j^*\Psi_j^*A^*A\Psi_j\Phi_j-\Phi_j^*\Phi_jA^*A$$

is of order $-1/2$ and type 1/2, 1/2, we obtain by summation

$$(A^*Au, u+Ru) \leqq (M+\varepsilon)^2(u+Ru, u)+C''_{\varepsilon, K}(\|u\|_{(-1/2)}\|u\|+\|u\|^2_{(-1)})$$

where

$$R = \Sigma\Phi_j^*\Phi_j-I$$

is of order -1 at K. This gives (1.3) since $\|u\|^2_{(-1/2)}\leqq C\|u\|_{(-1)}\|u\|$.

When $WF(\Psi(y, D))$ is in a small conic neighborhood of (y_0, η_0) then $WF'(A\Psi(y, D))$ is either empty or contained in a small conic neighborhood of a point $(x_0, \xi_0, y_0, \eta_0)\in C$. The principal symbol of $A\Psi(y, D)$ is $\sigma_A\psi$ so it satisfies (1.5). Changing notation so that $A\Psi(y, D)$ becomes A it is therefore sufficient to prove that (1.3) follows from (1.5) when A has support near $(0, 0)\in\mathbf{R}^{n_X}\times\mathbf{R}^{n_Y}$ and $WF'(A)$ is in a small conic neighborhood of $(0, \xi_0, 0, \eta_0)$. We can then write

$$A^*Au(z) = (2\pi)^{-n_Y}\int e^{i\tilde\varphi(z,\eta)}\tilde a(z, \eta)\hat u(\eta)\,d\eta$$

where $\tilde a\in S^0$ has support in a small conic neighborhood of (y_0, η_0) and $\psi(z, \eta)=\tilde\varphi(z, \eta)-\langle z, \eta\rangle$ satisfies (1.2). We have

$$\varlimsup_{t\to\infty}|\tilde a(z_0, t^2\eta_0)|\|B_0\| \leqq M^2 \quad \text{if} \quad \operatorname{Im}\psi(z_0, \eta_0) = 0, \tag{5.1}$$

which implies $\psi'(z_0, \eta_0)=0$ by (1.2); here

$$B_0u(z) = (2\pi)^{-n_Y}\int e^{iQ(z,\eta)}\hat u(\eta)\,d\eta, \quad u\in C_0^\infty(\mathbf{R}^{n_Y}), \tag{5.2}$$

$\tilde{Q}$ denoting the quadratic term in the Taylor expansion of $\tilde{\varphi}$ at (z_0, η_0). In the proof we may assume $z_0=0$ to simplify the notation. As in (2.2) we set

$$u_t(y) = e^{it^2\langle y, \eta_0\rangle} u(ty) t^{n_Y/2}, \quad u\in C_0^\infty(\mathbf{R}^{n_Y}).$$

Then

$$A^*Au_t(z) = (2\pi)^{-n_Y} \int e^{i\tilde{\varphi}(z,\eta)} \tilde{a}(z, \eta)\hat{u}((\eta-t^2\eta_0)/t) t^{-n_Y/2}\, d\eta$$
$$= (2\pi)^{-n_Y} t^{n_Y/2} \int e^{i\tilde{\varphi}(z, t^2\eta_0+t\eta)} \tilde{a}(z, t\eta+t^2\eta_0)\hat{u}(\eta)\, d\eta,$$

hence

$$(A^*Au_t, u_t) = (2\pi)^{-n_Y} \int \overline{u(z)}\, dz \int e^{i(\tilde{\varphi}(z/t, t^2\eta_0+t\eta)-t^2\langle z/t, \eta_0\rangle)} \tilde{a}(z/t, t^2\eta_0+t\eta)\hat{u}(\eta)\, d\eta.$$

Here the exponent $t^2(\tilde{\varphi}(z/t, \eta_0+\eta/t)-\langle z/t, \eta_0\rangle)$ converges to $\tilde{Q}(z, \eta)$ when $t\to\infty$ since $\tilde{\varphi}(z, \eta_0+\eta)-\langle z, \eta_0\rangle=\langle z, \eta\rangle+\psi(z, \eta_0+\eta)$ differs from $\tilde{Q}(z, \eta)$ by terms of higher order at $(0, 0)$. As in the proof of (2.2) it follows that

$$(A^*Au_t, u_t)-\tilde{a}(0, t^2\eta_0)(B_0u, u) \to 0$$

so (5.1) is a consequence of (1.5) and the remarks at the end of Section 4.

If q is any quadratic polynomial with principal part $\tilde{Q}$ and $\operatorname{Im} q\geqq 0$, and if b_0 is defined as B_0 with $\tilde{Q}$ replaced by q, then

$$\|b_0\| \leqq \|B_0\|. \tag{5.3}$$

In fact, we can find real z_0, η_0 such that $\operatorname{Im} q(z+z_0, \eta+\eta_0)$ has no first order terms in (z, η), hence is equal to $\operatorname{Im}\tilde{Q}(z, \eta)+t$ for some $t\geqq 0$. Thus

$$q(z, \eta) = \tilde{Q}(z-z_0, \eta-\eta_0)+\langle y_0, \eta\rangle+\langle z, \zeta_0\rangle+s+it$$

for some real y_0, ζ_0 and s, so b_0 is equal to B_0 multiplied by e^{is-t} and by unitary factors to the left and right, consisting of translation or multiplication by a character. Since $t\geqq 0$ the inequality (5.3) follows.

We can now prove (1.3) by a conventional localization argument. Let g be the metric in $T^*(\mathbf{R}^n)$, $n=\dim Y$, defined by

$$(1+|\eta|)|dy|^2+(1+|\eta|)^{-1}|d\eta|^2;$$

thus $S^m_{1/2,1/2}(\mathbf{R}^n\times\mathbf{R}^n)=S((1+|\eta|)^m, g)$ in the notation of [4]. Let R be a large positive number, and choose a standard partition of unity $\{\psi_j\}$ as in [4, Section 2] such that

$$\Sigma\psi_j(y, \eta)^2 = 1,$$

the diameter of the support of ψ_j is uniformly bounded in the metric g/R^2, only a fixed number of supports can overlap, and finally $\{\psi_j\}$ is a uniformly bounded symbol in $S^0(1, g/R^2)$ with values in l^2. Then the calculus gives that the symbol of

$$\Sigma\psi_j(y, D)^*\psi_j(y, D)-I$$

is bounded in $S(R^{-2}, g/R^2)$, so the norm is $O(R^{-2})$. We can regard A^*A as a pseudo-differential operator with symbol in $S(1, g)$. Hence a minor extension of the calculus results in [4] shows that

$$\psi_j(y, D)A^*A - A^*A\psi_j(y, D) = \varphi_j(y, D)$$

where $\{\varphi_j\}$ is bounded in $S(R^{-1}, g)$. (The gain in the weight here is the geometric mean of the gains in the calculi corresponding to g and to g/R^2.) Hence, writing ψ_j instead of $\psi_j(y, D)$ for the sake of brevity, we have

$$\|Au\|^2 = (A^*Au, u) \leqq \Sigma(A^*Au, \psi_j^*\psi_j u) + C\|A^*Au\|\,\|u\|/R^2 \tag{5.4}$$
$$\leqq \Sigma(A^*A\psi_j u, \psi_j u) + C_1\|u\|^2/R.$$

Choose $\chi_j \in C_0^\infty(\mathbf{R}^{2n})$ so that $\chi_j = 1$ in $\operatorname{supp}\psi_j$, the support has uniformly bounded diameter with respect to g/R^2, and $\{\chi_j\}$ is a vector valued symbol in $S(1, g/R^2)$ with uniform bounds. The estimate (1.3) will follow if we show that there is a constant C such that

$$\|A^*A\chi_j(y, D)\| \leqq M^2 + C/R, \quad j > j_R. \tag{5.5}$$

In fact, then we have

$$\left|\sum\nolimits_{j>j_R}(A^*A\chi_j(y, D)\psi_j(y, D)u, \psi_j(y, D)u)\right| \leqq (M^2 + C/R)\Sigma\|\psi_j(y, D)u\|^2$$
$$\leqq (M^2 + C/R)(1 + C_1/R^2)\|u\|^2.$$

Furthermore, $\{(1-\chi_j(y, D))\psi_j(y, D)\}$ has symbol bounded in $S(1/R^2, g/R^2)$, so

$$\left|\Sigma(A^*A(1-\chi_j(y, D))\psi_j(y, D)u, \ \psi_j(y, D)u)\right| \leqq C_2\|u\|^2/R,$$

and these estimates combined with (5.4) prove (1.3).

The operator $A^*A\chi_j(y, D)$ is the sum of the pseudo-differential operator with symbol

$$b_j(y, \eta) = \tilde{a}(y, \eta)e^{i\psi(y, \eta)}\chi_j(y, \eta) \tag{5.6}$$

and an operator of norm $O(1/R)$. Hence (5.5) follows if we show that there is a constant C' such that

$$\|b_j(y, D)\| \leqq M^2 + C'/R, \quad j > j_R. \tag{5.7}$$

Assume that (5.7) is false for some C' and R. Then we can find an infinite sequence J such that

$$\|b_j(y, D)\| > M^2 + C'/R, \quad j \in J. \tag{5.8}$$

Choose $(y_j, \eta_j) \in \operatorname{supp} b_j$. Since $\tilde{a}$ has compact support in y the sequence y_j

is bounded so passing to a subsequence if necessary we may assume that y_j and $\eta_j/|\eta_j|$ have limits y_0 and η_0 when $J\ni j\to\infty$. The sequence

$$(y,\eta)\to b_j(y_j+y/t_j,\ \eta_j+t_j\eta),\quad t_j=|\eta_j|^{1/2}$$

is bounded in C_0^∞ since $\tilde a e^{i\psi}\in S(1,g)$, so we may also assume that this sequence of functions has a limit $b\in C_0^\infty$. The norm of $b_j(y,D)$ is equal to the norm of $b_j(y_j+y/t_j,\eta_j+t_jD)$ so it follows from (5.8) that

$$\|b(y,D)\|\geqq M^2+C'/R \tag{5.9}$$

for convergence of the symbols in C_0^∞ implies norm convergence. We may also assume that the bounded sequense $\tilde a(y_j,\eta_j)$ has a limit c when $J\ni j\to\infty$. Then $\tilde a(y_j+y/t_j,\eta_j+t_j\eta)\to c$ in C^∞ since $\tilde a\in S^0$. By Taylor's formula

$$\psi(y_j+y/t_j,\eta_j+t_j\eta)=\sum_{|\alpha+\beta|\leqq 2}\psi_{(\beta)}^{(\alpha)}(y_j,\eta_j)t_j^{|\alpha|-|\beta|}y^\beta\eta^\alpha/\alpha!\beta!$$
$$+O((|y|+|\eta|)^3/t_j),\quad |y|+|\eta|<t_j/2.$$

The homogeneity of ψ gives

$$\psi_{(\beta)}^{(\alpha)}(y_j,\eta_j)t_j^{|\alpha|-|\beta|}=\psi_{(\beta)}^{(\alpha)}(y_j,\eta_j/|\eta_j|)t_j^{2-|\alpha|-|\beta|}\to\psi_{(\beta)}^{(\alpha)}(y_0,\eta_0)$$
$$\text{if}\quad |\alpha+\beta|=2,\ J\ni j\to\infty.$$

When $|\alpha+\beta|=1$ the square of this coefficient can be estimated by $C\operatorname{Im}\psi(y_j,\eta_j)$ in view of (1.2). If $\operatorname{Im}\psi(y_j,\eta_j)\to\infty$ it follows that $\operatorname{Im}\psi(y_j+y/t_j,\ \eta_j+t_j\eta)\to\infty$ which contradicts the fact that $b\neq 0$ by (5.9). Hence we may assume that $\operatorname{Im}\psi_j(y_j,\eta_j)$ has a finite limit as $J\ni j\to\infty$ and then we obtain

$$b(y,\eta)=ce^{i(q(y,\eta)-\langle y,\eta\rangle)}\chi(y,\eta)$$

where $\chi(y,\eta)$ is a limit of $\chi_j(y_j+y/t_j,\eta_j+t_j\eta)$ and q is a quadratic polynomial with $\operatorname{Im} q\geqq 0$ and principal part equal to the quadratic form in the Taylor expansion of $\tilde\varphi$ at (y_0,η_0). We have $\tilde a(y_0,t_j^2\eta_0)\to c$ since $y_j\to y_0$ and $|\eta_j-t_j^2\eta_0|=o(t_j^2)$.

From (5.3) it follows that $\|b_0\|\leqq M^2$ if

$$b_0u(y)=c(2\pi)^{-n}\int e^{iq(y,\eta)}\hat u(\eta)\,d\eta,\quad u\in C_0^\infty.$$

Now $0\leqq\chi\leqq 1$ and χ is uniformly bounded in $S(1,(dy^2+d\eta^2)/R^2)$. This implies that $\|\chi(y,D)\|\leqq 1+C_0/R$ by [4, Th. 6.11] for example. Hence

$$\|b_0\chi(y,D)\|\leqq M^2(1+C_0/R).$$

We may regard b_0 as a pseudo-differential operator, and the symbol $ce^{i(q(y,\eta)-\langle y,\eta\rangle)}$

is uniformly bounded in $S(1, dx^2+d\xi^2)$, for it is a limit of $ae^{i\psi}\in S(1, g)$ after a change of scales making g Euclidean. Hence the symbol of $b_0\chi(y, D)-b(y, D)$ is bounded in $S(1/R, dx^2+d\xi^2)$ which proves that

$$\|b(y, D)\| \leqq M^2+C_0'/R.$$

If $C'>C_0'$ this gives a contradiction with (5.9) which proves (5.7) and the theorem.

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Received December 20, 1982

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