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LABELLING GRAPHS WITH THE CIRCULAR DIFFERENCE*

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Abstract. For positive integers k and $d \geq 2$, a k-$S(d,1)$-*labelling* of a graph G is a function on the vertex set of G, $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$, such that

$$|f(u) - f(v)|_k \geq \begin{cases} d & \text{if } d_G(u,v) = 1; \\ 1 & \text{if } d_G(u,v) = 2, \end{cases}$$

where $|x|_k = \min\{|x|, k - |x|\}$ is the *circular difference* modulo k. In general, this kind of labelling is called the $S(d,1)$-*labelling*. The σ_d-number of G, $\sigma_d(G)$, is the minimum k of a k-$S(d,1)$-labelling of G. If the labelling is required to be injective, then we have analogous k-$S'(d,1)$-*labelling*, $S'(d,1)$-*labelling* and $\sigma'_d(G)$.

If the circular difference in the definition above is replaced by the absolute difference, then f is an $L(d,1)$-labelling of G. The *span* of an $L(d,1)$-labelling is the difference of the maximum and the minimum labels used. The λ_d-number of G, $\lambda_d(G)$, is defined as the minimum span among all $L(d,1)$-labellings of G. In this case, we have the corresponding $L'(d,1)$-labelling and $\lambda'_d(G)$ for the labelling with injective condition.

We will first study the relation between λ_d and σ_d as well λ'_d and σ'_d. Then we consider these parameters on cycles and trees. Finally, we study the join of graphs and the multipartite graphs.

1. Introduction

The distance-two labelling of graphs originated from a practical situation in the channel assignment problem introduced by Hale [6]. In 1988, Roberts [12] proposed the following problem. Suppose we need to assign television or

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radio channels to some stations or transmitters (each station gets one channel which is a nonnegative integer). In order to avoid interference, if two stations are too close, then the separation of the channels assigned to them has to be at least two. Moreover, if two stations are close (but not *too* close), then they must receive different channels.

Motivated by this problem, Griggs and Yeh [5] first proposed the $L(2, 1)$-labelling of a simple graph, which gives constraints on vertices within distance two. There are several articles (cf. [2, 4, 8, 9, 11, 13, 15, 16]) studying this labelling and its extensions (cf. [1, 3]) since then. More precisely, the $L(2, 1)$-*labelling* of a graph G is an integer assignment L from the vertex set $V(G)$ to the set of all nonnegative integers such that

$$|L(u) - L(v)| \geq \begin{cases} 2 & \text{if} \quad d_G(u, v) = 1; \\ 1 & \text{if} \quad d_G(u, v) = 2. \end{cases}$$

The k-$L(2, 1)$-labelling is an $L(2, 1)$-labelling such that no label is greater than k. The $L(2, 1)$-*labelling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2, 1)$-labelling.

Next, Georges and Mauro [3] generalized the $L(2, 1)$-labelling to the $L(i, j)$-labelling, $i \geq j$. They study $L(i, j)$-labellings on trees, cycles and product of paths. The definitions are similar to those for $L(2, 1)$-labellings. Namely, for any positive integers $i \geq j$, the $L(i, j)$-*labelling* of a graph G is an integer assignment L from the vertex set $V(G)$ to the set of all nonnegative integers such that

$$|L(u) - L(v)| \geq \begin{cases} i & \text{if} \quad d_G(u, v) = 1; \\ j & \text{if} \quad d_G(u, v) = 2. \end{cases}$$

In 1997, Chang *et al.* [1] study the $L(d, 1)$-labelling on serval classes of chordal graphs. They also derive an upper bound on general graphs in terms of maximum degree. The authors also introduced a related problem, which we call the $L'(d, 1)$-*labelling* problem. The definitions of the $L'(d, 1)$-*labelling* L, the $k-L'(d, 1)$-*labelling* L, and the $L'(d, 1)$-*labelling number* $\lambda'_d(G)$ are the same as those of the $L(d, 1)$-*labelling* L, the $k - L(d, 1)$-*labelling* L, and the $L(d, 1)$)-*labelling number* $\lambda_d(G)$, respectively, except that the integer assignment L is required to be one-to-one.

2. Circular Labellings

Definition 2.1. *For any positive integer k and $x, y \in \{0, 1, ..., k-1\}$, we define the circular difference of x and y modulo k by $|x-y|_k = \min\{|x-y|, k-|x-y|\}$.*

A convenient way to interpret Definition 2.1 is to think of the labels $0, 1, ..., k-1$ as consecutive vertices on a circuit. Then for any distinct pair $x, y \in \{0, 1, ..., k-1\}$, the distance between x and y, $|x-y|_k$, is the length of the shorter of the two paths connecting x and y on the circuit.

Using the circular difference, Vince [14] defined the *circular chromatic number* (or the *star chromatic number*) of a graph G.

Given positive integers d, k, a (d,k)-*coloring* c of G is a proper vertex-coloring of G such that $|c(u)-c(v)|_k \geq d$, where u and v are adjacent in G. The circular chromatic number of G, $\chi_c(G)$, is $\inf\{k/d$: G has a (d,k)-coloring$\}$.

Using the circular difference, we can then define the following labelling. We call it the *circular labelling.*

Definition 2.2. *Given a graph G and integers $d_1 \geq d_2 \geq \ldots \geq d_p \geq 1$, we say that f is a k-$S(d_1, d_2, \ldots, d_p)$-labelling of G if $|f(u)-f(v)|_k \geq d_i$, where $f : V(G) \to \{0, 1, \ldots, k-1\}$, whenever $d_G(u,v) = i$, for $i = 1, 2, \cdots, p$.*

The requirement $d_1 \geq d_2 \geq \cdots \geq d_p$ in Definition 2.2 arises from the assumption of a monotonic trade-off between distance and spectral separation.

Definition 2.3. *Given a graph G and integers $d_1 \geq d_2 \geq \cdots \geq d_p \geq 1$, the $S(d_1, d_2, ..., d_p)$-labelling number of G, denoted by $\sigma(G; d_1, d_2, ..., d_p)$, is the smallest k such that there exists a k-$S(d_1, d_2, ..., d_p)$-labelling of G. If no such k exists, we set $\sigma(G; d_1, d_2, ..., d_p) = \infty$. If $k = \sigma(G; d_1, d_2, ..., d_p)$, then any k-$S(d_1, d_2, ..., d_p)$-labelling of G is called optimal.*

The circular labelling defined above was first introduced by ven den Heuvel *et al.* [7]. They consider $S(d_1, d_2)$-labellings and $S(d_1, d_2, d_3)$-labellings on two infinite graphs. Liu [10] studies the $S(2,1)$-labelling by relating it to the Hamiltonicity and the path covering on general graphs.

This article will study the $S(d,1)$-labelling. We like to notice that results on trees and cycles are also obtained by Liu independently. The next section will consider $S(d,1)$-labellings on trees and cycles. The corresponding $\sigma(G; d, 1)$ is simplified as $\sigma_d(G)$. In the last section, we shall also introduce a related problem, which we call the $S'(d,1)$-*labelling* problem (or the *injective* $S(d,1)$-*labelling* problem). The definitions of the $S'(d,1)$-*labelling* f, the k-$S'(d,1)$-*labelling* f, and the $S'(d,1)$-*labelling number* $\sigma'_d(G)$ are the same as those of the $S(d,1)$-*labelling* f, the k-$S(d,1)$-*labelling* f, and the $S(d,1)$-labelling number $\sigma_d(G)$, respectively, except that the function f is required to be one-to-one.

It is clear to obtain the following results.

Proposition 2.4. *$\sigma_d(G) \le \sigma_d(H)$ and $\sigma'_d(G) \le \sigma'_d(H)$ for any subgraph G of a graph H.*

Proposition 2.5. *$\sigma_d(G) \le \sigma'_d(G)$ for any graph G. $\sigma_d(G) = \sigma'_d(G)$ if G is of diameter at most two.*

We then discuss relations between $\sigma_d(G)$ and $\lambda_d(G)$ for a given graph G. It is easy to observe that the following inequalities hold for any graph G.

Proposition 2.6 [7]. *$\lambda_d(G) + 1 \le \sigma_d(G) \le \lambda_d(G) + d$ for any graph G.*

3. Trees and Cycles

It was proved by Liu [10] that $\sigma_2(T) = \Delta + 3$, where T is any tree with maximum degree Δ. We shall generalize the result to the $S(d, 1)$-labelling. These results are also obtained by Liu independently.

Theorem 3.1. *Let T be any tree with maximum degree $\Delta \ge 3$. Then $\sigma_d(T) = 2d + \Delta - 1$.*

Proof. Let T be any tree with maximum degree $\Delta \ge 3$. It is easy to see that $\sigma_d(K_{1,\Delta}) = 2d + \Delta - 1$. (More precisely, see Proposition 4.4.) Since the maximum degree of T is Δ, T contains a $K_{1,\Delta}$ as a subgraph. Thus, $\sigma_d(T) \ge \sigma_d(K_{1,\Delta}) = 2d + \Delta - 1$ by Proposition 2.4.

On the other hand, we shall obtain the upper bound by a first-fit (greedy) labelling. Given a tree T, fix a vertex v_0 of maximum degree Δ as a root. First, label the vertex v_0 by 0 and its neighbors by $\{d, d+1, ..., d+\Delta-1\}$. Suppose v is a vertex that has been labelled with x. Let u be its ascendant with label y. Then label the unlabelled neighbors with numbers from $S = \{x+d, x+d+1, \cdots, x+d+\Delta-1 \pmod{2d+\Delta-1}\}$. Continue this process until all vertices have been labelled.

Claim 1. $|S| = \Delta$.

We see that $|S| \le \Delta$, by definition. Also it is easy to see that no two distinct elements in S are congruent modulo $2d + \Delta - 1$. Thus the equality holds.

Claim 2. There is a j, $1 \le j \le \Delta$, such that $y \equiv x+d+j-1 \pmod{2d+\Delta-1}$.

By the labelling process above, $x \equiv y + d + m - 1 \pmod{2d + \Delta - 1}$ for some $1 \le m \le \Delta$, since v is a decendant of u.

For any j, $1 \le j \le \Delta$, $x+d+j-1 \equiv y+d+m-1+d+j-1 \equiv y+2d+m+j-2 \pmod{2d + \Delta - 1}$.

Take $j = \Delta - m + 1$. Then $y + 2d + m + j - 1 \equiv y + 2d + \Delta - 1 \equiv y$ (mod $2d + \Delta - 1$. Hence $y \equiv x + d + j - 1$ (mod $2d + \Delta - 1$), where $j = \Delta - m + 1$ and $1 \leq j \leq \Delta$.

By both claims, the process is well-defined. According to this, we have a $(2d + \Delta - 1)$-$S(d, 1)$-labelling on T. Hence $\sigma_d(T) \leq 2d + \Delta - 1$. The theorem is then proved. ∎

We want to mention that the proof above can be extended to the general $S(d_1, d_2)$-labelling for finding $\sigma(T; d_1, d_2)$.

Next, we shall consider $S(d, 1)$-labellings on cycles. For $d = 2$, we have the following result. An *n-cycle* means a cycle of length n.

Theorem 3.2 [10]. *Let C_n be an n-cycle. Then we have*

$$\sigma_2(C_n) = \begin{cases} 5 & if \quad n \equiv 0 \text{ (mod 5)}; \\ 6 & if \quad n \not\equiv 0 \text{ (mod 5)}. \end{cases}$$

Now, we shall present the results for the general case. It was proved [3, 9] that $\lambda_d(C_n) \leq 2d$ for all n. Hence by Proposition 2.6, $\sigma_d(C_n) \leq \lambda_d(C_n) + d \leq 3d$ for all n. Also it is easy to see that $2d < \sigma_d(C_n)$ for all n. However we want to find the exact value of $\sigma_d(C_n)$ for different values of n. But first, we need to quote the following theorem.

Theorem 3.3 [14]. *Let C_n be an odd cycle. Then* $\chi_c(C_n) = 2 + 2/(n-1)$.

By definitions of the circular chromatic number and the σ_d number, we see that $d \cdot \chi_c(G) \leq \sigma_d(G)$ for any d and any graph G. Thus $d \cdot \chi_c(C_n) = 2d + 2d/(n-1) \leq \sigma_d(C_n)$. Since $\sigma_d(C_n)$ is an integer, we have $2d + \lceil 2d/(n-1) \rceil \leq \sigma_d(C_n)$ for n odd.

Theorem 3.4. *Let C_n be an n-cycle.*

1. *If* $n \equiv 0$ (mod $2d+1$), *then we have* $\sigma_d(C_n) = 2d + 1$.

2. *If* $n \not\equiv 0$ (mod $2d+1$), *then we have*

   (a) $\sigma_d(C_n) = 2d+2$, *for* $n \equiv 0$ (mod 2) *or* $n \equiv 1$ (mod 2) *but* $n > 2d+1$;

   (b) $\sigma_d(C_n) = 2d + \lceil \frac{2d}{n-1} \rceil$, *for* $n \equiv 1$ (mod 2) *and* $n < 2d + 1$.

Proof. Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1\}$.

Part 1. For $n \equiv 0 \pmod{2d+1}$, define f_1 when $1 \le i \le 2d+1$ by

$$f_1(v_i) = \begin{cases} \lfloor \frac{i-1}{2} \rfloor & \text{if} \quad i \equiv 1 \pmod 2 \text{ and } i \neq 2d+1; \\ d + \frac{i}{2} & \text{if} \quad i \equiv 0 \pmod 2 \text{ and } i \neq 2d; \\ 2d & \text{if} \quad i = 2d; \\ d & \text{if} \quad i = 2d+1. \end{cases}$$

For $j \ge 2d+2$, define $f_1(v_j) = f_1(v_i)$ if $i \equiv j \pmod{2d+1}$. That is, we continuously use $f_1(v_1), f_1(v_2), ..., f_1(v_{2d+1})$ to label all vertices.We see that f_1 is a $S(d,1)$-labelling of C_n. So $\sigma_d(C_n) \le 2d+1$.

On the other hand, we shall prove that $\sigma_d(C_n) \ge 2d+1$. It is easy to see that $\sigma_d(K_{1,2}) = 2d+1$. So $\sigma_d(C_n) \ge \sigma_d(K_{1,2}) = 2d+1$, since C_n contains $K_{1,2}$ as a subgraph. Thus, $\sigma_d(C_n) \ge 2d+1$. Hence $\sigma_d(C_n) = 2d+1$ if $n \equiv 0 \pmod{2d+1}$.

Part 2. Suppose $\sigma_d(C_n) = 2d+1$. Let f be an optimal labelling of C_n. Then we see that $f(v_{i+1}) = f(v_i) + d$ or $f(v_{i+1}) = f(v_i) + d + 1 \pmod{2d+1}$ for all i. (The indices are modulo n.) That is, we use $0, d, 2d, \cdots$ or $0, d+1, 2(d+1), \cdots \pmod{2d+1}$ consecutively to label C_n. Since d and $2d+1$ are relatively prime, the order of d in the cyclic group $(Z_{2d+1}; +)$ is $2d+1$. Hence $2d+1|n$ or $n \equiv 0 \pmod{2d+1}$. Notice that since $d+1 \equiv -d \pmod{2d+1}$, the other case is the same as this one. (Notice that $Z_{2d+1} = \{0, 1, \cdots, 2d\}$.)

Therefore, $n \equiv 0 \pmod{2d+1}$ if and only if $\sigma_d(C_n) = 2d+1$. Equivalently, if $n \not\equiv 0 \pmod{2d+1}$, then $\sigma_d(C_n) \neq 2d+1$.

Hence for $n \not\equiv 0 \pmod{2d+1}$, we have $2d+2 \le \sigma_d(C_n) \le 3d$ by the remark before the statement of this theorem.

(a) For $n \equiv 0 \pmod 2$ or $n \equiv 1 \pmod 2$ but $n > 2d+1$, we have $\sigma_d(C_n) \ge 2d+2$. On the other hand, we need to show that $\sigma_d(C_n) \le 2d+2$.

If $n \equiv 0 \pmod 4$, then f_2 defined by

$$f_2(v_i) = \begin{cases} 0 & \text{if} \quad i \equiv 1 \pmod 4; \\ d+1 & \text{if} \quad i \equiv 2 \pmod 4; \\ 1 & \text{if} \quad i \equiv 3 \pmod 4; \\ d+2 & \text{if} \quad i \equiv 0 \pmod 4) \end{cases}$$

is an $S(d,1)$-labelling of C_n. Thus, we have $\sigma_d(C_n) \le 2d+2$.

If $n \equiv 2 \pmod 4$, then f_3 defined by

$$f_3(v_i) = \begin{cases} d & \text{if} \quad i \equiv 1 \pmod 4 \text{ and } i \neq n-1; \\ 0 & \text{if} \quad i \equiv 2 \pmod 4 \text{ and } i \neq n; \\ d+1 & \text{if} \quad i \equiv 3 \pmod 4; \\ 2d+1 & \text{if} \quad i \equiv 0 \pmod 4; \\ d-1 & \text{if} \quad i = n-1; \\ 2d & \text{if} \quad i = n \end{cases}$$

is an $S(d,1)$-labelling of C_n. Thus, we have $\sigma_d(C_n) \le 2d+2$.

If $n \in \{x | x > 2d+1 \text{ and } x \equiv 2d+3 \pmod 4\}$, then f_4 defined by

$$f_4(v_i) = \begin{cases} \lfloor \frac{i-1}{2} \rfloor & \text{if} \quad i \equiv 1 \pmod 2 \text{ and } 3 \le i \le 2d+1; \\ d + \frac{i}{2} & \text{if} \quad i \equiv 0 \pmod 2 \text{ and } 4 \le i \le 2d; \\ 0 & \text{if} \quad i = 2d+2; \\ d+2 & \text{if} \quad i = 2d+3; \\ 2 & \text{if} \quad i \equiv 2d+4 \pmod 4 \text{ or } i = 1; \\ d+3 & \text{if} \quad i \equiv 2d+5 \pmod 4 \text{ or } i = 2; \\ 1 & \text{if} \quad i \equiv 2d+6 \pmod 4; \\ d+2 & \text{if} \quad i \equiv 2d+7 \pmod 4 \end{cases}$$

is an $S(d,1)$-labelling of C_n. Thus, we have $\sigma_d(C_n) \le 2d+2$.

If $n \in \{x | x > 2d+1 \text{ and } x \equiv 2d+5 \pmod 4\}$, then f_5 defined by

$$f_5(v_i) = \begin{cases} \lfloor \frac{i-1}{2} \rfloor & \text{if} \quad i \equiv 1 \pmod 2 \text{ and } \ i \le 2d+1; \\ d + \frac{i}{2} & \text{if} \quad i \equiv 0 \pmod 2 \text{ and } \ i \le 2d; \\ 0 & \text{if} \quad i \equiv 2d+2 \pmod 4; \\ d+1 & \text{if} \quad i \equiv 2d+3 \pmod 4; \\ 1 & \text{if} \quad i \equiv 2d+4 \pmod 4; \\ d+2 & \text{if} \quad i \equiv 2d+5 \pmod 4 \end{cases}$$

is an $S(d,1)$-labelling of C_n. Thus, we have $\sigma_d(C_n) \le 2d+2$.

Hence $\sigma_d(C_n) = 2d+2$ if $n \equiv 0 \pmod 2$ or $n \equiv 1 \pmod 2$ but $n > 2d+1$.

(b) For $n \equiv 1 \pmod 2$ and $n < 2d+1$, by the observation following Theorem 3.3, we have $\sigma_d(C_n) \ge 2d + \lceil 2d/(n-1) \rceil$. Thus it suffices to show that $\sigma_d(C_n) \le 2d + \lceil 2d/(n-1) \rceil$.

Now, define f_6 by

$$f_6(v_i) = \begin{cases} d + \lceil \frac{i+1}{n-1} d \rceil & \text{if } i \equiv 1 \pmod 2; \\ \lceil \frac{i}{n-1} d \rceil & \text{if } i \equiv 0 \pmod 2, \end{cases}$$

where $1 \le i \le n$. We see that f_6 is an $S(d,1)$-labelling of C_n. So $\sigma_d(C_n) \le 2d + \lceil 2d/(n-1) \rceil$. Hence $\sigma_d(C_n) = 2d + \lceil 2d/(n-1) \rceil$ if $n \equiv 1 \pmod 2$ and $n < 2d+1$. ∎

4. $S'(d,1)$-labellings on Joins of Graphs and Complete Multipartite Graphs

In this section, we shall focus on $S'(d,1)$-labellings on joins of graphs and complete multipartite graphs. Recall that the *join* of two graphs G and H,

$G \vee H$, is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{\{u, v\}\colon u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 4.1 [9]. *$\lambda_d(G \vee H) = \lambda_d{}'(G \vee H) = \lambda_d{}'(G) + \lambda_d{}'(H) + d$ for all graphs G and H.*

Theorem 4.2. *Let G and H be any graphs. Then $\sigma_d(G \vee H) = \sigma'_d(G \vee H) = \lambda_d{}'(G) + \lambda_d{}'(H) + 2d$.*

Proof. Since $G \vee H$ is of diameter at most two, $\sigma(G \vee H; d, 1) = \sigma'(G \vee H; d, 1)$ by Proposition 2.5.

It is clear that $\sigma'_d(G \vee H) \leq \lambda_d{}'(G \vee H) + d = \lambda_d{}'(G) + \lambda_d{}'(H) + 2d$, by Theorem 4.1. Next, we shall show that $\sigma'_d(G \vee H) \geq \lambda_d{}'(G) + \lambda_d{}'(H) + 2d$. Suppose $k = \sigma'_d(G \vee H)$ and f is a k-$S'(d, 1)$-labelling of $G \vee H$. Assume that $V(G) = \{v_1, v_2, ..., v_m\}$ and $V(H) = \{u_1, u_2, ..., u_n\}$. Reorder the vertices of G and H so that $f(v_1) < f(v_2) < ... < f(v_m)$ and $f(u_1) < f(u_2) < ... < f(u_n)$. If $f(v_i) < f(u_1) < f(u_2) < ... < f(u_j) < f(v_{i+1})$ for some i and j with $1 \leq i \leq m - 1$ and $1 \leq j \leq n$, then define f' by

$$f'(v_r) = \begin{cases} f(v_r) & \text{if } \ r \neq i + 1; \\ f(v_i) + d & \text{if } \ r = i + 1 \end{cases}$$

and

$$f'(u_s) = \begin{cases} f(u_s) + d & \text{if } \ s \leq j; \\ f(u_s) & \text{if } \ s > j. \end{cases}$$

It is straightforward to check that f' is a k'-$S'(d, 1)$-labelling of $G \vee H$ with $k' = k$. Continue this process until we have the k^*-$S'(d, 1)$-labelling f^* of $G \vee H$ with $k^* = k$ satisfying

$$f^*(v_1) < f^*(v_2) < ... < f^*(v_m) < f^*(v_m) + d \leq f^*(u_1) < f^*(u_2) < ... < f^*(u_n).$$

When restricting f^* on G, f^* is the $S'(d, 1)$-labelling of G. If we define $\bar{f}(u_i) = f^*(u_i) - f^*(u_1)$ for all $1 \leq i \leq n$, then $\bar{f}$ is the $S'(d, 1)$-labelling of H. Thus, we have $\sigma'_d(G \vee H) \geq \lambda_d{}'(G) + \lambda_d{}'(H) + 2d$. Hence $\sigma_d(G \vee H) = \sigma'_d(G \vee H) = \lambda_d{}'(G) + \lambda_d{}'(H) + 2d$. ■

Next, we shall generalize the above result to the result of k graphs $G_1, G_2, ..., G_k$. Then it is easy to verify that the following result holds by induction.

Corollary 4.3. *For $k \geq 2$, $G_1, G_2, ..., G_k$ are k graphs. If $G = G_1 \vee G_2 \vee ... \vee G_k$, then $\sigma_d(G) = \sigma'_d(G) = \sum_{i=1}^{k} [\lambda_d{}'(G_i) + d]$.*

At last, we have the following result on complete multipartite graphs.

Proposition 4.4. *For $k \geq 2$, if the graph G is $K_{n_1,n_2,\ldots,n_k}$, then $\sigma_d(G) = \sigma'_d(G) = |V(G)| + (d-1)k$.*

Proof. Since $G = K_{n_1,n_2,\ldots,n_k} = K_{n_1}{}^c \vee K_{n_2}{}^c \vee \ldots \vee K_{n_k}{}^c$ and $|V(G)| = \sum_{i=1}^{k} n_i$, we have $\sigma_d(G) = \sigma'_d(G) = \sum_{i=1}^{k}[\lambda'_d(K_{n_i}{}^c)+d] = \sum_{i=1}^{k}[(n_i-1)+d] = \sum_{i=1}^{k} n_i - k + dk = |V(G)| + (d-1)k$, by Corollary 4.3, where we know that $\lambda'_d(K^c_{n_i}) = n_i - 1$ for $i = 1, 2, \ldots, k$. ■

From Proposition 4.4, we can derive an upper bound on $\sigma_d(G)$ in terms of $|V(G)|$ and the chromatic number $\chi(G)$. That is, $\sigma_d(G) \leq |V(G)| + (d-1) \cdot \chi(G)$.

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