Laboratory wave generation

A second-order theory for regular and irregular waves in wave channels

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Executive's summary

The correct generation of a second order wave field in a laboratory wave tank is of importance in several experimental investigations, particularly those of nonlinear evolutions and sediment transport. In the present work, which was carried out under the MLTP (medium long-term planning) of DELFT HYDRAULICS for improving experimental techniques, expressions are found for the motion of a waveboard to generate a correct second order wave field. These expressions are valid for both regular and irregular waves. Two important features of the procedure adopted here are that the computing time for the motion of the waveboard is signifcantly smaller compared to a method based on the frequency domain and that the accuracy of the physical representation increases with decreasing spectral width. The assumption of a narrow band spectrum is sufficient for realistic sea states described by spectral shapes of JONSWAP and Pierson-Moskowitz types.

Results of some experimental investigations into the performance of the software based on the wave generation theory are also included in the report. Although the overall agreement between the theory and the experiment is good, some discrepancies are apparent from the limited analysis carried out so far, all of which cannot be attributed to the wave generation theory. Further analysis (and possibly a set of new experiments) is required in order to resolve all the discrepancies.

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1 Introduction

A wave generation theory correct up to second order is presented here for both regular and irregular waves. Wave-board motion based only on the first order theory creates a second order spectrum different from the one that exists in nature. The second order spectrum referred to here consists of both the superharmonic and subharmonic parts. In addition to the bound second order components (bound to the first order components through the inherent free surface nonlinearity) spurious waves, termed also as the free waves, are generated in the tank (see, for example, Buhr-Hansen & Svendsen, 1974; Flick & Guza, 1980) unless the motion of the waveboard calculated from the first order theory is corrected in order to minimize the generation of the spurious waves.

Second order corrections to the motion of a waveboard for reducing spurious waves have been proposed by Barthel *et. al* (1983) at the subharmonic range and by Sand & Mansard (1986) at the range of superharmonics. The analysis procedure used by them is based on the frequency domain. In the approach based on the frequency domain, the second order displacement χ^2 for irregular waves is expressed as a sum of the terms arising out of each combination of two first order components, *i.e.*,

$$\chi^2 = \sum_{p=1}^{N-1} \sum_{q=p+1}^{N} \chi^2_{0(p,q)} + \sum_{p=1}^{N} \sum_{q=p}^{N} \chi^2_{2(p,q)} \tag{1.1}$$

where $\chi^2_{0(p,q)}$ and $\chi^2_{2(p,q)}$ respectively represent the subharmonic and superharmonic part associated with components p and q, N being the total number of components. The required computing time for the generation of irregular waves (correct up to second order) is proportional to the square of the number of components compared to the time necessary for the first order signal. The computing time (specially on a PC) for the generation of second-order waves can sometimes be a critical factor since the determination of the coefficients associated with each $\chi^2_{(p,q)}$ is time consuming.

The theory presented here is based on a different mathematical approach. Instead of adopting a frequency domain analysis, we consider the time signal to be periodic with a slowly varying amplitude. We make use of the concepts of multiple-scale variations in space x and time t which have been earlier illustrated by Agnon & Mei (1985) in the study of the slow drift of an object subject to waves. Klopman and Van Leeuwen (1990) have shown the relevance of this approach for realistic sea-spectrum of the JONSWAP and Pierson-Moskowitz types in addition to presenting the subharmonic correction to the waveboard based on the multiple-scale perturbation approach. In this report, a complete second order solution (subharmonic, superharmonic and second order modulation of the first order field) is presented based on the same approach. The computing time necessary for the second order control signal is longer than that necessary for the first order only by a fraction.

2 Basic formulations

We consider a wavemaker of the piston type in translatory motion near $x = 0$. The waves are assumed to propagate from left to right over water of constant depth h. The wave generation problem in the velocity potential ϕ, surface elevation ζ and the wavemaker displacement $\mathcal{X}$ is given by the set:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (-h < z < \zeta) \qquad (2.1)$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad (z = -h) \qquad (2.2)$$

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial x}\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial z} \qquad (z = \zeta) \qquad (2.3)$$

$$g\zeta + \frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = 0 \qquad (z = \zeta) \qquad (2.4)$$

$$\frac{d\mathcal{X}}{dt} = \frac{\partial\phi}{\partial x} \qquad (x = \mathcal{X}(t)) \qquad (2.5)$$

requiring further that the waves are outgoing at infinity. The conditions, given by (2.3)-(2.5), are satisfied on the instanteneous position of the boundary (the free surface or the wave board). To express these conditions about the still water level and the zero position of the waveboard, we assume an expansion in the form of a perturbation series

$$(\phi,\ \zeta,\ \mathcal{X}) = \varepsilon(\phi_1,\ \zeta_1,\ \mathcal{X}_1) + \varepsilon^2(\phi_2,\ \zeta_2,\ \mathcal{X}_2) + \cdots \qquad (2.6)$$

with the parameter $\varepsilon = k_0 a$, k_0 and a being the typical wave number and amplitude respectively. Taylor expansions about $z = 0$ of (2.3) and (2.4) gives:

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial t}\left(-\frac{1}{2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right) + \frac{1}{g}\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial z\partial t}\right)$$
$$-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}\right) + O(\varepsilon^3) \qquad (z = 0) \qquad (2.7)$$

$$\zeta = -\frac{1}{g}\left(\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) - \frac{1}{g}(\phi_t\phi_{zt})\right) + O(\varepsilon^3) \qquad (z = 0) \qquad (2.8)$$

Taylor expansion about $x = 0$ of (2.5) gives

$$\frac{d\mathcal{X}}{dt} = \frac{\partial\phi}{\partial x} + \mathcal{X}\frac{\partial^2\phi}{\partial x^2} + O(\varepsilon^3) \qquad (x = 0) \qquad (2.9)$$

We assume the motions to be nearly periodic of angular frequency ω with slowly modulated amplitudes. The time and the length scales of the amplitude envelope are assumed to be $O(\varepsilon^{-1})$ times that of $(2\pi)/\omega$ and $(2\pi)/k_0$, with $\omega^2 = gk_0 \tanh(k_0 h)$. Following a procedure similar to that in Agnon & Mei (1985) (here after referred to as 'A & M'), we define the variables

$$x_0 = x,\; x_1 = \varepsilon x,\; t_0 = t,\; t_1 = \varepsilon t, \cdots \tag{2.10}$$

and express explicitly that

$$\begin{aligned} \phi^{(n)} &= \phi^{(n)}(x_0, z, t_0, x_1, t_1), \\ \zeta^{(n)} &= \zeta^{(n)}(x_0, z, t_0, x_1, t_1), \\ \mathcal{X}^{(n)} &= \mathcal{X}^{(n)}(t_0, t_1) \end{aligned} \tag{2.11}$$

From (2.1) we now find:

$$\phi^{(1)}_{x_0x_0} + \phi^{(1)}_{zz} = 0 \qquad (-h < z < 0) \tag{2.12}$$

$$\phi^{(2)}_{x_0x_0} + \phi^{(2)}_{zz} = -2\phi^{(1)}_{x_0x_1} \qquad (-h < z < 0) \tag{2.13}$$

At the bottom we find from (2.2) for all $n \in I\!N^+$:

$$\phi^{(n)}_z = 0 \qquad (z = -h) \tag{2.14}$$

At the free surface we get from (2.7) :

$$\phi^{(1)}_{t_0t_0} + g\phi^{(1)}_z = 0 \qquad (z = 0) \tag{2.15}$$

$$\begin{aligned} \phi^{(2)}_{t_0t_0} + g\phi^{(2)}_z = &-2\phi^{(1)}_{t_0t_1} - \left(\tfrac{1}{2}\left(\phi^{(1)\,2}_{x_0} + \phi^{(1)\,2}_z\right) + \frac{1}{g}\left(\phi^{(1)}_{t_0}\phi^{(1)}_z\right)_{t_0}\right)_{t_0} + \\ &- \left(\phi^{(1)}_{x_0}\phi^{(1)}_{t_0}\right)_{x_0} \qquad (z = 0) \end{aligned} \tag{2.16}$$

At the wave board we find from (2.9)

$$\mathcal{X}^{(1)}_{t_0} = \phi^{(1)}_{x_0} \qquad (x = 0) \tag{2.17}$$

$$\mathcal{X}^{(1)}_{t_1} + \mathcal{X}^{(2)}_{t_0} = \phi^{(1)}_{x_1} + \phi^{(2)}_{x_0} + \mathcal{X}^{(1)}\phi^{(1)}_{x_0x_0} \qquad (x = 0) \tag{2.18}$$

We seek solutions of the following form:

$$(\phi^{(n)}, \zeta^{(n)}, \mathcal{X}^{(n)}) = \sum_{m=-n}^{n} (\phi^{(n,m)}, \zeta^{(n,m)}, \mathcal{X}^{(n,m)}) \exp(-im\omega t_0) \tag{2.19}$$

where the short scale temporal variation is expressed by $\exp(-i\omega t_0)$. The long scale temporal variation with respect to t_1 and the spatial variation, both short and long scales, are contained in the terms $\phi^{(n,m)}$'s. In expressing the series from $m = -n$ to $m = n$ in (2.19), it is assumed that

$$\psi^{(n,m)} = \text{conj}\left(\psi^{(n,-m)}\right) \tag{2.20}$$

such that the resulting physical variable $\psi^{(n)}$ *etc.* is real, where ψ represents ϕ, ζ or $\mathcal{X}$.

Using (2.19) in (2.8) and grouping together the terms of the same order ε and same harmonic m, the following relations are established for $\zeta^{(n,m)}$:

$$\zeta^{(1,0)} = 0 \qquad (z = 0) \tag{2.21}$$

$$\zeta^{(1,1)} = \frac{i\omega}{g}\phi^{(1,1)} \qquad (z = 0) \tag{2.22}$$

$$\zeta^{(2,0)} = -\frac{1}{g}\left[\phi_{t_1}^{(1,0)} + |\phi_{x_0}^{(1,1)}|^2 + |\phi_z^{(1,1)}|^2 - \sigma\left(|\phi^{(1,1)}|^2\right)_z\right] \qquad (z = 0) \tag{2.23}$$

$$\zeta^{(2,1)} = -\frac{1}{g}\left[-i\omega\phi^{(2,1)} + \phi_{t_1}^{(1,1)}\right] \qquad (z = 0) \tag{2.24}$$

$$\zeta^{(2,2)} = -\frac{1}{g}\left[-2i\omega\phi^{(2,2)} + \tfrac{1}{2}\left(\phi_{x_0}^{(1,1)2} + \phi_z^{(1,1)2}\right) + \sigma\phi^{(1,1)}\phi_z^{(1,1)}\right] \qquad (z = 0) \tag{2.25}$$

with

$$\sigma = \frac{\omega^2}{g}.$$

3 Order (1,1) solution

For the first order and first harmonic we find:

$$\phi^{(1,1)}_{x_0x_0} + \phi^{(1,1)}_{zz} = 0 \qquad (-h < z < 0) \tag{3.1}$$

$$-\omega^2\phi^{(1,1)} + g\phi^{(1,1)}_{z} = 0 \qquad (z = 0) \tag{3.2}$$

$$\phi^{(1,1)}_{z} = 0 \qquad (z = -h) \tag{3.3}$$

$$-i\omega\mathcal{X}^{(1,1)} = \phi^{(1,1)}_{x_0} \qquad (x = 0, -h < z < 0) \tag{3.4}$$

The disturbances generated by the wave maker must be outgoing at infinity. The solution that satisfies equations (3.1)-(3.3) and the radiation condition can be expressed as follows:

$$\phi^{(1,1)} = a_0 f_0(z)\exp(ik_0x_0) + \sum_{n=1}^{\infty} b_n f_n(z)\exp(-k_nx_0) \qquad (x > 0) \tag{3.5}$$

where

$$f_0(z) = \frac{\sqrt{2}\cosh Q_0}{\sqrt{h + (g/\omega^2)\sinh^2 q_0}}$$

$$f_n(z) = \frac{\sqrt{2}\cos Q_n}{\sqrt{h - (g/\omega^2)\sin^2 q_n)}} \tag{3.6}$$

with the definitions

$$\begin{aligned} q_m &= k_m h \\ Q_m &= k_m (z + h) \end{aligned}$$

and the relations

$$\omega^2 = gk_0 \tanh q_0$$

and for $n > 0$ with $k_n > 0$:

$$\omega^2 = -gk_n \tan q_n$$

The real valued functions $(f_0, f_1, f_2, ...)$ constitute an orthonormal set with regard to the inner product:

$$(f \cdot g) = \int_{-h}^{0} f(z)g(z)dz \tag{3.7}$$

Furthermore we have $a_0 = a_0(x_1, t_1)$ and $b_n = b_n(x_1, t_1)$. From eq.(3.4) we can see that

$$\mathcal{X}^{(1,1)}(t_1) = -\frac{k_0}{\omega}a_0(0,t_1)f_0(z) - \frac{i}{\omega}\sum_{n=1}^{\infty} k_n b_n(0,t_1)f_n(z) \tag{3.8}$$

Multiplication with $f_m(z)$ and integrating from $-h$ to 0 yields for m=0:

$$\mathcal{X}^{(1,1)}(t_1) = -\frac{k_0}{F_0\omega}a_0(0,t_1) \tag{3.9}$$

and for $m > 0$

$$\mathcal{X}^{(1,1)}(t_1) = -i\frac{k_m}{F_m\omega}b_m(0,t_1) \tag{3.10}$$

where $F_0 = (1 \cdot f_0)$ and $F_m = (1 \cdot f_m)$. This leads to the conclusion that

$$b_n(0,t_1) = -i\frac{F_n k_0}{F_0 k_n}a_0(0,t_1) \quad \text{for} \quad n \in I\!N^+ \tag{3.11}$$

We note that a_0 is related to the 1st order complex surface amplitude A of the propagating mode through the relation

$$a_0 = -\frac{ig}{2\omega f_0(0)}A. \tag{3.12}$$

Similarly, each b_n is related to the surface amplitude of the evanescent mode n through

$$b_n = -\frac{ig}{2\omega f_n(0)}B_n. \tag{3.13}$$

In terms of A and B_n's, (3.5) expressing the potential $\phi^{(1,1)}(x_0, z, x_1, t_1)$ becomes

$$\begin{aligned} \phi^{(1,1)}(x_0,z,x_1,t_1) \quad = \quad & -\frac{ig\cosh Q_0}{2\omega\cosh q_0}A(x_1,t_1)\exp(ik_0x_0) \\ & -\frac{ig}{2\omega}\sum_{n=1}^{\infty}\frac{\cos Q_n}{\cos q_n}B_n(x_1,t_1)\exp(-k_nx_0) \end{aligned} \tag{3.14}$$

From (3.14) and (2.22), one has

$$\zeta^{(1,1)} = \frac{1}{2}A(x_1,t_1)\exp(ik_0x_0) + \frac{1}{2}\sum_{n=1}^{\infty}B_n(x_1,t_1)\exp(-k_nx_0) \tag{3.15}$$

The slow variations of the variables A and B_n's with respect to (x_1, t_1) in (3.14) are still implicit. We postpone this discussion to a later section.

Equations (3.9) and (3.10) can be modified to express $\mathcal{X}^{(1,1)}$ in terms of A and B_n's:

$$\begin{aligned} \mathcal{X}^{(1,1)} &= iI_0 \frac{gk_0}{2\omega^2} A(0, t_1), & (3.16) \\ &= -I_n \frac{gk_n}{2\omega^2} B_n(0, t_1) & (3.17) \end{aligned}$$

where

$$I_0 = \frac{\int_{-h}^{0} \left(\frac{f_0(z)}{f_0(0)}\right)^2 dz}{\int_{-h}^{0} \frac{f_0(z)}{f_0(0)} dz} = \frac{1}{2}\left[1 + \frac{2q_0}{\sinh 2q_0}\right], \qquad (3.18)$$

$$I_n = \frac{\int_{-h}^{0} \left(\frac{f_n(z)}{f_n(0)}\right)^2 dz}{\int_{-h}^{0} \frac{f_n(z)}{f_n(0)} dz} = \frac{1}{2}\left[1 + \frac{2q_n}{\sin 2q_n}\right] \qquad (3.19)$$

4 Order (2,1) problem

The equations are

$$\zeta^{(2,1)} = -\frac{1}{g}\left[-i\omega\phi^{(2,1)} + \phi^{(1,1)}_{t_1}\right]_{z=0} \tag{4.1}$$

and

$$\nabla^2\phi^{(2,1)} = -2\phi^{(1,1)}_{x_0x_1} \qquad [-h \le z \le 0], \tag{4.2}$$
$$-\omega^2\phi^{(2,1)} + g\phi^{(2,1)}_z = 2i\omega\phi^{(1,1)}_{t_1} \qquad (z = 0), \tag{4.3}$$
$$\phi^{(2,1)}_z = 0; \qquad (z = -h), \tag{4.4}$$
$$\phi^{(2,1)}_{x_0} = -i\omega\mathcal{X}^{(2,1)} - \phi^{(1,1)}_{x_1} + \mathcal{X}^{(1,1)}_{t_1} - \mathcal{X}^{(1,0)}\phi^{(1,1)}_{x_0x_0}; \qquad (x = 0) \tag{4.5}$$

Only those solutions of $\phi^{(2,1)}$ which are outgoing are permitted.

The nonhomogenity introduced in (4.2) and (4.3) require that certain solvability conditions be satisfied. This, in turn, determines the slow variation of $\phi^{(1,1)}$ with respect to x_1 and t_1. One may proceed to obtain the solvability conditions for the system (4.2) - (4.5) by using Green's theorem. However, this leads to a rather unwieldy form. Here, a different procedure is followed. We consider the problem in two parts.

Part 1:

$$\nabla^2\phi^{(2,1)}_a = -2\phi^{(1,1)}_{x_0x_1}; \qquad [-h \le z \le 0], \tag{4.6}$$
$$-\omega^2\phi^{(2,1)}_a + g\left(\phi^{(2,1)}_a\right)_z = 2i\omega\phi^{(1,1)}_{t_1}; \qquad (z = 0), \tag{4.7}$$
$$\left(\phi^{(2,1)}_a\right)_z = 0; \qquad (z = -h) \tag{4.8}$$

with no specified condition at $x = 0$.

Part 2:

$$\nabla^2\phi^{(2,1)}_b = 0; \qquad [-h \le z \le 0], \tag{4.9}$$
$$-\omega^2\phi^{(2,1)}_b + g\left(\phi^{(2,1)}_b\right)_z = 0; \qquad (z = 0), \tag{4.10}$$
$$\left(\phi^{(2,1)}_b\right)_z = 0; \qquad (z = -h), \tag{4.11}$$
$$\frac{\partial\phi^{(2,1)}_b}{\partial x_0} = -i\omega\mathcal{X}^{(2,1)} + \mathcal{X}^{(1,1)}_{t_1} - \phi^{(1,1)}_{x_1} - \left(\phi^{(2,1)}_a\right)_{x_0} - \mathcal{X}^{(1,0)}\phi^{(1,1)}_{x_0x_0}; \qquad (x = 0) \tag{4.12}$$

It is immediately clear that (4.9)-(4.12) represent the usual linearised wave maker problem and there exists a solution for any arbitrary function on the right hand of (4.12). We further note that $\left[\phi_a^{(2,1)} + \phi_b^{(2,1)}\right]$ satifies the complete problem given by (4.2)-(4.5).

4.1 Solution of Part 1

From the first order solution one has

$$\begin{aligned} -2\phi^{(1,1)}_{x_0x_1} &= -\frac{gk_0}{\omega}\frac{\partial A}{\partial x_1}\frac{\cosh k_0(h+z)}{\cosh k_0 h}\exp(ik_0x_0) + \\ &\quad -\frac{ig}{\omega}\sum_{n=1}^{\infty} k_n \frac{\partial B_n}{\partial x_1}\frac{\cos k_n(h+z)}{\cos k_n h}\exp(-k_n x_0). \end{aligned} \tag{4.13}$$

In order to facilitate the solution we consider $\phi_a^{(2,1)} = \sum_{n=0}^{\infty} \phi_{a,n}^{(2,1)}$ where the mode $n=0$ satisfies the forcing due to the propagating mode of $\phi^{(1,1)}$ and each other mode n, for $n \geq 1$, satisfies the corresponding evanescent mode of $\phi^{(1,1)}$.

The solution to $\phi_{a,n}^{(2,1)}$ satisfying the bottom condition and the Poisson equation is

$$\phi_{a,n}^{(2,1)} = -\frac{g}{2k_0\omega}\frac{\partial A}{\partial x_1}\left(\frac{Q_0 \sinh Q_0}{\cosh q_0}\right)\exp(ik_0x_0), \quad (n=0); \tag{4.14}$$

$$\phi_{a,n}^{(2,1)} = -\frac{ig}{2\omega}\sum_{n=1}^{\infty}\frac{1}{k_n}\frac{\partial B_n}{\partial x_1}\left(\frac{Q_n \sin Q_n}{\cos q_n}\right)\exp(-k_nx_0), \quad (n \geq 1). \tag{4.15}$$

The solvability conditions can now be obtained by requiring (4.15) to satisfy the free surface condition (4.7) for each n. The resulting conditions after some manipulations are

$$\frac{\partial A}{\partial t_1} + C_g\frac{\partial A}{\partial x_1} = 0 \tag{4.16}$$

$$\frac{\partial B_n}{\partial t_1} - iC_{g_n}\frac{\partial B_n}{\partial x_1} = 0; \quad n \in \mathbb{N}^+ \tag{4.17}$$

niet correct.

with C_g as the group velocity [*i.e.*, $C_g = CI_0$, and $C = \omega/k_0$] and

$$C_{g_n} = \frac{1}{2}\frac{\omega}{k_n}\left[1 + \frac{2q_n}{\sin 2q_n}\right]. \tag{4.18}$$

Expressions (4.16) and (4.17) govern the slow variations of $A(x_1, t_1)$ and $B_n(x_1, t_1)$'s respectively. We further note that the slow variations of $A(x_1,t_1)$ and $B_n(x_1,t_1)$'s at $x=0$ with respect to t_1 are also related to the first order wave maker motion $\mathcal{X}^{(1,1)}(t_1)$ through (3.16) and (3.17).

4.2 Solution of Part 2

The solution to $\phi_b^{(2,1)}$ is forced by $\phi^{(1,1)}$ and $\phi_a^{(2,1)}$ and the wave maker motions $\mathcal{X}^{(1,0)}$ and $\mathcal{X}^{(2,1)}$. The complete solution is of the form:

$$
\begin{aligned}
\phi_b^{(2,1)} &= -\frac{ig}{2\omega} A^{(2,1)}(x_1, t_1) \frac{\cosh k_0(h+z)}{\cosh k_0 h} \exp(ikx_0) - \\
&\quad \frac{ig}{2\omega} \sum_{n=1}^{\infty} B_n^{(2,1)}(x_1, t_1) \frac{\cos k_n(h+z)}{\cos k_n h} \exp(-k_n x_0).
\end{aligned} \tag{4.19}
$$

$A^{(2,1)}(0, t_1)$ and $B_n^{(2,1)}(0, t_1)$'s are explicitly obtained from the condition (4.12) at the wave maker, *i.e.*,

$$
\begin{aligned}
ik_0 A^{(2,1)} \frac{\cosh Q_0}{\cosh q_0} - \sum_{n=1}^{\infty} k_n B_n^{(2,1)} \frac{\cos Q_n}{\cos q_n} &= \\
\frac{2i\omega}{g} \left[-i\omega \mathcal{X}^{(2,1)} + \mathcal{X}_{t_1}^{(1,1)} \right] &+ \frac{\cosh Q_0}{\cosh q_0} \left[-\frac{\partial A}{\partial x_1} + \mathcal{X}^{(1,0)} k_0^2 A \right] + \\
\frac{Q_0 \sinh Q_0}{\cosh q_0} \left[-\frac{\partial A}{\partial x_1} \right] &\sum_{n=1}^{\infty} \frac{\cos Q_n}{\cos q_n} \left[-\frac{\partial B_n}{\partial x_1} - \mathcal{X}^{(1,0)} k_n^2 B_n \right] + \\
&\sum_{n=1}^{\infty} \frac{Q_n \sin Q_n}{\cos q_n} \left[\frac{\partial B_n}{\partial x_1} \right]
\end{aligned} \tag{4.20}
$$

$A^{(2,1)}$ and $B_n^{(2,1)}$'s can be obtained from (4.20) by utilizing the orthogonalities of $\cosh Q_0$ and $\cos Q_n$'s over the interval $[-h \le z \le 0]$. An interesting feature of (4.20) is that the amplitude of the propagating mode $A^{(2,1)}$ depends on the slow variation of the evanescent modes $B_n^{(1,1)}$'s (since $\cosh Q_0$ and $Q_n \sin Q_n$ are not orthogonal). Explicitly, one has

$$
\begin{aligned}
A^{(2,1)} &= 2\frac{C}{C_g} \tanh q_0 \left[-i\mathcal{X}^{(2,1)} + \frac{1}{\omega} \mathcal{X}_{t_1}^{(1,1)} \right] - \frac{i}{k_0} \left[-\frac{\partial A}{\partial x_1} + \mathcal{X}^{(1,0)} k_0^2 A \right] + \\
&\quad \frac{i}{2k_0} \left[1 + \frac{C}{C_g} (q_0 \tanh q_0 - 1) \right] \frac{\partial A}{\partial x_1} - i\frac{C}{C_g} \sum [\mathrm{ET}_n] \frac{\partial B_n}{\partial x_1}, \quad (x = 0)
\end{aligned} \tag{4.21}
$$

where

$$
\begin{aligned}
\mathrm{ET}_n &= \frac{q_n}{k_0^2 + k_n^2} \left(k_0 \tan q_n - \frac{k_n}{\tanh q_0} \right) - \frac{k_n k_0^2}{(k_0^2 + k_n^2)^2} \frac{\tan q_n}{\tanh q_0} \\
&\quad + 2\frac{k_n^2 k_0}{(k_0^2 + k_n^2)^2} + \frac{k_n^3}{(k_0^2 + k_n^2)^2} \frac{\tan q_n}{\tanh q_0} \\
&= \frac{k_0}{k_0^2 + k_n^2} \left[1 - \frac{\omega^2 h}{g} \left(1 + \frac{g^2 k_n^2}{\omega^4} \right) \right]
\end{aligned} \tag{4.22}
$$

It is convenient to express $A^{(2,1)}$ in terms of A_{t_1}. This is done as follows. The solvability condition (4.17) gives

$$\frac{\partial B_n}{\partial x_1} = -\frac{i}{C_{g_n}}\frac{\partial B_n}{\partial t_1} \tag{4.23}$$

and from the conditions (3.16) and (3.17) we have

$$B_n(0,t_1) = -i\frac{k_0}{k_n}\frac{I_0}{I_n}A(0,t_1). \tag{4.24}$$

Using (4.23) and (4.24) one gets

$$\frac{\partial B_n}{\partial x_1}(0,t_1) = -\frac{k_0 I_0}{k_n I_n C_{g_n}}\frac{\partial A}{\partial t_1}(0,t_1) = -\frac{\omega C_g}{C^2}\frac{1}{k_n I_n C_{g_n}}\frac{\partial A}{\partial t_1}(0,t_1) \tag{4.25}$$

Thus, in terms of A_{t_1}, (4.21) becomes

$$\begin{aligned} A^{(2,1)} &= \frac{C}{C_g}\left[-2i(\tanh q_0)\mathcal{X}^{(2,1)} + i\frac{I_0}{\omega}\frac{\partial A}{\partial t_1}\right] - \frac{i}{k_0}\left[\frac{1}{C_g}\frac{\partial A}{\partial t_1} + \mathcal{X}^{(1,0)}k_0^2 A\right] \\ &- \frac{i}{2C_g k_0}\left[1 + \frac{C}{C_g}(q_0\tanh q_0 - 1)\right]\frac{\partial A}{\partial t_1} + ik_0\frac{\partial A}{\partial t_1}\sum_{n=1}^{\infty}\frac{[\mathrm{ET}_n]}{k_n I_n C_{g_n}}, \quad (x=0) \end{aligned} \tag{4.26}$$

The expression for the term $[\mathrm{ET}_n]\,/\,(k_n I_n C_{g_n})$ in (4.26) can be simplified to be

$$\frac{[\mathrm{ET}_n]}{k_n I_n C_{g_n}} = \frac{2h}{\omega}\frac{q_0}{I_n\left(q_0^2 + q_n^2\right)}. \tag{4.27}$$

It is numerically more accurate to compute $[\mathrm{ET}_n]\,/\,(k_n I_n C_{g_n})$ through (4.27) than through computing $[\mathrm{ET}_n]$ and $(k_n I_n C_{g_n})$ separately.

4.3 Far field solution

The complete far field solution to $\phi^{(2,1)}$ is

$$\phi^{(2,1)} = -\left(\frac{ig}{2\omega}\right)\left[A^{(2,1)}\frac{\cosh k_0(h+z)}{\cosh k_0 h} - \frac{i}{k_0}\frac{\partial A}{\partial x_1}\frac{Q_0\sinh Q_0}{\cosh q_0}\right]\exp(ik_0 x_0) \tag{4.28}$$

where $A^{(2,1)}$ is given by (4.26). From (4.1), the second order surface elevation $\zeta^{(2,1)}$ far away from the wavemaker is

$$\zeta^{(2,1)} = \frac{1}{2}\left[A^{(2,1)} - \frac{i}{k_0}\frac{\partial A}{\partial x_1}q_0\tanh q_0\right]\exp(ik_0x_0) + \frac{i}{2\omega}\frac{\partial A}{\partial t_1}\exp(ik_0x_0) \tag{4.29}$$

4.4 Determination of $\mathcal{X}^{(2,1)}$

Since the free surface shape near the carrier frequency is assumed to be given by $\zeta^{(1,1)}$, we set the far-field condition

$$\zeta^{(2,1)} = 0. \tag{4.30}$$

With (4.30), (4.29) gives

$$A^{(2,1)} = \frac{i}{k_0}\frac{\partial A}{\partial x_1} q_0 \tanh q_0 - \frac{i}{\omega}\frac{\partial A}{\partial t_1} \tag{4.31}$$

$A^{(2,1)}$ can be expressed in terms of A_{t_1} by using (4.16) in (4.31):

$$A^{(2,1)} = -\frac{i}{\omega}\left[\frac{C}{C_g} q_0 \tanh q_0 + 1\right]\frac{\partial A}{\partial t_1} \tag{4.32}$$

The wave maker motion $\mathcal{X}^{(2,1)}$ is now determined by using the condition (4.32) in (4.26):

$$\begin{aligned} 2i\frac{C}{C_g}\tanh q_0 \mathcal{X}^{(2,1)} \quad &= \\ \frac{i}{\omega}\left[\frac{C}{C_g} q_0 \tanh q_0 + 1\right]\frac{\partial A}{\partial t_1} &+ i\frac{1}{\omega}\frac{\partial A}{\partial t_1} - \frac{i}{k_0}\left[\frac{1}{C_g}\frac{\partial A}{\partial t_1} + \mathcal{X}^{(1,0)}k_0^2 A\right] \\ -\frac{i}{2C_g k_0}\left[1 + \frac{C}{C_g}(q_0 \tanh q_0 - 1)\right]\frac{\partial A}{\partial t_1} &+ ik_0\frac{\partial A}{\partial t_1}\sum_{n=1}^{\infty}\frac{[\mathrm{ET}_n]}{k_n I_n C_{gn}}, \quad (x = 0) \end{aligned} \tag{4.33}$$

Or, in a slightly modified form as

$$\begin{aligned} \mathcal{X}^{(2,1)} \quad = \quad & \left[\frac{g}{2\omega^2 C}(q_0 \tanh q_0 - \frac{3}{2}) - \frac{g}{4\omega^2 C_g}(q_0 \tanh q_0 - 1)\right. \\ & \left. + \frac{gC_g}{\omega^2 C^2} + \frac{gC_g}{2C^3}\left(\sum_{n=1}^{\infty}\frac{[\mathrm{ET}_n]}{k_n I_n C_{gn}}\right)\right]\frac{\partial A}{\partial t_1} \\ & -\frac{gC_g}{2C^3}\mathcal{X}^{(1,0)} A \end{aligned} \tag{4.34}$$

4.5 Surface elevation $\zeta^{(2,1)}$

Surface elevation $\zeta^{(2,1)}$ is governed by (4.1). Because of the condition of $\zeta^{(2,1)}$ vanishing in the far-field, the coefficient of the term $\exp(ik_0x_0)$ is identically zero. From (4.1) one thus has

$$\zeta^{(2,1)} = \sum_{n=0}^{\infty}\left[\frac{1}{2}B_n^{(2,1)} + \frac{i}{2\omega}\left(\frac{\omega^2 h}{gI_n}+1\right)\frac{\partial B_n}{\partial t_1}\right]\exp(-k_n x_0), \quad x \geq 0. \tag{4.35}$$

The coefficient $B_n(0,t_1)$ is known in terms of A at the wave maker. The coefficient $B_n^{(2,1)}$ can be determined from (4.20) using the orthogonality of $\cos Q_n$'s and $\cosh Q_0$ over $[-h \leq z \leq 0]$. After long, but fairly straightforward, operations one gets

$+\frac{2k_0}{C_{g_n}(k_0^2+k_n^2)}A_{t_1}$

$$\begin{aligned} B_n^{(2,1)} &= \left[i\omega\mathcal{X}^{(2,1)} - \mathcal{X}_{t_1}^{(1,1)}\right]\frac{2i}{C_{g_n}}\frac{\tan q_n}{k_n} + \frac{1}{k_n}\left[-\frac{i}{C_{g_n}}\frac{\partial B_n}{\partial t_1} + \mathcal{X}^{(1,0)}k_n^2 B_n\right] \\ &\quad -\frac{\omega}{k_n}\frac{1}{C_g C_{g_n}\tan q_n}\frac{k_0}{k_0^2+k_n^2}\left[\tanh q_0 + q_0(1-\tanh^2 q_0)\right]\frac{\partial A}{\partial t_1} \\ &\quad +i\frac{\omega}{k_n C_{g_n}\tan q_n}\sum_{m=1}^{\infty}\frac{\mathrm{Cb}_{nm}}{C_{gm}}\frac{\partial B_m}{\partial t_1} \end{aligned} \tag{4.36}$$

where

$$\mathrm{Cb}_{nm} = \frac{1}{4k_n}\left[-q_n(1-\tan^2 q_n) + \tan q_n\right]; \quad (m = n) \tag{4.37}$$

and

$$\begin{aligned} \mathrm{Cb}_{nm} &= -\frac{q_m}{(k_m-k_n)^2}\left[k_m + k_n\tan q_n \tan q_m\right] + \\ &\quad \frac{k_m}{(k_m^2-k_n^2)^2}\left[(k_m^2+k_n^2)\tan q_m - 2k_m k_n \tan q_n\right]; \quad (m \neq n). \end{aligned} \tag{4.38}$$

Using the relations (3.16) and (3.17), $\mathcal{X}^{(1,1)}$ and B_n can be expressed in terms of A respectively in the form

$$\mathcal{X}^{(1,1)} = i\frac{gk}{2\omega^2}I_0 A(0,t_1) = \frac{i}{2}\frac{I_0}{\tanh q_0}A(0,t_1) \tag{4.39}$$

$$B_n = i\frac{I_0}{I_n}\frac{\tan q_n}{\tanh q_0}A(0,t_1) \tag{4.40}$$

Substitutions of the above two expressions in (4.36) lead to

$$
\begin{aligned}
B_n^{(2,1)}(0,t_1) \; = \; & 2\frac{\tan q_n}{I_n}\mathcal{X}^{(2,1)} - ik_0\frac{I_0}{I_n}\mathcal{X}^{(1,0)}A \\
& + \frac{k_0 I_0}{k_n I_n}\left(\frac{1}{\omega} - \frac{1}{k_n C_{gn}}\right)\frac{\partial A}{\partial t_1} \\
& \frac{k_0\omega I_0}{k_n C_{gn}\tan q_n}\left(\sum_{m=1}^{\infty}\frac{1}{k_m I_m}\frac{\mathrm{Cb}_{nm}}{C_{gm}}\right)\frac{\partial A}{\partial t_1}
\end{aligned}
\tag{4.41}
$$

Replacing the term containing B_n in (4.35) an expression for $\zeta^{(2,1)}$ <u>at $x = 0$</u> is obtained as

$$
\begin{aligned}
\zeta^{(2,1)} \; = \; & \sum_{n=1}^{\infty}\left[\frac{1}{2}B_n^{(2,1)}(0,t_1)\right. \\
& \left. + \frac{1}{2\omega}\frac{k_0 I_0}{k_n I_n}\left(\frac{\omega^2 h}{g I_n} + 1\right)\frac{\partial A}{\partial t_1}\right]; \quad x = 0.
\end{aligned}
\tag{4.42}
$$

or equivalently,

$$
\begin{aligned}
\zeta^{(2,1)}(0,t_1) \; = \; & \left(\sum_{n=1}^{\infty}\frac{\tan q_n}{I_n}\right)\mathcal{X}^{(2,1)} - \frac{ik_0 I_0}{2}\left(\sum_{n=1}^{\infty}\frac{1}{I_n}\right)\mathcal{X}^{(1,0)}A \\
& + \frac{\partial A}{\partial t_1}\sum_{n=1}^{\infty}\left[-\frac{1}{g}\left(-\frac{I_0\omega}{I_n k_n \tanh q_0} + \frac{\omega^2 I_0}{k_n^2 \tanh q_0 C_{gn} I_n}\right)\right. \\
& \left. - \frac{\omega I_0}{2\tanh q_0 C_{gn}}\left(\sum_{m=1}^{\infty}\frac{1}{k_m I_m}\frac{\mathrm{Cb}_{nm}}{C_{gm}}\right) + \frac{1}{2\omega}\frac{k_0 I_0}{k_n I_n}\left(\frac{\omega^2 h}{g I_n} + 1\right)\right]
\end{aligned}
\tag{4.43}
$$

5 Subharmonic solution

Because $\phi^{(1,0)}$ satisfies

$$\phi^{(1,0)}_{x_0x_0} + \phi^{(1,0)}_{zz} = 0 \qquad (-h < z < 0) \qquad (5.1)$$
$$\phi^{(1,0)}_{z} = 0 \qquad (z = 0, -h) \qquad (5.2)$$
$$\phi^{(1,0)}_{x_0} = 0 \qquad (x = 0) \qquad (5.3)$$

we conclude that $\phi^{(1,0)} = \phi^{(1,0)}(x_1, t_1)$ is independent of short scales. For the second order, zeroth harmonic we find the equations

$$\phi^{(2,0)}_{x_0x_0} + \phi^{(2,0)}_{zz} = 0 \qquad (-h < z < 0) \qquad (5.4)$$
$$\phi^{(2,0)}_{z} = 0 \qquad (z = -h) \qquad (5.5)$$
$$g\phi^{(2,0)}_{z} = \omega(i\phi^{(1,1)*}_{x_0}\phi^{(1,1)} + *)_{x_0} \qquad (z = 0) \qquad (5.6)$$
$$\mathcal{X}^{(1,0)}_{t_1} = \phi^{(1,0)}_{x_1} + \phi^{(2,0)}_{x_0} + (\mathcal{X}^{(1,1)*}\phi^{(1,1)}_{x_0x_0} + *) \qquad (x = 0) \qquad (5.7)$$

Our interest is the description of correct $\mathcal{X}^{(1,0)}_{t_1}$ which is related to $\phi^{(1,0)}_{x_1}$ and $\phi^{(2,0)}_{x_0}$ in addition to the first order quantities through (5.7). It is shown in Appendix B that an alternative formulation for $\mathcal{X}^{(1,0)}_{t_1}$ is possible without explicit dependence on $\phi^{(2,0)}$:

$$\mathcal{X}^{(1,0)}_{t_1} = \phi^{(1,0)}_{x_1} + \frac{\omega}{gh}\left(i\phi^{(1,1)*}_{x_0}\phi^{(1,1)} + *\right)\Big|_{z=0, x_0\to\infty} \qquad (5.8)$$

The solution of the long scale variation of $\phi^{(1,0)}$ is discussed in Appendix A. Allowing only the form that corresponds to a propagating bound long wave (*i.e.* we assume the motion of the waveboard is such that spurious long waves are absent), we get

$$\phi^{(1,0)} = \frac{g^2}{4\omega^2}\frac{2\omega k_0 + C_g\left(k_0^2 - \sigma^2\right)}{C_g^2 - gh}\left[B(x_1 - C_gt_1) + S \cdot (x_1 - C_gt_1) + P\right] \qquad (5.9)$$

where

$$B(\theta) = \int_0^\theta |A(\psi)|^2 d\psi \qquad (5.10)$$

and S and P are constants. After substituting (5.9) and (3.14) in (5.8) and integrating with respect to t_1, an expression for $\mathcal{X}^{(1,0)}$ results:

$$\begin{aligned}\mathcal{X}^{(1,0)}(t_1) \quad = \quad & \left(\frac{g^2\left[2\omega k_0 + C_g\left(k_0^2 - \sigma^2\right)\right]}{4\omega^2\left(C_g^2 - gh\right)} + \frac{k_0 g}{2\omega h}\right)\cdot\left[-\frac{1}{C_g}B(-C_gt_1) + K\right] \\ & + \frac{g^2\left[2\omega k_0 + C_g[k_0^2 - \sigma^2]\right]}{4\omega^2\left(C_g^2 - gh\right)}S\, t_1 \qquad (5.11)\end{aligned}$$

with two unknown constants S and K.

We set K to be zero corresponding to the initial position of the waveboard being at zero and determine S such that the the time average of $\mathcal{X}^{(1,0)}$ tends to zero. We recognize that $|A(\psi)|$ is a slowly modulated function making $B(\theta)$ to be oscillating about a linearly increasing function of time:

$$B(-C_g t_1) = - < |A|^2 > C_g t_1 + \text{oscillating function} \tag{5.12}$$

where $< |A|^2 >$ denotes the time average of $|A|^2$. We require therefore that

$$S = -\left(1 + \frac{2\omega\left(C_g^2 - gh\right)}{\left[2\omega k_0 + C_g\left(k_0^2 - \sigma^2\right)\right]}\frac{k_0}{gh}\right) < |A|^2 > \tag{5.13}$$

The subharmonic waveboard motion is then given by

$$\mathcal{X}^{(1,0)}(t_1) = -\frac{1}{C_g}\left(\frac{g^2\left[2\omega k_0 + C_g[k_0^2 - \sigma^2]\right]}{4\omega^2\left(C_g^2 - gh\right)} + \frac{k_0 g}{2\omega h}\right) \cdot \int_0^{-C_g t_1}\left(A^2(\theta) - < |A|^2 >\right) d\theta \tag{5.14}$$

The constant P in (5.9) can remain as an arbitrary additive constant. Using '$\hat{\ }$' to indicate that the expression is taken at $z = 0$, we find from eqs.(2.23) and (3.2) that

$$\zeta^{(2,0)} = -\frac{1}{g}\left[\phi_{t_1}^{(1,0)} + \left|\hat{\phi}_{x_0}^{(1,1)}\right|^2 - \sigma^2\left|\hat{\phi}^{(1,1)}\right|^2\right]$$

leading finally to

$$\zeta^{(2,0)} = -\frac{1}{g}\left[-C_g\left(\frac{g^2\left[2\omega k_0 + C_g[k_0^2 - \sigma^2]\right]}{4\omega^2\left(C_g^2 - gh\right)}\right)\left(|A|^2 - < |A|^2 >\right) + C_g\frac{k_0 g}{2\omega h} < |A|^2 > + \frac{g^2(k_0^2 - \sigma^2)}{4\omega^2}|A|^2\right] \tag{5.15}$$

6 Order (2,2) solution

The equations for $\phi^{(2,2)}$ are

$$\phi^{(2,2)}_{x_0x_0} + \phi^{(2,2)}_{zz} = 0 \qquad (-h < z < 0) \qquad (6.1)$$

$$-4\omega^2\phi^{(2,2)} + g\phi^{(2,2)}_z = i\omega\left(\phi^{(1,1)2}_{x_0} + \phi^{(1,1)2}_z\right) + $$
$$+2i\frac{\omega^3}{g}\phi^{(1,1)}\phi^{(1,1)}_z + i\omega\left(\phi^{(1,1)}\phi^{(1,1)}_{x_0}\right)_{x_0} \qquad (z=0) \qquad (6.2)$$

$$\phi^{(2,2)}_z = 0 \qquad (z=-h) \qquad (6.3)$$

$$\phi^{(2,2)}_{x_0} = -2i\omega\mathcal{X}^{(2,2)} - \mathcal{X}^{(1,1)}\phi^{(1,1)}_{x_0x_0} \qquad (x=0) \qquad (6.4)$$

If we write $\phi^{(1,1)} = \sum_{n=0}^{\infty} c_n$ with

$$c_0(x_0, z, x_1, t_1) = -\frac{igf_0(z)}{2\omega f_0(0)}A(x_1 - C_g t_1)\exp(ik_0x_0) \qquad (6.5)$$

and

$$c_n(x_0, z, t_1) = -\frac{k_0 g}{2\omega f_0(0)F_0}A(-C_g t_1)\frac{F_n}{k_n}f_n(z)\exp(-k_nx_0) \qquad n \in I\!N^+$$?

$$c_{0x_0} = ik_0c_0$$
$$c_{nx_0} = -k_nc_n \qquad n \in I\!N^+$$
$$c_{0z} = k_0\tanh Q_0 c_0$$
$$c_{nz} = -k_n\tan(Q_n)\, c_n \qquad n \in I\!N^+$$
$$c_{0x_0x_0} = -k_0^2c_0$$
$$c_{nx_0x_0} = k_n^2c_n \qquad n \in I\!N^+$$

The expression for the second equation for $\phi^{(2,2)}$ becomes:

$$-4\omega^2\phi^{(2,2)} + g\phi^{(2,2)}_z =$$
$$-3i\omega k_0^2\left(1 - \tanh^2 q_0\right)C_{00}A^2(x_1 - C_gt_1)\exp(2ik_0x_0)$$
$$+i\omega\sum_{n=1}^{\infty}\left[k_n^2 - 4ik_0k_n - k_0^2 + 6\sigma^2\right]C_{0n}A(-C_gt_1)\cdot$$
$$A(x_1 - C_gt_1)\exp((-k_n + ik_0)x_0)$$
$$+i\omega\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\left[2k_nk_p + k_p^2 + 3\sigma^2\right]C_{np}A^2(-C_gt_1)\exp(-(k_n+k_p)x_0) \qquad (6.6)$$

with

$$
\begin{aligned}
C_{00} &= \frac{-g^2}{4\omega^2}, \\
C_{0n} &= -\frac{ig^2k_0^3\left(h+\sigma^{-1}\sinh^2 q_0\right)\cos^2 q_n}{4\omega^2 k_n^3\left(h-\sigma^{-1}\sin^2 q_n\right)\cosh^2 q_0} \\
C_{np} &= \frac{g^2k_0^6\left(h+\sigma^{-1}\sinh^2 q_0\right)^2\cos^2 q_n\cos^2 q_p}{4\omega^2 k_n^3 k_p^3\left(h-\sigma^{-1}\sin^2 q_n\right)\left(h-\sigma^{-1}\sin^2 q_p\right)\cosh^4 q_0}
\end{aligned}
\tag{6.7}
$$

Where f and F have been eliminated using definition (3.6) and the relations

$$F_0 = (1\cdot f_0) = \frac{\sigma f_0(0)}{k_0^2} \tag{6.8}$$

$$F_n = (1\cdot f_n) = \frac{-\sigma f_n(0)}{k_n^2} \qquad n \in I\!N^+ \tag{6.9}$$

Functions that satisfy both eq.(6.1) and (6.3) look like:

$$
\begin{aligned}
S_{np} &= \left(\exp\left(i\alpha_{np}(z+h)\right)+\exp\left(-i\alpha_{np}(z+h)\right)\right)\cdot \\
&\quad \left(D_{np1}\exp\left(\alpha_{np}x_0\right)+D_{np2}\exp\left(-\alpha_{np}x_0\right)\right)
\end{aligned}
\tag{6.10}
$$

where D_{np1} and D_{np2} can be arbitrary functions of the slow variables, α_{np} is a constant. Let us assume that for the particular solution we have $\phi^{(2,2)P} = \phi^{(2,2)Q} + \phi^{(2,2)R}$ where $\phi^{(2,2)R}$ satisfies eqs.(6.1),(6.2) and (6.3), and $\phi^{(2,2)Q}$ satisfies the following four equations, given by (6.1), (6.3) and

$$-4\omega^2\phi^{(2,2)Q} + g\phi_z^{(2,2)Q} = 0 \qquad (z=0), \tag{6.11}$$

$$\phi_{x_0}^{(2,2)Q} = -2i\omega\mathcal{X}^{(2,2)} - \mathcal{X}^{(1,1)}\phi_{x_0x_0}^{(1,1)} - \phi_{x_0}^{(2,2)R} \qquad (x=0) \tag{6.12}$$

With

$$\phi^{(2,2)R} = S_{00}^R + \sum_{n=1}^{\infty} S_{n0}^R + \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} S_{np}^R. \tag{6.13}$$

From eq.(6.6) we see that the choice:

$$
\begin{aligned}
\alpha_{np}^R &= -k_n - k_p && (n>0, p>0) \\
\alpha_{n0}^R &= \alpha_{0n}^R = ik_0 - k_n && (n>0) \\
\alpha_{00}^R &= 2ik_0 && \\
D_{ij2}^R &= 0 && (i,j \in \{0,1,2,...\})
\end{aligned}
$$

is imperative.

Substitution in eq.(6.6) yields:

$$D^R_{001} = \frac{3i\left(k_0^4 - \sigma^4\right) C_{00} A^2 \left(x_1 - C_g t_1\right)}{8\omega\sigma^2 \cosh\left(2q_0\right)} \tag{6.14}$$

$$D^R_{n01} = \frac{-ik_0 k_n \left[k_n^2 - k_0^2 + 6\sigma^2 - 4ik_0 k_n\right] C_{0n} A\left(-C_g t_1\right) A\left(x_1 - C_g t_1\right)}{2\omega\left[2k_0 k_n + i\left(k_0^2 - k_n^2 - 4\sigma^2\right)\right] \cos q_n \cosh q_0} \qquad (n > 0) \tag{6.15}$$

$$D^R_{np1} = \frac{ik_n k_p \left[2k_n k_p + k_p^2 + 3\sigma^2\right] C_{np} A^2\left(-C_g t_1\right)}{2\omega\left[4\sigma^2 + \left(k_n - k_p\right)^2\right] \cos q_n \cos q_p} \qquad (n > 0, p > 0) \tag{6.16}$$

If we take:

$$\phi^{(2,2)Q} = \sum_{n=0}^{\infty} S_n^Q$$

with

$$S_n^Q = \left(\exp\left(i\alpha_n\left(z+h\right)\right) + \exp\left(-i\alpha_n\left(z+h\right)\right)\right)\left(D^Q_{n1}\exp\left(\alpha_n x_0\right) + D^Q_{n2}\exp\left(-\alpha_n x_0\right)\right),$$

we find that in order to satisfy eqs.(6.1), (6.11) and (6.3) combined with a radiation condition at infinity we must require:

$$\begin{array}{lll} 4\sigma = -\alpha_n \tan\left(\alpha_n h\right) & & \text{and } D^Q_{n1} = 0 \quad \text{for } n > 0 \\ \alpha_0 = i\beta_0 \quad \text{with } 4\sigma = \beta_0 \tanh\left(\beta_0 h\right) & & \text{and } D^Q_{02} = 0 \end{array}$$

Together with the boundednes of the solutions and the radiation condition at infinity this yields:

$$\phi^{(2,2)Q} = 2D^Q_{01}\cosh\left(\beta_0\left(z+h\right)\right)\exp\left(i\beta_0 x_0\right) + 2\sum_{n=1}^{\infty} D^Q_{n2}\cos\left(\alpha_n\left(z+h\right)\right)\exp\left(-\alpha_n x_0\right)$$

From eq.(6.12) we now find in $x = 0$:

$$\begin{aligned} \phi^{(2,2)Q}_{x_0} &= 2i\beta_0 D^Q_{01}\cosh\left(\beta_0\left(z+h\right)\right) - \sum_{n=1}^{\infty} 2\alpha_n D^Q_{n2}\cos\left(\alpha_n\left(z+h\right)\right) = \\ &= -2i\omega\mathcal{X}^{(2,2)} - i\frac{k_0^3 g^2\left(h + \sigma^{-1}\sinh^2 q_0\right) A\left(-C_g t_1\right)}{4\omega^4\cosh^2 q_0}\left[-k_0^2 c_0 + \sum_{n=1}^{\infty} k_n^2 c_n\right] \\ &\quad -4ik_0 D^R_{001}\cosh\left(2k_0\left(z+h\right)\right) + \\ &\quad -\sum_{n=1}^{\infty} 2\left(ik_0 - k_n\right) D^R_{n01}\left[\cos Q_n \cosh Q_0 + i\sin Q_n \sinh Q_0\right] + \\ &\quad +\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} 2\left(k_n + k_p\right) D^R_{np1}\cos\left(\left(k_n + k_p\right)\left(z+h\right)\right) \end{aligned} \tag{6.17}$$

After multiplication of this equation with $\cosh(\beta_0(z+h))$ and integrating from $-h$ to 0 we find after some manipulations where we make use of the integrals given in Appendix C:

$$
\begin{aligned}
& i\beta_0\left(h+\frac{1}{2\beta_0}\sinh(2\beta_0 h)\right)D^Q_{01} = -2i\omega\mathcal{X}^{(2,2)}\frac{1}{\beta_0}\sinh(\beta_0 h) + \\
& -i\frac{k_0^3 g^2\left(h-\sigma^{-1}\sinh^2 q_0\right)A(-C_g t_1)}{4\omega^4\cosh^2 q_0}\left[-\frac{3}{2}i\frac{k_0^2}{k_0^2-\beta_0^2}\cosh(\beta_0 h)\omega A(-C_g t_1) + \right. \\
& +\frac{3}{2}\frac{k_0^3\left(h+\sigma^{-1}\sinh^2 q_0\right)}{\cosh^2 q_0}\cosh(\beta_0 h)\omega A(-C_g t_1)\times \\
& \left.\times\sum_{n=1}^{\infty}\frac{\cos^2 q_n}{k_n\left(k_n^2+\beta_0^2\right)\left(h-\sigma^{-1}\sin^2 q_n\right)}\right] + \\
& -\left[4ik_0 D^R_{001}\frac{-4\sigma^3}{k_0^2\left(4k_0^2-\beta_0^2\right)}\cosh(\beta_0 h)\cosh^2 q_0 + \right. \\
& +\frac{2\sigma}{k_0}\cosh q_0\cosh(\beta_0 h)\sum_{n=1}^{\infty}\frac{ik_0-k_n}{k_n}D^R_{n01}\frac{-2k_0k_n+i\left(k_n^2-k_0^2+4\sigma^2\right)}{k_0^2-k_n^2-\beta_0^2+2ik_0k_n}\cos q_n + \\
& \left.+2\sigma\cosh(\beta_0 h)\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{k_n+k_p}{k_nk_p}D^R_{np1}\frac{(k_n-k_p)^2+4\sigma^2}{(k_n+k_p)^2+\beta_0^2}\cos q_n\cos q_p\right] \qquad (6.18) \\
& \qquad\qquad\qquad \text{for } x=0
\end{aligned}
$$

From this expression we can directly find D^Q_{01}. In order to avoid the occurance of waves with frequency 2ω we should choose $\mathcal{X}^{(2,2)}$ such that $D^Q_{01}=0$. This yields in $x=0$:

$$
\begin{aligned}
& \mathcal{X}^{(2,2)} = \frac{\beta_0}{2i\omega\sinh(\beta_0 h)} * \\
& \left\{ i\frac{k_0^3 g^2\left(h+\sigma^{-1}\sinh^2 q_0\right)A(-C_g t_1)}{4\omega^4\cosh^2 q_0}\left[\frac{3}{2}i\frac{k_0^2}{k_0^2-\beta_0^2}\cosh(\beta_0 h)\omega A(-C_g t_1) + \right.\right. \\
& -\frac{3}{2}\frac{k_0^3\left(h+\sigma^{-1}\sinh^2 q_0\right)}{\cosh^2 q_0}\cosh(\beta_0 h)\omega A(-C_g t_1)\times \\
& \left.\times\sum_{n=1}^{\infty}\frac{\cos^2 q_n}{k_n\left(k_n^2+\beta_0^2\right)\left(h-\sigma^{-1}sin^2 q_n\right)}\right] + \\
& -\left[4ik_0 D^R_{001}\frac{-4\sigma^3}{\left(4k_0^2-\beta_0^2\right)k_0^2}\cosh(\beta_0 h)\cosh^2 q_0 + \right. \\
& +\frac{2\sigma}{k_0}\cosh q_0\cosh(\beta_0 h)\sum_{n=1}^{\infty}\frac{ik_0-k_n}{k_n}D^R_{n01}\frac{-2k_0k_n+i\left(k_n^2-k_0^2+4\sigma^2\right)}{k_0^2-k_n^2-\beta_0^2+2ik_0k_n}\cos q_n \\
& \left.\left.+2\sigma\cosh(\beta_0 h)\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}D^R_{np1}\frac{k_n+k_p}{k_nk_p}\frac{(k_n-k_p)^2+4\sigma^2}{(k_n+k_p)^2+\beta_0^2}\cos q_n\cos q_p\right]\right\} \qquad (6.19)
\end{aligned}
$$

Multiplication of equation (6.17) with $\cos(\alpha_m(z+h))$ where $m \in I\!N^+$, and integrating from $-h$ to 0 (again making use of Appendix C), yields in $x=0$:

$$
\begin{aligned}
&-\alpha_m\left(h+\frac{1}{2\alpha_m}\sin(2\alpha_m h)\right)D^Q_{m2} = -2i\omega\mathcal{X}^{(2,2)}\frac{1}{\alpha_m}\sin(\alpha_m h) - \\
&i\frac{k_0^3\left(h+\sigma^{-1}\sinh^2 q_0\right)A(-C_g t_1)}{4\sigma^2\cosh^2 q_0}\left[-\frac{3}{2}i\frac{k_0^2}{k_0^2+\alpha_m^2}\omega A(-C_g t_1)\cos(\alpha_m h) + \right. \\
&+\frac{3}{2}\frac{\omega k_0^3\left(h+\sigma^{-1}\sinh q_0\right)}{\cosh^2 q_0}A(-C_g t_1)\cos(\alpha_m h)\cdot \\
&\left.\sum_{n=1}^{\infty}\frac{\cos^2 q_n}{k_n\left(k_n^2-\alpha_m^2\right)\left(h-\sigma^{-1}\sin^2 q_n\right)}\right] + \\
&-\left[iD^R_{001}\frac{-16\sigma^3}{k_0\left(4k_0^2+\alpha_m^2\right)}\cos(\alpha_m h)\cosh^2 q_0 + \right. \\
&\frac{2\sigma}{k_0}\cos(\alpha_m h)\cosh q_0\sum_{n=1}^{\infty}\frac{ik_0-k_n}{k_n}D^R_{n01}\frac{-2k_0k_n+i\left(k_n^2-k_0^2+4\sigma^2\right)}{k_0^2-k_n^2+\alpha_m^2+2ik_0k_n}\cos q_n + \\
&\left.-2\sigma\cos(\alpha_m h)\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{k_n+k_p}{k_nk_p}D^R_{np1}\frac{4\sigma^2+(k_n-k_p)^2}{\alpha_m^2-(k_n+k_p)^2}\cos q_n\cos q_p\right] \qquad (6.20)
\end{aligned}
$$

This yields an explicit expression for D^Q_{m2}, for $m = 1(1)\infty$. With equation (6.17) we can now determine $\phi^{(2,2)Q}$, from (6.13) we find $\phi^{(2,2)R}$; the sum of both should satisfy eqs. (6.1) to (6.4). Solutions of the homogeneous equations:

$$
\begin{aligned}
&\phi^{(2,2)}_{x_0x_0}+\phi^{(2,2)}_{zz}=0 && (-h<z<0) \quad (6.21)\\
&-4\omega^2\phi^{(2,2)}+g\phi^{(2,2)}_z=0 && (z=0) \quad (6.22)\\
&\phi^{(2,2)}_z=0 && (z=-h) \quad (6.23)\\
&\phi^{(2,2)}_{x_0}=0 && (x=0) \quad (6.24)
\end{aligned}
$$

that are of the form

$$\phi^{(2,2)H}=\sum_{n=0}^{\infty}S^H_n$$

with

$$S^H_n=\left(\exp(i\alpha_n(z+h))+\exp(-i\alpha_n(z+h))\right)\left(D^H_{n1}\exp(\alpha_n x_0)+D^H_{n2}\exp(-\alpha_n x_0)\right)$$

satisfy (6.21) and (3.5). Equation (6.22) supplies us with the conditions:

$$4\sigma=\alpha_n\tan(\alpha_n h),\, D^H_{n1}=0 \quad \text{for} \quad n=1(1)\infty$$

and

$$\alpha_0 = i\beta_0, 4\sigma = \beta_0 \tanh(\beta_0 h), D_{02}^H = 0$$

Furthermore we assume a radiation condition at infinity as well as boundednes of the solution for all $x_0 > 0$, whence:

$$\phi^{(2,2)H} = 2D_{01}^H \cosh(\beta_0 (z+h)) \exp(i\beta_0 x_0) + 2\sum_{n=1}^{\infty} D_{n2}^H \cos(\alpha_n (z+h)) \exp(-\alpha_n x_0)$$

From boundary condition (6.24) we find:

$$\phi_{x_0}^{(2,2)H}\Big|_{x=0} = 2i\beta_0 D_{01}^H \cosh(\beta_0 (z+h)) - 2\sum_{n=1}^{\infty} \alpha_n D_{n2}^H \cos(\alpha_n (z+h)) = 0$$

By making use of the orthogonality of the functions $\cosh(\beta_0 (z+h))$ and $\cos(\alpha_n (z+h))$ for $n = 1(1)\infty$ on the interval from $-h$ to 0, as given in Appendix C, we find that:

$$\phi^{(2,2)H} = 0$$

for all x_0 and z.

7 Summary of the first- and second-order solution

The waveboard displacement $\mathcal{X}$ is given by

$$\mathcal{X} = \mathcal{X}^{(1,0)} + \left[\left(\mathcal{X}^{(1,1)} + \mathcal{X}^{(2,1)}\right)\exp(-i\omega_0 t) + *\right] + \left[\mathcal{X}^{(2,2)}\exp(-2i\omega_0 t) + *\right] \tag{7.1}$$

where

$$\mathcal{X}^{(1,0)} = -\frac{1}{C_g}\left(\frac{g^2\left[2\omega k_0 + C_g[k_0^2 - \sigma^2]\right]}{4\omega^2\left(C_g^2 - gh\right)} + \frac{k_0 g}{2\omega h}\right) \cdot$$

$$\cdot \int_0^{-C_g t}\left(A^2(\theta) - < A^2 >\right) d\theta, \tag{7.2}$$

$$\mathcal{X}^{(1,1)} = i I_0 \frac{g k_0}{2\omega^2} A(0,t), \tag{7.3}$$

$$\mathcal{X}^{(2,1)} = \Bigg[\frac{g}{2\omega^2 C}(q_0 \tanh q_0 - \frac{3}{2}) - \frac{g}{4\omega^2 C_g}(q_0 \tanh q_0 - 1) +$$

$$+\frac{gC_g}{\omega^2 C^2} + \frac{gC_g}{2C^3}\left(\sum_{n=1}^{\infty}\frac{[\mathrm{ET}_n]}{k_n I_n C_{gn}}\right)\Bigg]\frac{\partial A}{\partial t} - \frac{gC_g}{2C^3}\mathcal{X}^{(1,0)} A, \tag{7.4}$$

$$\mathcal{X}^{(2,2)} = \frac{\beta_0}{2i\omega \sinh(\beta_0 h)} \cdot$$

$$\cdot\Bigg\{ i\frac{k_0^3 g^2\left(h + \sigma^{-1}\sinh^2 q_0\right) A(-C_g t)}{4\omega^4 \cosh^2 q_0}\Bigg[\frac{3}{2} i \frac{k_0^2}{k_0^2 - \beta_0^2}\cosh(\beta_0 h)\,\omega A(-C_g t) +$$

$$-\frac{3}{2}\frac{k_0^3\left(h + \sigma^{-1}\sinh^2 q_0\right)}{\cosh^2 q_0}\cosh(\beta_0 h)\,\omega A(-C_g t) \times$$

$$\times \sum_{n=1}^{\infty}\frac{\cos^2 q_n}{k_n\left(k_n^2 + \beta_0^2\right)\left(h - \sigma^{-1} sin^2 q_n\right)}\Bigg] +$$

$$-\Bigg[4ik_0 D_{001}^R \frac{-4\sigma^3}{\left(4k_0^2 - \beta_0^2\right)k_0^2}\cosh(\beta_0 h)\cosh^2 q_0 +$$

$$\frac{2\sigma}{k_0}\cosh q_0 \cosh(\beta_0 h)\sum_{n=1}^{\infty}\frac{ik_0 - k_n}{k_n} D_{n01}^R \frac{-2k_0 k_n + i\left(k_n^2 - k_0^2 + 4\sigma^2\right)}{k_0^2 - k_n^2 - \beta_0^2 + 2ik_0 k_n}\cos q_n$$

$$+2\sigma\cosh(\beta_0 h)\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} D_{np1}^R \frac{k_n + k_p}{k_n k_p}\frac{(k_n - k_p)^2 + 4\sigma^2}{(k_n + k_p)^2 + \beta_0^2}\cos q_n \cos q_p\Bigg]\Bigg\}. \tag{7.5}$$

The surface elevation ζ at $x = 0$ is given by

$$\zeta = \zeta^{(2,0)} + \left[\left(\zeta^{(1,1)} + \zeta^{(2,1)}\right)\exp(-i\omega_0 t) + *\right] + \left[\zeta^{(2,2)}\exp(-2i\omega_0 t) + *\right] \quad (7.6)$$

where

$$\begin{aligned}
\zeta^{(2,0)} &= -\frac{1}{g}\left[-C_g\left(\frac{g^2\left[2\omega k_0 + C_g[k_0^2-\sigma^2]\right]}{4\omega^2\left(C_g^2 - gh\right)}\right)\left(|A|^2 - < A^2 >\right)\right. \\
&\quad \left. + C_g\frac{k_0 g}{2\omega h} < A^2 > + \frac{g^2(k_0^2-\sigma^2)}{4\omega^2}|A|^2\right], \qquad (7.7) \\
\zeta^{(1,1)} &= \frac{1}{2}A + \frac{1}{2}\sum_{n=1}^{\infty} B_n, \qquad \text{and} \qquad (7.8) \\
\zeta^{(2,1)} &= \sum_{n=1}^{\infty}\left[\frac{1}{2}B_n^{(2,1)}(0,t) + \frac{1}{2\omega}\frac{k_0 I_0}{k_n I_n}\left(\frac{\omega^2 h}{g I_n} + 1\right)\frac{\partial A}{\partial t}\right]. \qquad (7.9)
\end{aligned}$$

8 Experimental measurements

A series of tests were performed to verify the second order wave generation theory as described earlier in this report. Two different sets of experiments (similar to those reported in Kostense, 1984) were performed, the first set to analyse the generated subharmonic motion due to a bichromatic signal and the second to analyse the superharmonic elevation due to a monochromatic incident field.

The experiments were conducted in a flume (Scheldegoot) which is 1m wide, 1.2m deep and 55m long. During the experiments a beach of slope 1 : 5 existed with its toe about 43m away from the mean position of the wave maker. Maximum value of the reflection coefficient of the primary waves was found to be 20% over the entire range of the frequencies tested. Resistance-type wave gauges were used to collect the time record of the surface elevation and a probe was fixed near the wave maker to record the displacement of the wave maker. The wave gauges and the data-acquistion system were tested prior to the experiments to ascertain their reliability. In all cases, waves were generated for sufficiently long time (for about 5 minutes) and the reflection compensation mechanism was activated to reduce reflections from the waveboard.

Detailed measurements of surface elevation were also done during a later test (Klopman; 1993) using the second order wave generation theory. The results of subharmonic analyses of these tests are also included in this report.

8.1 Measurements and analyses: Subharmonic elevation

The experiments are described in three groups: 'ba', 'be' and '#wbo, #wbn'. Surface elevation was recorded at four locations (table 8.1) during the 'ba' and 'be' tests and at six locations (table 8.2) during the '#wbo, #wbn' tests. The analysis procedure is as follows:

1. Amplitudes and phases at the primary frequencies f_1, f_2 and the subharmonic excitation $|f_1 - f_2|$ at each location are obtained from the time record of the measured elevation by Fourier analysis.
2. The incident and reflected amplitudes of the carrier waves are determined from the amplitudes at two 'suitably' chosen wave gauges. ('Suitably' chosen wave gauges mean that the relative locations of these wave gauges give the best resolution of the different components). These measured amplitudes of the

x_1(m)	x_2(m)	x_3(m)	x_4(m)	Test
7.00	11.25	15.50	12.50	ba-1, ba-2
7.00	15.00	23.00	16.25	ba-3, ba-4
7.00	11.25	15.50	12.00	be-1, be-2, be-3
7.00	15.00	23.00	15.75	be-4

Table 8.1: Locations of the wave gauges from the wave maker for the 'ba' and the 'be' tests.

x_1(m)	x_2(m)	x_3(m)	x_4(m)	x_5(m)	x_6(m)	Test
12.50	18.50	22.15	22.85	26.32	34.50	#wbo
10.50	16.50	22.15	22.85	28.50	34.50	#wbn01-19

Table 8.2: Locations of the wave gauges from the wave maker for the '#wbo' and '#wbn' tests.

incident carrier waves are used in getting the theoretical amplitudes of the bound long waves.

3. The harmonic amplitudes at $\Delta f = |f_1 - f_2|$ are analysed to give amplitudes of free long waves and bound long waves assuming that the subharmonic surface elevation consists of the following components:

   (a) incident bound wave
   (b) incident free wave
   (c) reflected free wave

   Three wave gauges are needed for the analysis. These three wave gauges are 'suitably' chosen out of the four or six wave gauges used during the experiments.

Bound long waves associated with the reflected primary waves are assumed to be negligible in the present analysis. This is based on the ground that the amplitude of the bound waves under a group is proportional to the product of the 1st-harmonic amplitudes of the carrier waves. Thus, the bound long waves associated with a maximum of 20% reflection of the primary waves can only be as large as 4% of the incident bound long waves. Results of the analyses are shown in tables 8.3 and 8.4. a_{lf} represents

		(expt. measurements) (mm)				a_{lb}^{th}
test	f_1, f_2	a_1	a_2	a_{lb}	a_{lf}	(mm)
ba-1	0.48, 0.33	54.4	11.4	4.7	1.4	5.9
ba-2	0.48, 0.36	54.7	11.1	4.7	1.1	5.3
ba-3	0.48, 0.39	54.3	12.4	4.7	1.6	5.3
ba-4	0.48, 0.42	54.3	11.8	5.3	1.7	4.7
be-1	0.69, 0.54	34.7	28.2	3.8	0.2	3.9
be-2	0.69, 0.57	34.4	28.2	3.6	0.3	3.7
be-3	0.69, 0.60	34.5	28.4	3.8	0.3	3.5
be-4	0.69, 0.63	34.1	28.3	3.1	0.6	3.3

Table 8.3: Measured amplitudes of subharmonic waves due to a bichromatic signal. $h = 0.5$m. a_{lb}^{th} denotes the theoretical value of the amplitude of the bound long waves (Laing, 1986).

the amplitude of the free long waves propagating away from the wavemaker. The amplitudes of the free long waves from the beach are not shown in the tables. In the series 'ba' and 'be', wave gauge 4 was optimally located relative to gauge 2 in order to analyse the the carrier waves. The incident and reflected amplitudes of the carrier waves in table 8.3 are obtained from these two gauges. The amplitudes of the subharmonic surface elevations are obtained from three wave gauges which give the largest determinant of the sytem of equations.

Comments on the results

Two aspects of interest in tables 8.3 and 8.4 are the comparison of the analysed and the predicted value of the bound long waves and the amplitude of the incident free long waves. The ratio of amplitude of free waves to that of bound waves is largest in the ba-tests, being about 29%, and smallest in the be-tests, being about 6%. There are a few factors which can contribute to the deviations of the analysed results from the expected values, *i.e.* amplitude of the bound long wave is as predicted by the second order Laing theory and the incident free wave is zero. These factors are:

1. difficulty in the analysis of long waves.
2. the amplitude of the incident free wave is not only a result of the second order wave-generation, but also depends on the reflection compensation mechanism of the waveboard.
3. higher order effects, more pronounced in ba-tests.

The difficulty in the analysis of long waves in a wave flume can be explained by considering the length scales of modulation L_1, L_2, L_3 associated with the components to be anlysed, *i.e.*;

- $L_1 = \lambda_{lf}/4$ (long free waves from the wavemaker + beach)
- $L_2 = \lambda_{lf}\lambda_{lb}/\{2\,(\lambda_{lf}+\lambda_{lb})\}$ (long free waves from the beach + incident long bound waves)
- $L_3 = \lambda_{lf}\lambda_{lb}/\{2\,|\lambda_{lf}-\lambda_{lb}|\}$ (incident long free waves + incident long bound waves)

where λ_{lf} and λ_{lb} denote respectively the lengths of the long free waves and the long bound waves. The wave gauges should have separation distances of roughly L_1, L_2 and L_3 for a good resolution of the subharmonic components. These length scales for the series 'ba', 'be' and '#wbo, #wbn' are presented in table 8.5. It is clear that the wave gauges cannot be ideally located in a wave flume of effective length less than 43m to resolve the free and bound components for quite a few of the conditions in 'ba' and 'be'. A measure of the soundness of the analysis for various separation distances of the wave gauges can be given by the determinant of the system. We show in table 8.6 the values of the determinant D for a few specific cases.

		(expt. measurements) (mm)				a_{lb}^{th}
test	$f_1,\ f_2$	a_1	a_2	a_{lb}	a_{lf}	(mm)
#wbo03	0.588, 0.735	48.8	35.0	5.1	0.6	5.9
#wbn17	0.606, 0.758	48.4	35.1	5.1	0.4	5.5

Table 8.4: Measured amplitudes of subharmonic waves due to a bichromatic signal. $h = 0.5$m. a_{lb}^{th} denotes the theoretical value of the amplitude of the bound long waves (Laing, 1986).

series	L_1 (m)	L_2 (m)	L_3 (m)
ba-1	3.7	3.4	42.2
ba-4	9.2	8.3	80.5
be-1	3.7	3.0	15.7
be-4	9.2	7.2	32.7
#wbo, #wbn	3.7	2.9	13.8

Table 8.5: Length scales of modulations of subharmonic surface elevation for the series 'ba', 'be', '#wbo', '#wbn'.

test	gauge 1	gauge 2	gauge 3	D
ba-1	7	11.25	15.5	1.14
ba-1	7	11.25	12.5	0.49
ba-1	11.25	12.5	15.5	0.35
ba-1 (*)	7	10.6	49.0	4.25
be-1	7	11.25	15.5	2.5
be-1 (*)	7	10.0	20.0	4.6
be-4	7	15.0	23.0	2.9
be-4 (*)	7	15.0	33.0	4.6

Table 8.6: Values of the determinant D depending on the locations of the wave gaues (in meters from the wavemaker). The '(*)' denotes a sort of ideal configuraton of the gauges based on the length scales of modulations.

8.2 Measurements and analyses: Superharmonic elevation

In the tests conducted to analyze the performance of the generated superharmonic field, only monochromatic incident wave field is considered. The nondimensional wave number kh ranges from 2.7 (deep) till 0.5 (intermediate depth). Surface elevation is again recorded at four locations as shown in table 8.7. Amplitudes of the incident and

x_1(m)	x_2(m)	x_3(m)	x_4(m)	Test no.
14.16	15.00	15.92	15.46	sh-1
14.58	15.00	16.30	15.65	sh-2
14.16	15.00	17.10	16.05	sh-3
14.30	15.00	16.40	15.70	sh-4

Table 8.7: Locations of the wave gauges from the wavemaker for the 'sh'-tests.

reflected carrier waves are analyzed from surface elevation at stations 1 & 2. Table 8.8 shows the analyzed values of the superharmonic amplitudes, the bound component a_{2s} and the incident free component a_{2f}, for a given first order amplitude a. Magnitudes of the components a_{2s} & a_{2f} show slight variations depending on which three gauges are considered for the analysis. The listed values of the superharmonic components are based on the three locations for which the determinant D of the system is the largest. It is seen from table 8.8 that the wave gauge locations are far from ideal, particularly for the test case sh-4. Further, there is no clear trend of reduction of the amplitude of the free waves. Besides the location of the wave gauges there are a few factors which

	kh	(expt. measurements) (m) a	a_{2s}	a_{2f}	(theoretical) (m) a_{2s}^{th}	a_{2f}^{th}	D
sh-1	2.7	0.04878	0.008582	0.001105	0.007423	0.002722	3.6
sh-2	2	0.04981	0.006472	0.002133	0.006172	0.002263	4.3
sh-3	1.1	0.04091	0.004099	0.000598	0.004125	0.001925	1.4
sh-4	0.5	0.02981	0.005428	0.005310	0.005849	0.004679	0.9

Table 8.8: Measurement of the superharmonic components. a_{2f} denotes the measured amplitude of the superharmonic free waves from the wavemaker and a_{2f}^{th} denotes the amplitude produced by a sinusoidally moving piston wavemaker (Flick & Guza, 1980)..

can contribute to this problem:

1. actual superharmonic correction to the waveboard in the the 'wave generation' software may not have been updated from the old version to the one described in this report.
2. amplitude of the incident free wave depends not only on the wave generation theory, but also on the reflection compensation mechanism.

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A Derivation of $\phi^{(1,0)}$

Up to the third order in ε, we find:

$$\int_{-h}^{\zeta} u dz = \int_{-h}^{0} \phi_x dz + \int_{0}^{\zeta} (\hat{\phi}_x + z\hat{\phi}_{xz}) dz =$$

$$\int_{-h}^{0} (\varepsilon\phi_{x_0}^{(1)} + \varepsilon^2\phi_{x_0}^{(2)} + \varepsilon^3\phi_{x_0}^{(3)} + \varepsilon^2\phi_{x_1}^{(1)} + \varepsilon^3\phi_{x_1}^{(2)}) dz +$$

$$\varepsilon^2\zeta^{(1)}\hat{\phi}_{x_0}^{(1)} + \varepsilon^3\zeta^{(1)}\hat{\phi}_{x_0}^{(2)} + \varepsilon^3\zeta^{(1)}\hat{\phi}_{x_1}^{(1)} + \varepsilon^3\zeta^{(2)}\hat{\phi}_{x_0}^{(1)} + \tfrac{1}{2}\varepsilon^3\zeta^{(1)2}\hat{\phi}_{x_0 z}^{(1)} + O(\varepsilon^4)$$

where ' $\hat{}$ ' indicates that the expression is taken in $z = 0$. This implies that the continuity equation

$$\frac{\partial}{\partial t}\zeta + \frac{\partial}{\partial x}\int_{-h}^{\zeta} u dz = 0 \tag{A.1}$$

can, up to third order in ε, be written as:

$$\int_{-h}^{0} (\varepsilon\phi_{x_0x_0}^{(1)} + 2\varepsilon^2\phi_{x_0x_1}^{(1)} + \varepsilon^2\phi_{x_0x_0}^{(2)} + 2\varepsilon^3\phi_{x_0x_1}^{(2)} + \varepsilon^3\phi_{x_0x_0}^{(3)} + \varepsilon^3\phi_{x_1x_1}^{(1)}) dz +$$

$$\varepsilon^2\zeta_{x_0}^{(1)}\hat{\phi}_{x_0}^{(1)} + \varepsilon^2\zeta^{(1)}\hat{\phi}_{x_0x_0}^{(1)} + \varepsilon^3\zeta_{x_1}^{(1)}\hat{\phi}_{x_0}^{(1)} + 2\varepsilon^3\zeta^{(1)}\hat{\phi}_{x_0x_1}^{(1)} + \varepsilon^3\zeta_{x_0}^{(1)}\hat{\phi}_{x_1}^{(1)} +$$

$$\varepsilon^3\zeta_{x_0}^{(2)}\hat{\phi}_{x_0}^{(1)} + \varepsilon^3\zeta^{(2)}\hat{\phi}_{x_0x_0}^{(1)} + \varepsilon^3\zeta_{x_0}^{(1)}\hat{\phi}_{x_0}^{(2)} + \varepsilon^3\zeta^{(1)}\hat{\phi}_{x_0x_0}^{(2)} + \varepsilon^3\zeta^{(1)}\zeta_{x_0}^{(1)}\hat{\phi}_{x_0z}^{(1)} +$$

$$\frac{1}{2}\varepsilon^3\zeta^{(1)2}\hat{\phi}_{x_0x_0z}^{(1)} + \varepsilon\zeta_{t_0}^{(1)} + \varepsilon^2\zeta_{t_1}^{(1)} + \varepsilon^2\zeta_{t_0}^{(2)} + \varepsilon^3\zeta_{t_1}^{(2)} + \varepsilon^3\zeta_{t_0}^{(3)} = O(\varepsilon^4)$$

We now have for the first order, zeroth harmonic:

$$\int_{-h}^{0} \phi_{x_0x_0}^{(1,0)} dz = 0$$

as was to be expected from eqs.(5.1) to (5.3). For the second order zeroth harmonic we find:

$$\int_{-h}^{0} \phi_{x_0x_0}^{(2,0)} dz + (\zeta_{x_0}^{(1,1)*}\hat{\phi}_{x_0}^{(1,1)} + *) + (\zeta^{(1,1)*}\hat{\phi}_{x_0x_0}^{(1,1)} + *) = 0 \tag{A.2}$$

as $\zeta^{(1,0)} = 0$, and for the third order zeroth harmonic:

$$\zeta_{t_1}^{(2,0)} + \int_{-h}^{0} (2\phi_{x_0x_1}^{(2,0)} + \phi_{x_0x_0}^{(3,0)})dz + h\phi_{x_1x_1}^{(1,0)} + (\zeta_{x_1}^{(1,1)*}\hat{\phi}_{x_0}^{(1,1)} + *) +$$
$$2(\zeta^{(1,1)*}\hat{\phi}_{x_0x_1}^{(1,1)} + *) + (\zeta_{x_0}^{(1,1)*}\hat{\phi}_{x_1}^{(1,1)} + *) + (\zeta_{x_0}^{(2,1)*}\hat{\phi}_{x_0}^{(1,1)} + *) +$$
$$(\zeta^{(2,1)*}\hat{\phi}_{x_0x_0}^{(1,1)} + *) + (\zeta^{(1,1)*}\hat{\phi}_{x_0x_0}^{(2,1)} + *) + (\zeta_{x_0}^{(1,1)*}\hat{\phi}_{x_0}^{(2,1)} + *) = 0 \qquad \text{(A.3)}$$

because $\phi^{(1,0)}$ is independent of the short scales. With the use of eq.(2.23) we find the following wave equation for $\phi^{(1,0)}$:

$$\phi_{t_1t_1}^{(1,0)} - gh\phi_{x_1x_1}^{(1,0)} = -\left(|\hat{\phi}_{x_0}^{(1,1)}|^2\right)_{t_1} - \left(|\hat{\phi}_{z}^{(1,1)}|^2\right)_{t_1} + \sigma\left(|\hat{\phi}^{(1,1)}|^2\right)_{zt_1}$$
$$+g\int_{-h}^{0} (2\phi_{x_0x_1}^{(2,0)} + \phi_{x_0x_0}^{(3,0)})dz + g(\zeta_{x_1}^{(1,1)*}\hat{\phi}_{x_0}^{(1,1)} + *) +$$
$$2g(\zeta^{(1,1)*}\hat{\phi}_{x_0x_1}^{(1,1)} + *) + g(\zeta_{x_0}^{(1,1)*}\hat{\phi}_{x_1}^{(1,1)} + *) + g(\zeta_{x_0}^{(2,1)*}\hat{\phi}_{x_0}^{(1,1)} + *) +$$
$$g(\zeta^{(2,1)}\hat{\phi}_{x_0x_0}^{(1,1)*} + *) + g(\zeta^{(1,1)*}\hat{\phi}_{x_0x_0}^{(2,1)} + *) + g(\zeta_{x_0}^{(1,1)*}\hat{\phi}_{x_0}^{(2,1)} + *)$$
$$= -\left(|\hat{\phi}_{x_0}^{(1,1)}|^2\right)_{t_1} - \left(|\hat{\phi}_{z}^{(1,1)}|^2\right)_{t_1} + \sigma\left(|\hat{\phi}^{(1,1)}|^2\right)_{t_1z} +$$
$$g\int_{-h}^{0} (2\phi_{x_0x_1}^{(2,0)} + \phi_{x_0x_0}^{(3,0)})dz - 2\omega(i\hat{\phi}^{(1,1)*}\hat{\phi}_{x_0x_1}^{(1,1)} + *) - \omega(i\hat{\phi}^{(1,1)*}\hat{\phi}_{x_0x_0}^{(2,1)} + *)$$
$$-\omega(i\hat{\phi}_{x_0}^{(1,1)*}\hat{\phi}_{x_0}^{(2,1)} + *) + g(\zeta^{(2,1)*}\hat{\phi}_{x_0}^{(1,1)} + *)_{x_0}$$

After the terms in the right-hand side that are third order in the wave height have been neglected, the wave equation becomes:

$$\phi_{t_1t_1}^{(1,0)} - gh\phi_{x_1x_1}^{(1,0)} = -\left(|\hat{\phi}_{x_0}^{(1,1)}|^2\right)_{t_1} - \left(|\hat{\phi}_{z}^{(1,1)}|^2\right)_{t_1} + \sigma\left(|\hat{\phi}^{(1,1)}|^2\right)_{t_1z} +$$
$$-2\omega(i\hat{\phi}^{(1,1)*}\hat{\phi}_{x_0x_1}^{(1,1)} + *)$$

By neglecting the higher order terms in the wave height, we have reintroduced the x_0 dependence of the right-hand side. In order to restore this we only take that part that does not depend on x_0, so we leave out the influence of the evanescent modes. The equation now becomes with the use of (3.14):

$$\phi_{t_1t_1}^{(1,0)} - gh\phi_{x_1x_1}^{(1,0)} = \frac{g^2}{4\omega^2}\left[C_g(k_0^2 - \sigma^2) + 2\omega k_0\right]\left(|A|^2\right)_{x_1}. \qquad \text{(A.4)}$$

To find the solutions of this equation that only depend on $(x_1 - C_g t_1)$ we can use the fact that the right-hand side is a function of $(x_1 - C_g t_1)$. We find:

$$\left(C_g^2 - gh\right) \phi_{x_1 x_1}^{(1,0)} = \frac{g^2}{4\omega^2} \left[C_g(k_0^2 - \sigma^2) + 2\omega k_0\right] \left(|A|^2\right)_{x_1} . \tag{A.5}$$

With the definition $B(\theta) = \int_0^\theta |A(\psi)|^2 d\psi$, we find the following expression:

$$\phi^{(1,0)} = \frac{g^2}{4\omega^2} \frac{2\omega k_0 + C_g\left(k_0^2 - \sigma^2\right)}{C_g^2 - gh} \left(B(x_1 - C_g t_1) + S \cdot (x_1 - C_g t_1) + P\right) \tag{A.6}$$

where S and P are constants.

B Derivation of $\mathcal{X}^{(1,0)}$

If we apply Green's theorem on $\phi^{(1,0)}$ and $\phi^{(2,0)}$ we find, with

$$G = \{(x_0, z) \in \mathbb{R}_2 | 0 < x_0 < L \wedge -h < z < 0\}$$

and ∇ as the gradient operator in x_0 and z:

$$\begin{aligned}
0 \;=\; & \int\int_G \phi^{(1,0)}\nabla^2\phi^{(2,0)} - \phi^{(2,0)}\nabla^2\phi^{(1,0)} dx dz = \\
& \int_{\partial G} \phi^{(1,0)}\frac{\partial}{\partial n}\phi^{(2,0)} - \phi^{(2,0)}\frac{\partial}{\partial n}\phi^{(1,0)} dl = \\
& \phi^{(1,0)} \int_{x_0=0}^{L} \left.\frac{\partial\phi^{(2,0)}}{\partial z}\right|_{z=0} dx_0 + \phi^{(1,0)} \int_{z=-h}^{0} \left.\frac{\partial\phi^{(2,0)}}{\partial x_0}\right|_{x_0=L} dz + \\
& \qquad -\phi^{(1,0)} \int_{z=-h}^{0} \left.\frac{\partial\phi^{(2,0)}}{\partial x_0}\right|_{x_0=0} dz = \\
& \phi^{(1,0)} \left(\int_{z=-h}^{0} \left.\frac{\partial\phi^{(2,0)}}{\partial x_0}\right|_{x_0=L} dz - \int_{z=-h}^{0} \left.\frac{\partial\phi^{(2,0)}}{\partial x_0}\right|_{x_0=0} dz + \right. \\
& \qquad \left. + \frac{\omega}{g} \int_{x_0=0}^{L} \left.\left(i\phi_{x_0}^{(1,1)*}\phi^{(1,1)} + *\right)_{x_0}\right|_{z=0} dx_0 \right)
\end{aligned}$$

resulting in

$$\begin{aligned}
\int_{z=-h}^{0} \left.\frac{\partial\phi^{(2,0)}}{\partial x_0}\right|_{x_0=0} dz + \frac{\omega}{g}\left.\left(i\phi_{x_0}^{(1,1)*}\phi^{(1,1)} + *\right)\right|_{z=0,x_0=0} = \\
\frac{\omega}{g}\left.\left(i\phi_{x_0}^{(1,1)*}\phi^{(1,1)} + *\right)\right|_{z=0,x_0=L}
\end{aligned} \tag{B.1}$$

after we discarded $\int_{z=-h}^{0} \left.\frac{\partial\phi^{(2,0)}}{\partial x_0}\right|_{x_0=L} dz$ for $L \to \infty$.
This is allowed as for large x_0 we have

$$\phi_{x_0}^{(1,1)} \to ik_0\phi^{(1,1)} \quad \text{for} \quad x_0 \to \infty \tag{B.2}$$

This implies

$$(i\phi_{x_0}^{(1,1)*}\phi^{(1,1)} + *) \to 2k_0 \left|\phi^{(1,1)}\right|^2 \quad \text{for} \quad x_0 \to \infty \tag{B.3}$$

and this is a function of x_1 and t_1 only! From eqs. (5.4), (5.5) and (5.6) we now find that for $x_0 \to \infty$ we have $\phi^{(2,0)} = \phi^{(2,0)}(x_1, t_1)$ and this implies that $\int_{z=-h}^{0} \left.\frac{\partial\phi^{(2,0)}}{\partial x_0}\right|_{x_0=L} dz \to 0$ for $x_0 \to \infty$.

Integration of eq.(5.7) yields using eqs. (3.4) and (3.2)

$$h\mathcal{X}_{t_1}^{(1,0)} = h\phi_{x_1}^{(1,0)} + \int_{z=-h}^{0} \left.\frac{\partial\phi^{(2,0)}}{\partial x_0}\right|_{x_0=0} dz + \frac{\omega}{g}\left.\left(i\phi_{x_0}^{(1,1)*}\phi^{(1,1)} + *\right)\right|_{z=0,x_0=0}$$

with eq.(B.1) we can now find

$$\mathcal{X}_{t_1}^{(1,0)} = \phi_{x_1}^{(1,0)} + \frac{\omega}{gh}\left.\left(i\phi_{x_0}^{(1,1)*}\phi^{(1,1)} + *\right)\right|_{z=0,x_0\to\infty} \tag{B.4}$$

yielding with eqs. (5.9), (3.14) and (4.16)

$$\mathcal{X}_{t_1}^{(1,0)} = \frac{g^2\left[2\omega k_0 + C_g[k_0^2 - \sigma^2]\right]}{4\omega^2\left(C_g^2 - gh\right)}\left(|A(-C_g t_1)|^2 + S\right) + \frac{k_0 g}{2\omega h}|A(-C_g t_1)|^2$$

With this result we can find an expression for $\mathcal{X}^{(1,0)}$:

$$\begin{aligned}\mathcal{X}^{(1,0)}(t_1) \quad = \quad & \left(\frac{g^2\left[2\omega k_0 + C_g[k_0^2 - \sigma^2]\right]}{4\omega^2\left(C_g^2 - gh\right)} + \frac{k_0 g}{2\omega h}\right) * \\ & \left(-\frac{1}{C_g}B(-C_g t_1) + K\right) + \frac{g^2\left[2\omega k_0 + C_g[k_0^2 - \sigma^2]\right]}{4\omega^2\left(C_g^2 - gh\right)} S\, t_1\end{aligned}$$

C Some integral expressions

In the derivation of the expression for $\phi^{(2,2)}$ the several integrals have to be determined. With the relations

$$
\begin{aligned}
\sigma &= k_0 \tanh q_0 \\
\sigma &= -k_n \tan q_n \\
4\sigma &= \beta_0 \tanh(\beta_0 h) \\
4\sigma &= -\alpha_n \tan(\alpha_n h)
\end{aligned}
$$

the integrals become:

$$\int_{-h}^{0} \cosh(\beta_0 (z+h))\, dz = \frac{1}{\beta_0} \sinh(\beta_0 h)$$

$$\int_{-h}^{0} \cosh^2(\beta_0 (z+h))\, dz = \frac{1}{2}\left(h + \frac{1}{2\beta_0}\sinh(2\beta_0 h)\right)$$

$$\int_{-h}^{0} \cosh(\beta_0 (z+h)) \cos(\alpha_n (z+h))\, dz = 0 \qquad (n \in I\!N^+)$$

$$\int_{-h}^{0} \cosh(\beta_0 (z+h)) \cos Q_n dz = \frac{3\sigma}{k_n^2 + \beta_0^2} \cosh(\beta_0 h) \cos q_n$$

$$(n \in I\!N^+)$$

$$\int_{-h}^{0} \cosh(\beta_0 (z+h)) \cosh Q_0 dz = \frac{-3\sigma}{k_0^2 - \beta_0^2} \cosh(\beta_0 h) \cosh q_0$$

$$\int_{-h}^{0} \cosh(\beta_0 (z+h)) \cosh(2Q_0)\, dz = \frac{-4\sigma^3}{k_0^2 (4k_0^2 - \beta_0^2)} \cosh(\beta_0 h) \cosh^2 q_0$$

$$\int_{-h}^{0} \cosh(\beta_0 (z+h)) \left[\cosh Q_0 \cos Q_n + i \sinh Q_0 \sin Q_n\right] dz =$$

$$\frac{\sigma}{k_0 k_n} \frac{-2k_0 k_n + i\left(k_n^2 - k_0^2 + 4\sigma^2\right)}{k_0^2 - k_n^2 - \beta_0^2 + 2ik_0 k_n} \cosh q_0 \cos q_n \cosh(\beta_0 h)$$

$$(n \in I\!N^+)$$

$$\int_{-h}^{0} \cosh(\beta_0 (z+h)) \cos(Q_n + Q_p)\, dz =$$

$$-\frac{\sigma}{k_n k_p} \frac{(k_n - k_p)^2 + 4\sigma^2}{(k_n + k_p)^2 + \beta_0^2} \cosh(\beta_0 h) \cos q_n \cos q_p \qquad (n, p \in I\!N^+)$$

$$\int_{-h}^{0} \cos\left(\alpha_n (z+h)\right) dz = \frac{1}{\alpha_n} \sin\left(\alpha_n h\right) \qquad (n \in I\!N^+)$$

$$\int_{-h}^{0} \cos^2\left(\alpha_n (z+h)\right) dz = \frac{1}{2}\left(h + \frac{1}{2\alpha_n}\sin\left(2\alpha_n h\right)\right) \qquad (n \in I\!N^+)$$

$$\int_{-h}^{0} \cos\left(\alpha_n (z+h)\right) \cos\left(\alpha_p (z+h)\right) dz = 0 \qquad (n, p \in I\!N^+, n \neq p)$$

$$\int_{-h}^{0} \cos\left(\alpha_n (z+h)\right) \cosh Q_0 = \frac{-3\sigma}{\alpha_n^2 + k_0^2} \cos\left(\alpha_n h\right) \cosh q_0$$

$$(n \in I\!N^+)$$

$$\int_{-h}^{0} \cos\left(\alpha_n (z+h)\right) \cos Q_p dz = \frac{3\sigma}{k_p^2 - \alpha_n^2} \cos\left(\alpha_n h\right) \cos q_p$$

$$(n, p \in I\!N^+)$$

$$\int_{-h}^{0} \cos\left(\alpha_n (z+h)\right) \cosh\left(2Q_0\right) dz = \frac{-4\sigma^3}{k_0^2\left(4k_0^2 + \alpha_n^2\right)} \cos\left(\alpha_n h\right) \cosh^2 q_0$$

$$(n \in I\!N^+)$$

$$\int_{-h}^{0} \cos\left(\alpha_n (z+h)\right) \left[\cosh Q_0 \cos Q_p + i \sinh Q_0 \sin Q_p\right] dz =$$

$$\frac{\sigma}{k_0 k_p} \frac{-2k_0 k_p + i\left(k_p^2 - k_0^2 + 4\sigma^2\right)}{k_0^2 - k_p^2 + \alpha_n^2 + 2ik_0 k_p} \cos\left(\alpha_n h\right) \cosh q_0 \cos q_p$$

$$(n, p \in I\!N^+)$$

$$\int_{-h}^{0} \cos\left(\alpha_m (z+h)\right) \cos\left(Q_n + Q_p\right) dz =$$

$$\frac{\sigma}{k_n k_p} \frac{4\sigma^2 + \left(k_n - k_p\right)^2}{\alpha_m^2 - \left(k_n + k_p\right)^2} \cos\left(\alpha_m h\right) \cos q_n \cos q_p \qquad (n, m, p \in I\!N^+)$$