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LACUNARY STATISTICAL CONVERGENCE

JOHN ALBERT FRIDY AND CIHAN ORHAN

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## LACUNARY STATISTICAL CONVERGENCE

J. A. FRIDY AND C. ORHAN

The sequence  $x$  is statistically convergent to  $L$  provided that for each  $\varepsilon > 0$ ,

$$\lim_n n^{-1} \{\text{the number of } k \leq n: |x_k - L| \geq \varepsilon\} = 0.$$

In this paper we study a related concept of convergence in which the set  $\{k: k \leq n\}$  is replaced by  $\{k: k_{r-1} < k \leq k_r\}$ , for some lacunary sequence  $\{k_r\}$ . The resulting summability method is compared to statistical convergence and other summability methods, and questions of uniqueness of the limit value are considered.

**1. Introduction.** A complex number sequence  $x$  is said to be *statistically convergent* to the number  $L$  if for every  $\varepsilon > 0$ ,

$$(1) \quad \lim_n \frac{1}{n} |\{k \leq n: |x_k - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write  $S\text{-}\lim x = L$  or  $x_k \rightarrow L(S)$ . We shall also use  $S$  to denote the set of all statistically convergent sequences. The idea of statistical convergence was introduced by Fast [4] and studied by several authors [2], [3], [5], [6], [11]. There is a natural relationship [2] between statistical convergence and strong Cesàro summability:

$$|\sigma_1| := \left\{ x: \text{for some } L, \lim_n \left( \frac{1}{n} \sum_{k=1}^n |x_k - L| \right) = 0 \right\}.$$

By a *lacunary sequence* we mean an increasing integer sequence  $\theta = \{k_r\}$  such that  $k_0 = 0$  and  $h_r := k_r - k_{r-1} \rightarrow \infty$  as  $r \rightarrow \infty$ . Throughout this paper the intervals determined by  $\theta$  will be denoted by  $I_r := (k_{r-1}, k_r]$ , and the ratio  $k_r/k_{r-1}$  will be abbreviated by  $q_r$ . There is a strong connection [7] between  $|\sigma_1|$  and the sequence space  $N_\theta$ , which is defined by

$$N_\theta := \left\{ x: \text{for some } L, \lim_r \left( \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \right) = 0 \right\}.$$

The purpose of this paper is to introduce and study a concept of convergence that is related to statistical convergence (1) in the same way that  $N_\theta$  is related to  $|\sigma_1|$ .

DEFINITION. Let  $\theta$  be a lacunary sequence; the number sequence  $x$  is  $S_\theta$ -convergent to  $L$  provided that for every  $\varepsilon > 0$ ,

$$(2) \quad \lim_r \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write  $S_\theta\text{-lim } x = L$  or  $x_k \rightarrow L(S_\theta)$ , and we define

$$S_\theta := \{x : \text{for some } L, S_\theta\text{-lim } x = L\}.$$

The limits in (1) and (2) can be expressed using matrix transformations of the characteristic function  $\chi_K$  of the set

$$K = K(x, L, \varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}.$$

The limit in (1) is  $\lim_n (C_1 \chi_K)_n = 0$ , where  $C_1$  is the Cesàro mean; the limit in (2) is  $\lim_n (C_\theta \chi_K)_n = 0$ , where  $C_\theta$  is the matrix given by

$$C_\theta[n, k] := \begin{cases} \frac{1}{h_r}, & \text{if } k \in I_r, \\ 0, & \text{if } k \notin I_r. \end{cases}$$

In this form  $S_\theta$ -convergence is seen to be a part of “A-density convergence” as defined in [8] and [3].

In the next section we establish inclusion relations between  $S_\theta$  and  $N_\theta$  and also between  $S_\theta$  and  $S$ . In §3 we show that the  $S_\theta$ -limit of a given sequence  $x$  is not necessarily unique for different  $\theta$ 's, but different  $S_\theta$ -limits cannot occur if  $x \in S$ . In the final section we get a relationship between  $S_\theta$ -convergence and strong almost convergence, a concept introduced by Maddox [10] and (independently) by Freedman et al. [7].

**2. Inclusion theorems.** In this section we first give some inclusion relations between  $N_\theta$ - and  $S_\theta$ -convergence and show that they are equivalent for bounded sequences. We also study the inclusions  $S \subseteq S_\theta$  and  $S_\theta \subseteq S$  under certain restrictions on  $\theta = \{k_r\}$ .

**THEOREM 1.** *Let  $\theta = \{k_r\}$  be a lacunary sequence; then*

- (i) (a)  $x_k \rightarrow L(N_\theta)$  implies  $x_k \rightarrow L(S_\theta)$ , and
- (b)  $N_\theta$  is a proper subset of  $S_\theta$ ;
- (ii)  $x \in l_\infty$  and  $x_k \rightarrow L(S_\theta)$  imply  $x_k \rightarrow L(N_\theta)$ ;
- (iii)  $S_\theta \cap l_\infty = N_\theta \cap l_\infty$ ,

where  $l_\infty$  denotes the set of bounded sequences.

Before proving this theorem we remark that this result is included by Theorem 8 in [3], where Connor bases the proof on the concept of ideals in  $l_\infty$ ; we give a direct proof.

*Proof.* (a) If  $\varepsilon > 0$  and  $x_k \rightarrow L(N_\theta)$  we can write

$$\sum_{k \in I_r} |x_k - L| \geq \sum_{\substack{k \in I_r \\ |x_k - L| \geq \varepsilon}} |x_k - L| \geq \varepsilon |\{k \in I_r : |x_k - L| \geq \varepsilon\}|,$$

which yields the result.

(b) In order to establish that the inclusion  $N_\theta \subseteq S_\theta$  in (i) is proper, let  $\theta$  be given and define  $x_k$  to be  $1, 2, \dots, [\sqrt{h_r}]$  at the first  $[\sqrt{h_r}]$  integers in  $I_r$ , and  $x_k = 0$  otherwise. Note that  $x$  is not bounded. We have, for every  $\varepsilon > 0$ ,

$$\frac{1}{h_r} |\{k \in I_r : |x_k - 0| \geq \varepsilon\}| = \frac{[\sqrt{h_r}]}{h_r} \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

i.e.,  $x_k \rightarrow 0(S_\theta)$ . On the other hand,

$$\frac{1}{h_r} \sum_{k \in I_r} |x_k - 0| = \frac{1}{h_r} \frac{[\sqrt{h_r}]( [\sqrt{h_r}] + 1 )}{2} \rightarrow \frac{1}{2} \neq 0;$$

hence  $x_k \not\rightarrow 0(N_\theta)$ .

(ii) Suppose that  $x_k \rightarrow L(S_\theta)$  and  $x \in l_\infty$ , say  $|x_k - L| \leq M$  for all  $k$ . Given  $\varepsilon > 0$ , we get

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |x_k - L| \geq \varepsilon}} |x_k - L| + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |x_k - L| < \varepsilon}} |x_k - L| \\ &\leq \frac{M}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| + \varepsilon, \end{aligned}$$

from which the result follows.

We remark that the example given in (i) shows that the boundedness condition cannot be omitted from the hypothesis of Theorem 1 (ii).

(iii) This is an immediate consequence of (i) and (ii).

Since any  $N_\theta$ -summable sequence is  $C_\theta$ -summable, we conclude from Theorem 1 (ii) that any bounded  $S_\theta$ -summable sequence is also  $C_\theta$ -summable.

**LEMMA 2.** *For any lacunary sequence  $\theta$ ,  $S\text{-}\lim x = L$  implies  $S_\theta\text{-}\lim x = L$  if and only if  $\liminf_r q_r > 1$ . If  $\liminf_r q_r = 1$ , then there exists a bounded  $S_\theta$ -summable sequence that is not  $S$ -summable (to any limit).*

*Proof.* Suppose first that  $\liminf_r q_r > 1$ ; then there exists a  $\delta > 0$  such that  $q_r \geq 1 + \delta$  for sufficiently large  $r$ , which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1 + \delta}.$$

If  $x_k \rightarrow L(S)$ , then for every  $\varepsilon > 0$  and for sufficiently large  $r$ , we have

$$\begin{aligned} \frac{1}{k_r} |\{k \leq k_r : |x_k - L| \geq \varepsilon\}| &\geq \frac{1}{k_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| \\ &\geq \frac{\delta}{1 + \delta} \cdot \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}|; \end{aligned}$$

this proves the sufficiency.

Conversely, suppose that  $\liminf_r q_r = 1$ . Proceeding as in [7; p. 510] we can select a subsequence  $\{k_{r(j)}\}$  of the lacunary sequence  $\theta$  such that

$$\frac{k_{r(j)}}{k_{r(j)-1}} < 1 + \frac{1}{j} \quad \text{and} \quad \frac{k_{r(j)-1}}{k_{r(j-1)}} > j, \quad \text{where } r(j) \geq r(j-1) + 2.$$

Now define a bounded sequence  $x$  by  $x_i = 1$  if  $i \in I_{r(j)}$  for some  $j = 1, 2, \dots$  and  $x_i = 0$  otherwise. It is shown in [7; p. 510] that  $x \notin N_\theta$  but  $x \in |\sigma_1|$ . The above Theorem 1 (ii) implies that  $x \notin S_\theta$ , but it follows from Theorem 2.1 of [2] that  $x \in S$ . Hence  $S \not\subseteq S_\theta$ , and the proof is complete.

**LEMMA 3.** *For any lacunary sequence  $\theta$ ,  $S\text{-}\lim x = L$  implies  $S_\theta\text{-}\lim x = L$  if and only if  $\limsup_r q_r < \infty$ . If  $\limsup_r q_r = \infty$ , then there exists a bounded  $S$ -summable sequence that is not  $S_\theta$ -summable (to any limit).*

*Proof.* If  $\limsup_r q_r < \infty$ , then there is an  $H > 0$  such that  $q_r < H$  for all  $r$ . Suppose that  $x_k \rightarrow L(S_\theta)$ , and let  $N_r := |\{k \in I_r : |x_k - L| \geq \varepsilon\}|$ . By (2), given  $\varepsilon > 0$ , there is an  $r_0 \in \mathbb{N}$  such that

$$(3) \quad \frac{N_r}{h_r} < \varepsilon \quad \text{for all } r > r_0.$$

Now let  $M := \max\{N_r : 1 \leq r \leq r_0\}$  and let  $n$  be any integer satisfying

$k_{r-1} < n \leq k_r$ ; then we can write

$$\begin{aligned}
 \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| &\leq \frac{1}{k_{r-1}} |\{k \leq k_r : |x_k - L| \geq \varepsilon\}| \\
 &= \frac{1}{k_{r-1}} \{N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r\} \\
 &\leq \frac{M}{k_{r-1}} \cdot r_0 + \frac{1}{k_{r-1}} \left\{ h_{r_0+1} \frac{N_{r_0+1}}{h_{r_0+1}} + \cdots + h_r \frac{N_r}{h_r} \right\} \\
 &\leq \frac{r_0 \cdot M}{k_{r-1}} + \frac{1}{k_{r-1}} \left( \sup_{r > r_0} \frac{N_r}{h_r} \right) \{h_{r_0+1} + \cdots + h_r\} \\
 &\leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon \cdot \frac{k_r - k_{r_0}}{k_{r-1}}, \quad \text{by (3),} \\
 &\leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon \cdot q_r \leq \frac{r_0 \cdot M}{k_{r-1}} + \varepsilon H,
 \end{aligned}$$

and the sufficiency follows immediately.

Conversely, suppose that  $\limsup_r q_r = \infty$ . Following the idea in [7; p. 511] we can select a subsequence  $\{k_{r(j)}\}$  of the lacunary sequence  $\theta = \{k_r\}$  such that  $q_{r(j)} > j$ , and define a bounded sequence by  $x_i = 1$  if  $k_{r(j)-1} < i \leq 2k_{r(j)-1}$  for some  $j = 1, 2, \dots$ , and  $x_i = 0$  otherwise. It is shown in [7; p. 5.11] that  $x \in N_\theta$  but  $x \notin |\sigma_1|$ . By Theorem 1 (i) we conclude that  $x \in S_\theta$ , but Theorem 2.1 of [2] implies that  $x \notin S$ . Hence,  $S_\theta \not\subseteq S$ .

Combining Lemma 2 and Lemma 3 we get

**THEOREM 4.** *Let  $\theta$  be a lacunary sequence; then  $S = S_\theta$  if and only if*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty;$$

*then  $S$ -lim  $x = L$  implies  $S_\theta$ -lim  $x = L$ .*

For an example of a lacunary sequence satisfying the conditions of Theorem 4, we can take  $k_r = 2^r$  for  $r > 0$ , whence  $S_{\{2^r\}} = S$ . We remark that the examples given in Lemmas 2 and 3 illustrate the difference between  $S$ -convergence and  $S_\theta$ -convergence.

We conclude this section with the following observation. Buck [1, Theorem 3.2] proved that if a real sequence is  $C_1$ -summable to its finite limit inferior, then the sequence “converges to that point for almost all  $n$ ” (i.e., it is statistically convergent to its limit inferior [2]). Note that this result remains true if we replace limit inferior by

limit superior. For each subset  $K$  of  $\mathbb{N}$ , define

$$D(K) := \lim_r (C_\theta \chi_K)_r = \lim_r \frac{|K \cap I_r|}{h_r};$$

then  $D$  is a density [8; p. 296], and it is not hard to get a result for  $S_\theta$ -convergence that is analogous to Buck's. To be precise, the following result is such an analogue.

**PROPOSITION 5.** *If the real number sequence  $x$  is  $C_\theta$ -summable to either its finite limit inferior or finite limit superior, then  $x$  is  $S_\theta$ -convergent to that value.*

**3. Uniqueness of  $S_\theta$ -limit and lacunary refinements.** It is easy to see that, for any fixed  $\theta$ , the  $S_\theta$ -limit is unique. It is possible, however, for a sequence—even a bounded one—to have different  $S_\theta$ -limits for different  $\theta$ 's. This can be seen by applying Theorem 1 (i) to the sequence  $x$  given in [7, proof of Theorem 2.1] for which  $N_{\theta_1}$ - $\lim x = 0$  and  $N_{\theta_2}$ - $\lim x = 1$ . The next theorem shows that this situation cannot occur if  $x \in S$ ; in other words, every  $S_\theta$  method is consistent with the  $S$ -method.

**THEOREM 6.** *If  $x \in S \cap S_\theta$ , then  $S_\theta$ - $\lim x = S$ - $\lim x$ .*

*Proof.* Suppose  $S$ - $\lim x = L$  and  $S_\theta$ - $\lim x = L'$ , and  $L \neq L'$ . For  $\varepsilon < \frac{1}{2}|L - L'|$  we get

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L'| \geq \varepsilon\}| = 1.$$

Consider the  $k_m$ th term of the statistical limit expression  $n^{-1}|\{k \leq n : |x_k - L'| \geq \varepsilon\}|$ :

$$(4) \quad \frac{1}{k_m} \left| \left\{ k \in \bigcup_{r=1}^m I_r : |x_k - L'| \geq \varepsilon \right\} \right| \\ = \frac{1}{k_m} \sum_{r=1}^m |\{k \in I_r : |x_k - L'| \geq \varepsilon\}| = \frac{1}{\sum_{r=1}^m h_r} \sum_{r=1}^m h_r t_r,$$

where  $t_r = h_r^{-1} |\{k \in I_r : |x_k - L'| \geq \varepsilon\}| \rightarrow 0$  because  $x_k \rightarrow L'(S_\theta)$ . Since  $\theta$  is a lacunary sequence, (4) is a regular weighted mean transform of  $t$ , and therefore it, too, tends to zero as  $m \rightarrow \infty$ . Also, since this is a subsequence of  $\{n^{-1}|\{k \leq n : |x_k - L'| \geq \varepsilon\}|\}_{n=1}^\infty$ , we infer that

$$\frac{1}{n} |\{k \leq n : |x_k - L'| \geq \varepsilon\}| \rightarrow 1,$$

and this contradiction shows that we cannot have  $L \neq L'$ .

We now consider the inclusion of  $S_{\theta'}$  by  $S_{\theta}$ , where  $\theta'$  is a lacunary refinement of  $\theta$ . Recall [7] that the lacunary sequence  $\theta' = \{k'_r\}$  is called a *lacunary refinement* of the lacunary sequence  $\theta = \{k_r\}$  if  $\{k_r\} \subseteq \{k'_r\}$ .

**THEOREM 7.** *If  $\theta'$  is a lacunary refinement of  $\theta$  and  $x_k \rightarrow L(S_{\theta'})$ , then  $x_k \rightarrow L(S_{\theta})$ .*

*Proof.* Suppose each  $I_r$  of  $\theta$  contains the points  $\{k'_{r,i}\}_{i=1}^{\nu(r)}$  of  $\theta'$  so that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < \cdots < k'_{r,\nu(r)} = k_r, \quad \text{where } I'_{r,i} = (k'_{r,i-1}, k'_{r,i}].$$

Note that for all  $r$ ,  $\nu(r) \geq 1$  because  $\{k_r\} \subseteq \{k'_r\}$ . Let  $\{I_j^*\}_{j=1}^{\infty}$  be the sequence of abutting intervals  $\{I'_{r,i}\}$  ordered by increasing right end points. Since  $x_k \rightarrow L(S_{\theta'})$ , we get, for each  $\varepsilon > 0$ ,

$$(5) \quad \lim_j \sum_{I_j^* \subset I_r} \frac{1}{h_r^*} |\{k \in I_j^* : |x_k - L| \geq \varepsilon\}| = 0.$$

As before we write,  $h_r = k_r - k_{r-1}$ ,  $h'_{r,i} = k'_{r,i} - k'_{r,i-1}$ , and  $h'_{r,1} = k'_{r,1} - k_{r-1}$ . For each  $\varepsilon > 0$  we have

$$(6) \quad \begin{aligned} & \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| \\ &= \frac{1}{h_r} \sum_{I_j^* \subset I_r} h_j^* \frac{1}{h_j^*} |\{k \in I_j^* : |x_k - L| \geq \varepsilon\}| \\ &= \frac{1}{h_r} \sum_{I_j^* \subset I_r} h_j^* (C_{\theta'} \chi_K)_j, \end{aligned}$$

where  $\chi_K$  is the characteristic function of the set  $K := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ . By (5),  $C_{\theta'} \chi_K$  is a null sequence, and (6) is a regular weighted mean transform of  $C_{\theta'} \chi_K$ . Hence, the transform (6) also tends to zero as  $r \rightarrow \infty$ .

We conclude this section by observing that Theorem 7 establishes inclusion between two lacunary methods *only* when one sequence is a lacunary refinement of the other. The example cited at the beginning of this section shows that  $S_{\theta}$  can be inconsistent with  $S_{\theta'}$ . A general description of inclusion between two arbitrary lacunary methods is left as an open problem.



**4. Strong almost convergence and  $S_\theta$ -convergence.** The idea of almost convergence was introduced by Lorentz [9]: the sequence  $x$  is said to be *almost convergent* to  $L$  if

$$\lim_n \frac{1}{n} \sum_{i=m+1}^{m+n} (x_i - L) = 0, \quad \text{uniformly in } m.$$

Maddox [10] and (independently) Freedman et al. [7] introduced the notion of strong almost convergence: the sequence  $x$  is said to be *strongly almost convergent* to  $L$  if

$$\lim_n \frac{1}{n} \sum_{i=m+1}^{m+n} |x_i - L| = 0, \quad \text{uniformly in } m.$$

Let  $c$ ,  $AC$  and  $[AC]$ , respectively, denote the sets of all convergent, almost convergent, and strongly almost convergent sequences. It is known [10] that

$$(7) \quad c \subsetneq [AC] \subsetneq AC \subsetneq l_\infty.$$

**THEOREM 8.** *If  $\mathcal{L}$  denotes the set of all lacunary sequences, then*

$$[AC] = l_\infty \cap \left( \bigcap_{\theta \in \mathcal{L}} S_\theta \right).$$

*Proof.* By [7, Theorem 3.1], the relations (7) and Theorem 1 (iii), we have

$$\begin{aligned} l_\infty \supset [AC] &= \bigcap_{\theta \in \mathcal{L}} N_\theta = l_\infty \cap \left( \bigcap_{\theta \in \mathcal{L}} N_\theta \right) \bigcap_{\theta \in \mathcal{L}} (l_\infty \cap N_\theta) \\ &= \bigcap_{\theta \in \mathcal{L}} (l_\infty \cap S_\theta) = l_\infty \cap \left( \bigcap_{\theta \in \mathcal{L}} S_\theta \right). \end{aligned}$$

Finally we remark that in contrast to [7, Theorem 3.1] where it was proved that  $[AC] = \bigcap N_\theta$ , the factor  $l_\infty$  cannot be omitted from Theorem 8. For,  $\bigcap S_\theta \not\subseteq l_\infty$  and  $\bigcap N_\theta = [AC]$  is a proper subset of  $\bigcap S_\theta$ . To see this consider the sequence  $x$  defined by  $x_k = m$ , if  $k = m^2$  for  $m = 1, 2, \dots$ , and  $x_k = 0$  otherwise. Observe that  $x$  is not bounded, so it is not strongly almost convergent. On the other hand, for any lacunary sequence  $\theta$ , we have

$$\frac{1}{h_r} |\{k \in I_r : x_k \neq 0\}| \leq \frac{\sqrt{h_r}}{h_r} \rightarrow 0, \quad \text{as } r \rightarrow \infty;$$

hence,  $x_k \rightarrow O(S_\theta)$ .

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KENT STATE UNIVERSITY  
KENT, OH 44242  
U.S.A.

AND

ANKARA UNIVERSITY  
ANKARA, 06100  
TURKEY



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