LACUNARY STRONG CONVERGENCE OF DIFFERENCE SEQUENCES WITH RESPECT TO A MODULUS FUNCTION

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Abstract. **A sequence $\theta = (k_r)$ of positive integers is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$.**

Let ω be the set of all sequences of complex numbers and f be a modulus function. Then we define

$$N_\theta(\Delta^m, f) = \{x \in \omega : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f(|\Delta^m x_k - l|) = 0 \text{ for some } l\}.$$

$$N_\theta^0(\Delta^m, f) = \{x \in \omega : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f(|\Delta^m x_k|) = 0\}.$$

$$N_\theta^\infty(\Delta^m, f) = \{x \in \omega : \sup_r \frac{1}{h_r} \sum_{k \in I_r} f(|\Delta^m x_k|) < \infty\}.$$

where $\Delta x_k = x_k - x_{k+1}$, $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ and m is a fixed positive integer.

In this study we give various properties and inclusion relations on these sequence spaces.

1. INTRODUCTION

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c, and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$, where $k \in N = \{1, 2, ...\}$ the set of positive integers.

The difference sequence space $X(\Delta)$ was introduced by Kizmaz [4] as follows:

$$X(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in X\} \text{ for } X = \ell_\infty, c \text{ and } c_0,$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

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The difference sequence spaces were generalized by Et and Çolak [2] as follows:

$X(\Delta^m) = \{x = (x_k) \in w : \Delta^m x = (\Delta^m x_k) \in X\}$ for $X = \ell_\infty$, c and c_0,

where $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, ...$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. We denote by $I_r = (k_{r-1}, k_r]$ the intervals determined by θ and $q_r = \frac{k_r}{k_{r-1}}$ for $r = 0, 1, 2, ...$. The space of lacunary convergent sequences N_θ was defined by Freedman et all [3] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{k \in I_r} |x_k - \ell| = 0, \text{ for some } \ell \right\}. \tag{1.1}$$

The space N_θ is a BK-space with the norm

$$\|x\|_\theta = \sup_r h_r^{-1} \sum_{k \in I_r} |x_k|. \tag{1.2}$$

N_θ^0 denotes the subset of N_θ those sequences for which $\ell = 0$ in the definition of N_θ. $(N_\theta^0, \|\cdot\|_\theta)$ is also a BK-space. There is a relation (see for instance [3]) between N_θ and the space $|\sigma_1|$ of strongly Cesaro summable sequences defined by

$$|\sigma_1| = \left\{ x = (x_k) : \lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} |x_k - \ell| = 0, \text{ for some } \ell \right\}. \tag{1.3}$$

In the special case $\theta = (2^r)$ we have $N_\theta = |\sigma_1|$.

The notion of a modulus function was introduced by Nakano [7]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$

(ii) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$

(iii) f increasing,

(iv) f continuous from the right at zero.

Since $|f(x) - f(y)| \leq f|(x - y)|$, it follows from (iv) that f is continuous on $[0, \infty)$. Furthermore, we have $f(nx) \leq nf(x)$ for all $n \in N$ from condition (ii), and so $f(x) = f\left(nx\frac{1}{n}\right) \leq nf(x/n)$, hence $\frac{1}{n} f(x) \leq f(x/n)$ for all $n \in N$.

A modulus may be bounded or unbounded. For example, $f(x) = x^p$ for $0 < p \leq 1$ is unbounded, but $f(x) = \frac{x}{x+1}$ is bounded.

Subsequently modulus function have been studied by Ruckle [9], Connor [1], Maddox [5], Pehlivan and Fisher [8], Malkowsky and Savaş [6].

2. MAIN RESULTS

In this section, we extend Δ^m-lacunary strongly convergent sequences which were defined by B.C. Tripathy, M. Et and S. Mahanta [10], and give the relation between Δ^m-lacunary strong convergence and Δ^m-lacunary strong convergence with respect to a modulus.

Definition 2.1. Let f be a modulus function. We define the following sets

$$\begin{aligned} N_\theta^0\left(\Delta^m, f\right) &= \left\{x \in w : \lim_r h_r^{-1} \sum_{k \in I_r} f\left(\left|\Delta^m x_k\right|\right) = 0\right\} \\ N_\theta\left(\Delta^m, f\right) &= \left\{x \in w : \lim_r h_r^{-1} \sum_{k \in I_r} f\left(\left|\Delta^m x_k - \ell\right|\right) = 0, \text{ for some } \ell\right\} \\ N_\theta^\infty\left(\Delta^m, f\right) &= \left\{x \in w : \sup_r h_r^{-1} \sum_{k \in I_r} f\left(\left|\Delta^m x_k\right|\right) < \infty\right\}. \end{aligned}$$

If we take $f(x) = x$ then we obtain the sequence spaces $N_\theta^0\left(\Delta^m\right), N_\theta\left(\Delta^m\right)$ and $N_\theta^\infty\left(\Delta^m\right)$ from the above sequence spaces.

Theorem 2.2. *The sets $N_\theta^0\left(\Delta^m, f\right), N_\theta\left(\Delta^m, f\right)$ and $N_\theta^\infty\left(\Delta^m, f\right)$ are linear spaces.*

Proof. We consider only $N_\theta\left(\Delta^m, f\right)$. Let $x, y \in N_\theta\left(\Delta^m, f\right)$ and $\alpha, \gamma \in C$. Then there exist positive integers K_α and M_γ such that $|\alpha| \leq K_\alpha$ and $|\gamma| \leq M_\gamma$. From the definition of f and Δ^m we have

$$\begin{aligned} & h_r^{-1} \sum_{k \in I_r} f\left(\left|\Delta^m\left(\alpha x_k + \gamma y_k\right) - \left(\alpha \ell_1 + \gamma \ell_2\right)\right|\right) \leq \\ & K_\alpha h_r^{-1} \sum_{k \in I_r} f\left(\left|\Delta^m x_k - \ell_1\right|\right) + M_\gamma h_r^{-1} \sum_{k \in I_r} f\left(\left|\Delta^m y_k - \ell_2\right|\right) \rightarrow 0, \quad r \rightarrow \infty. \end{aligned}$$

Thus $N_\theta\left(\Delta^m, f\right)$ is a linear space. □

Lemma 2.3. [8] *Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.*

Theorem 2.4. *Let f be a modulus function. Then $N_\theta\left(\Delta^m\right) \subset N_\theta\left(\Delta^m, f\right)$.*

Proof. Let $x \in N_\theta\left(\Delta^m\right)$. Then we have

$$\tau_r = h_r^{-1} \sum_{k \in I_r} \left|\Delta^m x_k - \ell\right| \rightarrow 0 \quad as \quad r \rightarrow \infty, \text{ for some } \ell. \tag{2.1}$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every u with $0 \leq u \leq \delta$. Then we can write

$$h_r^{-1} \sum_{k \in I_r} f(|\Delta^m x_k - \ell|) =$$

$$h_r^{-1} \sum_{k \in I_r, |\Delta^m x_k - \ell| \leq \delta} f(|\Delta^m x_k - \ell|) + h_r^{-1} \sum_{k \in I_r, |\Delta^m x_k - \ell| > \delta} f(|\Delta^m x_k - \ell|) \leq$$

$$h_r^{-1}(h_r \varepsilon) + h_r^{-1} 2 f(1) \delta^{-1} h_r \tau_r$$

from Lemma 2.3. Therefore $x \in N_\theta(\Delta^m, f)$. □

Theorem 2.5. *Let f be a modulus function, if $\lim_{t\to\infty} \frac{f(t)}{t} = \beta > 0$, then $N_\theta(\Delta^m, f) = N_\theta(\Delta^m)$.*

Proof. By Theorem 2.4, we need only show that $N_\theta(\Delta^m, f) \subset N_\theta(\Delta^m)$. Let $\beta > 0$ and $x \in N_\theta(\Delta^m, f)$. Since $\beta > 0$, we have $f(t) \geq \beta t$ for all $t \geq 0$. Hence we have

$$h_r^{-1} \sum_{k \in I_r} f(|\Delta^m x_k - \ell|) \geq h_r^{-1} \sum_{k \in I_r} \beta |\Delta^m x_k - \ell| = \beta h_r^{-1} \sum_{k \in I_r} |\Delta^m x_k - \ell|.$$

Therefore we have $x \in N_\theta(\Delta^m)$. □

In Theorem 2.5, the condition $\beta > 0$ can not be omitted. For this consider the following example.

Example 2.6. Let $f(x) = \ln(1 + x)$. Then $\beta = 0$. Define $\Delta^m x_k$ to be h_r at the $(k_{r-1} + 1)-$th term in I_r for every $r \geq 1$ and $x_i = 0$ otherwise. Note that $x \notin \ell_\infty(\Delta^m)$. Then we have

$$h_r^{-1} \sum_{k \in I_r} f(|\Delta^m x_k|) = h_r^{-1} \ln(1 + h_r) \to 0, \quad as \quad r \to \infty$$

and so $x \in N_\theta(\Delta^m, f)$, but

$$h_r^{-1} \sum_{k \in I_r} |\Delta^m x_k| = h_r^{-1} h_r \to 1, \quad as \quad r \to \infty,$$

and so $x \notin N_\theta(\Delta^m)$.

Theorem 2.7. *Let $m \geq 1$ be a fixed integer, then*
(i) $N_\theta^0(\Delta^{m-1}, f) \subset N_\theta^0(\Delta^m, f)$;
(ii) $N_\theta(\Delta^{m-1}, f) \subset N_\theta(\Delta^m, f)$;
(iii) $N_\theta^\infty(\Delta^{m-1}, f) \subset N_\theta^\infty(\Delta^m, f)$;
and the inclusions are strict.

Proof. The proof of the inclusions follows from the following inequality

$$h_r^{-1} \sum_{k \in I_r} f(|\Delta^m x_k|) \leq h_r^{-1} \sum_{k \in I_r} f\left(\left|\Delta^{m-1} x_k\right|\right) + h_r^{-1} \sum_{k \in I_r} f\left(\left|\Delta^{m-1} x_{k+1}\right|\right).$$

To show the inclusions are strict, let $\theta = (2^r)$ and $x = k^m$. Then $x \in N_\theta^\infty(\Delta^m, f)$, but $x \notin N_\theta^\infty(\Delta^{m-1}, f)$. If $x = (k^m)$, then $\Delta^m x = (-1)^m m!$ and $\Delta^{m-1} x_k = (-1)^{m+1} r! \left(k + \frac{(m-1)}{2}\right)$. □

Theorem 2.8. *Let $\theta = (k_r)$ be a lacunary sequence. If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then $|\sigma_1|(\Delta^m, f) = N_\theta(\Delta^m, f)$, where*

$$|\sigma_1|(\Delta^m, f) = \left\{ x \in w : \frac{1}{n} \sum_{k=1}^{n} f(|\Delta^m x_k - \ell|) = 0, \text{ for some } \ell \right\}.$$

Proof. Suppose that $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$ for all $r \geq 1$. Furthermore we have $\frac{k_r}{h_r} \leq \frac{(1+\delta)}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$, for all $r \geq 1$. Then we may write

$$\frac{1}{h_r} \sum_{i \in I_r} f(|\Delta^m x_i|) = \frac{1}{h_r} \sum_{i=1}^{k_r} f(|\Delta^m x_i|) - \frac{1}{h_r} \sum_{i=1}^{k_{r-1}} f(|\Delta^m x_i|) \tag{2.2}$$

$$= \frac{k_r}{h_r} \left(k_r^{-1} \sum_{i=1}^{k_r} f(|\Delta^m x_i|) - \frac{k_{r-1}}{h_r} \left(k_{r-1}^{-1} \sum_{i=1}^{k_{r-1}} f|\Delta^m x_i| \right) \right).$$

Now suppose that $\limsup_r q_r < \infty$ and let $\varepsilon > 0$ be given. Then there exists j_0 such that for every $j \geq j_0$

$$A_j = \frac{1}{h_j} \sum_{i \in I_j} [f(|\Delta^m x_i|)] < \varepsilon.$$

We can also choose a number $M > 0$ such that $A_j \leq M$ for all j. If $\limsup_r q_r < \infty$, then there exists a number $\beta > 0$ such that $q_r < \beta$ for every r. Now let n be any integer with $k_{r-1} < n < k_r$. Then

$$\begin{aligned}
n^{-1} \sum_{i=1}^{n} f(|\Delta^m x_i|) &\leq k_{r-1}^{-1} \sum_{i=1}^{k_r} f(|\Delta^m x_i|) \\
&= k_{r-1}^{-1} \left\{ \sum_{i \in I_1} f(|\Delta^m x_i|) + \ldots + \sum_{i \in I_r} f(|\Delta^m x_i|) \right\} \\
&= k_{r-1}^{-1} \left\{ \sum_{j=1}^{j_0} \sum_{i \in I_j} f(|\Delta^m x_i|) + \sum_{j=j_0+1}^{r} \sum_{i \in I_j} f(|\Delta^m x_i|) \right\} \\
&\leq k_{r-1}^{-1} \left\{ \sum_{j=1}^{j_0} \sum_{i \in I_j} f(|\Delta^m x_i|) + \varepsilon (k_r - k_{j_0}) k_{r-1}^{-1} \right\} \\
&= k_{r-1}^{-1} \{ h_1 A_1 + h_2 A_2 + \ldots + h_{j_0} A_{j_0} \} \\
&\quad + \varepsilon (k_r - k_{j_0}) k_{r-1}^{-1}
\end{aligned}$$

$$\begin{aligned} &\leq k_{r-1}^{-1}\left(\sup_{1\leq i\leq j_0} A_j\right)k_{j_0}+\varepsilon\left(k_r-k_{j_0}\right)k_{r-1}^{-1} \\ &< Mk_{r-1}^{-1}k_{j_0}+\varepsilon\beta \end{aligned}$$

which yields that $x\in|\sigma_1|(\Delta^m,f)$. □

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