Applied Mathematical Sciences, Vol. 5, 2011, no. 49, 2443 - 2450

Lagrange Functions Method for Solving Nonlinear Hammerstein Fredholm-Volterra Integral Equations

A. Shahsavaran

Department of Mathematics, Islamic Azad University
Borujerd Branch, Borujerd, Iran
a.shahsavaran@iaub.ac.ir
ahmad_shahsavar2000@yahoo.com

Abstract

A numerical method for solving nonlinear Fredholm-Volterra integral equations is presented. The method is based upon Lagrange functions approximations. These functions together with the Gaussian quadrature rule are then utilized to reduce the Fredholm-Volterra integral equations to the solution of algebraic equations. Some examples are included to demonstrate the validity and applicability of the technique.

Keywords: Fredholm-Volterra integral equations; Lagrange functions; Gaussian Quadrature rule

1 Introduction

Integral equations of the Hammerstein type have been one of the most important domains of applications of the ideas and methods of nonlinear functional analysis and in particular of the theory of nonlinear operators of monotone type. Various applied problems arrising in mathematical physics, mechanics and control theory leads to multivalued analogs of the Hammerstein integral equations [4]. Several numerical methods for approximating the solution of linear and nonlinear integral equations and specially Fredholm-Volterra integral equations are known [1-12]. The classical method of successive approximation for Fredholm-Hammerstein integral equations was introduced in [10]. Brunner in [3] applied a collocation type method and Ordokhani in [9] applied rationalized Haar function to nonlinear Volterra-Fredholm-Hammerstein integral equations. A variation of the Nystrom method was presented in [8]. A collocation type method was developed in [7]. The asymptotic error expansion of a collocation type method for Volterra-Hammerstein integral equations has been considered in [6]. Yousefi in [12] applied Legendre wavelets to a special type of nonlinear Volterra-Fredholm integral equations of the form

$$u(t) = f(t) + \lambda_1 \int_0^t K_1(t,x)F(u(x))dx + \lambda_2 \int_0^1 K_2(t,x)G(u(x))dx, \quad 0 \le x, t \le 1, \tag{1}$$

Yalcinbas in [11] used Taylor polynomials for solving equation (1) with $F(u) = u^p$ and $G(u) = u^q$. In this paper, we are concerned with the application of Lagrange polynomials to approximate the solution of the nonlinear Fredholm-Volterra integral equations of the form

$$u(t) = f(t) + \lambda_1 \int_0^t K_1(t,x)\phi_1(x,u(x))dx + \lambda_2 \int_0^1 K_2(t,x)\phi_2(x,u(x))dx, \quad 0 \le x, t \le 1, \tag{2}$$

where $f(t)$, and $K_1(t,x)$ and $K_2(t,x)$ are assumed to be in $L^2(R)$ on the interval $0 \le x, t \le 1$. We assume that Eq. (2) has a unique solution u to be determined.

2 Integral and Function Approximation

In this paper, since we use Gaussian quadrature rule we approximate the integral of f on $[-1, 1]$ as:

$$\int_{-1}^1 f(x)dx \approx \sum_{j=0}^k w_j f(x_j), \tag{3}$$

also, function $u(t)$ defined over [0, 1) may be expanded as

$$u(t) \simeq \sum_{i=0}^n u_i L_i(t), \tag{4}$$

with $u_i = u(t_i)$ and

$$L_i(t) = \prod_{j=0, j\neq i}^n \left(\frac{t - x_j}{x_i - x_j}\right).$$

with the property, $L_i(x_j) = \delta_{ij}$ where δ_{ij} is the *Kronecker* delta.
We can write (4) in the matrix form

$$u(t) \simeq \mathbf{u}^t \mathbf{L}(t),$$

where, $\mathbf{u} = [u_0, u_1, \ldots, u_n]^t$ and $\mathbf{L}(t) = [L_0(t), L_1(t), \ldots, L_n(t)]^t$.
In this work we consider (3) with $k = n$.

3 Nonlinear Fredholm-Volterra integral equations of Hammerstein type

Now consider the nonlinear Fredholm-Volterra integral equations given in Eq. (2). In order to use Lagrange functions, we first approximate $u(t)$ as

$$u(t) \simeq \mathbf{u}^t \mathbf{L}(t), \tag{5}$$

equations (2) and (5) gives

$$\mathbf{u}^t \mathbf{L}(t) = f(t) + \lambda_1 \int_0^t K_1(t,x)\phi_1(x, \mathbf{u}^t \mathbf{L}(x))dx + \lambda_2 \int_0^1 K_2(t,x)\phi_2(x, \mathbf{u}^t \mathbf{L}(x))dx, \tag{6}$$

for simplicity we let

$$F_1(t,x) = K_1(t,x)\phi_1(x, \mathbf{u}^t \mathbf{L}(x)), \quad F_2(t,x) = K_2(t,x)\phi_2(x, \mathbf{u}^t \mathbf{L}(x)).$$

Now by collocating transformed Eq. (6) at the $n+1$ points $t = x_i, \quad i = 0, 1, ..., n$ which are the same points of quadrature rule, we get

$$u_i = f(x_i) + \lambda_1 \int_0^{x_i} F_1(x_i, x)dx + \lambda_2 \int_0^1 F_2(x_i, x)dx. \tag{7}$$

In order to use the Gaussian quadrature rule for Eq. (7), we transfer the intervals $[0, x_i]$ and $[0, 1]$ into interval $[-1, 1]$ by transformations

$$y_1 = \frac{2}{x_i}x - 1, \quad y_2 = 2x - 1.$$

So, Eq. (7) may then be restated as

$$u_i = f(x_i) + \lambda_1 \frac{x_i}{2} \int_{-1}^1 F_1\left(x_i, \frac{x_i}{2}(y_1 + 1)\right) dy_1 + \frac{\lambda_2}{2} \int_{-1}^1 F_2\left(x_i, \frac{1}{2}(y_2 + 1)\right) dy_2, \tag{8}$$

now by using Gaussian quadrature rule, Eq. (8) may be approximated as

$$u_i \approx f(x_i) + \lambda_1 \frac{x_i}{2} \sum_{r=0}^{n_1} w_{1r} F_1 \left(x_i, \frac{x_i}{2}(y_{1r}+1) \right) + \frac{\lambda_2}{2} \sum_{r=0}^{n_2} w_{2r} F_2 \left(x_i, \frac{1}{2}(y_{2r}+1) \right), \tag{9}$$

for $i = 0, 1, ..., n$.
The system (9) including $n+1$ nonlinear equations which can be solved by usual iterative method such as Newton's method.

4 Illustrative examples

In this section we consider some nonlinear Fredholm and Volterra integral equations and also Hammerstein Fredholm-Volterra integral equations. For comparision the results, we choose the examples from [2] which have solved by Chebyshev approximation method.

5 Numerical Examples

Example 1:

$$u(t) = t^3 - (6-2e)e^t + \int_0^1 e^{(t+x)} u(x)dx, \quad 0 \le t < 1,$$

with exact solution $u(t) = t^3$.

Example 2:

$$u(t) = 2\cos(t) - 2 + 3\int_0^t \sin(t-x)(u(x))^2 dx + \frac{6}{7-6\cos(1)} \int_0^1 (1-x)\cos^2(t)(x+u(x))dx,$$

$0 \le t < 1$, with exact solution $u(t) = \cos(t)$.

Example 3:

$$u(t) = et + 1 - \int_0^1 (t+x)e^{u(x)}dx =, \quad 0 \le t < 1,$$

with exact solution $u(t) = t$.

Example 4:

$$u(t)=e^t-\frac{1}{3}e^{3t}+\frac{1}{3}+\int_0^t (u(x))^3dx, \quad 0\le t<1,$$

with exact solution $u(t)=e^t$.

Table 1-3 shows the computed error $\|e\|=\|u(t)-u_n(t)\|$ for the examples 1-3 by introduced method in [2] and presented method with $n=4$.

Table 1

t	approximated by the method introduced in [2](n=4)	approximated by presented method(n=4)
0.0	0.001×10^{-4}	3.003×10^{-9}
0.2	0.156×10^{-2}	5.500×10^{-9}
0.4	0.349×10^{-2}	7.650×10^{-9}
0.6	0.567×10^{-2}	8.300×10^{-9}
0.8	0.797×10^{-2}	5.600×10^{-9}
1.0	0.010×10^{-2}	2.100×10^{-9}

Table 2

t	approximated by the method introduced in [2](n=5)	approximated by presented method(n=4)
0.0	0.306×10^{-3}	0.468×10^{-4}
0.1	0.305×10^{-3}	0.443×10^{-4}
0.2	0.304×10^{-3}	0.387×10^{-4}
0.3	0.311×10^{-3}	0.349×10^{-4}
0.4	0.336×10^{-3}	0.394×10^{-4}
0.5	0.391×10^{-3}	0.578×10^{-4}
0.6	0.485×10^{-3}	0.913×10^{-4}
0.7	0.620×10^{-3}	0.132×10^{-3}
0.8	0.785×10^{-3}	0.161×10^{-3}
0.9	0.953×10^{-3}	0.135×10^{-3}
1.0	0.107×10^{-3}	0.142×10^{-3}

Table 3

t	approximated by the method introduced in [2](n=7)	approximated by presented method(n=4)
0.0	0.258×10^{-5}	2.340×10^{-9}
0.2	0.735×10^{-5}	3.200×10^{-9}
0.4	0.793×10^{-5}	3.700×10^{-9}
0.6	0.255×10^{-5}	4.100×10^{-9}
0.8	0.398×10^{-5}	5.000×10^{-9}
1.0	0.264×10^{-5}	4.800×10^{-9}

Table 4 shows the computed errors for example 4 by the presented method with $n = 4, 5$.

Table 4

t	approximated by presented method(n=4)	approximated by presented method(n=5)
0.0	0.0	0.0
0.1	0.432×10^{-4}	0.799×10^{-5}
0.2	0.319×10^{-4}	0.783×10^{-5}
0.3	0.468×10^{-4}	0.474×10^{-5}
0.4	0.304×10^{-3}	0.119×10^{-5}
0.5	0.304×10^{-3}	0.846×10^{-5}
0.6	0.313×10^{-3}	0.134×10^{-4}
0.7	0.244×10^{-3}	0.947×10^{-4}
0.8	0.836×10^{-3}	0.140×10^{-4}
0.9	0.264×10^{-3}	0.135×10^{-4}
1.0	0.953×10^{-3}	0.730×10^{-4}

6 Conclusion

The aim of presented work is to develop an efficient method for solving the nonlinear Fredholm-Volterra integral equations. As shown the method reduced to solving nonlinear system of algebraic equations. Illustrative examples are selected from [2]. Comparing the results states that more accuracy of the presented technique is obtained.

References

[1] A. Akyuz-Dascioglu, H. Cerdik Yaslan, An approximation method for the solution of nonlinear integral equations, Appl. Math. Comput. 174 (2006) 619-629.

[2] E. Babolian, F. Fattahzadeh, E. Golpar Raboky, A Chebyshev approximation for solving nonlinear integral equations of Hammerstein type, Appl. Math. Comput. 189 (2007) 641-646.

[3] H. Brunner, Implicity linear collocation method for nonlinear Volterra equations, J. Appl. Numer. Math. 9 (1982) 235-247.

[4] T. Cardinali, N. S. Papageorgiou, Hammerstein integral inclusions in reflexive banach spaces, American Mathematical Society, Vol. 127, No. 1, (1999) 95-103.

[5] L. M. Delves, J. L. Mohamad, Computational Methods for Integral Equations, Cambridge University Press, 1985.

[6] H. Guoqiang, Asymptotic error expansion variation of a collocation method for Volterra-Hammerstein equations, J. Appl. Numer. Math. 13 (1993) 357-369.

[7] S. Kumar, I. H. Sloan, A new collocation-type method for Hammerstein integral equations, J. Math. Comput. 48 (1987) 123-129.

[8] L. J. Lardy, A variation of Nysrtom's method for Hammerstein integral equations, J. Integral Eq. 3 (1982) 123-129.

[9] Y. Ordokhani, Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via rationalized Haar functions, Appl. Math. Comput. 180 (2006) 436-443.

[10] F. G. Tricomi, Integral Equations, Dover, 1982.

[11] S. Yashilbas, Taylor polynomial solution of nonlinear Volterra-Fredholm integral equations, Appl. Math. Comput. 127 (2002) 195-206.

[12] S. Yousefi, M. Razzaghi, Legendre wavelet method for the nonlinear Volterra-Fredholm integral equations, Math. Comp. Simul. 70 (2005) 1-8.

Received: February, 2011