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Lagrangian Submersions from Normal Almost Contact Manifolds

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Abstract. We study Lagrangian submersions from Sasakian and Kenmotsu manifolds onto Riemannian manifolds. We prove that the horizontal distribution of a Lagrangian submersion from a Sasakian manifold onto a Riemannian manifold admitting vertical Reeb vector field is integrable, but the one admitting horizontal Reeb vector field is not. We also show that the horizontal distribution of a such submersion is integrable when the total manifold is Kenmotsu. Moreover, we give some applications of these results.

1. Introduction

The theory of Riemannian submersions was initiated by O'Neill [13] and Gray [6]. In [23], the Riemannian submersions were considered between almost Hermitian manifolds by Watson under the name of almost Hermitian submersions. In this case, the Riemannian submersion is also an almost complex mapping and consequently the vertical and horizontal distribution are invariant with respect to the almost complex structure of the total manifold of the submersion. Afterwards, almost Hermitian submersions have been actively studied between different subclasses of almost Hermitian manifolds, for example, see [5]. Also, Riemannian submersions were extended to several subclasses of almost contact manifolds under the name of contact Riemannian submersions. Most of the studies related to Riemannian, almost Hermitian or contact Riemannian submersions can be found in the book [4]. The study of anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds was initiated by Şahin [17]. In this case, the fibres are anti-invariant with respect to the almost complex structure of the total manifold. He studied such submersions from a Kählerian manifold onto a Riemannian manifold. Recently, Shahid and Tanveer [15] extended this notion to the case when the total manifold is nearly Kählerian. A Lagrangian submersion is a special case of an anti-invariant Riemannian submersion such that the almost complex structure of the total manifold reverses the vertical and horizontal distributions. In [20], we studied Lagrangian submersions in detail. Anti-invariant Riemannian submersions have been extend to several subclasses of almost contact manifolds such as Cosympletic [12], Sasakian [9, 10] and Kenmotsu [2]. Recently, it has been defined that there are several types of Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds such as semi-invariant submersions [18], generic submersions [16], slant submersions [19], semi-slant submersions [14], pointwise slant submersions [11], hemi-slant submersions [21].

In the present paper, we consider Lagrangian submersions from Sasakian and Kenmotsu manifolds onto Riemannian manifolds and focus on the investigation of the integrability of the horizontal distributions of

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such submersions. In section 2, we present the basic background about Riemannian submersions which is needed for the further study. In section 3, we recall the well known definitions and notions of Sasakian and Kenmotsu structures. In section 4, we give the fundamental definitions of anti-invariant Riemannian and Lagrangian submersions from almost contact metric manifolds onto Riemannian manifolds. We begin to study Lagrangian submersions from Sasakian manifolds onto Riemannian manifolds admitting vertical Reeb vector field in section 5. In this section, we also prove that the horizontal distribution of a such submersion is integrable. In section 6, we observe that the O'Neill's tensor $\mathcal{A}$ of a Lagrangian submersion admitting horizontal Reeb vector field has some restrictions when the total manifold is Sasakian. Consequently, the horizontal distribution of a such submersion cannot be integrable. In section 7, we study Lagrangian submersions from Kenmotsu manifolds and prove that the horizontal distribution of a such submersion is integrable. In the last section, we obtain several applications for Lagrangian submersions studied in this paper.

2. Riemannian Submersions

In this section, we give necessary background for Riemannian submersions.

Let (M, g) and (N, g_N) be Riemannian manifolds, where $dim(M) > dim(N)$. A surjective mapping $\pi : (M, g) \to (N, g_N)$ is called a *Riemannian submersion* [13] if:

(S1) π has maximal rank, and

(S2) π_*, restricted to $(ker\pi_*)^\perp$, is a linear isometry.

In this case, for each $y \in N$, $\pi^{-1}(y)$ is a k-dimensional submanifold of M and called *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e., $\pi_* X_x = X_{*\pi(x)}$ for all $x \in M$. As usual, we denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\pi_*$ and the horizontal distribution $(ker\pi_*)^\perp$, respectively. The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{1}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection of g. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let V, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{4}$$

On the other hand, from (1) and (2), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{5}$$

$$\nabla_V X = \mathcal{T}_V X + \mathcal{H}\nabla_V X, \tag{6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{8}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. Moreover, if X is basic, then we have $\mathcal{H}\nabla_V X = \mathcal{A}_X V$. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form while $\mathcal{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on the Riemannian submersions, we refer to O'Neill's paper [13] and to the book [4].

Finally, we recall that the notion of the second fundamental form of a map between Riemannian manifolds. Let (M, g) and (N, g_N) be Riemannian manifolds and $f : (M, g) \to (N, g_N)$ be a smooth map. Then, the second fundamental form of f is given by

$$(\nabla f_*)(E, F) = \nabla^f_E f_* F - f_*(\nabla_E F)$$

for $E, F \in \Gamma(TM)$, where ∇^f is the pull back connection and we denote for convenience by ∇ the Riemannian connections of the metrics g and g_N. It is well known that the second fundamental form is symmetric . Moreover, f is said to be *totally geodesic* if $(\nabla f_*)(E, F) = 0$ for all $E, F \in \Gamma(TM)$ (see [1, page 119]), and f is called a *harmonic* map if $trace(\nabla f_*) = 0$ (see [1, page 73]).

3. Almost contact metric, Sasakian and Kenmotsu structures

Let (M, g) be a $(2m + 1)$-dimensional Riemannian manifold and denote by TM the tangent bundle of M. Then M is called an *almost contact metric manifold* [3] if there exists a tensor φ of type $(1, 1)$ and global vector field ξ which is called the *Reeb vector field* or the *characteristic vector field* such that, if η is the dual 1-form of ξ, then we have

$$\varphi\xi = 0\,, \quad \eta(\xi) = 1\,, \quad \varphi^2 = -I + \eta \otimes \xi \quad \textit{and} \quad g(\varphi E, \varphi F) = g(E, F) - \eta(E)\eta(F),$$

where $E, F \in \Gamma(TM)$. Also, it can be deduced from the above axioms that $\eta \circ \varphi = 0$ and $\eta(E) = g(E, \xi)$. In this case, (φ, ξ, η, g) is called the *almost contact metric structure* of M. The almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called a *contact metric manifold* if

$$\Phi(E, F) = d\eta(E, F)$$

for any $E, F \in \Gamma(TM)$, where Φ is a 2-form in M defined by $\Phi(E, F) = g(E, \varphi F)$. The 2-form Φ is called the *fundamental 2-form* of M. A contact metric structure of M is said to be *normal* if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0\,,$$

where $[\varphi, \varphi]$ is Nijenhuis tensor of φ. Any normal contact metric manifold is called a *Sasakian manifold.* It is not difficult to prove that a contact metric manifold M is a Sasakian manifold if and only if

$$(\nabla_E \varphi)F = g(E, F)\xi - \eta(F)E \tag{9}$$

for any $E, F \in \Gamma(TM)$, where ∇ denotes the Levi-Civita connection of M. On a Sasakian manifold, we always have

$$\nabla_E \xi = -\varphi E. \tag{10}$$

A *Kenmotsu manifold* M [8] is a normal almost contact metric manifold satisfying

$$(\nabla_E \varphi)F = g(\varphi E, F)\xi - \eta(F)\varphi E. \tag{11}$$

for all $E, F \in \Gamma(TM)$. From (11), we have

$$\nabla_E \xi = E - \eta(E)\xi. \tag{12}$$

4. Anti-invariant Riemannian and Lagrangian submersions from almost contact metric manifolds

Definition 4.1. *([9, 10]) Let M be a $(2m+1)$-dimensional almost contact metric manifold with almost contact metric structure (φ, ξ, η, g) and N be a Riemannian manifold with Riemannian metric g_N. Suppose that there exists a Riemannian submersion $\pi : M \to N$ such that the vertical distribution $ker\pi_*$ is anti-invariant with respect to φ, i.e., $\varphi ker\pi_* \subseteq ker\pi_*^\perp$. Then the Riemannian submersion π is called an* anti-invariant Riemannian submersion.

In this case, the horizontal distribution $ker\pi_*^\perp$ is decomposed as

$$ker\pi_*^\perp = \varphi ker\pi_* \oplus \mu\,, \tag{13}$$

where μ is the orthogonal complementary distribution of $\varphi ker\pi_*$ in $ker\pi_*^\perp$ and it is invariant with respect to φ.

We say that an anti-invariant Riemannian submersion $\pi : M \to N$ *admits vertical Reeb vector field* if the Reeb vector field ξ is tangent to $ker\pi_*$ and it *admits horizontal Reeb vector field* if the Reeb vector field ξ is normal to $ker\pi_*$. It is easy to see that μ contains the Reeb vector field ξ in the case of $\pi : M \to N$ admits horizontal Reeb vector field ξ. For any $X \in ker\pi_*^\perp$, we write

$$\varphi X = \mathcal{B}X + \mathcal{C}X\,, \tag{14}$$

where $\mathcal{B}X \in \Gamma(ker\pi_*)$ and $\mathcal{C}X \in \Gamma(ker\pi_*^\perp)$. At first, we examine how the Sasakian structure on M has effects on the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of an anti-invariant Riemannian submersion $\pi : (M, \varphi, \xi, \eta, g) \to (N, g_N)$.

Lemma 4.2. *Let π be an anti-invariant Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting vertical or horizontal Reeb vector field. Then we have*

$$\mathcal{T}_V\varphi W - g(V,W)\mathcal{V}\xi = \mathcal{B}\mathcal{T}_V W - \eta(W)V\,, \tag{15}$$

$$\mathcal{H}\nabla_V\varphi W - g(V,W)\mathcal{H}\xi = \mathcal{C}\mathcal{T}_V W + \varphi\hat{\nabla}_V W\,, \tag{16}$$

$$\hat{\nabla}_V\mathcal{B}X + \mathcal{T}_V\mathcal{C}X = \mathcal{B}\mathcal{H}\nabla_V X - \eta(X)V\,, \tag{17}$$

$$\mathcal{T}_V\mathcal{B}X + \mathcal{H}\nabla_V\mathcal{C}X = \mathcal{C}\mathcal{H}\nabla_V X + \varphi\mathcal{T}_V X\,, \tag{18}$$

$$\mathcal{A}_X\varphi V = \mathcal{B}\mathcal{A}_X V\,, \tag{19}$$

$$\mathcal{H}\nabla_X\varphi V + \eta(V)X = \varphi(\mathcal{V}\nabla_X V) + \mathcal{C}\mathcal{A}_X V\,, \tag{20}$$

$$\mathcal{V}\nabla_X\mathcal{B}Y + \mathcal{A}_X\mathcal{C}Y = \mathcal{B}\mathcal{H}\nabla_X Y + g(X,Y)\mathcal{V}\xi\,, \tag{21}$$

$$\mathcal{A}_X\mathcal{B}Y + \mathcal{H}\nabla_X\mathcal{C}Y = \mathcal{C}\mathcal{H}\nabla_X Y + \varphi\mathcal{A}_X Y + g(X,Y)\mathcal{H}\xi - \eta(Y)X\,, \tag{22}$$

where $V, W \in \Gamma(ker\pi_*)$ and $X, Y \in \Gamma(ker\pi_*^\perp)$.

Proof. For any $V, W \in \Gamma(ker\pi_*)$, from (9), we have

$$\nabla_V\varphi W = \varphi\nabla_V W + g(V,W)\xi - \eta(W)V.$$

Hence, using (5), (6) and (14), we obtain

$$\mathcal{H}\nabla_V\varphi W + \mathcal{T}_V\varphi W = \mathcal{B}\mathcal{T}_V W + \mathcal{C}\mathcal{T}_V W + \varphi\hat{\nabla}_V W + g(V,W)\xi - \eta(W)V. \tag{23}$$

Taking the vertical and horizontal parts of (23), we get (15) and (16), respectively.

Now, let X and Y be any horizontal vector fields. Again, from (9), we have

$$\nabla_X \varphi Y = \varphi \nabla_X Y + g(X, Y)\xi - \eta(Y)X.$$

Hence, using (7), (8) and (14), we obtain

$$\mathcal{A}_X \mathcal{B}Y + \mathcal{V}\nabla_X \mathcal{B}Y + \mathcal{H}\nabla_X CY + \mathcal{A}_X CY = \mathcal{B}\mathcal{H}\nabla_X Y + C\mathcal{H}\nabla_X Y + \varphi\mathcal{A}_X Y + g(X, Y)\xi - \eta(Y)X. \quad (24)$$

If we take the vertical and horizontal parts of (24), we easily get (21) and (22), respectively. The other assertions can be obtained in a similar method. □

For some details and examples of the anti-invariant Riemannian submersions from an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N); see [2, 9, 10, 12].

Definition 4.3. *Let π be an anti-invariant Riemannian submersion from an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). If $\mu = \{0\}$ or $\mu = span\{\xi\}$, i.e., $ker\pi_*^\perp = \varphi(ker\pi_*)$ or $ker\pi_*^\perp = \varphi(ker\pi_*) \oplus < \xi >$, respectively, then we call π a* Lagrangian submersion.

Remark 4.4. *This case has been studied partially as a special case of an anti-invariant Riemannian submersion; see [2, 9, 10, 12] for some details and examples.*

5. Lagrangian Submersions Admitting Vertical Reeb Vector Field from Sasakian Manifolds

In this section, we study Lagrangian submersions admitting vertical Reeb vector field from Sasakian manifolds onto Riemannian manifolds. Since $CX = 0$ for any $X \in \Gamma(ker\pi_*^\perp)$, for a Lagrangian submersion, we easily obtain the following result.

Corollary 5.1. *Let π be a Lagrangian submersion admitting vertical Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{T}_V \xi = -\varphi V, \quad (25)$$

$$\mathcal{T}_V \varphi W - g(V, W)\xi = \varphi \mathcal{T}_V W - \eta(W)V, \quad (26)$$

$$\mathcal{T}_V \varphi X = \varphi \mathcal{T}_V X, \quad (27)$$

$$\mathcal{A}_X \varphi E = \varphi \mathcal{A}_X E, \quad (28)$$

for $V, W \in \Gamma(ker\pi_*)$ $X \in \Gamma(ker\pi_*^\perp)$ and $E \in \Gamma(TM)$.

Proof. (25) follows immediately from (5) and (10). The others assertions follows from Lemma 4.2. □

We note that the first equation (25) was also obtained in the proof of Theorem 2 of [9].

Remark 5.2. *For a Riemannian submersion, the integrability and totally geodesicness of the horizontal distribution are equivalent to each other. This fact can be seen from (4) and (8). In this case, $\mathcal{A}_X = 0$ for any horizontal vector field X.*

In [9], the authors gave the necessary and sufficient conditions for the integrability and totally geodesicness of the horizontal distribution of a Lagrangian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting vertical Reeb vector field. We now improve their result as follows.

Theorem 5.3. *Let π be a Lagrangian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting vertical Reeb vector field. Then the horizontal distribution of the submersion π is totally geodesic.*

Proof. For any horizontal vector fields X, Y and Z, we have

$$g(\mathcal{A}_X\varphi Y, Z) = g(\varphi\mathcal{A}_X Y, Z) \tag{29}$$

from (28). By (4), we conclude that

$$g(\mathcal{A}_X\varphi Y, Z) = -g(\mathcal{A}_Y\varphi X, Z) \tag{30}$$

from (29). On the other hand, we easily have

$$g(\nabla_X\varphi Y, Z) = g(\varphi\nabla_X Y, Z) \tag{31}$$

from (9). Thus, by (7), (8), (30) and (31), we get

$$g(\mathrm{A}_X\varphi Y, Z) = -g(\mathrm{A}_Y\varphi X, Z) = -g(\nabla_Y\varphi X, Z) = -g(\varphi\nabla_Y X, Z) = g(\nabla_Y X, \varphi Z) = -g(\mathrm{A}_X Y, \varphi Z)$$

$$= g(\mathrm{A}_X\varphi Z, Y) = -g(\mathrm{A}_Z\varphi X, Y) = g(\mathrm{A}_Z Y, \varphi X) = -g(\mathrm{A}_Y Z, \varphi X) = g(\mathrm{A}_Y\varphi X, Z) = -g(\mathrm{A}_X\varphi Y, Z).$$

Hence, we deduce that

$$g(\mathcal{A}_X\varphi Y, Z) = 0 \tag{32}$$

for all $X, Y, Z \in \Gamma(ker\pi_*^\perp)$. Then, for any horizontal vector field X, it follows that $\mathcal{A}_X = 0$ from (32). □

We note that the technique used in the proof of Teorem 5.3 was also used in the proof of Theorem 4.5 of [20].

6. Lagrangian Submersions Admitting Horizontal Reeb Vector Field from Sasakian Manifolds

In this section, we study Lagrangian submersions admitting horizontal Reeb vector field from Sasakian manifolds onto Riemannian manifolds. First of all, we give the following result.

Corollary 6.1. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{T}_V\xi = 0\,, \tag{33}$$

$$\mathcal{T}_V\varphi E = \varphi\mathcal{T}_V E\,, \tag{34}$$

$$\mathcal{A}_X\varphi V = \varphi\mathcal{A}_X V\,, \tag{35}$$

$$\mathcal{A}_X\varphi Y = \varphi\mathcal{A}_X Y + g(X, Y)\xi - \eta(Y)X\,, \tag{36}$$

for $V \in \Gamma(ker\pi_*)$, $X, Y \in \Gamma(ker\pi_*^\perp)$ and $E \in \Gamma(TM)$.

Proof. (33) follows immediately from (4) and (10). The second assertion follows from (15) and (18). The third one comes from (19) and the last one comes from (22). □

We remark that the equation (33) was also given in Lemma 6 of [9].

Now, let π be a Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). For any horizontal vector fields X, Y and Z orthogonal to ξ, we have

$$g(\nabla_X\varphi Y, Z) = g(\varphi\nabla_X Y, Z) \tag{37}$$

and

$$g(\mathcal{A}_X\varphi Y, Z) = g(\varphi\mathcal{A}_X Y, Z) \tag{38}$$

from (9) and (22), respectively. By (4), we deduce that

$$g(\mathcal{A}_X\varphi Y, Z) = -g(\mathcal{A}_Y\varphi X, Z) \quad (39)$$

from (38). Thus, by (37), (39) and the same technique in the proof of Theorem 5.3, we get

$$g(\mathcal{A}_X\varphi Y, Z) = 0 \quad (40)$$

for $X, Y, Z \in \Gamma(\varphi ker\pi_*)$. Since any vertical vector field V can be written in the form $V = \varphi Y$ for $Y \in \Gamma(\varphi ker\pi_*)$, by virtue of (40), we have

$$g(\mathcal{A}_X V, Z) = 0. \quad (41)$$

By (13), we deduce from (41) that $\mathcal{A}_X V$ must be linearly dependent on the Reeb vector field ξ. So, we can write

$$\mathcal{A}_X V = g(\mathcal{A}_X V, \xi)\xi. \quad (42)$$

Thus, we obtain the following restrictions for behavior of the O'Neill's tensor $\mathcal{A}$ of a such submersion.

Theorem 6.2. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{A}_X Y = 0\,, \quad (43)$$

$$\mathcal{A}_X V = -\Phi(X, V)\xi\,, \quad (44)$$

$$\mathcal{A}_\xi X = \varphi X\,, \quad (45)$$

$$\mathcal{A}_\xi V = \varphi V\,, \quad (46)$$

where $V \in \Gamma(ker\pi_*)$, $X, Y \in \Gamma(\varphi ker\pi_*)$ and ξ the Reeb vector field.

Proof. The first assertion comes from (41). The second assertion follows from (42) after some calculation. Both the third one and last assertion follows from (44). □

We remark that in the above Theorem (45) and (46) follow from (10) and (36), respectively.

One can see that the O'Neill's tensor $\mathcal{A}$ cannot vanish for a such submersion from Theorem 6.2. Thus, we get the following result.

Theorem 6.3. *There is no Lagrangian submersion with integrable horizontal distribution from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting horizontal Reeb vector field.*

7. Lagrangian Submersions from Kenmotsu Manifolds

In this section, we will study Lagrangian submersions from Kenmotsu manifolds onto Riemannian manifolds. In this case, we can suppose that the Reeb vector field ξ is horizontal, because Beri et al. [2] proved the non-existence of (anti-invariant) Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds such that the Reeb vector field ξ is vertical.

Let examine how the Kenmotsu structure on M has effects on the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of an anti-invariant Riemannian submersion $\pi : (M, \varphi, \xi, \eta, g) \to (N, g_N)$.

Lemma 7.1. *Let π be an anti-invariant Riemannian submersion from a Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N), where the Reeb vector field ξ is necessarily horizontal. Then we have*

$$\mathcal{T}_V \xi = V\,, \tag{47}$$

$$\mathcal{T}_V \varphi W = \mathcal{B}\mathcal{T}_V W\,, \tag{48}$$

$$\mathcal{H}\nabla_V \varphi W = \mathcal{C}\mathcal{T}_V W + \varphi\hat{\nabla}_V W\,, \tag{49}$$

$$\hat{\nabla}_V \mathcal{B}X + \mathcal{T}_V \mathcal{C}X = \mathcal{B}\mathcal{H}\nabla_V X\,, \tag{50}$$

$$\mathcal{T}_V \mathcal{B}X + \mathcal{H}\nabla_V \mathcal{C}X = \mathcal{C}\mathcal{H}\nabla_V X + \varphi\mathcal{T}_V X + g(\varphi V, X)\xi - \eta(X)\varphi V\,, \tag{51}$$

$$\mathcal{A}_X \varphi V = \mathcal{B}\mathcal{A}_X V\,, \tag{52}$$

$$\mathcal{H}\nabla_X \varphi V = \varphi\mathcal{V}\nabla_X V + \mathcal{C}\mathcal{A}_X V + g(\varphi X, V)\xi\,, \tag{53}$$

$$\mathcal{V}\nabla_X \mathcal{B}Y + \mathcal{A}_X \mathcal{C}Y = \mathcal{B}\mathcal{H}\nabla_X Y - \eta(Y)\varphi\mathcal{B}X\,, \tag{54}$$

$$\mathcal{A}_X \mathcal{B}Y + \mathcal{H}\nabla_X \mathcal{C}Y = \mathcal{C}\mathcal{H}\nabla_X Y + \varphi\mathcal{A}_X Y - \eta(Y)\mathcal{C}X\,, \tag{55}$$

$$\mathcal{A}_X \xi = 0\,, \tag{56}$$

where $V, W \in \Gamma(ker\pi_*)$ and $X, Y \in \Gamma(ker\pi_*^\perp)$.

Proof. The first assertion (47) follows immediately from (6) and (12), and the last assertion (56) follows immediately from (8) and (12). Now, let X and Y be any horizontal vector fields. From (11), we have

$$\nabla_X \varphi Y = \varphi\nabla_X Y + g(\varphi X, Y)\xi - \eta(Y)\varphi X.$$

Hence, using (7), (8) and (14), we obtain

$$\mathcal{A}_X \mathcal{B}Y + \mathcal{V}\nabla_X \mathcal{B}Y + \mathcal{H}\nabla_X \mathcal{C}Y + \mathcal{A}_X \mathcal{C}Y = \mathcal{B}\mathcal{H}\nabla_X Y + \mathcal{C}\mathcal{H}\nabla_X Y + \varphi\mathcal{A}_X Y - \eta(Y)\mathcal{B}X - \eta(Y)\mathcal{C}X. \tag{57}$$

If we take the vertical and horizontal parts of (57), we easily get (54) and (55), respectively. In a similar way, the other assertions can be obtained . □

Note that the first equation (47) and the last equation (56) of Lemma 7.1 were also proved in Lemma 3 of [2].

Corollary 7.2. *Let π be a Lagrangian submersion from a Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\mathcal{T}_V \varphi W = \varphi\mathcal{T}_V W\,, \tag{58}$$

$$\mathcal{T}_V \varphi X = \varphi\mathcal{T}_V X + g(\varphi V, X)\xi - \eta(X)\varphi V\,, \tag{59}$$

$$\mathcal{A}_X \varphi E = \varphi\mathcal{A}_X E\,, \tag{60}$$

for $V, W \in \Gamma(ker\pi_*)$, $X \in \Gamma(ker\pi_*^\perp)$ and $E \in \Gamma(TM)$.

Proof. First assertion follows from (48). The second assertion follows from (51) and the third one follows from (52) and (55) . □

In [2, Corollaries 1 and 2], the authors obtained the necessary and sufficient conditions for the integrability and totally geodesicness of the horizontal distribution of a Lagrangian submersion from a Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). We now improve their result as follows.

Theorem 7.3. *Let π be a Lagrangian submersion from a Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then the horizontal distribution of the submersion π is totally geodesic, i.e., $\mathcal{A} \equiv 0$.*

Proof. For any horizontal vector fields X, Y and Z, we have

$$g(\mathcal{A}_X \varphi Y, Z) = g(\varphi \mathcal{A}_X Y, Z) \tag{61}$$

from (60). By (4), we conclude that

$$g(\mathcal{A}_X \varphi Y, Z) = -g(\mathcal{A}_Y \varphi X, Z) \tag{62}$$

from (61). On the other hand, we easily have

$$g(\nabla_X \varphi Y, Z) = g(\varphi \nabla_X Y, Z) \tag{63}$$

from (11). Thus, by (62), (63) and the same method in the proof of Theorem 5.3, we get

$$g(\mathcal{A}_X \varphi Y, Z) = 0 \tag{64}$$

for all $X, Y, Z \in \Gamma(ker\pi_*^{\perp})$. It follows that $\mathcal{A}_X = 0$ for any horizontal vector field X. □

8. Applications

A smooth map $\pi : (M, g) \to (N, g_N)$ between Riemannian manifolds is called a *totally geodesic map* if π_* preserves parallel translation. Vilms [22] classified totally geodesic Riemannian submersions and proved that a Riemannian submersion $\pi : (M, g) \to (N, g_N)$ is totally geodesic if and only if both O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ vanish. On the other hand, it is well known that the fibers of a Riemannian submersion are totally geodesic if and only if the O'Neill's tensor $\mathcal{T}$ vanishes. For a Lagrangian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting vertical Reeb vector field, it is seen from (25) and (26) that the O'Neill's tensor $\mathcal{T}$ cannot vanish. Thus, we have the following result.

Theorem 8.1. *Let π be a Lagrangian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting vertical Reeb vector field. Then the fibers of π cannot be totally geodesic. Consequently, the submersion π cannot be a totally geodesic map.*

Moreover, we easily see from (47) that the O'Neill's tensor $\mathcal{T}$ cannot vanish for an anti-invariant Riemannian submersion whose total manifold is Kenmotsu. Thus, we obtain the following result.

Theorem 8.2. *Let π be an anti-invariant Riemannian submersion from a Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N). Then the fibers of π cannot be totally geodesic. Consequently, the submersion π cannot be a totally geodesic map.*

By Theorem 6.3, we get the following result.

Theorem 8.3. *There is no Lagrangian submersion which is totally geodesic map from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting horizontal Reeb vector field.*

Now, let π be a Lagrangian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting horizontal Reeb vector field. Then the mean curvature vector field H [4] of the fiber of π is defined by

$$H = \frac{1}{n} \sum_{k=1}^{n} \mathcal{T}_{V_k} V_k \,,$$

where $\{V_1, ..., V_n\}$ is a local orthonormal frame of the vertical distribution $ker\pi_*$. Then the skew-symmetric $\mathcal{T}$ and (33) give

$$g(H,\xi) = \frac{1}{n}\sum_{k=1}^{n} g(\mathcal{T}_{V_k}V_k,\xi) = -\frac{1}{n}\sum_{k=1}^{n} g(\mathcal{T}_{V_k}\xi, V_k) = 0.$$

Hence, it follows that $H \perp \xi$ or equivalently $H \in \Gamma(\varphi ker\pi_*)$ from (13). Thus, we get the following.

Corollary 8.4. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold (M,φ,ξ,η,g) onto a Riemannian manifold (N,g_N). If H is the mean curvature vector field of the fiber, then*

$$H \in \Gamma(\varphi ker\pi_*)\,. \tag{65}$$

In [9] and [10], the authors independently gave the following necessary and sufficient condition for the harmonicity of a Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold (M,φ,ξ,η,g) onto a Riemannian manifold (N,g_N).

Theorem 8.5. *([9, 10]) Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold (M,φ,ξ,η,g) onto a Riemannian manifold (N,g_N). Then π is harmonic if and only if $Trace(\varphi\mathcal{T}_V) = 0$ for $V \in \Gamma(ker\pi_*)$.*

We next improve this result as follows.

Theorem 8.6. *Let π be a Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold (M,φ,ξ,η,g) onto a Riemannian manifold (N,g_N). Then π cannot be a harmonic map.*

Proof. Let V be any vertical vector field, then using (3) and (34), we obtain

$$Trace(\varphi\mathcal{T}_V) = \sum_{k=1}^{n} g(\varphi\mathcal{T}_V V_k, V_k) = \sum_{k=1}^{n} g(\varphi\mathcal{T}_{V_k}V, V_k) = \sum_{k=1}^{n} g(\mathcal{T}_{V_k}\varphi V, V_k) = -\sum_{k=1}^{n} g(\mathcal{T}_{V_k}V_k, \varphi V) = -ng(H,\varphi V).$$

But, by virtue of (65), the equality $g(H,\varphi V) = 0$ does not hold for all $V \in \Gamma(ker\pi_*)$. By Theorem 8.5, the submersion π cannot be a harmonic map. □

On a Sasakian manifold (M,φ,ξ,η,g), for all $E,F \in \Gamma(TM)$, the following equality [3]

$$(\nabla_E\varphi)F = \mathcal{R}(\xi,E)F \tag{66}$$

holds, where $\mathcal{R}$ is the curvature tensor [24] of M. Thus, from (36) and (66), we have that:

Corollary 8.7. *Let π be a Lagrangian submersion from a Sasakian manifold (M,φ,ξ,η,g) onto a Riemannian manifold (N,g_N) admitting horizontal Reeb vector field. Then, for all horizontal vector fields X and Y, the equality*

$$\mathcal{R}(\xi,X)Y = \mathcal{A}_X\varphi Y - \varphi\mathcal{A}_X Y$$

holds.

Corollary 8.8. *Let π be a Lagrangian submersion from a Sasakian manifold (M,φ,ξ,η,g) onto a Riemannian manifold (N,g_N) admitting horizontal Reeb vector field. Then, for any unit vertical vector field V, we have*

$$g((\nabla_\xi\mathcal{T})_V V,\xi) = 0.$$

Proof. Let V be a unit vertical vector field, then from the 2_{nd} equation of Corollary 1 of [13], we have

$$K(\xi,V) = g((\nabla_\xi\mathcal{T})_V V,\xi) + \|\mathcal{A}_\xi V\|^2 - \|\mathcal{T}_V\xi\|^2, \tag{67}$$

where $K(\xi,V)$ is the sectional curvature of the plane section spanned by ξ and V. Here, by (33) and (46) we know $\|\mathcal{T}_V\xi\| = 0$ and $\|\mathcal{A}_\xi V\| = 1$, respectively. On the other hand, on a Sasakian manifold, the sectional curvature of any plane section containing the Reeb vector field ξ is equal to 1. Thus, the assertion follows from (67). □

Now, let π be a Lagrangian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold (N, g_N) admitting horizontal Reeb vector field ξ. Assume that the total manifold M has constant sectional curvature c_1 and the base manifold N has constant sectional curvature c. Here, we have $c_1 = 1$ (see [7]). Then for any orthonormal horizontal vector fields X and Y, we have

$$1 = c - 3\|\mathcal{A}_X Y\|^2, \tag{68}$$

from the 3_{nd} equation of Corollary 1 of [13]. In which case, if we choose the orthonormal set $\{X, Y\}$ orthogonal to ξ, then by (43), we deduce that $c = 1$ from (68). But, by (45), for ξ and any unit horizontal X orthogonal to ξ, (68) gives $c = 4$, which is a contradiction. Thus, we get the following result.

Theorem 8.9. *There is no Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ of constant sectional curvature onto a Riemannian manifold (N, g_N) of constant sectional curvature.*

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