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LAGRANGIAN SURFACES IN COMPLEX EUCLIDEAN PLANE VIA SPHERICAL AND HYPERBOLIC CURVES

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Abstract. We present a method to construct a large family of Lagrangian surfaces in complex Euclidean plane $\boldsymbol{C}^2$ by using Legendre curves in the 3-sphere and in the anti de Sitter 3-space or, equivalently, by using spherical and hyperbolic curves, respectively. Among this family, we characterize minimal, constant mean curvature, Hamiltonian-minimal and Willmore surfaces in terms of simple properties of the curvature of the generating curves. As applications, we provide explicitly conformal parametrizations of known and new examples of minimal, constant mean curvature, Hamiltonian-minimal and Willmore surfaces in $\boldsymbol{C}^2$.

1. Introduction. An immersion $\phi : M^n \to \tilde{M}^n$ of an n-manifold M^n into a Kaehler n-manifold $\tilde{M}^n$ is called a *Lagrangian immersion* if the complex structure J of $\tilde{M}^n$ interchanges each tangent space of M^n with its corresponding normal space. Lagrangian submanifolds appear naturally in several contexts of mathematical physics. A very important problem in this setting is to find nontrivial examples of Lagrangian submanifolds with some given geometric properties. In this line, we find many papers (see the survey article [5]) where the different authors investigate intrinsic and extrinsic geometric properties related mainly with the intrinsic curvatures and the mean curvature vector of the submanifolds, respectively.

An important problem in the theory of Lagrangian surfaces is to find non-trivial examples with some given special geometric properties. For instance, a method was given in [8] to construct an important family of special Lagrangian submanifolds in $\boldsymbol{C}^n$ with large symmetric groups. Also, a spinor-like representation formula for Lagrangian surfaces in $\boldsymbol{C}^2$ which parameterizes immersions through two complex functions F_1, F_2 and a real one (the Lagrangian angle) was introduced in [1]. This formula is useful to construct examples of Lagrangian surfaces in $\boldsymbol{C}^2$.

In this article, we present a simple specific new method to construct Lagrangian surfaces in complex Euclidean plane $\boldsymbol{C}^2$ with nice properties that only involves two Legendre curves; one in the 3-sphere S^3 and the other in the anti De Sitter 3-space H^3_1.

Recall that a regular curve $\gamma : I_1 \to S^3$ (resp. $\alpha : I_2 \to H^3_1$) is called a *Legendre curve* if $\langle \gamma'(s), i\gamma(s) \rangle = 0$ (resp. $\langle \alpha'(t), i\alpha(t) \rangle = 0$) holds identically. For each such pair of

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Legendre curves (γ, α), we consider the map:

$$\phi : I_1 \times I_2 \to \boldsymbol{C}^2 = \boldsymbol{C} \times \boldsymbol{C}; \quad (t, s) \mapsto (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s)) . \tag{1.1}$$

In Section 2, we show that the map ϕ defined by (1.1) is a Lagrangian immersion. We also study geometric properties of such Lagrangian surfaces. In particular, we investigate the close relationship of such a Lagrangian surface with the curve in the 2-sphere S^2 and the curve in the hyperbolic 2-plane H^2 given by the projections of γ and α via their corresponding Hopf fibrations.

In Section 3 we prove an useful "additive formula" (see Theorem 1) involving the Lagrangian angle map of ϕ and the Legendre angles of γ and α for the Lagrangian immersions. As a consequence, we establish a simple relationship (see Corollary 1) between the mean curvature vector of ϕ and the curvature functions of γ and α.

The last section provides several nice applications of the results obtained in Section 3. First we characterize the minimal Lagrangian surfaces obtained by our construction in terms of geodesics in S^3 and H_1^3. In such a way we are able to provide explicit expressions of the minimal Lagrangian conformal immersions in $\boldsymbol{C}^2$ in terms of some elementary functions. We also determine Lagrangian surfaces constructed by our method with constant mean curvature and, in particular, with parallel mean curvature vector. This enables us to obtain interesting new examples of Lagrangian tori in $\boldsymbol{C}^2$ with constant mean curvature. Next we characterize Hamiltonian-minimal Lagrangian surfaces among the family of Lagrangian surfaces using our construction with Legendre curves such that their curvature functions (in terms of the arclength parameter) are linear. As a by-product, we are able to establish the explicit expressions of some new Hamiltonian-minimal Lagrangian conformal immersions in $\boldsymbol{C}^2$ in terms of elementary functions as well. Finally, we apply our construction method to provide new examples of Willmore Lagrangian surfaces in $\boldsymbol{C}^2$. Our result states that the Lagrangian surfaces constructed by the pair (γ, α) of Legendre curves are Willmore surfaces if and only if γ and α are elastic curves in S^3 and H_1^3, respectively.

2. A new construction method of Lagrangian surfaces. In the complex Euclidean plane $\boldsymbol{C}^2$ we consider the bilinear Hermitian product defined by

$$(z, w) = z_1\bar{w}_1 + z_2\bar{w}_2, \quad z, w \in \boldsymbol{C}^2 .$$

Then $\langle\ ,\ \rangle = \Re(\ ,\ )$ is the Euclidean metric on $\boldsymbol{C}^2$ and $\omega = -\Im(\ ,\ )$ is the Kaehler two-form given by $\omega(\cdot\,,\,\cdot) = \langle J\cdot, \cdot\rangle$, where J is the complex structure on $\boldsymbol{C}^2$.

Let $\phi : M \to \boldsymbol{C}^2$ be an isometric Lagrangian immersion of a surface M into $\boldsymbol{C}^2$, i.e., an immersion satisfying $\omega_{|M} \equiv 0$. We denote the Riemannian connections of M and $\boldsymbol{C}^2$ by ∇ and $\bar{\nabla}$, respectively. We also denote by $\langle\ ,\ \rangle$ the induced metric on M. Then we have $\phi^* T\boldsymbol{C}^2 = \phi_* TM \oplus T^\perp M$, where TM and $T^\perp M$ are the tangent and normal bundles of M, respectively. The second fundamental form σ is given by $\sigma(x, y) = JA_{Jx}y$, where A is the shape operator. Thus, the (0,3)-tensor $C(x, y, z) = \langle\sigma(x, y), Jz\rangle$ is totally symmetric.

The space of oriented Lagrangian planes in $\boldsymbol{C}^2$ can be identified with the symmetric space $U(2)/SO(2)$, so the determinant map, det : $U(2)/SO(2) \to S^1$, is well-defined. If M

is an orientable Lagrangian surface in $\boldsymbol{C}^2$ and $\nu : M \to U(2)/SO(2)$ is its Gauss map, then $\det\circ\nu : M \to S^1$ can be expressed as $\det\circ\nu = e^{i\beta_\phi}$ for some function $\beta_\phi : M \to \boldsymbol{R}/2\pi\boldsymbol{Z}$. This function β_ϕ is called the *Lagrangian angle map* of ϕ. The Lagrangian angle map β_ϕ satisfies

$$J\nabla\beta_\phi = 2H\,, \tag{2.1}$$

where H is the mean curvature of ϕ, defined by $H = (1/2)\operatorname{trace}\sigma$.

The Lagrangian immersion ϕ is called minimal if $H = 0$ identically, or equivalently, the Lagrangian angle map β_ϕ is constant. The minimality on the surfaces means that the surface is a critical point of the area functional over any compactly supported variation. On the other hand, Hamiltonian-minimal Lagrangian surfaces are Lagrangian surfaces which are critical points of the area functional with respect to a special class of infinitesimal variations preserving the Lagrangian constraint; namely, the class of compactly supported Hamiltonian vector fields (see [13]). Such Lagrangian surfaces are characterized by the harmonicity of the Lagrangian angle map β_ϕ (cf. [9]).

Let S^3 and H_1^3 denote the unit hypersphere and the unit anti De Sitter space in $\boldsymbol{C}^2$ given respectively by

$$S^3 = \{(z,w)\in\boldsymbol{C}^2\,;\,|z|^2+|w|^2=1\}\,,\quad H_1^3 = \{(z,w)\in\boldsymbol{C}^2\,;\,|z|^2-|w|^2=-1\}\,.$$

Let $\gamma := \gamma(s) = (\gamma_1,\gamma_2) : I_1 \to S^3$ be a unit speed Legendre curve in S^3 and $\alpha = \alpha(t) = (\alpha_1,\alpha_2) : I_2 \to H_1^3$ a unit speed Legendre curve in H_1^3. Then γ and α satisfy

$$|\gamma_1|^2+|\gamma_2|^2=1\,,\quad |\gamma_1'|^2+|\gamma_2'|^2=1\,,\quad \gamma_1'\bar\gamma_1+\gamma_2'\bar\gamma_2=0\,, \tag{2.2}$$

$$|\alpha_1|^2-|\alpha_2|^2=-1\,,\quad |\alpha_1'|^2-|\alpha_2'|^2=1\,,\quad \alpha_1'\bar\alpha_1-\alpha_2'\bar\alpha_2=0\,. \tag{2.3}$$

PROPOSITION 2.1. *Let γ be a unit speed Legendre curve in S^3 and α be a unit speed Legendre curve in H_1^3. Consider the map*: $\phi : I_1\times I_2\subset\boldsymbol{R}^2\to\boldsymbol{C}^2=\boldsymbol{C}\times\boldsymbol{C}$ *defined by*

$$\phi(t,s) = (\alpha_1(t)\gamma_1(s),\alpha_2(t)\gamma_2(s))\,. \tag{2.4}$$

Then ϕ is a Lagrangian conformal immersion in $\boldsymbol{C}^2$ such that the induced metric is given by

$$\langle\ ,\ \rangle = (|\gamma_1|^2+|\alpha_1|^2)(dt^2+ds^2) \tag{2.5}$$

and the intrinsic tensor $C(x,y,z)=\langle\sigma(x,y),Jz\rangle$ is given by

$$\begin{aligned} C_{ttt} &= \langle\alpha_1'',J\alpha_1'\rangle|\gamma_1|^2+\langle\alpha_2'',J\alpha_2'\rangle|\gamma_2|^2\,,\\ C_{tts} &= \langle\gamma_1',J\gamma_1\rangle\,,\\ C_{tss} &= \langle\alpha_1',J\alpha_1\rangle\,,\\ C_{sss} &= |\alpha_1|^2\langle\gamma_1'',J\gamma_1'\rangle+|\alpha_2|^2\langle\gamma_2'',J\gamma_2'\rangle\,, \end{aligned} \tag{2.6}$$

where $C_{ttt}=C(\partial_t,\partial_t,\partial_t)$, $C_{tts}=C(\partial_t,\partial_t,\partial_s),\ldots$, etc. The J in (2.6) *is the $+\pi/2$-rotation acting on* $\mathbf{C}\equiv\boldsymbol{R}^2$.

PROOF. From (2.4) we get

$$\phi_t = (\alpha_1'(t)\gamma_1(s), \alpha_2'(t)\gamma_2(s)), \quad \phi_s = (\alpha_1(t)\gamma_1'(s), \alpha_2(t)\gamma_2'(s)). \tag{2.7}$$

Thus, by applying (2.2) and (2.3), we find

$$\begin{aligned} |\phi_t|^2 &= |\alpha_1'|^2|\gamma_1|^2 + |\alpha_2'|^2|\gamma_2|^2 \\ &= |\alpha_1'|^2|\gamma_1|^2 + (|\alpha_1'|^2 - 1)(1 - |\gamma_1|^2) \\ &= |\alpha_1'|^2 + |\gamma_1|^2 - 1 \\ &= |\alpha_2'|^2 + |\gamma_1|^2 . \end{aligned} \tag{2.8}$$

On the other hand, from the last equation of (2.3), we have

$$\begin{aligned} |\alpha_1|^2(1 + |\alpha_2'|^2) &= |\alpha_1|^2|\alpha_1'|^2 \\ &= |\alpha_2|^2|\alpha_2'|^2 = |\alpha_2'|^2(1 + |\alpha_1|^2) . \end{aligned} \tag{2.9}$$

Thus, we obtain $|\alpha_1|^2 = |\alpha_2'|^2$. Substituting this into (2.8) gives $|\phi_t|^2 = |\alpha_1|^2 + |\gamma_1|^2$. Similarly, we also have $|\phi_s|^2 = |\alpha_1|^2 + |\gamma_1|^2$. By the last equations in (2.2) and (2.3), we have

$$(\phi_t, \phi_s) = \alpha_1'\bar{\alpha}_1\gamma_1\bar{\gamma}_1' + \alpha_2'\bar{\alpha}_2\gamma_2\bar{\gamma}_2' = 0 . \tag{2.10}$$

Thus, by taking the imaginary part in (2.10), we see that ϕ is a Lagrangian immersion whose induced metric via ϕ is given by (2.5).

It follows from (2.2) and (2.4) that

$$\begin{aligned} (\phi_{tt}, \phi_s) &= ((\alpha_1''\gamma_1, \alpha_2''\gamma_2), (\alpha_1\gamma_1', \alpha_2\gamma_2')) \\ &= \alpha_1''\bar{\alpha}_1\gamma_1\bar{\gamma}_1' + \alpha_2''\bar{\alpha}_2\gamma_2\bar{\gamma}_2' \\ &= \gamma_1\bar{\gamma}_1'(\alpha_1''\bar{\alpha}_1 - \alpha_2''\bar{\alpha}_2) \\ &= -\gamma_1\bar{\gamma}_1' , \end{aligned} \tag{2.11}$$

where we have applied the identity: $\alpha_1''\bar{\alpha}_1 - \alpha_2''\bar{\alpha}_2 = -1$ deduced from the last equation of (2.3). Similarly, we also have

$$(\phi_{tt}, \phi_t) = \alpha_1''\bar{\alpha}_1'|\gamma_1|^2 + \alpha_2''\bar{\alpha}_2'|\gamma_2|^2 . \tag{2.12}$$

By taking the imaginary parts in (2.11) and (2.12), we obtain the first two equations of (2.6). Similarly, we also have the last two equations of (2.6). □

In the same spirit as the proof of (2.6), we find by taking the real parts that

$$\begin{aligned} \langle \nabla_{\partial_t}\partial_t , \partial_t \rangle &= \langle \alpha_1'', \alpha_1' \rangle |\gamma_1|^2 + \langle \alpha_2'', \alpha_2' \rangle |\gamma_2|^2 , \\ \langle \nabla_{\partial_t}\partial_t , \partial_s \rangle &= -\langle \nabla_{\partial_t}\partial_s , \partial_t \rangle = -\langle \gamma_1', \gamma_1 \rangle , \\ \langle \nabla_{\partial_t}\partial_s , \partial_s \rangle &= -\langle \nabla_{\partial_s}\partial_s , \partial_t \rangle = \langle \alpha_1', \alpha_1 \rangle , \\ \langle \nabla_{\partial_s}\partial_s , \partial_s \rangle &= |\alpha_1|^2\langle \gamma_1'', \gamma_1' \rangle + |\alpha_2|^2\langle \gamma_2'', \gamma_2' \rangle . \end{aligned} \tag{2.13}$$

Since ϕ is a conformal Lagrangian immersion, the Laplacian of the Lagrangian surface with the induced metric (2.5) is given by

$$\Delta = e^{-2u}\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}\right), \tag{2.14}$$

where $e^{2u} = |\gamma_1|^2 + |\alpha_1|^2$.

Via the Hopf fibration, Legendre curves in S^3 and H_1^3 are projected into curves in S^2 and H^2, respectively. Hence, it is interesting to describe the geometry of the Lagrangian surfaces obtained in Proposition 2.1 by using the geometry of the curves in S^2 and H^2. We study this as follows:

Let $S^2(1/2) := \{(x_1, x_2, x_3) \in \boldsymbol{R}^3 \,;\, x_1^2 + x_2^2 + x_3^2 = 1/4\}$ which is the 2-sphere with radius $1/2$ in $\boldsymbol{R}^3$. The Hopf fibration $\pi : S^3 \to S^2(1/2) \equiv CP^1(4)$ is given by

$$\pi(z, w) = \frac{1}{2}(2z\bar{w}, |z|^2 - |w|^2), \quad (z, w) \in S^3 \subset \boldsymbol{C}^2. \tag{2.15}$$

Notice that (2.15) is well-defined, since $|2z\bar{w}|^2 + (|z|^2 - |w|^2)^2 = (|z|^2 + |w|^2)^2 = 1$.

For each Legendre curve $\gamma = \gamma(s)$ in S^3, the projection $\xi = \pi \circ \gamma$ is a curve in $S^2(1/2)$. Conversely, each curve ξ in $S^2(1/2)$ gives rise to a horizontal lift $\tilde{\xi}$ in S^3 via π which is unique up to a factor $e^{i\theta_1}$, $\theta_1 \in \boldsymbol{R}$. Notice that each horizontal lift of ξ is a Legendre curve in S^3.

Since the Hopf fibration π is a Riemannian submersion, each unit speed Legendre curve γ in S^3 is projected to a unit speed curve ξ in $S^2(1/2)$ with the same curvature function. From (2.15), it is not difficult to see that

$$|\gamma_1|^2 = \frac{1}{2} + \xi_3, \quad \langle \gamma_1', J\gamma_1 \rangle = (\xi \times \xi')_3, \tag{2.16}$$

where $\times$ denotes the cross product in $\boldsymbol{R}^3$ and $(\xi \times \xi')_3$ is the third coordinate of $\xi \times \xi'$ in the 3-space $\boldsymbol{R}^3$ containing $S^2(1/2)$.

Similarly, let $H^2(-1/2) = \{(x_1, x_2, x_3) \in \boldsymbol{R}^3 \,;\, x_1^2 + x_2^2 - x_3^2 = -1/4,\ x_3 \geq 1/2\}$ which is the model of the real hyperbolic plane of curvature -4. The Hopf fibration $\pi : H_1^3 \to H^2(-1/2) \equiv CH^1(-4)$ is then given by

$$\pi(z, w) = \frac{1}{2}(2z\bar{w}, |z|^2 + |w|^2), \quad (z, w) \in H_1^3 \subset \boldsymbol{C}_1^2. \tag{2.17}$$

Notice that (2.17) is well-defined, since $|2z\bar{w}|^2 - (|z|^2 + |w|^2)^2 = -(|z|^2 - |w|^2)^2 = -1$.

For each Legendre curve $\alpha = \alpha(t)$ in H_1^3, the projection $\eta = \pi \circ \alpha$ is a curve in $H^2(-1/2)$. Conversely, each curve η in $H^2(-1/2)$ gives rise to a horizontal lift $\tilde{\eta}$ in H_1^3 via π that is unique up to a factor $e^{i\theta_2}$, $\theta_2 \in \boldsymbol{R}$. Each horizontal lift $\tilde{\eta}$ is a Legendre curve in H_1^3.

In the same way as γ, if α is a unit speed Legendre curve in H_1^3, then the projection η is also a unit speed curve in $H^2(-1/2)$ with the same curvature function. It follows from (2.17) that

$$|\alpha_1|^2 = -\frac{1}{2} + \eta_3, \quad \langle \alpha_1', J\alpha_1 \rangle = (\eta \times \eta')_3. \tag{2.18}$$

Taking the above considerations into account, our construction of the Lagrangian conformal surfaces in Proposition 2.1 can also be obtained by using a unit speed curve ξ in $S^2(1/2)$ and a unit speed curve η in $H^2(-1/2)$ as follows:

$$\phi(t,s) = (\tilde{\eta}_1(t)\tilde{\xi}_1(s), \tilde{\eta}_2(t)\tilde{\xi}_2(s)) . \tag{2.19}$$

Notice that if we choose different horizontal lifts, say $\hat{\tilde{\xi}}$ and $\hat{\eta}$ of ξ and η, then we have $\hat{\tilde{\xi}} = e^{i\theta_1}\tilde{\xi}$ and $\hat{\eta} = e^{i\theta_2}\tilde{\eta}$ for some $\theta_1, \theta_2 \in \boldsymbol{R}$. Hence, the corresponding Lagrangian conformal immersion

$$\psi(t,s) = (\hat{\eta}_1(t)\hat{\tilde{\xi}}_1(s), \hat{\eta}_2(t)\hat{\tilde{\xi}}_2(s))$$

is related with (2.19) by $\psi = e^{i(\theta_1+\theta_2)}\phi$. Therefore, the two Lagrangian conformal immersions ϕ and ψ via different horizontal lifts are always congruent.

In fact, the geometry of the Lagrangian conformal immersion ϕ depends essentially on the initial curves ξ and η. For example, it follows from (2.16) and (2.18) that the induced metric of ϕ is given by $\langle\ ,\ \rangle = (\eta_3(t) + \xi_3(s))(dt^2 + ds^2)$ and the intrinsic tensor C satisfies $C_{tts} = (\xi \times \xi')_3$, $C_{tss} = (\eta \times \eta')_3, \dots$, etc. These show that the third coordinates of the position of ξ and η in $\boldsymbol{R}^3$ are relevant in the geometry of our construction. More precisely, any rotation around the x_3-axis of $\boldsymbol{R}^3$ acting on the generating curves ξ and η gives rise to a congruent Lagrangian immersion, since we have

$$(e^{i\varphi_1}(\xi_1 + i\xi_2); \xi_3) = \pi(e^{i\varphi_1}\tilde{\xi}_1, \tilde{\xi}_2) \tag{2.20}$$

and

$$(e^{i\varphi_2}(\eta_1 + i\eta_2); \eta_3) = \pi(e^{i\varphi_2}\tilde{\eta}_1, \tilde{\eta}_2) . \tag{2.21}$$

As an illustration, the totally geodesic Lagrangian planes can be obtained by taking any meridian in $S^2(1/2)$ passing through the north and south poles and any meridian passing through the vertex $(0,0,1/2)$ in $H^2(-1/2)$. Therefore, up to congruence, the totally geodesic Lagrangian planes can be given by $\phi(t,s) = (\cos s \sinh t, \sin s \cosh t)$.

3. Additive formula of Lagrangian angle map. For a unit speed Legendre curve $\gamma = (\gamma_1, \gamma_2)$ in S^3, we define the *Legendre angle* θ_γ of γ by

$$e^{i\theta_\gamma} = \det{}_{\boldsymbol{C}}(\gamma, \gamma') = \gamma_1\gamma_2' - \gamma_1'\gamma_2 . \tag{3.1}$$

For instance, the Legendre angle of $\gamma(s) = (\cos s, \sin s)$ is 0 (mod 2π).

LEMMA 3.1. *Let $\gamma : I_1 \to S^3 \subset \boldsymbol{C}^2$ be a unit speed curve. Then we have*

(1a) *If γ is a Legendre curve, then it is a solution of the second order differential equation*:

$$\gamma'' - ik_\gamma\gamma' + \gamma = 0 , \tag{3.2}$$

where k_γ is the curvature of γ in S^3.

(1b) *If γ satisfies* (3.2), *then γ is a Legendre curve if and only if* $\langle\gamma'(0), i\gamma(0)\rangle = 0$ $(0 \in I_1)$.

(2) *The Legendre angle θ_γ satisfies $\theta_\gamma' = k_\gamma$.*

PROOF. Statement (1a) has been proved in [6]. Statement (1b) follows from the constancy of the function $s \mapsto \langle \gamma'(s), i\gamma(s)\rangle$ using (3.2). Finally, from (3.2) and (3.1) we find

$$
\begin{aligned}
ie^{i\theta_\gamma}\theta'_\gamma &= (\gamma_1\gamma'_2 - \gamma'_1\gamma_2)' \\
&= \gamma_1(ik_\gamma\gamma'_2 - \gamma_2) - (ik_\gamma\gamma'_1 - \gamma_1)\gamma_2 \\
&= ik_\gamma(\gamma_1\gamma'_2 - \gamma'_1\gamma_2) = ik_\gamma e^{i\theta_\gamma}\,,
\end{aligned} \tag{3.3}
$$

which implies statement (2). □

Similarly, we define the *Legendre angle* θ_α of a unit speed Legendre curve α in H^3_1 by

$$e^{i\theta_\alpha} = \det_{\boldsymbol{C}}(\alpha, \alpha') = \alpha_1\alpha'_2 - \alpha'_1\alpha_2\,. \tag{3.4}$$

For instance, the Legendre angle of $\alpha(t) = (\sinh t, \cosh t)$ is 0 (mod 2π).

We also have the following.

LEMMA 3.2. *Let $\alpha : I_2 \to H^3_1 \subset \boldsymbol{C}^2$ be a unit speed curve. Then we have*

(1a) *α is a solution of the second order differential equation*:

$$\alpha'' - ik_\alpha\alpha' - \alpha = 0\,, \tag{3.5}$$

where k_α is the curvature of α in H^3_1.

(1b) *If α satisfies* (3.5), *then α is a Legendre curve if and only if* $\langle \alpha'(0), i\alpha(0)\rangle = 0$ $(0 \in I_2)$.

(2) *The Legendre angle θ_α satisfies $\theta'_\alpha = k_\alpha$.*

PROOF. This can be done in the same way as Lemma 3.1. □

REMARK 3.3. If (γ_1, γ_2) is a Legendre curve in S^3, $(e^{i\theta}\gamma_1, \gamma_2)$ and $(\gamma_1, e^{i\theta}\gamma_2)$ are also Legendre curves in S^3. The same happens to a Legendre curve (α_1, α_2) in H^3_1.

Using this fact, up to congruences in $\boldsymbol{C}^2$, we can restrict our attention in our construction (2.4) of Lagrangian surfaces to consider the initial conditions

$$\gamma(0) = (\cos\psi, \sin\psi)\,,\ \gamma'(0) = e^{ia}(\sin\psi, -\cos\psi)\,,\quad 0 \le \psi \le \pi/2,\ -\pi < a \le \pi\,, \tag{3.6}$$

and

$$\alpha(0) = (\sinh\delta, \cosh\delta)\,,\ \alpha'(0) = e^{ib}(\cosh\delta, \sinh\delta)\,,\quad \delta \ge 0,\ -\pi < b \le \pi\,. \tag{3.7}$$

We consider now the Lagrangian conformal immersion $\phi : I_1 \times I_2 \to \boldsymbol{C}^2$ defined by $\phi(t, s) = (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$, where $\gamma = (\gamma_1, \gamma_2)$ is a unit speed Legendre curve in $S^3 \subset \boldsymbol{C}^2$ and $\alpha = (\alpha_1, \alpha_2)$ is a unit speed Legendre curve in $H^3_1 \subset \boldsymbol{C}^2$.

The Lagrangian angle map β_ϕ of ϕ (see Section 2) can be computed by

$$e^{i\beta_\phi} = \det_{\boldsymbol{C}}(\phi_* e_1, \phi_* e_2)\,, \tag{3.8}$$

where e_1, e_2 is any oriented orthonormal basis of the Lagrangian surface.

We now prove a useful interesting additive formula which relates the Lagrangian angle map of our Lagrangian surfaces with the Legendre angle of the generating curves.

THEOREM 3.4. *Let γ be a unit speed Legendre curve in S^3 and α a unit speed Legendre curve in H_1^3. Then the Lagrangian angle map β_ϕ of the Lagrangian conformal immersion $\phi(t,s) = (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$ and the Legendre angles θ_γ and θ_α of γ and α are related by*

$$\beta_\phi(t,s) = \theta_\gamma(s) + \theta_\alpha(t) + \pi \pmod{2\pi}\,. \tag{3.9}$$

PROOF. From (2.2), (2.3), (3.1) and (3.4), we have

$$\begin{aligned} e^{i(\theta_\gamma+\theta_\alpha)} &= (\gamma_1\gamma_2' - \gamma_1'\gamma_2)(\alpha_1\alpha_2' - \alpha_1'\alpha_2) \\ &= \gamma_1\gamma_2'\alpha_1\alpha_2' - \gamma_1'\gamma_2\alpha_1\alpha_2' - \gamma_1\gamma_2'\alpha_1'\alpha_2 + \gamma_1'\gamma_2\alpha_1'\alpha_2 \\ &= \gamma_1\gamma_2'|\alpha_1|^2\frac{\alpha_1'}{\bar\alpha_2} + \frac{\gamma_2'}{\bar\gamma_1}|\gamma_2|^2|\alpha_1|^2\frac{\alpha_1'}{\bar\alpha_2} - \gamma_1\gamma_2'\alpha_1'\alpha_2 - \frac{\gamma_2'}{\bar\gamma_1}|\gamma_2|^2\alpha_1'\alpha_2 \\ &= \frac{\alpha_1'\gamma_2'}{\bar\alpha_2\bar\gamma_1}(|\gamma_1|^2|\alpha_1|^2 + |\gamma_2|^2|\alpha_1|^2 - |\gamma_1|^2|\alpha_2|^2 - |\gamma_2|^2|\alpha_2|^2) \\ &= -\frac{\alpha_1'\gamma_2'}{\bar\alpha_2\bar\gamma_1}\,. \end{aligned} \tag{3.10}$$

On the other hand, from (2.2), (2.3) and (3.8) we find

$$\begin{aligned} (|\alpha_1|^2 + |\gamma_1|^2)e^{i\beta_\phi} &= \alpha_1'\alpha_2\gamma_1\gamma_2' - \alpha_1\alpha_2'\gamma_1'\gamma_2 \\ &= \alpha_1'\alpha_2\gamma_1\gamma_2' + \alpha_1\frac{\alpha_1'\bar\alpha_1}{\bar\alpha_2}\frac{\gamma_2'\bar\gamma_2}{\bar\gamma_1}\gamma_2 \\ &= \frac{\alpha_1'\gamma_2'}{\bar\alpha_2\bar\gamma_1}(|\alpha_2|^2|\gamma_1|^2 + |\alpha_1|^2|\gamma_2|^2) \\ &= \frac{\alpha_1'\gamma_2'}{\bar\alpha_2\bar\gamma_1}((1+|\alpha_1|^2)|\gamma_1|^2 + |\alpha_1|^2(1-|\gamma_1|^2)) \\ &= \frac{\alpha_1'\gamma_2'}{\bar\alpha_2\bar\gamma_1}(|\alpha_1|^2 + |\gamma_1|^2)\,, \end{aligned} \tag{3.11}$$

which implies that

$$e^{i\beta_\phi} = \frac{\alpha_1'\gamma_2'}{\bar\alpha_2\bar\gamma_1}\,. \tag{3.12}$$

Combining (3.10) and (3.12) yields (3.9). □

COROLLARY 3.5. *Let γ be a unit speed Legendre curve in S^3 and α be a unit speed Legendre curve in H_1^3. Consider the Lagrangian conformal immersion $\phi : I_1 \times I_2 \to \boldsymbol{C}^2$ defined by $\phi(t,s) = (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$. Then the mean curvature vector field of ϕ is given by*

$$H = \frac{e^{-2u}}{2}(k_\alpha J\phi_t + k_\gamma J\phi_s)\,, \tag{3.13}$$

where $e^{2u} = |\gamma_1|^2 + |\alpha_1|^2$ and k_α and k_γ are the curvature functions of α and γ, respectively.

PROOF. According to (2.1), we have to compute the gradient of the Lagrangian angle β_ϕ. If $e_1 := e^{-u}\partial_t$ and $e_2 := e^{-u}\partial_s$, it is clear that $e_1(\beta_\phi) = e^{-u}k_\alpha$ and $e_2(\beta_\phi) = e^{-u}k_\gamma$ and so (3.13) follows immediately. $\square$

4. Applications. In this section we are devoted to study several families of Lagrangian surfaces of our construction; those characterized by different geometric properties related with the behaviour of the mean curvature vector.

4.1. Minimal Lagrangian immersions. As the first consequence we can obtain from Corollary 3.5 is the following.

THEOREM 4.1. *Let γ be a unit speed Legendre curve in S^3 and let α be a unit speed Legendre curve in H_1^3. Then the Lagrangian conformal immersion ϕ defined by $\phi(t,s) = (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$ is minimal if and only if the Legendre curves γ and α are geodesics in S^3 and H_1^3, respectively.*

Theorem 4.1 also follows directly from Theorem 3.4 by using Lemmas 3.1(2) and 3.2(2) and by taking into account that the minimality of ϕ is equivalently to the constancy of β_ϕ.

Using the statements (1) of Lemmas 3.1 and 3.2, the unit speed Legendre curves that are geodesic of S^3 and H_1^3 can be written as $\gamma(s) = \cos s\,\gamma(0) + \sin s\,\gamma'(0)$ and $\alpha(t) = \cosh t\,\alpha(0) + \sinh t\,\alpha'(0)$. After choosing the initial conditions given in (3.6) and (3.7), we arrive at the explicit expressions of the minimal Lagrangian surfaces in $\boldsymbol{C}^2$ that can be constructed by our method taking

$$\gamma(s) = (c_\psi \cos s + e^{ia} s_\psi \sin s, s_\psi \cos s - e^{ia} c_\psi \sin s)\,, \tag{4.1}$$

where $c_\psi := \cos\psi$ and $s_\psi := \sin\psi$, and

$$\alpha(t) = (sh_\delta \cosh t + e^{ib} ch_\delta \sinh t, ch_\delta \cosh t + e^{ib} sh_\delta \sinh t)\,, \tag{4.2}$$

where $sh_\delta := \sinh\delta$ and $ch_\delta := \cosh\delta$.

The Legendre geodesics (4.1) project by the Hopf fibration in the great circles of $S^2(1/2)$ contained in the planes $c_a x_2 = s_a(s_{2\psi}x_3 - c_{2\psi}x_1)$. The Legendre geodesics (4.2) project by the Hopf fibration in the geodesics of $H^2(-1/2)$ contained in the planes $c_b x_2 = s_b(ch_{2\delta}x_1 - sh_{2\delta}x_3)$. So, if $(a,\psi) \in \{(0,0),(0,\pi/2),(\pi,0),(\pi,\pi/2)\}$ and $(b,\delta) \in \{(0,0),(\pi,0)\}$, we arrive at totally geodesic Lagrangian planes.

Topologically all these surfaces are $\boldsymbol{R} \times S^1$ and it is possible to prove that they correspond, after changing suitably the complex structure in $\boldsymbol{C}^2$ (cf. [7]), to the family of complex surfaces in $\boldsymbol{C}^2$ with finite total curvature -4π (including the Lagrangian catenoid of [3]) given by Hoffman and Osserman in Proposition 6.6, case 2, of [10].

In conclusion, if we choose great circles in $S^2(1/2)$ and geodesics in $H^2(-1/2)$, our construction provides us examples of minimal Lagrangian surfaces in $\boldsymbol{C}^2$.

4.2. Lagrangian surfaces with constant mean curvature. The easiest (non minimal) examples of Lagrangian surfaces with constant mean curvature, i.e., $|H| \equiv \rho > 0$, are those with parallel mean curvature vector.

THEOREM 4.2. *Let γ be a unit speed Legendre curve in S^3 and let α be a unit speed Legendre curve in H_1^3. Then the Lagrangian conformal immersion ϕ defined by $\phi(t,s) = (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$ has parallel (non null) mean curvature vector if and only if the Legendre curves γ and α in S^3 and H_1^3 respectively, satisfy that $|\gamma_1|$ and $|\alpha_1|$ are constant.*

PROOF. Using Corollary 3.5, it is easy to check that $JH = -(e^{-2u}/2)(k_\alpha\partial_t + k_\gamma\partial_s)$ is a parallel vector field if and only if

$$k'_\alpha - u_t k_\alpha + u_s k_\gamma = 0\,, \quad u_t k_\gamma + u_s k_\alpha = 0\,, \quad k'_\gamma - u_s k_\gamma + u_t k_\alpha = 0\,, \tag{4.3}$$

where $e^{2u} = |\gamma_1|^2 + |\alpha_1|^2$. From the first and third equation of (4.3) we deduce that $k'_\alpha + k'_\gamma = 0$ and so $k_\alpha(t) = at + b$ and $k_\gamma(s) = -as + c$, with $a, b, c \in \boldsymbol{R}$. We distinguish three cases: We first suppose that $u_s = 0$. It is equivalent to $|\gamma_1|$ is constant. Using (4.3) and that ϕ is non minimal, we obtain that $u_t = 0$ what means that $|\alpha_1|$ is constant. If $u_t = 0$, we make a similar reasoning. Finally, if $u_t \neq 0$ and $u_s \neq 0$, from the second equation of (4.3), there exists $c_1 \in \mathbf{R}^*$ such that $k_\gamma = -c_1 u_s$ and $k_\alpha = c_1 u_t$. So $u_s = (as - c)/c_1$ and $u_t = (at + b)/c_1$. Putting this in (4.3), we arrive at $a = b = c = 0$, which is a contradiction. □

If we call a small circle ξ in $S^2(1/2)$ (resp. in $H^2(-1/2)$) *horizontal* when it is orthogonal to the x_3-coordinate, then we can easily show that a unit speed Legendre curve γ in S^3 is a horizontal lift of a horizontal circle in $S^2(1/2)$ if and only if $|\gamma_1|$ is a nonzero constant. Moreover, such Legendre curves can be parametrized by

$$\gamma(s) = (\cos\psi\, e^{i\tan\psi\, s}, \sin\psi\, e^{-i\cot\psi\, s})\,, \quad \psi \in (0, \pi/2)\,, \tag{4.4}$$

where $\pi/2 - 2\psi$ is the latitude of the parallel $\pi \circ \gamma$.

Similarly, a unit speed Legendre curve α in H_1^3 is a horizontal lift of a horizontal circle in $H^2(-1/2)$ if and only if $|\alpha_1|$ is a nonzero constant. Moreover, such Legendre curves can be parametrized by

$$\alpha(t) = (\sinh\delta\, e^{i\coth\delta\, t}, \cosh\delta\, e^{i\tanh\delta\, t})\,, \quad \delta > 0\,. \tag{4.5}$$

In conclusion, using both Legendre curves given in (4.4) and in (4.5) in our construction we obtain conformal parametrizations of the examples of Lagrangian surfaces with parallel mean curvature vector in $\boldsymbol{C}^2$. They correspond to flat tori $S^1 \times S^1$ in the 3-sphere of radius $\sqrt{\sin^2\psi + \sinh^2\delta}$.

THEOREM 4.3. *Let γ be a unit speed Legendre curve in S^3 and let α be a unit speed Legendre curve in H_1^3. Then the Lagrangian conformal immersion ϕ defined by $\phi(t,s) = (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$ has constant mean curvature $|H| \equiv \rho > 0$ if and only if the Legendre curves γ and α in S^3 and H_1^3, respectively, satisfy that $k_\gamma^2 = 4\rho^2|\gamma_1|^2 - \lambda$ and $k_\alpha^2 = 4\rho^2|\alpha_1|^2 + \lambda$ with $\lambda \in \boldsymbol{R}$.*

PROOF. Using Corollary 3.5, we have that $4\rho^2(|\gamma_1|^2 + |\alpha_1|^2) = k_\alpha^2 + k_\gamma^2$. Since γ depends on s and α depends on t, we obtain the result. □

Now we see how the condition on γ and α in Theorem 4.3 determine both curves. Let define $r(s) := |\gamma_1(s)|$. Using the Legendre character of γ and that it is parametrized by arclength and satisfies (3.2), it is not difficult to get that γ can be expressed in terms of r in the following way:

(4.6)

$$\gamma(s)=\left(r(s)\exp\left(i\int_0^s\frac{\sqrt{1-r^2-r'^2}}{r}ds\right),\sqrt{1-r(s)^2}\exp\left(i\int_0^s\frac{r\sqrt{1-r^2-r'^2}}{r^2-1}ds\right)\right).$$

We observe that when $r(s)=\cos s$, we get the geodesic $\gamma(s)=(\cos s,\sin s)$, and if we take r constant, say $r\equiv\cos\psi$, we arrive at the expression of (4.4). We can also compute the curvature of γ in terms of $r=r(s)$ obtaining

$$r''-\frac{1-r^2-r'^2}{r}+r+k_\gamma\sqrt{1-r^2-r'^2}=0\,. \tag{4.7}$$

If we use a similar argument, a unit speed Legendre curve in H_1^3 can be written in terms of $r(t)=|\alpha_1(t)|$ as

(4.8)

$$\alpha(t)=\left(r(t)\exp\left(i\int_0^t\frac{\sqrt{1+r^2-r'^2}}{r}dt\right),\sqrt{1+r(t)^2}\exp\left(i\int_0^t\frac{r\sqrt{1+r^2-r'^2}}{1+r^2}dt\right)\right).$$

We note that $r(t)=\sinh t$ give us the geodesic $\alpha(t)=(\sinh t,\cosh t)$ and that $r(t)\equiv\sinh\delta$ leads to (4.5). Moreover, the curvature of α is given by

$$r''-\frac{1+r^2-r'^2}{r}-r+k_\alpha\sqrt{1+r^2-r'^2}=0\,. \tag{4.9}$$

We study the case of Theorem 4.3. If $k_\gamma(s)^2=4\rho^2r(s)^2-\lambda$ and $k_\alpha(t)^2=4\rho^2r(t)^2+\lambda$, we are able to obtain first integrals of the differential equations (4.7) and (4.9):

$$\frac{(4\rho^2r^2-\lambda)^{3/2}}{12\rho^2}+\mu_1=r\sqrt{1-r^2-r'^2}\,,\quad r=r(s)\,, \tag{4.10}$$

and

$$\frac{(4\rho^2r^2+\lambda)^{3/2}}{12\rho^2}+\mu_2=r\sqrt{1+r^2-r'^2}\,,\quad r=r(t)\,, \tag{4.11}$$

where λ,μ_1,μ_2 are arbitrary constants.

This shows that the family of Lagrangian surfaces with constant mean curvature $\rho>0$ in our construction with Legendre curves is quite big. In general, the solutions of (4.10) and (4.11) are not easy to control, appearing hyperelliptic functions in most cases. We finish this section considering the following illustrative situation.

Particular case: Let $\lambda=\mu_1=\mu_2=0$. Up to dilations, we can suppose $\rho=3/2$. Then equations (4.10) and (4.11) reduce to $r'^2+r^2+r^4=1$ and $r'^2-r^2+r^4=1$, respectively. After solving the differential equation $r'^2+r^2+r^4=1$, we know that, up to translations on s, its solution is given by

$$r(s)=\sqrt{\frac{\sqrt5-1}{2}}\,\mathrm{cn}(\sqrt[4]{5}s,k),\quad k=\sqrt{\frac{5-\sqrt5}{10}}\,, \tag{4.12}$$

where cn is a Jacobi elliptic function usually known as the cosine amplitude and k is its modulus (cf., for instance, [12]). Hence, using standard formulae on elliptic functions and a straightforward long computation, (4.6) and (4.12) imply that, up to rotations, γ is given by

$$\begin{aligned}\gamma(s) = (\mathrm{dn}(\sqrt[4]{5}s,k) + i\,k\,\mathrm{sn}(\sqrt[4]{5}s,k))\Bigg(&\sqrt{\tfrac{\sqrt{5}-1}{2}}\mathrm{cn}(\sqrt[4]{5}s,k),\\ \sqrt{1+(1/2)(1-\sqrt{5})\mathrm{cn}^2(\sqrt[4]{5}s,k)}\,&\frac{2\mathrm{dn}(\sqrt[4]{5}s,k) - \sqrt{5+2\sqrt{5}}i\,\mathrm{sn}(\sqrt[4]{5}s,k)}{\sqrt{4\mathrm{dn}^2(\sqrt[4]{5}s,k) + (5+2\sqrt{5})\mathrm{sn}^2(\sqrt[4]{5}s,k)}}\Bigg),\end{aligned} \tag{4.13}$$

where dn and sn are the Jacobi elliptic function known as the delta amplitude and the sine amplitude with modulus k.

Similarly, up to translations in t, the solution of $r'^2 - r^2 + r^4 = 1$ is given by

$$r(t) = \sqrt{\tfrac{\sqrt{5}+1}{2}}\,\mathrm{cn}(\sqrt[4]{5}t,\hat{k}), \quad \hat{k} = \sqrt{(5+\sqrt{5}\,)/10}\,. \tag{4.14}$$

Thus, in an analogous way, it follows from (4.8), (4.14) and a long computation that, up to rotations, α is given by

$$\begin{aligned}\alpha(t) = (\mathrm{dn}(\sqrt[4]{5}t,\hat{k}) + i\,\hat{k}\,\mathrm{sn}(\sqrt[4]{5}t,\hat{k}))\Bigg(&\sqrt{\tfrac{\sqrt{5}+1}{2}}\mathrm{cn}(\sqrt[4]{5}t,\hat{k}),\\ \sqrt{1+(1/2)(1+\sqrt{5})\mathrm{cn}^2(\sqrt[4]{5}t,\hat{k})}\,&\frac{2\mathrm{dn}(\sqrt[4]{5}t,\hat{k}) - \sqrt{5-2\sqrt{5}}i\,\mathrm{sn}(\sqrt[4]{5}t,\hat{k})}{\sqrt{4\mathrm{dn}^2(\sqrt[4]{5}t,\hat{k}) + (5-2\sqrt{5})\mathrm{sn}^2(\sqrt[4]{5}t,\hat{k})}}\Bigg).\end{aligned} \tag{4.15}$$

We remark that both Legendre curves γ and α given in (4.13) and (4.15) are periodic on account of the periodicity of the elliptic functions cn, sn and dn. So they provide an interesting example of a Lagrangian torus with constant mean curvature in complex Euclidean plane.

4.3. New examples of Hamiltonian-minimal Lagrangian surfaces. By applying Theorem 3.4, we have the following

THEOREM 4.4. *Let γ and α be unit speed Legendre curves in S^3 and H_1^3, respectively. Then the Lagrangian conformal immersion $\phi(t,s) = (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$ is Hamiltonian-minimal if and only if the curvature functions k_α and k_γ of α and γ are given by $k_\alpha(t) = at + b$ and $k_\gamma(s) = -as + c$ with $a, b, c \in \boldsymbol{R}$.*

PROOF. It is known that the Lagrangian surface ϕ is Hamiltonian-minimal if and only if the Lagrangian angle map β_ϕ is harmonic, i.e., $\Delta\beta_\phi = 0$ (see Section 2). Thus, Theorem 3.4 implies that ϕ is Hamiltonian-minimal if and only if we have $\theta''_\gamma + \theta''_\alpha = 0$. Therefore, by Lemmas 3.1 and 3.2, we know that ϕ is Hamiltonian-minimal if and only if the curvature function k_α and k_γ of α and γ satisfy $k'_\alpha + k'_\gamma = 0$. We put then $k'_\alpha = a = -k_\gamma$ and the proof is finished. □

We distingish two essential cases in this family:

Case (i): $a = 0$, i.e., k_α and k_γ are constant.

From Lemma 3.1 (1) we know that unit speed Legendre curves in S^3 with constant curvature $k_\gamma \equiv c$ can be parametrized by

$$\gamma(s) = e^{i(c+\sqrt{c^2+4})s/2}A_1 + e^{i(c-\sqrt{c^2+4})s/2}B_1$$

for suitable $A_1, B_1 \in \boldsymbol{C}^2$ that can be expressed in terms on the initial conditions given in (3.6).

Similarly, from Lemma 3.2 (1) we also know that unit speed Legendre curves in H_1^3 with constant curvature $k_\alpha \equiv b$ can be parametrized by

(1) If $|b| > 2$,

$$\alpha(t) = e^{ibt/2}(e^{i\sqrt{b^2-4}\,t/2}A_2 + e^{-i\sqrt{b^2-4}\,t/2}B_2)\,;$$

(2) if $|b| < 2$,

$$\alpha(t) = e^{ibt/2}(e^{\sqrt{4-b^2}\,t/2}A_2 + e^{-\sqrt{4-b^2}\,t/2}B_2)\,;$$

(3) if $b = 2$,

$$\alpha(t) = e^t A_2 + te^t B_2\,;$$

(4) if $b = -2$,

$$\alpha(t) = e^{-t}A_2 + te^{-t}B_2\,,$$

for suitable $A_2, B_2 \in \boldsymbol{C}^2$ that can be expressed in terms on the initial conditions given in (3.7).

REMARK 4.5. In this context, it is not difficult to check that the Hamiltonian-minimal Lagrangian tori of [2] are constructed by using horizontal small circles in $S^2(1/2)$ (see (4.4)) with closed horizontal lift (i.e., $\tan^2\psi$ is a rational number) and the projections to $H^2(-1/2)$ of the above α's for certain b's such that $|b| > 2$.

Case (ii): $a \neq 0$, i.e., k_α and k_γ are certain linear functions of the arc parameter.

In this case, after applying suitable translations, we have $\kappa_\alpha = at$ and $\kappa_\gamma = -as$. Thus, by Lemmas 3.1 and 3.2, we know that the Legendre curves α and γ satisfy

$$\alpha''(t) - iat\alpha'(t) - \alpha(t) = 0, \quad \gamma''(s) + ias\gamma'(s) + \gamma(s) = 0\,. \tag{4.16}$$

Therefore, after solving these differential equations, we know that the unit speed Legendre curves α in H_1^3 with curvature $\kappa_\alpha = at$ and the unit speed Legendre curves γ in S^3 with $\kappa_\gamma = -as$ can be expressed in terms of Hermite polynomials and hypergeometric functions (see [14]) by

$$\alpha(t) = \mathrm{HermiteH}(i/a, \sqrt{ia/2}\,t\big)A_1 + {}_1F_1\big(1/(2ai), 1/2, ait^2/2)B_1\,, \tag{4.17}$$

$$\begin{aligned}\gamma(s) = {} & e^{-ais^2/2}\{\mathrm{HermiteH}(1/(ai) - 1, \sqrt{ia/2}\,s)A_2 \\ & + {}_1F_1((ai-1)/(2ai), 1/2, ais^2/2)B_2\}\end{aligned} \tag{4.18}$$

for suitable $A_1, B_1 \in \boldsymbol{C}_1^2$ and $A_2, B_2 \in \boldsymbol{C}^2$ depending on the initial conditions given in (3.6) and (3.7), where HermiteH is the Hermite polynomial and ${}_1F_1$ is the Kummer confluent hypergeometric function.

4.4. Willmore Lagrangian surfaces. Consider the Willmore functional

$$W = \int_{\Sigma} |H|^2 dA \tag{4.19}$$

for a surface Σ in a Euclidean space.

For a unit speed Legendre curve γ in S^3 and a unit speed Legendre curve α in H_1^3, the Willmore functional of the Lagrangian conformal immersion $\phi : I_1 \times I_2 \to \boldsymbol{C}^2$; $(t,s) \mapsto (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$ is given by

$$W_\phi = \frac{1}{4}\int_{\phi(I_1\times I_2)} |\nabla \beta_\phi|^2 dA\,. \tag{4.20}$$

Hence, it follows from Theorem 3.4 and Lemmas 3.1 and 3.2 that the Willmore functional associated with ϕ is given by

$$\begin{aligned} W_\phi &= \frac{1}{4}\int_{I_1\times I_2} (k_\alpha^2 + k_\gamma^2)\,dt\,ds \\ &= \frac{L(\gamma)}{4}\int_{I_1} k_\alpha^2 dt + \frac{L(\alpha)}{4}\int_{I_2} k_\gamma^2 ds\,, \end{aligned} \tag{4.21}$$

where $L(\gamma)$ and $L(\alpha)$ denote the length of γ and of α, respectively.

THEOREM 4.6. *Let γ and α be unit speed Legendre curves in S^3 and H_1^3, respectively. Then the Lagrangian conformal immersion $\phi(t,s) = (\alpha_1(t)\gamma_1(s), \alpha_2(t)\gamma_2(s))$ is a critical point of the Willmore functional W_ϕ (with fixed lengths $L(\alpha)$ and $L(\gamma)$) if and only if the Legendre curves α and γ are elastic curves.*

PROOF. From (4.21), we see that the critical points of the Willmore functional W_ϕ (with fixed $L_1 = L(\alpha)$ and $L_2 = L(\gamma)$) are given by the Lagrangian conformal immersions constructed with Legendre curves α and γ that are critical points of the functionals $\int_0^{L_1} k_\alpha^2 dt$ and $\int_0^{L_2} k_\gamma^2 ds$, respectively. But these are precisely free elastic curves according to [11]. $\square$

REMARK 4.7. As corollary of Theorem 4.6, using free elastica in $S^2(1/2)$ and $H^2(-1/2)$ our construction provides new examples of Willmore Lagrangian surfaces in $\boldsymbol{C}^2$. In [11] we can find explicitly examples of (closed) free elastica on the sphere and in the Poincaré disk.

A different construction of Willmore Lagrangian surfaces in $\boldsymbol{C}^2$ can be found in [4].

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