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# Lagrangian theory of gravitational instability of Friedman-Lemaître cosmologies and the 'Zel'dovich approximation'

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**SUMMARY**

The aim of this paper is to clarify the connection between the so-called 'Zel'dovich approximation' and perturbative solutions of the Euler–Poisson system for the motion of a self-gravitating dust continuum, evaluated in the Lagrangian picture of fluid dynamics. Solutions of the Lagrangian equations, linearized at an isotropically expanding background, are derived. This approximation is investigated; it contains the 'Zel'dovich approximation' as well as a generalized form of it as subclasses, and allowed treatment of the non-linear evolution of vortical perturbations consistently within the framework of self-gravitating motions. In contrast to the prediction of the standard linear approximation, vorticity is coupled to the density enhancement and is amplified in the present approximation. The transport equation for vorticity is examined and applied. Besides the generalization to vortical flows, this work gives a straightforward derivation and consistent definition of Zel'dovich's approach. Relations to the fully non-linear equations are discussed in terms of the eigenvalues of the velocity gradient of the inhomogeneous deformation. In particular, the approximation, although based on linearized equations, provides the exact solution for the evolution of plane-symmetric inhomogeneities as well as a class of three-dimensional solutions, as was shown in an earlier paper.



## 1 OVERVIEW

In this paper we strictly confine our attention to the Lagrangian picture describing a fluid moving in its own Newtonian gravitational self-field without additional forces. I investigate the Lagrangian theory of gravitational instability of homogeneous model universes, a problem which is commonly studied in the Eulerian picture. Since the density is not considered as a dynamical variable in the Lagrangian representation, we are led to an approximation, which, in contrast to, e.g., the known linear theory of gravitational instability formulated in the Eulerian picture, does not require linearization of this variable. Thus the present approach allows us to follow the fluid's motion into the non-linear regime, still conserving mass and linear momentum according to the construction of this theory. Since the standard linear approximation allows for vortical flows, it is desirable to obtain an approximation which also conserves the angular momentum until shell crossing, i.e., until the fields are no longer single-valued functions of the Eulerian coordinates. We examine the transport equation for vorticity and give an expression for the evolution of vorticity in the general case and in the approximation presented.

The study of the formation of large-scale structure in the Universe has been successfully described by the so-called 'Zel'dovich approximation' (Zel'dovich 1970a, b, c, 1978), which is shown to be contained in a subclass. However, this approximation is restricted to potential flows, as will be demonstrated. The approximation presented allows us to study the non-linear evolution of vortical perturbations preserving the potential character of the gravitational field. The impact of vorticity on the formation of large-scale structure is presently being studied in two spatial dimensions by a numerical simulation (Buchert & Alimi, in preparation). The study of vortical pancake formation gains interest in connection with reconstruction methods of the density field based on the observed velocity distribution (Dekel 1991), and is important for the interpretation of structures in redshift space.

The validity of the approximation is supported by numerical simulations on the one hand (see Doroshkevich, Ryabenkii & Shandarin, 1973, 1975), and the study of exact solutions on the other. A related class of solutions has been investigated earlier (Buchert 1989a, henceforth abbreviated by B89). Explicit solutions have been obtained for the evolution of inhomogeneities in background models with non-vanishing curvature parameter, as well as with non-vanishing cosmological constant (B89; Bildhauer, Buchert & Kasai 1991). Limitations of this approach have been discussed by Peebles (1987), Grinstein & Wise (1987) and myself (B89).

## 2 THE EULER–POISSON SYSTEM IN EULERIAN AND LAGRANGIAN FORM

Let us characterize the state of a 'dust' medium (i.e. a medium without pressure) at the time $t_0$ by a velocity and a density field $\mathring{\boldsymbol{v}}(\boldsymbol{x})$ and $\mathring{\rho}(\boldsymbol{x})$. We are interested in the evolution of the medium governed by the Eulerian evolution equations for the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ and the density field $\rho(\boldsymbol{x}, t)$:

$$\boldsymbol{v}_{,t} = -(\boldsymbol{v}\cdot\nabla_x)\,\boldsymbol{v}+\boldsymbol{g}, \qquad \boldsymbol{v}(\boldsymbol{x}, t_0)=\mathring{\boldsymbol{v}}(\boldsymbol{x}), \tag{1a}$$

$$\rho_{,t} = -\nabla_x\cdot(\rho\boldsymbol{v}), \qquad \rho(\boldsymbol{x}, t_0)=\mathring{\rho}(\boldsymbol{x}). \tag{1b}$$

The evolution of the medium is governed by Newton's field equations for the gravitational field strength $\boldsymbol{g}(\boldsymbol{x}, t)$ (which is equal to the acceleration field according to the equivalence principle of inertial and gravitational mass):

$$\nabla_x\times\boldsymbol{g}=\boldsymbol{0}, \tag{1c}$$

$$\nabla_x\cdot\boldsymbol{g}=\Lambda-4\pi G\rho, \tag{1d}$$

$$\rho>0. \tag{1e}$$

$\Lambda$ is the cosmological constant.

In the Lagrangian description we concentrate on integral curves $\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{X}, t)$ of the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$:

$$\frac{d\boldsymbol{f}}{dt}=\boldsymbol{v}(\boldsymbol{f}, t), \qquad \boldsymbol{f}(\boldsymbol{X}, t_0)=:\boldsymbol{X}. \tag{2}$$

Introducing these curves, which are labelled by the Lagrangian coordinates $X_i$, we can express all vector fields like the velocity $\boldsymbol{v}$, the gravitational field strength $\boldsymbol{g}$, and the density $\rho$, in terms of the field of trajectories $\boldsymbol{f}(\boldsymbol{X}, t)$ as follows:

$$\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{X}, t), \tag{3a}$$

$$\boldsymbol{v}=\dot{\boldsymbol{f}}(\boldsymbol{X}, t), \tag{3b}$$

$$\boldsymbol{g}=\ddot{\boldsymbol{f}}(\boldsymbol{X}, t), \tag{3c}$$

$$\rho=\mathring{\rho}(\boldsymbol{X})\,\{\det[f_{i,k}(\boldsymbol{X}, t)]\}^{-1}. \tag{3d}$$

The comma denotes partial differentiation with respect to the Lagrangian coordinates, and the dot denotes Lagrangian time derivative

$$\frac{d}{dt}:=\partial_t|_x+\boldsymbol{v}\cdot\nabla_x=\partial_t|_X;$$

comma and dot commute.

The spatial coordinates $x_i$ (independent in the Eulerian picture) are now construed as an additional vector field on Lagrange space (3a). Equation (3c) is just the definition of the acceleration field, and equation (3d) represents the general solution of the Eulerian continuity equation determining the density field in terms of derivatives of the field of trajectories (compare Serrin 1959 for the derivation). We have to use the identity

$$\frac{d}{dt}\det(f_{i,k})=\det(f_{i,k})\,\nabla_x\cdot\boldsymbol{v}$$

to obtain

$$\dot{\rho}/\rho=-\nabla_x\cdot\boldsymbol{v}\Rightarrow\frac{d}{dt}[\rho\det(f_{i,k})]=0 \quad\Leftrightarrow\quad (3\text{d}).$$

Thus mass conservation is guaranteed, irrespective of *any* equations which the trajectories $\boldsymbol{f}$ might obey. In fact, both evolution equations (1a) and (1b) are solved in *any* case in the Lagrangian description.

With the help of the transformation (3a), we transform the field equations (1c) and (1d). The definition of $\boldsymbol{g}$ (3c), and the density expression (3d), are inserted into the transformed field equations. We then obtain the following system of equations which the integral curves must obey [for details see also Buchert & Götz (1987) ($\Lambda=0$), and B89 ($\Lambda\neq0$)]:

$$\frac{1}{2}\,\varepsilon_{pq[j}\,\frac{\partial(\ddot{f}_{i]}, f_p, f_q)}{\partial(X_1, X_2, X_3)}=0, \qquad i\neq j, \tag{4a, b, c}$$

$$\sum_{a,b,c}\frac{1}{2}\,\varepsilon_{abc}\,\frac{\partial(\ddot{f}_a, f_b, f_c)}{\partial(X_1, X_2, X_3)}-\Lambda\det[f_{i,k}(\boldsymbol{X}, t)]=-4\pi G\mathring{\rho}(\boldsymbol{X}), \tag{4d}$$

$$\mathring{\rho}(\boldsymbol{X})>0, \qquad \det[f_{i,k}(\boldsymbol{X}, t)]\neq0. \tag{4e, f}$$

In order to show the equivalence of the two systems (1) and (4), we have to use the relations (3) to express the Eulerian vector fields in terms of Eulerian coordinates. This can be done by inserting the inverse of the transformation (3a), $\boldsymbol{X}=\boldsymbol{h}(\boldsymbol{x}, t)$, into (3b, c, d). This is possible as long as the transformation (3a) is invertible, i.e., as long as $\det(f_{i,k})\neq0$. We conclude that equation (1a) is solved by $\boldsymbol{f}$ via (3b) and (3c). Equation (3d) is the general solution of the continuity equation (1b). Thus the set of solutions $[\boldsymbol{v}(\boldsymbol{x}, t), \rho(\boldsymbol{x}, t)]$ of the Euler–Poisson system (1) can be transformed one-to-one on to the set of solutions $\boldsymbol{f}(\boldsymbol{X}, t)$ of the Lagrangian system (4). In the Lagrangian description, the Euler–Poisson system consists of four second-order equations for the *single* dynamical variable $\boldsymbol{f}$. This corresponds to seven first-order equations, whereas in the Eulerian description one has eight first-order equations to determine the evolution of the *two* dynamical fields $\boldsymbol{v}$ and $\rho$. Besides this reduction to one dynamical field, the Lagrangian picture has further advantages, as will be discussed in this paper. On the other hand, existence and uniqueness theorems have so far been established on the basis of the Eulerian formulation only (see, e.g. Brauer 1991).

## 3 LAGRANGIAN THEORY OF GRAVITATIONAL INSTABILITY – SOLUTION OF THE LINEARIZED EQUATIONS

### 3.1 Linearizing the Euler–Poisson system in the Lagrangian framework

We consider the equations (4) for the single dynamical variable $\boldsymbol{f}$ and linearize these equations on the homogeneous and isotropic background solutions.

Let $\boldsymbol{f}$ be a superposition of a homogeneous isotropic deformation and a vector function $\boldsymbol{p}$ for the inhomogeneous deformation of the medium:

$$\boldsymbol{f}(\boldsymbol{X}, t)=a(t)\,\boldsymbol{X}+\boldsymbol{p}(\boldsymbol{X}, t), \qquad a(t_0):=1, \qquad \boldsymbol{p}(\boldsymbol{X}, t_0):=\boldsymbol{0}. \tag{5}$$

Inserting this ansatz into the system of equations (4), we obtain to first order in the field $\boldsymbol{p}(\boldsymbol{X}, t)$ (the higher order terms are appended):

$$\ddot{a}a\nabla_0\times\boldsymbol{p}-a^2\nabla_0\times\ddot{\boldsymbol{p}}=\boldsymbol{0}, \tag{6a, b, c}$$

$$(3\ddot{a}a^2-a^3\Lambda)+(2\ddot{a}a-a^2\Lambda)\nabla_0\cdot\boldsymbol{p}+a^2\nabla_0\cdot\ddot{\boldsymbol{p}}=-4\pi G\mathring{\rho}(\boldsymbol{X}). \tag{6d}$$

($\nabla_0$ denotes the nabla operator with respect to the Lagrangian frame which commutes with the dot.)

If the homogeneous deformation separately solves the system of equations (4), i.e., if we insert the ansatz (5) for $\boldsymbol{p}=\boldsymbol{0}$, then the function $a(t)$ obeys the single equation:

$$3\ddot{a}a^2-a^3\Lambda=-4\pi G\mathring{\rho}_{\rm H};\qquad \mathring{\rho}_{\rm H}>0, \tag{7a}$$

the first integral of which is given by Friedman's differential equation:

$$\frac{\dot{a}^2+\text{const.}}{a^2}=\frac{8\pi G\rho_{\rm H}+\Lambda}{3}, \tag{7b}$$

where $\rho_{\rm H}=\mathring{\rho}_{\rm H}a^{-3}$ is the background density. Using (6), the remaining equations to be solved are

$$\ddot{a}a\nabla_0\times\boldsymbol{p}-a^2\nabla_0\times\ddot{\boldsymbol{p}}=\boldsymbol{0}, \tag{8a, b, c}$$

$$(2\ddot{a}a-a^2\Lambda)\nabla_0\cdot\boldsymbol{p}+a^2\nabla_0\cdot\ddot{\boldsymbol{p}}=-4\pi G(\mathring{\rho}-\mathring{\rho}_{\rm H}). \tag{8d}$$

To solve the equations (8), we first use the field equations (1c, d) at the initial time $t_0$ in order to express the source term in equation (8d) by the divergence of the initial field strength perturbation $\ddot{\boldsymbol{p}}(t_0)$. We obtain

$$\nabla_0\times\left(\frac{\ddot{a}}{a}\boldsymbol{p}-\ddot{\boldsymbol{p}}\right)=\boldsymbol{0}, \tag{9a, b, c}$$

$$\nabla_0\cdot\left(\left(2\frac{\ddot{a}}{a}-\Lambda\right)\boldsymbol{p}+\ddot{\boldsymbol{p}}-\ddot{\boldsymbol{p}}(t_0)\,a^{-2}\right)=0. \tag{9d}$$

We now split the vector perturbation $\boldsymbol{p}$ into a curl-free part $\boldsymbol{p}^{\rm D}$ and a divergence-less part $\boldsymbol{p}^{\rm R}$ (note that this decomposition is not unique, unless we impose boundary conditions):

$$\boldsymbol{p}=\boldsymbol{p}^{\rm D}+\boldsymbol{p}^{\rm R}. \tag{10}$$

Inserting this ansatz into the equations (9), we conclude that $\boldsymbol{p}^{\rm D}$ and $\boldsymbol{p}^{\rm R}$ obey the following equations ($\psi$ is an arbitrary scalar potential, $\boldsymbol{K}$ an arbitrary vector potential):

$$\frac{\ddot{a}}{a}\boldsymbol{p}^{\rm R}-\ddot{\boldsymbol{p}}^{\rm R}=\nabla_0\psi,\qquad \Delta_0\psi=0, \tag{11a}$$

$$\left(2\frac{\ddot{a}}{a}-\Lambda\right)\boldsymbol{p}^{\rm D}+\ddot{\boldsymbol{p}}^{\rm D}-\ddot{\boldsymbol{p}}^{\rm D}(t_0)\,a^{-2}=\nabla_0\times\boldsymbol{K},\quad \nabla_0\times(\nabla_0\times\boldsymbol{K})=\boldsymbol{0}. \tag{11b}$$

We now give a class of solutions of the system (11) assuming $\nabla_0\psi=\boldsymbol{0}$ and $\nabla_0\times\boldsymbol{K}=\boldsymbol{0}$. Since both of the resulting equations are linear and of second order, and one of the equations (11b) is inhomogeneous, a class of solutions of the set of equations (11) can be written as a superposition of five solutions, which separate with respect to space and time (four of them are linearly independent).

*Theorem 1*

A large class of linear inhomogeneous deformations has the form

$$\boldsymbol{p}=q_1^{\rm D}(t)\,\boldsymbol{Q}_1^{\rm D}(\boldsymbol{X})+q_2^{\rm D}(t)\,\boldsymbol{Q}_2^{\rm D}(\boldsymbol{X})+q_{\rm p}^{\rm D}(t)\,\boldsymbol{P}(\boldsymbol{X})+q_1^{\rm R}(t)\,\boldsymbol{Q}_1^{\rm R}(\boldsymbol{X})+q_2^{\rm R}(t)\,\boldsymbol{Q}_2^{\rm R}(\boldsymbol{X}), \tag{12}$$

with

$$\frac{\ddot{a}}{a}q_l^{\rm R}-\ddot{q}_l^{\rm R}=0,\qquad l=1,2, \tag{13a}$$

$$\left(2\frac{\ddot{a}}{a}-\Lambda\right)q_l^{\rm D}+\ddot{q}_l^{\rm D}=0,\qquad l=1,2, \tag{13b}$$

$$\left(2\frac{\ddot{a}}{a}-\Lambda\right)q_{\rm p}^{\rm D}+\ddot{q}_{\rm p}^{\rm D}=\ddot{q}_{\rm p}^{\rm D}(t_0)\,a^{-2}. \tag{13c}$$

The functions $q_l^{\rm D}$ and $q_l^{\rm R}$ are linearly independent solutions of the homogeneous equations in (11) obeying (13a, b), and the function $q_{\rm p}^{\rm D}$ is any particular solution of the equation (11b) and obeys (13c). The vector function $\boldsymbol{P}$ is a linear combination of the vector functions $\boldsymbol{Q}_l^{\rm D}$ and is determined by $\boldsymbol{p}^{\rm D}(t_0)=\boldsymbol{0}$. The vector functions $\boldsymbol{Q}_l^{\rm D}$ are gradients of arbitrary scalar potentials, and the vector functions $\boldsymbol{Q}_l^{\rm R}$ are curls of arbitrary vector potentials. All five vector functions can be expressed in terms of the initial conditions $\mathring{\rho}(\boldsymbol{X})=\rho(\boldsymbol{X},t_0)$ and $\mathring{\boldsymbol{v}}(\boldsymbol{X})=\boldsymbol{v}(\boldsymbol{X},t_0)$ after solving the equations (13). [This is explicitly done in Section 4 for a special case. This example also demonstrates that the solution (12) is *general*, see Section 4.]

### 3.2 The approximate theory

In the Lagrangian picture, the basic dynamical variable is the deformation field $\boldsymbol{f}$. If we linearize the Euler–Poisson system (4), formulated as equations for only this variable, we merely linearize with respect to $\boldsymbol{p}$. The density field appears not as a dynamical variable and can be integrated fully in the Lagrangian picture (equation 3d). Strictly speaking, in order to obtain a *linear* theory in the Lagrangian description, we would have to linearize the integral expression (3d) for the density. If $\rho$ (as a functional of the linearized field $\boldsymbol{f}^{\rm L}$) is not linearized, we are led to a Lagrangian approximate theory, which satisfies the mass conservation law exactly, and allows us to follow the fluid's motion into the non-linear regime of structure formation. The validity of this approximation if compared to the fully non-linear problem will be discussed in Section 7. This approximation is not constructable out of the Eulerian linear theory within the equivalence class of *linear* transformations.

The formalism presented can be used to investigate a higher order perturbation analysis in the Lagrangian framework. An effort in this direction has been undertaken by Moutarde *et al.* (1991).

### 3.3 The Lagrangian theory in a frame comoving with the background

Pancake models (see, e.g. Buchert 1989a, b and references therein) deal with the evolution of inhomogeneities superimposed on an isotropically expanding background, considered in a frame comoving with that background. This corresponds to a choice of Eulerian coordinates $\boldsymbol{q}:=\boldsymbol{x}/a(t)$. In the Lagrangian picture, we introduce the 'comoving' orbit $\boldsymbol{F}$ (not to be confused with the Lagrangian position, which is comoving with the inhomogeneously deformed fluid):

$$\boldsymbol{q}=\boldsymbol{F}(\boldsymbol{X},t):=\frac{1}{a(t)}\,\boldsymbol{f}(\boldsymbol{X},t)=\boldsymbol{X}+\frac{\boldsymbol{p}(\boldsymbol{X},t)}{a(t)}. \tag{14}$$

All fields are still represented in the same Lagrangian frame $\boldsymbol{X}$ if we define $a(t_o):=1$ (equation 5). In this *comoving* picture, we introduce the *peculiar velocity* $\boldsymbol{u}$ and the *peculiar acceleration* $\boldsymbol{w}$ as usual:

$$\boldsymbol{v}=\dot{\boldsymbol{f}}=\dot{a}\boldsymbol{F}+\boldsymbol{u},\qquad \boldsymbol{u}:=a\dot{\boldsymbol{F}}, \tag{15a}$$

$$\boldsymbol{g}=\ddot{\boldsymbol{f}}=\ddot{a}\boldsymbol{F}+\boldsymbol{w},\qquad \boldsymbol{w}:=2\dot{a}\dot{\boldsymbol{F}}+a\ddot{\boldsymbol{F}}=\dot{\boldsymbol{u}}+\frac{\dot{a}}{a}\boldsymbol{u}. \tag{15b}$$

Note that the convective derivative with respect to $\boldsymbol{v}$ is equivalent to the convective derivative with respect to $\boldsymbol{u}/a$ in the comoving picture:

$$\frac{d}{dt}:=\partial_t|_x+\boldsymbol{v}\cdot\nabla_x=\partial_t|_q+\frac{\boldsymbol{u}}{a}\nabla_q. \tag{16}$$

The example presented in Section 4 will be evaluated in this representation.

## 4 EXAMPLE: VORTICAL PANCAKE MODEL IN A FLAT UNIVERSE

In this example, we consider the linear inhomogeneous deformation (12) in a flat background universe. Setting (const. = 0; $\Lambda=0$) in equation (7b), we obtain

$$a(t)=\left(\frac{t}{t_o}\right)^{2/3}. \tag{17}$$

With the ansatz $q^{\mathrm{D}}(t)=(t/t_o)^n$, $q_{\mathrm{p}}^{\mathrm{D}}(t)=(t/t_o)^p$, $q^{\mathrm{R}}(t)=(t/t_o)^m$, we seek solutions for the time-dependent functions in (13). We obtain

$$n_1=\tfrac{4}{3},\qquad n_2=-\tfrac{1}{3},\qquad p=\tfrac{2}{3},\qquad m_1=\tfrac{2}{3},\qquad m_2=\tfrac{1}{3}. \tag{18}$$

The linear inhomogeneous deformation (12) then reads

$$\boldsymbol{p}=\left(\frac{t}{t_o}\right)^{4/3}\boldsymbol{Q}_1^{\mathrm{D}}(\boldsymbol{X})+\left(\frac{t}{t_o}\right)^{-1/3}\boldsymbol{Q}_2^{\mathrm{D}}(\boldsymbol{X})+\left(\frac{t}{t_o}\right)^{2/3}\boldsymbol{P}(\boldsymbol{X})$$
$$+\left(\frac{t}{t_o}\right)^{2/3}\boldsymbol{Q}_1^{\mathrm{R}}(\boldsymbol{X})+\left(\frac{t}{t_o}\right)^{1/3}\boldsymbol{Q}_2^{\mathrm{R}}(\boldsymbol{X}). \tag{19}$$

Calculating the velocity $\boldsymbol{v}_{\mathrm{rel}}$ and the gravitational field strength, or the acceleration $\boldsymbol{g}_{\mathrm{rel}}$ *relative* to the Hubble flow, we have

$$\boldsymbol{v}_{\mathrm{rel}}:=\dot{\boldsymbol{f}}-\dot{a}\boldsymbol{X}=\dot{\boldsymbol{p}}(\boldsymbol{X},t),\qquad \boldsymbol{V}_{\mathrm{rel}}:=\dot{\boldsymbol{p}}(\boldsymbol{X},t_o), \tag{20a}$$

$$\boldsymbol{g}_{\mathrm{rel}}:=\ddot{\boldsymbol{f}}-\ddot{a}\boldsymbol{X}=\ddot{\boldsymbol{p}}(\boldsymbol{X},t),\qquad \boldsymbol{G}_{\mathrm{rel}}:=\ddot{\boldsymbol{p}}(\boldsymbol{X},t_o). \tag{20b}$$

In comoving coordinates, commonly used in applications of pancake models, we have to transform the relative fields to the peculiar fields $\boldsymbol{u}$ and $\boldsymbol{w}$ according to (15). The initial fields are again split into their rotational and irrotational parts:

$$\boldsymbol{u}=\boldsymbol{v}_{\mathrm{rel}}-\frac{\dot{a}}{a}\boldsymbol{p},\qquad \boldsymbol{U}^{\mathrm{D}}=\boldsymbol{V}_{\mathrm{rel}}^{\mathrm{D}},\qquad \boldsymbol{U}^{\mathrm{R}}=\boldsymbol{V}_{\mathrm{rel}}^{\mathrm{R}}, \tag{21a}$$

$$\boldsymbol{w}=\boldsymbol{g}_{\mathrm{rel}}-\frac{\ddot{a}}{a}\boldsymbol{p},\qquad \boldsymbol{W}^{\mathrm{D}}=\boldsymbol{G}_{\mathrm{rel}}^{\mathrm{D}},\qquad \boldsymbol{W}^{\mathrm{R}}=\boldsymbol{G}_{\mathrm{rel}}^{\mathrm{R}}=\boldsymbol{0}. \tag{21b}$$

Using (19), we express the initial perturbation fields in terms of the initial conditions $\boldsymbol{V}_{\mathrm{rel}}$, $\boldsymbol{G}_{\mathrm{rel}}$, or $\boldsymbol{U}$, $\boldsymbol{W}$, respectively:

$$\boldsymbol{Q}_1^{\mathrm{D}}=\tfrac{3}{5}\boldsymbol{U}^{\mathrm{D}}t_o+\tfrac{9}{10}\boldsymbol{W}^{\mathrm{D}}t_o^2, \tag{22a}$$

$$\boldsymbol{Q}_2^{\mathrm{D}}=-\tfrac{3}{5}\boldsymbol{U}^{\mathrm{D}}t_o+\tfrac{3}{5}\boldsymbol{W}^{\mathrm{D}}t_o^2, \tag{22b}$$

$$\boldsymbol{P}=-\boldsymbol{Q}_1^{\mathrm{D}}-\boldsymbol{Q}_2^{\mathrm{D}}=-\tfrac{3}{2}\boldsymbol{W}^{\mathrm{D}}t_o^2, \tag{22c}$$

$$\boldsymbol{Q}_1^{\mathrm{R}}=3\boldsymbol{U}^{\mathrm{R}}t_o, \tag{22d}$$

$$\boldsymbol{Q}_2^{\mathrm{R}}=-3\boldsymbol{U}^{\mathrm{R}}t_o. \tag{22e}$$

We finally write down the full solution $\boldsymbol{f}^{\mathrm{L}}$ in terms of initial conditions for the peculiar velocity and acceleration:

$$\boldsymbol{f}^{\mathrm{L}}(\boldsymbol{X},t)=\left(\frac{t}{t_o}\right)^{2/3}\boldsymbol{X}+\left[\frac{3}{5}\left(\frac{t}{t_o}\right)^{4/3}-\frac{3}{5}\left(\frac{t}{t_o}\right)^{-1/3}\right]\boldsymbol{U}^{\mathrm{D}}(\boldsymbol{X})\,t_o$$
$$+\left[3\left(\frac{t}{t_o}\right)^{2/3}-3\left(\frac{t}{t_o}\right)^{1/3}\right]\boldsymbol{U}^{\mathrm{R}}(\boldsymbol{X})\,t_o$$
$$+\left[\frac{9}{10}\left(\frac{t}{t_o}\right)^{4/3}-\frac{3}{2}\left(\frac{t}{t_o}\right)^{2/3}+\frac{3}{5}\left(\frac{t}{t_o}\right)^{-1/3}\right]\boldsymbol{W}^{\mathrm{D}}(\boldsymbol{X})\,t_o^2. \tag{23a}$$

According to the general solution (3d), the density is given by

$$\rho(\boldsymbol{X},t)=\mathring{\rho}(\boldsymbol{X})\{\det[f_{i,k}^{\mathrm{L}}(\boldsymbol{X},t)]\}^{-1}. \tag{23b}$$

For given initial conditions

$$\mathring{\rho}=\mathring{\rho}_{\mathrm{H}}-\frac{1}{4\pi G}\nabla_o\cdot\boldsymbol{W}^{\mathrm{D}},\qquad \boldsymbol{U}=\boldsymbol{U}^{\mathrm{D}}+\boldsymbol{U}^{\mathrm{R}},$$

the solution (23) provides the *general* solution, since four functions of three variables can be given independently, which is the required number in the Euler–Poisson system.

## 5 THE ZEL'DOVICH APPROXIMATION EMBEDDED INTO THE LAGRANGIAN THEORY; DISCUSSION

The Zel'dovich approximation (Zel'dovich 1970a, b, c, 1978) is a special case of the solutions (23). There are various different interpretations of this approximation in the literature, e.g., as the solution of a system of particles moving under inertia with a transformation of the time variable, and others. The former version has been generalized to the 'adhesion approximation' (Gurbatov, Saichev & Shandarin 1989). Some versions involve inconsistencies with regard to the density for a given displacement mapping. We now discuss two possibilities to define the subclass such that the initial conditions only involve *one* function, which is in conformity with the common use of this approximation. However, the reader may impose his own assumptions on the initial conditions starting from the general form (23). Note that the normalization of the spectrum for the density contrast is related to the acceleration field only [in the flat model presented in Section 4 we have $\mathring{\delta} = -\nabla_0 \cdot \boldsymbol{W}/(4\pi G \mathring{\rho}_H) = -\frac{3}{2} t_0^2 \nabla_0 \cdot \boldsymbol{W}$]. It is interesting to note that in Zel'dovich's original work (1970a), the initial data are supposed to consist of two initial perturbation fields, one for the velocity, *and* one for the density. In an initial value problem of self-gravitating motions, these fields have to be given *independently*, which is possible in the general form (23).

One way of restricting the solutions (23) is to require the following:

$$\boldsymbol{U}^{\mathrm{R}}(\boldsymbol{X}) = \boldsymbol{0}, \qquad \boldsymbol{W}(\boldsymbol{X}) = \boldsymbol{0}. \tag{24a}$$

Inserting (24a) into the general orbit (23), we obtain

$$\boldsymbol{F}^{\mathrm{Z1}}(\boldsymbol{X}, t) = \boldsymbol{X} + \left[\left(\frac{t}{t_0}\right)^{2/3} - \left(\frac{t}{t_0}\right)^{-1}\right] \frac{3}{5} \boldsymbol{U}^{\mathrm{D}}(\boldsymbol{X})\, t_0. \tag{24b}$$

The expression for the density can then be written as

$$\rho^{\mathrm{Z1}} = \rho_{\mathrm{H}}(t)/\det[F^{\mathrm{Z1}}_{i,k}(\boldsymbol{X}, t)], \tag{24c}$$

i.e., the density is initially given by the homogeneous background density, and the fluctuations are produced solely by velocity perturbations. Thus the mapping (24b) can be considered as a consistent form of Zel'dovich's mapping.

A second possibility is to require the following condition to hold at the initial time:

$$\boldsymbol{U}(\boldsymbol{X}) = \boldsymbol{W}(\boldsymbol{X})\, t_0. \tag{25a}$$

This relation between the initial peculiar velocity field and the initial peculiar acceleration field implicitly holds in Zel'dovich's ansatz. Initially, there is a density perturbation proportional to the velocity perturbation. The congruence of the initial peculiar fields implies

$$\boldsymbol{U}^{\mathrm{D}}(\boldsymbol{X}) = \boldsymbol{W}^{\mathrm{D}}(\boldsymbol{X})\, t_0, \qquad \boldsymbol{U}^{\mathrm{R}}(\boldsymbol{X}) = \boldsymbol{W}^{\mathrm{R}}(\boldsymbol{X})\, t_0 = \boldsymbol{0}. \tag{25b}$$

Inserting (25a) into the general orbit (23), we again obtain Zel'dovich's mapping:

$$\boldsymbol{F}^{\mathrm{Z2}}(\boldsymbol{X}, t) = \boldsymbol{X} + \left[\left(\frac{t}{t_0}\right)^{2/3} - 1\right] \frac{3}{2} \boldsymbol{U}^{\mathrm{D}}(\boldsymbol{X})\, t_0. \tag{25c}$$

The integration constant appearing in (25c) (in contrast to Zel'dovich's ansatz) is due to our choice of Lagrangian coordinates (2). The mapping (25c) is formally equivalent to Zel'dovich's mapping if a perturbed Lagrangian frame $\boldsymbol{Y} := \boldsymbol{X} + \boldsymbol{P}(\boldsymbol{X}) = \boldsymbol{X} - \frac{3}{2} \boldsymbol{W}^{\mathrm{D}}(\boldsymbol{X})\, t_0^2$ is used. I emphasize that the expression for the density is different from $\rho^{\mathrm{Z1}}$ and is given by

$$\rho^{\mathrm{Z2}} = \frac{\rho_{\mathrm{H}}(t)}{\rho_{\mathrm{H}}(t_0)} \mathring{\rho}(\boldsymbol{X})/\det[F^{\mathrm{Z2}}_{i,k}(\boldsymbol{X}, t)]$$

$$= \frac{\rho_{\mathrm{H}}(t)}{\rho_{\mathrm{H}}(t_0)} \mathring{\rho}(\boldsymbol{Y}) \det[F^{\mathrm{Z2}}_{i,k}(\boldsymbol{Y}, t_0)]/\det[F^{\mathrm{Z2}}_{i,k}(\boldsymbol{Y}, t)]. \tag{25d}$$

[For a related discussion of the requirement (25) see B89, section 4.2.1.]

According to (24a) and (25b), vorticity is absent in Zel'dovich's approach. Introduction of vorticity artificially into the single initial function (here, the function $\boldsymbol{U}^{\mathrm{D}}$) would violate the integrability condition of the gravitational field at the initial time and for all times $\nabla_0 \times \boldsymbol{W} = \boldsymbol{0}$, $\nabla_x \times \boldsymbol{w} = \boldsymbol{0}$. In particular, the assumption (25) implies that the peculiar velocity field is parallel to the peculiar acceleration field for all times, which is only asymptotically valid in the general form (23), and only for $\boldsymbol{U}^{\mathrm{R}} = \boldsymbol{0}$ (see also Bildhauer & Buchert 1991). The coincidence or parallelism of the two fields is such that, already initially, the velocities of the particles point into the direction of enhanced density. If the fields are initially not congruent, as in the general mapping (23), the particles could flow away from high-density peaks. Then, later on, they are caught by the gravitational action. For irrotational initial conditions, and if the amplitudes of $\boldsymbol{U}^{\mathrm{D}}$ and $\boldsymbol{W}^{\mathrm{D}}$ are comparable, the particles will then move towards high-density peaks as in the restricted case.

We now discuss the vortical part of the solution (23) in terms of the vorticity transport equation. I first remind the reader of some basic equations for the evolution of the vorticity field $\boldsymbol{\omega} := \nabla_x \times \boldsymbol{v}$. The following evolution equation is valid for any conservative force acting on the fluid (see e.g. Serrin 1959):

$$\boldsymbol{\omega}_{,t}|_x + (\boldsymbol{v} \cdot \nabla_x)\, \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla_x)\, \boldsymbol{v} - \boldsymbol{\omega}(\nabla_x \cdot \boldsymbol{v}). \tag{26a}$$

In the comoving frame,

$$\boldsymbol{\omega} = \nabla_x \times \boldsymbol{v} = \frac{1}{a} \nabla_q \times \boldsymbol{u},$$

we accordingly have

$$\dot{\boldsymbol{\omega}} + \frac{\dot{a}}{a} \boldsymbol{\omega} = \boldsymbol{\omega}_{,t}|_q + \left(\frac{\boldsymbol{u}}{a} \cdot \nabla_q\right) \boldsymbol{\omega} + 2 \frac{\dot{a}}{a} \boldsymbol{\omega} = \left(\boldsymbol{\omega} \cdot \nabla_q\right) \frac{\boldsymbol{u}}{a} - \boldsymbol{\omega}\left(\nabla_q \cdot \frac{\boldsymbol{u}}{a}\right). \tag{26b}$$

If we linearize the equation (26b) at the background in Eulerian space, we obtain the solution in the linear regime $\boldsymbol{\omega}^{\mathrm{l}}$:

$$\boldsymbol{\omega}^{\mathrm{l}}_{,t}|_q + 2 \frac{\dot{a}}{a} \boldsymbol{\omega}^{\mathrm{l}} = \boldsymbol{0}, \qquad \boldsymbol{\omega}^{\mathrm{l}} = \frac{\mathring{\boldsymbol{\omega}}^{\mathrm{l}}(\boldsymbol{q})}{a^2}. \tag{26b$^{\mathrm{l}}$}$$

Using the mass conservation law (3d), we obtain from (26a) the diffusion equation of Beltrami:

$$\left(\frac{\boldsymbol{\omega}}{\rho}\right)^{\dot{}} = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla_x\right) \boldsymbol{v}. \tag{26c}$$

In the Lagrangian representation, this equation can be integrated along the particle trajectories (Cauchy's integral, see Serrin 1959):

$$\frac{\boldsymbol{\omega}}{\rho} = \left(\frac{\mathring{\boldsymbol{\omega}}}{\mathring{\rho}}(\boldsymbol{X}) \cdot \nabla_0\right) \boldsymbol{f}(\boldsymbol{X}, t), \qquad \det(f_{i,k}) \neq 0, \tag{26d}$$

or, again using (3d):

$$\boldsymbol{\omega}(\boldsymbol{X}, t) = (\mathring{\boldsymbol{\omega}} \cdot \nabla_0)\, \boldsymbol{f}(\boldsymbol{X}, t) \{\det[f_{i,k}(\boldsymbol{X}, t)]\}^{-1}. \tag{26e}$$

Equation (26c) tells us that the vorticity is coupled to the density enhancement (see also the relativistic treatment by

Ellis, Bruni & Hwang 1990). I emphasize that, in the Lagrangian description, the vorticity couples to the density, irrespective of *any* equations the trajectories $\boldsymbol{f}$ might obey. This is true even for the linearized field, since the vortex modes are *transported with the full deformation of the fluid*, i.e., they also couple to the growing density mode. This fact is not respected in the standard Eulerian picture, where vorticity decouples from the density evolution [see (26b)$^{\mathrm{l}}$].

From (26c, d) we draw another important conclusion: as the density field develops singularities, the vorticity will blow up simultaneously. This means that, e.g., high-density regions (such as clusters and superclusters in the pancake theory) are associated with strong vortex flows. We illustrate this by discussing the angular momentum conservation, which is related to Kelvin's circulation theorem (see, e.g. Serrin 1959). The circulation flow around any (orientable) surface $A$ in the fluid can be expressed via Stokes' theorem by the vector flux of $\boldsymbol{\omega}$ through that surface:

$$\boldsymbol{C} := \frac{1}{2}\int_{\partial A} \boldsymbol{v}\cdot d\boldsymbol{f} = \int_{A} \boldsymbol{\omega}\cdot\boldsymbol{n}\, dA \tag{27}$$

($\boldsymbol{n}$ is the unit normal on the surface). The theorem tells us that, for conservative forces acting on the fluid, the quantity $\boldsymbol{C}$ is conserved along integral curves of the velocity field, i.e., along $\boldsymbol{f}$: $d/dt\, \boldsymbol{C} = \boldsymbol{0}$; the circuit $\partial A = \partial A(t)$ is moving with the fluid. ($\boldsymbol{C}$ is also conserved along integral curves of the vorticity field $\boldsymbol{\omega}$.) As long as the surface area $A$ remains finite during a collapse process, the vorticity does not blow up. It does, however, if the surface element $dA$ degenerates. We now show that this surface element degenerates for almost all initial conditions, if the volume element, measured by $\det(f_{i,k})$, degenerates. This proves the statement above that high-density regions are always associated with strong vortex flows, even if the collapse proceeds dominantly along one eigendirection of the deformation tensor, as is the case in the formation of 'pancakes'.

We first consider the special case where the directional derivative of $\boldsymbol{v}$ with respect to $\boldsymbol{\omega}$ vanishes for all times: $(\boldsymbol{\omega}\cdot\nabla_x)\,\boldsymbol{v} = \boldsymbol{0}$. This is always true in the two-dimensional case, where the fluid is restricted to a plane and the vorticity vector is orthogonal to the plane of motion. The vorticity is still amplified in proportion to the density enhancement before shell-crossing: $\boldsymbol{\omega}^{2\mathrm{D}} = \hat{\boldsymbol{\omega}}[\det(f_{i,k})]^{-1}$. The surface element $dA$ here is proportional to the determinant and consequently degenerates simultaneously. In the three-dimensional case, we can imagine a situation in which the collapsing direction is oriented along the unit normal $\boldsymbol{n}$ at the critical point $(\boldsymbol{X}_{\mathrm{c}}, t_{\mathrm{c}})$ at the time of degeneracy of the Jacobian $(f_{i,k})$. While the volume element degenerates, the motion orthogonal to $\boldsymbol{n}$ could still be expanding in the remaining two directions. Since, in that case, $dA$ lies in the plane orthogonal to $\boldsymbol{n}$, the vorticity vector is orthogonal to the plane of motion: $(\hat{\boldsymbol{\omega}}\cdot\nabla_0)\,\boldsymbol{p} = \boldsymbol{0}$ at $(\boldsymbol{X}_{\mathrm{c}}, t_{\mathrm{c}})$. For generic initial conditions, this situation is realized for a set of measure zero only. In that case, part of the coupling to the inhomogeneous deformation vanishes [compare the numerator in equation (26e)] and the general formula reads: $\boldsymbol{\omega}^{3\mathrm{D}}(\boldsymbol{X}_{\mathrm{c}}, t_{\mathrm{c}}) = \hat{\boldsymbol{\omega}}[\det(f_{i,k}(\boldsymbol{X}_{\mathrm{c}}, t_{\mathrm{c}}))]^{-1}$, i.e., vorticity still tends to infinity at caustics as in all other cases. (Note that deformations, in general, remain finite.)

We now return to the special solution (23). If initial vorticity is taken into account, the solution (23) shows that there are *two* rotational solutions in the Lagrangian theory. One of these solutions behaves like the Eulerian linear mode (Peebles 1980, section II.15); the other just compensates for the damping influence of the background expansion in the comoving deformation field $\boldsymbol{F}^{\mathrm{L}} = \boldsymbol{f}^{\mathrm{L}}/a$. However, both Lagrangian solutions depend on the Lagrangian coordinates, i.e., follow the full deformation of the fluid. This implies that the vorticity is coupled to the density evolution, i.e., is also coupled to the *growing* density mode. The general formula (26e) clearly shows this behaviour. We express (26e) in the comoving frame:

$$\boldsymbol{\omega}(\boldsymbol{X}, t) = \frac{1}{a}(\hat{\boldsymbol{\omega}}\cdot\nabla_0)\,\boldsymbol{F}(\boldsymbol{X}, t)\,\{\det[F_{i,k}(\boldsymbol{X}, t)]\}^{-1}, \tag{28}$$

and insert the solution $\boldsymbol{F}^{\mathrm{L}}$ into (28) to obtain

$$\boldsymbol{\omega}^{\mathrm{L}}(\boldsymbol{X}, t) = \frac{1}{a^2}[\hat{\boldsymbol{\omega}} + g_1(\hat{\boldsymbol{\omega}}\cdot\nabla_0)\,\boldsymbol{U}^{\mathrm{R}} + g_2(\hat{\boldsymbol{\omega}}\cdot\nabla_0)\,\boldsymbol{U}^{\mathrm{D}} + g_3(\hat{\boldsymbol{\omega}}\cdot\nabla_0)\,\boldsymbol{W}^{\mathrm{D}}][\det(F^{\mathrm{L}}_{i,k})]^{-1}. \tag{29}$$

For the example presented in Section 4, the coefficient functions are given by

$$a(t) := \left(\frac{t}{t_0}\right)^{2/3}, \qquad g_1(t) := 3t_0 - 3t_0\left(\frac{t}{t_0}\right)^{-1/3},$$

$$g_2(t) := \frac{3}{5}\,t_0\left(\frac{t}{t_0}\right)^{2/3} - \frac{3}{5}\,t_0\left(\frac{t}{t_0}\right)^{-1},$$

$$g_3(t) := \frac{9}{10}\,t_0^2\left(\frac{t}{t_0}\right)^{2/3} + \frac{3}{5}\,t_0^2\left(\frac{t}{t_0}\right)^{-1} - \frac{3}{2}\,t_0^2.$$

The terms proportional to the initial perturbation fields express the coupling to the homogeneous deformation of the fluid, while the first term only takes the coupling to the homogeneous deformation into account. In addition, however, the Jacobian couples $\boldsymbol{\omega}$ to the inhomogeneous deformation and counteracts the damping influence of the expansion for *each* term. Thus there is no way of arguing that initial vorticity perturbations die out, which is the commonly held view. This prediction of the Eulerian linear theory is therefore misleading if incorrectly applied to the non-linear regime. In contrast, the Lagrangian theory offers a reasonable description of the vorticity evolution in the early non-linear regime. However, the impact of vorticity on the formation of large-scale structure is strongest in the fully developed non-linear regime, which has to be studied, e.g., with numerical simulation (Buchert & Alimi, in preparation).

Linearizing the expression (29), we recover the Eulerian linear vorticity mode (26b)$^{\mathrm{l}}$:

$$\boldsymbol{\omega}^{\mathrm{l}} \approx \frac{1}{a^2}\left(\frac{\hat{\boldsymbol{\omega}}}{1 + \mathrm{Trace}\left(\frac{p^{\mathrm{L}}_{i,k}}{a}\right) + \dots}\right)$$

$$\approx \frac{1}{a^2}\left[\hat{\boldsymbol{\omega}}\left(1 - \mathrm{Trace}\left(\frac{p^{\mathrm{L}}_{i,k}}{a}\right)\right)\right] \approx \frac{\hat{\boldsymbol{\omega}}}{a^2}, \tag{29$^{\mathrm{l}}$}$$

where, in the limit, the function $\hat{\omega}(X)$ has to be replaced by $\hat{\omega}(q)$, $q = F(X, t) \approx X$.

Below, I summarize those non-linear features of the general mapping which are not described by Zel'dovich's approach.

(i) The general theory involves *two* initial perturbation fields for the peculiar velocity and the density (or the peculiar acceleration, respectively) *independently* according to the initial value problem of Newtonian Gravity. This implies that the total initial fluctuation field contains twice as many harmonics on the same wavenumber range compared to Zel'dovich's mapping. The two fields have (even for equal amplitudes of the two fields) different coefficients. The combined Gaussian field has a different total amplitude as that in the restricted cases (24, 25), which is important for the normalization of the model. The parameter freedom of the initial conditions is increased from one to three parameters for the amplitudes of $U^D$, $U^R$ and $W^D$.

(ii) Non-linear coupling of perturbation modes: the decaying mode itself is numerically small and could be neglected in the deformation field. This does not imply that (23) reduces to the Zel'dovich approximation (24, 25). In three dimensions, however, the non-linear coupling of modes involves a growing function $\propto t^{-1} t^{2/3} t^{2/3} = t^{1/3}$ in the density, if the mode is not neglected *a priori*.

(iii) The description of vortex perturbation modes and the non-linear coupling of vorticity to the inhomogeneous deformation of the fluid: according to the conservation of the angular momentum, the Lagrangian theory shows that vorticity is amplified in proportion to the density enhancement before shell-crossing. The location of singularities of the mapping (23) in Eulerian space (caustic) is the location of points of infinite vorticity.

(iv) While Zel'dovich's mapping (24, 25), considered as a one-parameter family of mappings from Lagrangian to Eulerian space, is *Lagrangian*, the mapping (23), in general, is *non-Lagrangian* in the following sense.

In our problem, the motion of the fluid naturally describes a manifold as the three-dimensional hypersurface in the six-dimensional phase space, which is generated by the flow, i.e., by the collection of all trajectories. Let us introduce the Lagrangian coordinates $X_i$ as the local coordinates on this manifold, and the Eulerian coordinates as local coordinates in real space. The one-parameter family of mappings

$$\pi_t : \Re^3 \to \Re^3, \qquad X \mapsto x = f(X; t) \tag{30}$$

defines a family of *Lagrangian* mappings, if the two-form $\Omega = \Sigma_i \, dx_i \wedge dX_i$ vanishes on the manifold $\pi_t$. In that case, the manifold generated by the flow lines is *Lagrangian* according to Arnol'd's theory (Arnol'd, Shandarin & Zel'dovich 1982; Arnol'd, Gusein-Zade & Varchenko 1985). Thus the specialized mappings (24) and (25) are Lagrangian, since they can be written as a one-parameter family of *gradients* $x = \nabla_0 \Phi(X; t)$, which implies vanishing of the form $\Omega$. This form does not vanish in the case of rotational flows, as allowed in the general mapping (23). Note that Bruce and collaborators (Bruce 1986) have generalized the Lagrange-singularity theory to solenoidal velocity fields. They find that the most ubiquitous 'pancake singularities' are again stable. However, the singularities associated with umbilics in the potential case have to be removed from the list of stable generic patterns in the vortical case.

## 6 RELATION TO THE FULLY NON-LINEAR EQUATIONS

A striking property of the Zel'dovich approximation has been noticed earlier by Doroshkevich *et al.* (1973), who have shown that the Zel'dovich approximation provides a (special) exact solution to the fully non-linear equations for the evolution of one-dimensional inhomogeneities. They have shown this with the help of a self-consistency test comparing the density as derived from the acceleration field and the density as given by the integral of the continuity equation (compare also B89 for a discussion of this test). However, the fully non-linear equations can be integrated in that case, demonstrating that (12) provides the *general* exact solution for plane-symmetric perturbations on a three-dimensional background (flat background: Zentsova & Chernin 1980a, b; non-flat backgrounds: B89). Moreover, the three-dimensional orbit (12) provides a special class of three-dimensional solutions, which is even more astonishing. This class, elaborated in (B89), exists, since the non-linearities neglected involve only spatial derivatives of the fields (see below). The vanishing of these non-linearities, however, implies strong restrictions on the initial conditions. The kinematics of the fluid (with respect to the inhomogeneous deformation) is restricted to locally one-dimensional contractions or expansions along only one eigendirection of the fluid's (inhomogeneous part of the) Eulerian velocity gradient $(\partial u_i / \partial q_j)$, $i, j = 1, 2, 3$. The vanishing of two eigenvalues of this peculiar velocity gradient is an invariant characterization of the solution class. I have to add a technical remark here, because the solution class mentioned is investigated only for potential flows. If we insert the general ansatz (12) into the system of equations (4), the constraining equations (B89, appendix B), which involve the vortical part of the peculiar velocity, are compensated by the vortical part of the ansatz (12). Consequently, the class allows for special vortical flows which, however, preserve the one-dimensionality of the local peculiar motion [see also Buchert & Götz (1987) for a discussion of special vortical flows in a class with similar kinematical properties].

In the sequel I demonstrate briefly the relation of the theory presented (or, equivalently, the corresponding three-dimensional solution class) to the fully non-linear equations using an equation for the eigenvalues of the Eulerian peculiar-velocity gradient (with respect to the comoving frame) $u_{ij} := (\partial u_i / \partial q_j)(X, t)$, $i, j = 1, 2, 3$, considered as a tensor function of Lagrangian coordinates. We first write down an equation derived in [B89, equation (7e)] relating the equation for the density as solved by the theory presented (which is a linear equation for $\Delta := (\rho_H - \rho)/\rho$) to the remaining non-linearities:

$$\ddot{\Delta} + 2 \frac{\dot{a}}{a} \Delta - 4\pi G \rho_H \Delta = -\frac{1}{a^2} (1 + \Delta) \nabla_q \cdot [(u \cdot \nabla_q) \, u - u (\nabla_q \cdot u)]. \tag{31}$$

The expression for the peculiar velocity on the right-hand side is proportional to the second scalar invariant of the comoving Eulerian peculiar velocity gradient:

$$\boldsymbol{II}(\boldsymbol{u}_{ij}) := -\frac{1}{2}\nabla_q\cdot[(\boldsymbol{u}\cdot\nabla_q)\,\boldsymbol{u}-\boldsymbol{u}(\nabla_q\cdot\boldsymbol{u})]$$

$$=\frac{\partial u_1}{\partial q_1}\frac{\partial u_2}{\partial q_2}+\frac{\partial u_2}{\partial q_2}\frac{\partial u_3}{\partial q_3}+\frac{\partial u_3}{\partial q_3}\frac{\partial u_1}{\partial q_1}$$

$$-\frac{\partial u_1}{\partial q_2}\frac{\partial u_2}{\partial q_1}-\frac{\partial u_2}{\partial q_3}\frac{\partial u_3}{\partial q_2}-\frac{\partial u_3}{\partial q_1}\frac{\partial u_1}{\partial q_3}. \qquad (32)$$

Introducing the first scalar invariant, the trace of the peculiar velocity gradient, and relating it to the density by the continuity equation:

$$\boldsymbol{I}(\boldsymbol{u}_{ij}) := \frac{\partial u_1}{\partial q_1}+\frac{\partial u_2}{\partial q_2}+\frac{\partial u_3}{\partial q_3}=a\frac{d}{dt}\ln(1+\Delta), \qquad (33)$$

we obtain [from (31), (32), (33)] the following.

*Theorem 2*

For Euler–Poisson 'dust' on a Friedman background, the following equation, which relates the first and second scalar invariants, $\boldsymbol{I}$ and $\boldsymbol{II}$, of the comoving peculiar velocity gradient $\boldsymbol{u}_{ij} := (\partial u_i/\partial q_j)(\boldsymbol{X}, t)$, $i, j = 1, 2, 3$, holds in general:

$$\dot{\boldsymbol{I}}+\frac{\dot a}{a}\boldsymbol{I}+\frac{1}{a}(\boldsymbol{I}^2-2\boldsymbol{II})=-4\pi G\rho_{\rm H}\left(\exp\left(-\int_{t_0}^{t}\frac{\boldsymbol{I}}{a}\,dt\right)-1\right). \qquad (34a)$$

In the case of potential flows ($\nabla_q\times\boldsymbol{u}=\boldsymbol{0}$), but not only then, we can introduce the eigenvalues of the peculiar velocity gradient $\mu_i(\boldsymbol{X}, t)$, $i = 1, 2, 3$. With $\boldsymbol{I}=\mu_1+\mu_2+\mu_3$ and $\boldsymbol{II}=\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1$, we obtain the following relation among the eigenvalues in the fully non-linear problem:

$$(\dot\mu_1+\dot\mu_2+\dot\mu_3)+\frac{\dot a}{a}(\mu_1+\mu_2+\mu_3)+\frac{1}{a}(\mu_1^2+\mu_2^2+\mu_3^2)$$

$$=-4\pi G\rho_{\rm H}\left(\exp\left(-\int_{t_0}^{t}\frac{\mu_1+\mu_2+\mu_3}{a}\,dt\right)-1\right). \qquad (34b)$$

The trajectories $\boldsymbol{F}^{\rm L}$ can be derived from (34) by putting $\boldsymbol{II}(\boldsymbol{u}_{ij})$ to zero. We learn that these trajectories can also be obtained from the following equation for the peculiar velocity $\boldsymbol{u} =: \boldsymbol{u}^{\rm D}+\boldsymbol{u}^{\rm R}=a\dot{\boldsymbol{F}}^{\rm L}$, which is a linear equation in the Lagrangian frame (see B89 for the derivation):

$$\ddot{\boldsymbol{u}}+3\frac{\dot a}{a}\dot{\boldsymbol{u}}+\left(\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right)\boldsymbol{u}-4\pi G\rho_{\rm H}(\boldsymbol{u}-\boldsymbol{u}^{\rm R})=\boldsymbol{0}. \qquad (35)$$

We already noted that the contrast density $\Delta=(\rho_{\rm H}-\rho)/\rho$ obeys, for the approximation presented, a linear equation [31, $\boldsymbol{II}(\boldsymbol{u}_{ij})=0$], if considered as a function of Lagrangian coordinates. This contrast density should not be confused with the density contrast as usually defined by $\delta := (\rho-\rho_{\rm H})/\rho_{\rm H}$. It has been noted in (B89) that solutions of the Eulerian linear theory for the density contrast $\delta^{\rm l}$ and the peculiar velocity $\boldsymbol{u}^{\rm l}$ obey the *linearized* equations (31) and (35), if $\boldsymbol{u}$ and $\delta$ are considered as functions of Eulerian coordinates:

$$\left.\frac{\partial^2}{\partial_t^2}\right|_q\boldsymbol{u}+\left(\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}\right)\left.\frac{\partial}{\partial_t}\right|_q\boldsymbol{u}-4\pi G\rho_{\rm H}(\boldsymbol{u}-\boldsymbol{u}^{\rm R})=\boldsymbol{0}, \qquad (36a)$$

$$\left.\frac{\partial^2}{\partial_t^2}\right|_q\delta+2\frac{\dot a}{a}\left.\frac{\partial}{\partial_t}\right|_q\delta-4\pi G\rho_{\rm H}\delta=0. \qquad (36b)$$

The equations (36) form the basis of the Eulerian linear theory of gravitational instability (Peebles 1980, section II.10). The fact that both equations, (35) and (36a), are structurally identical except for the transformation from Eulerian to Lagrangian coordinates, has given rise to an *Extrapolation Theorem*, (B89), giving an algorithm for the construction of 3D solutions from given solutions of the Eulerian linear theory. The non-trivial fact of structural identity is the reason for the success of Zel'dovich's formalism.

For locally one-dimensional peculiar motions ($\mu_2=0$, $\mu_3=0$), i.e., $\boldsymbol{II}(\boldsymbol{u}_{ij})=0$, the special solution class investigated in (B89) is recovered providing the 'exact body' of the Lagrangian approximate theory investigated in this paper. Note that, in general, this theory is *approximate* only, since it does not assume *a posteriori* that two of the three eigenvalues have to vanish, which is the case for the solution class. This is, however, always true in the one-dimensional case. The equations (34) may be used to make different assumptions for the eigenvalues yielding different approaches to our problem. An effort considering assumptions for eigenvalues of the velocity gradient can be reported (Nusser, Dekel & Lynden-Bell, in preparation).

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## APPENDIX

In what follows, $I(p_{i,k})$, $II(p_{i,k})$ and $III(p_{i,k})$ denote the three scalar invariants of the tensor $(p_{i,k})$. Inserting the ansatz (8) into the system of equations (4), we obtain from the integrability conditions (4a, b, c), $i, j = 1, 2, 3$; $i \neq j$:

$$0 = \ddot{a}ap_{[j,i]} - a^2\ddot{p}_{[j,i]} \qquad \text{(linear)}$$

$$+\ddot{a}\varepsilon_{pq[j}\frac{\partial(p_{i]}, p_p, X_q)}{\partial(X_1, X_2, X_3)} - a\varepsilon_{pq[j}\frac{\partial(\ddot{p}_{i]}, p_p, X_q)}{\partial(X_1, X_2, X_3)} \qquad \text{(quadratic)}$$

$$-\varepsilon_{pq[j}\frac{\partial(\ddot{p}_{i]}, p_p, p_q)}{\partial(X_1, X_2, X_3)} \qquad \text{(cubic)}$$

and from the source equation (4d):

$$-4\pi G\mathring{\rho} = 3\ddot{a}a^2 - a^3\Lambda + (2\ddot{a}a - a^2\Lambda)\, I(p_{i,k}) + a^2 I(\ddot{p}_{i,k}) \qquad \text{(linear)}$$

$$+(\ddot{a} - a\Lambda)\, II(p_{i,k}) + a\sum_{a,b,c}\varepsilon_{abc}\frac{\partial(\ddot{p}_a, p_b, X_c)}{\partial(X_1, X_2, X_3)} \qquad \text{(quadratic)}$$

$$+\sum_{a,b,c}\frac{1}{2}\varepsilon_{abc}\frac{\partial(\ddot{p}_a, p_b, p_c)}{\partial(X_1, X_2, X_3)} - \Lambda\, III(p_{i,k}). \qquad \text{(cubic)}$$

For the determinant of the whole tensor $III(f_{i,k})$ we have

$$\det[f_{i,k}(\mathbf{X}, t)] = a^3 + a^2 I(p_{i,k}) + aII(p_{i,k}) + III(p_{i,k}).$$

All equations in this paper were checked with the algebraic manipulation system REDUCE. A program to compute the equations algebraically can be obtained from the author via electronic mail: Bitnet: TOB @ DGAIPP1s.