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Lakshmikantham Monotone Iterative Principle for Hybrid Atangana-Baleanu-Caputo Fractional Differential Equations

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Abstract In this paper, we study the following fractional differential equation involving the Atangana-Baleanu-Caputo fractional derivative:

$$\begin{cases} {}^{ABC}{}_aD^{\theta}_{\tau}\left[x(\vartheta) - F(\vartheta, x(\vartheta))\right] = G(\vartheta, x(\vartheta)), \quad \vartheta \in J := [a, b], \\ x(a) = \varphi_a \in \mathbb{R}. \end{cases}$$

The result is based on a Dhage fixed point theorem. Further, an example is provided for the justification of our main result.

AMS Subject Classification (2020) 26A33; 34A08; 34A12
Keywords Atangana-Baleanu-Caputo fractional derivative; existence; uniqueness; lower solution; Dhage iteration principle; partially ordered Banach space; nonlinear D-contraction operator

1 Introduction

Fractional differential equations are increasingly being used to model complex real-world problems in various fields such as visco-elasticity, electromagnetic, chemistry, biology, finance, and engineering, see ([1–3, 5–7, 9, 22, 23, 26, 28]). The Atangana-Baleanu-Caputo derivative is a fractional derivative that has recently garnered popularity due to its ability to capture the memory impact of a system being modeled. Atangana and Baleanu first proposed the derivative in their works [10–13] as an extension of the Caputo-Fabrizio derivative [14] to account for the non-singularity and non-locality of the kernel linked with the fractional derivative. Unlike the normal Caputo derivative, which lacks a fractional integral, the Atangana-Baleanu derivative has a fractional integral as its anti-derivative. Because of this property, the Atangana-Baleanu derivative has a more robust mathematical structure and a wider variety of utility in describing complicated physical systems (see [4, 24, 25]).

The Dhage iteration approach is a mathematical method for approximating solutions to nonlinear differential equations with maxima, such as hybrid functional differential equations and quadratic integral equations. The technique is based on an iterative procedure that converges to the solution of the problem and is based on a hybrid fixed point theory established by Bapurao Dhage in 2014, see [15–19]. The technique has been used to establish the existence and approximation of various kinds of differential equations, such as neutral functional differential equations with delay and maxima, quadratic fractional integral equations with maxima, and hybrid functional differential equations. For more results, see [8, 20, 21].

In [8], the authors discussed via a new version of Kransoselskii-type fixed-point theorem under a nonlinear D-contraction condition (see Dhage's version of Kransoselskii-type fixed-point theorem [15]) the following fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators of orders $0 < \lambda < 1$ and $\gamma > 0$:

$$\begin{cases} \mathbb{D}^{\lambda}[x(\varepsilon) - \Phi(\varepsilon, x(\varepsilon))] = \Psi\left(\varepsilon, x(\varepsilon), \mathbb{I}^{\gamma}(x(\varepsilon))\right), \text{ a.e. } \varepsilon \in J, \gamma > 0, \\ x\left(\varepsilon_0\right) = x_0, \end{cases}$$

where $J = [\varepsilon_0, \varepsilon_0 + \ell]$, for some fixed $\varepsilon_0 \in \mathbb{R}$ and $\ell > 0$ and $\Phi \in C(J \times \mathbb{R}, \mathbb{R}), \Psi \in \mathcal{C}(J \times \mathbb{R}^2, \mathbb{R})$. In [21], the authors considered the functional integro-differential equations of fractional order

$$\frac{d^{\varrho}}{d\varepsilon^{\varrho}}[x(\varepsilon) - \Phi(\varepsilon, x(\varepsilon))] = \Psi\left(\varepsilon, \int_0^{\varepsilon} \Upsilon\left(s, x_s\right)\right) ds, \qquad \varepsilon \in \mathbb{R}_+,$$

where $0 < \varrho < 1$, $x_{\varepsilon} : \mathbb{R}_+ \to \mathbb{R}, \Phi(\varepsilon, x) = \Phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \Psi(\varepsilon, x) = \Psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. Dhage and Jadhav [20] studied the existence of solution for hybrid differential equation:

$$\begin{cases} \dfrac{d}{d}[x(\varepsilon) - \Phi(\varepsilon, x(\varepsilon))] = \Psi(\varepsilon, x(\varepsilon)), \varepsilon \in J, \\ x\left(\varepsilon_0\right) = x_0 \in \mathbb{R}, \end{cases}$$

where $\Phi, \Psi \in C(J \times \mathbb{R}, \mathbb{R} \backslash \{0\})$. In [27], Lu et al. established, under the φ -Lipschitz contraction condition, the existence result for the following fractional hybrid differential problems via the Riemann-Liouville derivative of order $0 < \xi < 1$:

$$\begin{cases} \mathbb{D}^{\xi}[x(\varepsilon) - \Phi(\varepsilon, x(\varepsilon))] = \Psi(\varepsilon, x(\varepsilon)), \text{ a.e. } \varepsilon \in J, \\ x\left(\varepsilon_0\right) = x_0 \in \mathbb{R}, \end{cases}$$

where $\Phi, \Psi \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$.

In this article, we apply the Lakshmikantham monotone iterative technique to study the following class of Atangana-Baleanu-Caputo fractional differential equation:

$$\begin{cases} {}^{ABC}{}_{a}D^{\theta}_{\tau}\left[x(\vartheta) - F(\vartheta, x(\vartheta))\right] = G(\vartheta, x(\vartheta)), \quad \vartheta \in J, \\ x(a) = \varphi_a, \end{cases} \tag{1.1}$$

where $F, G : J\times\mathbb{R} \to \mathbb{R}$ are given continuous functions, ${}^{ABC}{}_aD^{\theta}_{\vartheta}$ is the Atangana-Baleanu-Caputo derivative of order $\theta \in (0, 1)$.

The paper is organized as follows: In Section 2, we present some useful definitions and lemmas. In Section 3, we develop the monotone iterative technique and prove the existence of solution for the problem (1.1) by using Dhage fixed point theorem (Theorem 2.2). In the last section, we give an example to illustrate the applicability of our main results.

2 Preliminaries and Auxiliary results

Let $J := [a, b]$, $(b > a)$, be a finite interval of the real line $\mathbb{R}$ and $C := C(J,\mathbb{R})$ be the Banach space of all continuous functions v from J into $\mathbb{R}$ with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{\vartheta\in J} |v(\vartheta)|.$$

Definition 2.1 ([12, 13, 24, 25]). *Let $\phi \in H^1(a, b),\ a < b,\ \theta \in (0, 1]$. The left Atangana-Baleanu fractional derivative of Caputo type (for shot* **ABC***) of a function ϕ of order θ is defined by*

$$ {}^{ABC}{}_aD^{\theta}_{\tau}\phi(\tau) := \frac{\Theta(\theta)}{1-\theta}\int_a^\tau \mathbb{E}_\theta\left[-\frac{\theta}{1-\theta}(\tau-\sigma)^\theta\right]\phi'(\sigma)d\sigma, \tag{2.1}$$

where $\Theta(\theta)$ is a normalization function satisfying $\Theta(0) = \Theta(1) = 1$ and $\Theta(\theta) > 0$, and $\mathbb{E}_\theta$ is the Mittag-Leffler function defined by

$$\mathbb{E}_\theta(z) = \sum_{n=0}^{n=\infty} \frac{z^n}{\Gamma(n\theta+1)}, \tag{2.2}$$

and $H^p(\Omega)$ the Sobolev space defined by

$$H^p(\Omega) = \left\{f \in L^2(\Omega) : D^\beta f \in L^2(\Omega),\ \textit{for all } |\beta| \le p\right\}. \tag{2.3}$$

The Riemann-Atangana-Baleanu fractional derivative of a function ϕ of order θ is defined by

$$ {}^{ABR}{}_aD^{\theta}_{\tau}\phi(\tau) := \frac{\Theta(\theta)}{1-\alpha}\frac{d}{d\tau}\int_a^\tau \mathbb{E}_\theta\left[-\theta\frac{(\tau-\sigma)^\theta}{1-\theta}\right]\phi(\sigma)d\sigma \tag{2.4}$$

The associative fractional integral is defined by

$$ {}^{AB}{}_aI^{\theta}_{\tau}\phi(\tau) := \frac{1-\theta}{\Theta(\theta)}\phi(\tau) + \frac{\theta}{\Theta(\theta)}{}_a\mathbb{I}^{\theta}_{\tau}\phi(\tau), \tag{2.5}$$

where

$$ {}_a\mathbb{I}^{\theta}_{\tau}\phi(\tau) := \int_a^\tau \frac{(\tau-\sigma)^{\theta-1}}{\Gamma(\theta)}\phi(\sigma)d\sigma \tag{2.6}$$

is the left Riemann-Liouville fractional integral.

Lemma 2.1. *Let $0 \leq \theta < 1$. Then, ${}^{AB}{}_aI^{\theta}_{\tau}\left({}^{ABC}{}_aD^{\theta}_{\tau}\phi(\tau)\right) = \phi(\tau) - \phi(a)$.*

Definition 2.2. *Let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. E is* regular *if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$.*

Definition 2.3 (Partially continuous mapping [17, 18])**.** *Let $K : E \to E$ a mapping. $K : E \to E$ is called **partially continuous (p.c.)** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|Kx - Ka\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. It is said to be partially continuous on E if it is partially continuous at any point of E.*

Definition 2.4 (Chain or subset totally ordered)**.** *A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable.*

Remark 2.1. If K is p.c. on E, then it is continuous on every chain C contained in E.

Definition 2.5 ([16, 17])**.** *Let K an operator on a partially normed linear space E into itself.*

*(i) K is called **partially bounded** if $K(C)$ is bounded for every chain C in E.*

*(ii) K is called **partially compact** if $K(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E.*

Definition 2.6 (Partially nonlinear D-Lipschitz mapping [18])**.** *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space and $K : E \to E$ a mapping.*

*(i) K is said to be **partially nonlinear D-Lipschitz** if there is a non-decreasing semi-continuous upper function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(0) = 0$ and*

$$\|Kx - Ky\| \leq \psi(\|x - y\|) \tag{2.7}$$

for all comparable elements $x, y \in E$.

*(ii) K is called **nonlinear D-contraction** if it is a nonlinear D-Lipschitz with $\psi(r) < r$ for $r > 0$.*

Theorem 2.2. *[17] Let $\left(E, \preceq, \|\cdot\|\right)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible in E. Let $\Psi, \Phi : E \to E$ be two nondecreasing operators such that*

(i) Ψ is partially bounded and partially nonlinear D-contraction,

(ii) Φ is partially continuous and partially compact, and

(iii) there exists an element $x_0 \in E$ such that $x_0 \preceq \Psi x_0 + \Phi x_0$.

Then:

(i) the operator equation $\Psi x + \Phi x = x$ has a solution x^* in E, and

(ii) the sequence $\{x_n\}$ of successive iterations defined by

$$x_{n+1} = \Psi x_n + \Phi x_n, \quad n = 0, 1, \dots,$$

converges monotonically to x^*.

Define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|u\| = \sup_{\vartheta \in J} |u(\vartheta)|, \tag{2.8}$$

$$u \leq v \iff u(\vartheta) \leq v(\vartheta) \tag{2.9}$$

for all $\vartheta \in J$.

Remark 2.2. With the norm defined in (2.8), $C(J, \mathbb{R})$ is a Banach space, and with a partially order relation $\leq$ in (2.9) it is partially ordered.

3 Existence of Solutions

Definition 3.1. *A function $\upsilon \in C(J, \mathbb{R})$ is said to be a lower solution of the HFDE (1.1) if it satisfies*

$$\begin{cases} {}^{ABC}{}_{a}D^{\theta}_{\tau} \left[\upsilon(\vartheta) - F(\vartheta, \upsilon(\vartheta))\right] \leq G(\vartheta, \upsilon(\theta)), \\ \upsilon(a) \leq \varphi_a, \end{cases} \tag{3.1}$$

for all $\tau \in J$.

Assumptions:

($\mathbf{C_I}$) The functions $F, G : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

($\mathbf{C_{II}}$) The functions F, G are nondecreasing functions in x for all $\vartheta \in J$.

($\mathbf{C_{III}}$) There exist constants $\Delta_F, \nabla_G > 0$ such that

$$\begin{aligned} |F(\vartheta, x)| &\leq \Delta_F, \\ |G(\vartheta, x)| &\leq \nabla_G, \end{aligned} \tag{3.2}$$

for all $\vartheta \in J$ and $x \in \mathbb{R}$.

($\mathbf{C_{IV}}$) There exists a $D-$contraction ζ such that

$$0 \leq F(\vartheta, x) - F(\vartheta, \varpi) \leq \zeta(x - \varpi),$$

for all $\vartheta \in J$ and $x, \varpi \in \mathbb{R}$ with $x \geq \varpi$.

($\mathbf{C_V}$) The exists a lower solution $\upsilon \in C(J, \mathbb{R})$ of problem (1.1), that is

$$\begin{cases} {}^{ABC}{}_{a}D^{\theta}_{\tau} \left[\upsilon(\vartheta) - F(\vartheta, \upsilon(\vartheta))\right] \leq G(\vartheta, \upsilon(\vartheta)), \\ \upsilon(a) \leq \varphi_a. \end{cases} \tag{3.3}$$

Put $E = C(J, \mathbb{R})$.

Lemma 3.1. *Let $\alpha \in (0,1]$ and a continuous function $H : G \to \mathbb{R}$, where $H(a) = 0$. Then the Cauchy problem*

$$\begin{cases} {}^{ABC}{}_{a}D_{\vartheta}^{\theta} x(\vartheta) = H(\vartheta), \quad \vartheta \in J, \\ x(a) = \varphi_a, \end{cases} \tag{3.4}$$

has a unique solution given by

$$x(\vartheta) = \varphi_a + (1-\theta)\overline{\Theta}_\theta H(\vartheta) + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \int_a^{\vartheta} (\vartheta - \sigma)^{\theta-1} H(\sigma) d\sigma, \tag{3.5}$$

where $\overline{\Theta}_\theta := \dfrac{1}{\Theta(\theta)}$.

Theorem 3.2. *Suppose that ($\boldsymbol{C_I}$)$-$($\boldsymbol{C_V}$) are satisfied. Then problem (1.1) has a solution x^* defined on J. Moreover, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations :*

$$\begin{cases} x_0(\vartheta) = u(\vartheta), \\ x_{n+1}(\vartheta) = F(\vartheta, x_n(\vartheta)) + \varphi_a - F(a, \varphi_a) + (1-\theta)\overline{\Theta}_\theta G(\vartheta, x_n(\vartheta)) \\ \qquad\qquad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \displaystyle\int_a^{\vartheta} (\vartheta - \sigma)^{\theta-1} G(\sigma, x_n(\sigma)) d\sigma, \end{cases} \tag{3.6}$$

converges to the solution x^*.

Proof. Clearly, E is partially ordered Banach space. Consider the equivalent operator equation

$$\Psi x(\vartheta) + \Phi x(\vartheta) = x(\vartheta),$$

where

$$\Psi x(\vartheta) = F(\vartheta, x(\vartheta)),$$

and

$$\Phi x(\vartheta) = \varphi_a - F(a, \varphi_a) + (1-\theta)\overline{\Theta}_\theta G(\vartheta, x(\vartheta)) + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \int_a^{\vartheta} (\vartheta - \sigma)^{\theta-1} G(\sigma, x(\sigma)) d\sigma,$$

for $\vartheta \in J$.

Step I: Ψ *and* Φ *are nondecreasing operators.*
Let $x, \varpi \in E$ where $x \geq \varpi$. Then by hypothesis ($\mathbf{C_{II}}$) we get, for $\vartheta \in J$:

$$\begin{aligned} x(\vartheta) \geq \varpi(\vartheta) &\Rightarrow F(\vartheta, x(\vartheta)) \geq F(\vartheta, \varpi(\vartheta)) \\ &\Rightarrow \Psi x(\vartheta) \geq \Psi\varpi(\vartheta). \end{aligned}$$

This shows that Ψ is nondecreasing operator on E into E. And, for $\vartheta \in J$, we get by ($\mathbf{C_{II}}$),

$$\begin{aligned} \Phi x(\vartheta) - \Phi\varpi(\vartheta) &= (1-\theta)\overline{\Theta}_\theta \left[G(\vartheta, x(\vartheta)) - G(\vartheta, \varpi(\vartheta))\right] \\ &\quad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \int_a^{\vartheta} (\vartheta - \sigma)^{\theta-1} \left[G(\sigma, x(\sigma)) - G(\sigma, \varpi(\sigma))\right] d\sigma \geq 0. \end{aligned}$$

Then, Φ is nondecreasing operator on E into E.

Step II: Ψ *is a partially bounded and partially nonlinear D-contraction operator.* Let $x \in E$, then for $\vartheta \in J$, we get by $(\mathbf{C_{III}})$:

$$|\Psi x(\vartheta)| = |F(\vartheta, x(\vartheta))| \leq \Delta_F.$$

Thus,

$$\| \Psi x \| \leq \Delta_F.$$

Then, Ψ is bounded on E and so partially bounded.
On the other hand, let $x, \varpi \in E$ where $x \geq \varpi$. Then for $\vartheta \in J$, by hypothesis $(\mathbf{C_{IV}})$ we get:

$$\begin{aligned}|\Psi x(\vartheta) - \Psi \varpi(\vartheta)| &= |F(\vartheta, x(\vartheta)) - F(\vartheta, \varpi(\vartheta))| \\ &\leq \zeta(|x(\vartheta) - \varpi(\vartheta)|) \\ &\leq \zeta(\|x - \varpi\|).\end{aligned}$$

Then, for each $x, \varpi \in E$ where $x \geq \varpi$, we get

$$\|\Psi x - \Psi \varpi\| \leq \zeta(\|x - \varpi\|).$$

Thus, Ψ is a partially nonlinear $D-$contraction on E and, thus partially continuous.

Step III: Φ *is partially continuous.*
Let C a chain in E, and $\{x_n\}$ a sequence of points of C in E where $x_n \to x^*$ for each $n \in \mathbb{N}$. By $(\mathbf{C_I})$ and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} (\Phi x_n)(\vartheta) &= \lim_{n\to\infty} \Big[\varphi_a - F(a, \varphi_a) + (1-\theta)\overline{\Theta}_\theta G(\vartheta, x_n(\vartheta)) \\ &\quad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \int_a^\vartheta (\vartheta - \sigma)^{\theta-1} G(\sigma, x_n(\sigma)) d\sigma\Big] \\ &= \varphi_a - F(a, \varphi_a) + (1-\theta)\overline{\Theta}_\theta \left[\lim_{n\to\infty} G(\vartheta, x_n(\vartheta))\right] \\ &\quad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \int_a^\vartheta (\vartheta - \sigma)^{\theta-1} \left[\lim_{n\to\infty} G(\vartheta, x_n(\tau))\right] d\sigma \\ &= \varphi_a - F(a, \varphi_a) + (1-\theta)\overline{\Theta}_\theta G(\vartheta, x^*(\vartheta)) \\ &\quad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \int_a^\vartheta (\vartheta - \sigma)^{\theta-1} G(\sigma, x^*(\sigma)) d\sigma \\ &= (\Phi x^*)(\vartheta),\end{aligned}$$

for all $\vartheta \in J$. This shows that $\{\Phi x_n\}$ converges to Φx^* pointwise on J and and the convergence is monotonic by the property of G.
Next, we will show that $\{\Phi x_n\}$ is an equicontinuous sequence of functions in E.
Let $\vartheta_1, \vartheta_2 \in J$ be arbitrary with $\vartheta_1 < \vartheta_2$. Then, by $(\mathbf{C_{III}})$ we have

$$\begin{aligned}&|\Phi x_n(\vartheta_2) - \Phi x_n(\vartheta_1)| \\ &\quad \leq (1-\theta)\overline{\Theta}_\theta |G(\vartheta_2, x_n(\vartheta_2)) - G(\vartheta_1, x_n(\vartheta_1))| \\ &\qquad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \left| \int_a^{\vartheta_2} \left[(\vartheta_2 - \sigma)^{\theta-1} - (\vartheta_1 - \sigma)^{\theta-1}\right] G(\sigma, x_n(\sigma)) d\sigma \right|\end{aligned}$$

$$
\begin{aligned}
&+\frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)}\left|\int_a^{\vartheta_2}(\vartheta_1-\sigma)^{\theta-1}G(\sigma,x_n(\sigma))\,d\sigma-\int_a^{\vartheta_1}(\vartheta_1-\sigma)^{\theta-1}G(\sigma,x_n(\sigma))\,d\sigma\right|\\
&\leq(1-\theta)\overline{\Theta}_\theta\left|G(\vartheta_2,x_n(\vartheta_2))-G(\vartheta_1,x_n(\vartheta_1))\right|\\
&\quad+\frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)}\int_a^{\vartheta_2}\left|(\vartheta_2-\sigma)^{\theta-1}-(\vartheta_1-\sigma)^{\theta-1}\right|\left|G(\sigma,x_n(\sigma))\right|d\sigma\\
&\quad+\frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)}\left|\int_{\vartheta_1}^{\vartheta_2}(\vartheta_1-\sigma)^{\theta-1}G(\sigma,x_n(\sigma))\,d\sigma\right|\\
&\leq(1-\theta)\overline{\Theta}_\theta\left|G(\vartheta_2,x_n(\vartheta_2))-G(\vartheta_1,x_n(\vartheta_1))\right|\\
&\quad+\frac{\theta\overline{\Theta}_\theta\nabla_G}{\Gamma(\theta)}\int_a^{\vartheta_2}\left|(\vartheta_2-\sigma)^{\theta-1}-(\vartheta_1-\sigma)^{\theta-1}\right|d\sigma+\frac{\theta\overline{\Theta}_\theta\nabla_G}{\Gamma(\theta)}\int_{\vartheta_1}^{\vartheta_2}\left|(\vartheta_1-\sigma)^{\theta-1}\right|d\sigma\\
&\leq(1-\theta)\overline{\Theta}_\theta\left|G(\vartheta_2,x_n(\vartheta_2))-G(\vartheta_1,x_n(\vartheta_1))\right|\\
&\quad+\frac{\theta\overline{\Theta}_\theta\nabla_G}{\Gamma(\theta)}\int_a^{b}\left|(\vartheta_2-\sigma)^{\theta-1}-(\vartheta_1-\sigma)^{\theta-1}\right|d\sigma+\frac{\theta\overline{\Theta}_\theta\nabla_G}{\Gamma(\theta)}\left|\Upsilon(\vartheta_2)-\Upsilon(\vartheta_1)\right|,
\end{aligned}
$$

where $\Upsilon(\vartheta)=\displaystyle\int_a^{\vartheta}\left|(b-\sigma)^{\theta-1}\right|d\sigma$. Since the functions $\vartheta\mapsto(\vartheta-\sigma)^{\theta-1}$ and $\vartheta\mapsto\Upsilon(\vartheta)$ are uniformly continuous on compact $J=[a,b]$, we have that

$$
\left|\Phi x_n(\vartheta_2)-\Phi x_n(\vartheta_1)\right|\to 0 \text{ as } \vartheta_2\to\vartheta_1
$$

uniformly for each $n\in\mathbb{N}$. Then, the convergence $\Phi x_n\to\Phi x^*$ is uniform. Thus Φ is partially continuous on E.

Step IV: Φ *is partially compact.*

Let C be a chain in E. We shall show that $\Phi(C)$ is uniformly bounded and equicontinuous in E. Let $\widetilde{x}\in\Phi(C)$ be arbitrary. We have $\widetilde{x}=\Phi(x)$ for some $x\in C$, and by $(\mathbf{C_{III}})$, we get

$$
\begin{aligned}
\widetilde{x}(\vartheta)&=|\Phi x(\vartheta)|\\
&\leq|\varphi_a-F(a,\varphi_a)|\\
&\quad+(1-\theta)\overline{\Theta}_\theta\left|G(\vartheta,x(\vartheta))\right|+\frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)}\int_a^{\vartheta}(\vartheta-\sigma)^{\theta-1}\left|G(\sigma,x(\sigma))\right|d\sigma\\
&\leq|\varphi_a-F(a,\varphi_a)|\\
&\quad+(1-\theta)\overline{\Theta}_\theta\left|G(\vartheta,x(\vartheta))\right|+\frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)}\int_a^{\vartheta}(\vartheta-\sigma)^{\theta-1}\left|G(\sigma,x(\sigma))\right|d\sigma\\
&\leq|\varphi_a-F(a,\varphi_a)|+(1-\theta)\overline{\Theta}_\theta\nabla_G+\frac{\overline{\Theta}_\theta\nabla_G(b-a)^\theta}{\Gamma(\theta)}\\
&:=\mathbf{M},
\end{aligned}
$$

for all $\vartheta\in J$. Thus,

$$
\|\widetilde{x}\|=\|\Phi x\|\leq\mathbf{M},
$$

for each $\widetilde{x}\in\Phi(C)$. Consequently, $\Phi(C)$ is a uniformly bounded subset of E.

Let us show that $\Phi(C)$ is an equicontinuous set in E. Let $\vartheta_1,\vartheta_2\in J$ be arbitrary with

$\vartheta_1 < \vartheta_2$. Then, by $(\mathbf{C_{III}})$ we get

$$\begin{aligned}
&|\Phi x(\vartheta_2) - \Phi x(\vartheta_1)| \\
&\quad \le (1-\theta)\overline{\Theta}_\theta |G(\vartheta_2, x(\vartheta_2)) - G(\vartheta_1, x(\vartheta_1))| \\
&\qquad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \left| \int_a^{\vartheta_2} \left[(\vartheta_2-\sigma)^{\theta-1} - (\vartheta_1-\sigma)^{\theta-1} \right] G(\sigma, x(\sigma))\, d\sigma \right| \\
&\qquad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \left| \int_a^{\vartheta_2} (\vartheta_1-\sigma)^{\theta-1} G(\sigma, x(\sigma))\, d\sigma - \int_a^{\vartheta_1} (\vartheta_1-\sigma)^{\theta-1} G(\sigma, x(\sigma))\, d\sigma \right| \\
&\quad \le (1-\theta)\overline{\Theta}_\theta |G(\vartheta_2, x(\vartheta_2)) - G(\vartheta_1, x(\vartheta_1))| \\
&\qquad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \int_a^{\vartheta_2} \left| (\vartheta_2-\sigma)^{\theta-1} - (\vartheta_1-\sigma)^{\theta-1} \right| |G(\sigma, x(\sigma))|\, d\sigma \\
&\qquad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \left| \int_{\vartheta_1}^{\vartheta_2} (\vartheta_1-\sigma)^{\theta-1} G(\sigma, x(\sigma))\, d\sigma \right| \\
&\quad \le (1-\theta)\overline{\Theta}_\theta |G(\vartheta_2, x(\vartheta_2)) - G(\vartheta_1, x(\vartheta_1))| \\
&\qquad + \frac{\theta\overline{\Theta}_\theta\nabla_G}{\Gamma(\theta)} \int_a^{\vartheta_2} \left| (\vartheta_2-\sigma)^{\theta-1} - (\vartheta_1-\sigma)^{\theta-1} \right| d\sigma + \frac{\theta\overline{\Theta}_\theta\nabla_G}{\Gamma(\theta)} \int_{\vartheta_1}^{\vartheta_2} \left| (\vartheta_1-\sigma)^{\theta-1} \right| d\sigma \\
&\quad \le (1-\theta)\overline{\Theta}_\theta |G(\vartheta_2, x(\vartheta_2)) - G(\vartheta_1, x(\vartheta_1))| \\
&\qquad + \frac{\theta\overline{\Theta}_\theta\nabla_G}{\Gamma(\theta)} \int_a^{b} \left| (\vartheta_2-\sigma)^{\theta-1} - (\vartheta_1-\sigma)^{\theta-1} \right| d\sigma + \frac{\theta\overline{\Theta}_\theta\nabla_G}{\Gamma(\theta)} |\Upsilon(\vartheta_2) - \Upsilon(\vartheta_1)|.
\end{aligned}$$

Since the functions $\vartheta \mapsto (\vartheta - s)^{\theta-1}$ and $\vartheta \mapsto \Upsilon(\vartheta)$ are uniformly continuous on compact $J = [a, b]$, we have

$$|\Phi x(\vartheta_2) - \Phi x(\vartheta_1)| \to 0 \quad \text{as} \quad \vartheta_2 \to \vartheta_1$$

uniformly for each $x \in C$. Then, $\Phi(C)$ is an equicontinuous set in E. Hence $\Phi(C)$ is compact subset of E and consequently Φ is a partially compact operator on E into itself.

Step V: *υ satisfies the operator inequality $\upsilon \le \Phi\upsilon$.*
By condition $(\mathbf{C_V})$, υ is a lower solution of (1.1) defined on J, i.e

$$\begin{cases} {}^{ABC}{}_aD_\tau^\theta [\upsilon(\vartheta) - F(\vartheta, \upsilon(\vartheta))] \le G(\vartheta, \upsilon(\vartheta)), \\ \upsilon(a) \le \varphi_a, \end{cases}$$

for all $\vartheta \in J$. By integrating of inequality

$$ {}^{ABC}{}_aD_\tau^\theta [\upsilon(\vartheta) - F(\vartheta, \upsilon(\vartheta))] \le G(\vartheta, \upsilon(\vartheta)), \tag{3.7}$$

from a to ϑ, we get

$$\begin{aligned}
\upsilon(\vartheta) &\le F(\vartheta, \upsilon(\vartheta)) + \varphi_a - F(a, \varphi_a) + (1-\theta)\overline{\Theta}_\theta G(\vartheta, \upsilon(\vartheta)) \\
&\quad + \frac{\theta\overline{\Theta}_\theta}{\Gamma(\theta)} \int_a^{\vartheta} (\vartheta-\sigma)^{\theta-1} G(\sigma, \upsilon(\sigma)) d\sigma,
\end{aligned}$$

for all $\vartheta \in J$. Then, υ is a lower solution of the operator equation $\upsilon = \Psi\upsilon + \Phi\upsilon$. Consequently, Ψ and Φ satisfies all conditions in Theorem 2.2. Thus, the operator equation $\Psi x + \Phi x = x$ has a solution. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.6) converges monotonically to x^*. □

4 An Example

Consider the Cauchy problem:

$$ {}^{ABC}{}_{0}D^{\frac{1}{2}}\left[x(\vartheta)-\frac{1}{100}\left(\frac{x(\vartheta)}{10+x(\vartheta)}+1\right)\right]=\frac{1}{2}e^{-\vartheta}\arctan x(\vartheta),\ \vartheta\in J=[0,1], \tag{4.1}$$

$$ x(0)=\varphi_0\in\mathbb{R}. \tag{4.2}$$

Set

$$ F(\vartheta,x)=\frac{1}{100}\left(\frac{x}{10+x}+1\right), $$

and

$$ G(\vartheta,x)=\frac{1}{2}e^{-\vartheta}\arctan x, $$

for all $\vartheta\in J,\ x\in\mathbb{R}$. Clearly, the functions F and G are jointly continuous and nondecreasing in x for all $\vartheta\in J$. Then conditions $(\mathbf{C_I})$ and $(\mathbf{C_{II}})$ are satisfied. Furthermore, the functions F and G satisfy the condition $(\mathbf{C_{III}})$ with $\Delta_F=\dfrac{3}{275}$ and $\nabla_G=\dfrac{\pi}{4}$. On the other hand, let $x,\varpi\in\mathbb{R}$ where $x\geq\varpi$, and $\vartheta\in J$, then:

$$\begin{aligned} 0\leq F(\vartheta,x)-F(\vartheta,\varpi) &= \frac{1}{100}\left[\frac{x}{10+x}-\frac{\varpi}{10+\varpi}\right]\\ &= \frac{1}{10}\left[\frac{x-\varpi}{(10+x)(10+\varpi)}\right]\\ &\leq \frac{1}{10}(x-\varpi)=\zeta(x-\varpi), \end{aligned}$$

for all $\vartheta\in J$, where $\zeta:\mathbb{R}^+\to\mathbb{R}^+$ defined by $\zeta(\vartheta)=\dfrac{1}{10}\vartheta<\vartheta,\ \vartheta>0$, is a $D-$contraction. Then, the function F satisfies the condition $(\mathbf{C_{IV}})$.

Finally, if $\vartheta\in[0,1]$, $\upsilon(\vartheta)=0$ is a lower solution of (4.1). Indeed,

$$\begin{aligned} 0=\upsilon(\vartheta) &\leq F(\vartheta,0)+\varphi_0-F(0,\varphi_0)+\left(1-\frac{1}{2}\right)\overline{\Theta}_{\frac{1}{2}}G(\vartheta,0)\\ &\quad+\frac{\frac{1}{2}\overline{\Theta}_{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\int_0^{\vartheta}(\vartheta-\sigma)^{\frac{1}{2}-1}G(\sigma,0)d\sigma\\ &\leq \varphi_0-\frac{\varphi_0}{100(10+\varphi_0)}, \end{aligned}$$

for some $\varphi_0\in\mathbb{R}$. Then, the condition $(\mathbf{C_V})$ is true. Thus, all conditions, $(\mathbf{C_I})-(\mathbf{C_V})$, are satisfied. It follows from Theorem 3.2 that the problem (4.1)-(4.2) has a solution x^* on $J=[0,1]$, which is a limit of the monotone sequence $(x_n),\ n=0,1,\dots,$ defined by

$$ x_0(\vartheta)=\varphi_0,\ \text{for}\ \vartheta\in[0,1], $$

where $\varphi_0-\dfrac{\varphi_0}{100(10+\varphi_0)}\geq 0$ and

$$\begin{aligned} x_{n+1}(\vartheta) &= F(\vartheta,x_n(\vartheta))+\varphi_0-F(0,\varphi_0)\\ &\quad+\frac{1}{2}\overline{\Theta}_{\frac{1}{2}}G(\vartheta,x_n(\vartheta))+\frac{\frac{1}{2}\overline{\Theta}_{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\int_0^{\vartheta}(\vartheta-\sigma)^{\frac{1}{2}-1}G(\sigma,x_n(\sigma))d\sigma. \end{aligned}$$

Declarations

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