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Laminar entry flow of Herschel-Bulkley Fluids in a circular pipe

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Abstract

Fargie and Martin used an elegant approach to obtain the entry region flow of Newtonian fluids in a circular pipe. They combined the differential and integral momentum equation in such a way that elimination of pressure gradient leads to a closed form solution. In this paper we extend this procedure to study the laminar entry region flow of Herschel-Bulkley fluids in a circular pipe. The solution procedure involves certain approximations. Applicability of these approximations to Herschel-Bulkley fluids has been discussed and the flow description has been obtained for various values of the flow behavior index and the Herschel-Bulkley number. Results have been illustrated graphically and compared with other available solutions: momentum integral and momentum energy integral solutions of the problem. Data displayed in the paper should be useful for design of any flow intake device involving Herschel-Bulkley fluids.

1 Introduction

Non-Newtonian fluid flow in a circular pipe is of considerable importance due to their natural occurrence in various real life applications, e.g., chemical, biomedical and food processes. Whenever a constant property viscous fluid enters a circular pipe, its velocity pattern changes from some initial distribution to a fully developed profile which thereafter does not vary with the axial distance from the entry plane. The region along the pipe where the velocity changes take place is called the entry region. Knowledge of a reliable flow description of the entry region flow is vital for proper design of

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any flow intake device.

Since the governing differential equations are not amenable to an exact solution, various approximate methods of solution have been proposed [1]. Fargie and Martin [1] described a simple and elegant approach for predicting entry region flow development in a circular pipe. They combined the differential and integral momentum equations in such a way that elimination of pressure gradient leads to a closed form solution.

Herschel-Bulkley fluids represent an empirical combination of Bingham plastic and power-law fluids and can be used to represent a wide range of fluids such as greases, starch pastes, colloidal suspensions, blood etc. [2]. Such fluids are also known as yield power-law fluids.

In this paper we use the procedure of Fargie and Martin [1] to obtain the entry region flow of Herschel-Bulkley fluids in a circular pipe. Herschel-Bulkley fluids are described by the relation

$$\tau = -\left(\eta|\dot{\gamma}|^{n-1} + \tau_0|\dot{\gamma}|^{-1}\right)\dot{\gamma}, \tag{1}$$

where τ is the shear stress, τ_0 is the yield stress, η is the coefficient of fluidity, $\dot{\gamma}$ is the rate of strain and n is the flow behaviour index.

2 Statement of the problem

Consider the steady, laminar, and isothermal flow of an incompressible constant property Herschel-Bulkley fluid entering a horizontal circular pipe as shown in figure 1. In view of axial symmetry of the flow, we use cylindrical polar coordinates with the $\hat{z}$-axis selected to coincide with the axis of the pipe and the radial coordinate $\hat{r}$ measured from the $\hat{z}$-axis, and the origin selected at the entrance on the axis of the pipe.

In terms of nondimensional quantities

$$u = \hat{u}/V, v = 2R_e\hat{v}/V, p = 2\hat{p}/\rho V^2, z = \hat{z}/2RR_e, \tag{2}$$

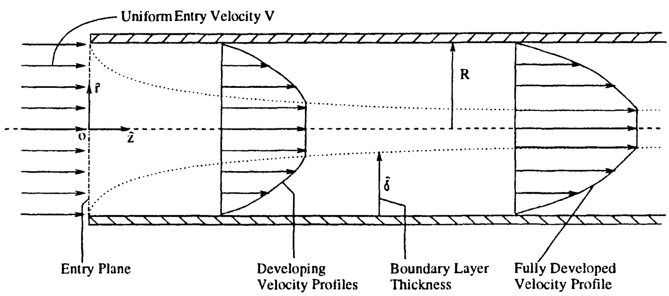

Figure 1: Herschel-Bulkley fluid flow in a horizontal circular pipe.

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the boundary layer equations governing the flow are

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial u}{\partial z} = 0, \tag{3}$$

$$v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = -\frac{1}{2}\frac{dp}{dz} + \frac{4}{r}\frac{\partial}{\partial r}\left[r\left(\left|\frac{\partial u}{\partial r}\right|^{n-1} + H_n\left|\frac{\partial u}{\partial r}\right|^{-1}\right)\frac{\partial u}{\partial r}\right]. \tag{4}$$

Here $R_e(=\frac{2\rho R^n V^{2-n}}{\eta})$ and $H_n(=\frac{\tau_0 R^n}{\eta V^n})$ respectively represent the Reynolds and the Herschel-Bulkley numbers. The boundary conditions are

$$\begin{aligned} &u = v = 0 && \text{when} \quad r = 1, \quad z \geq 0; \\ &u = 1, v = 0 && \text{when} \quad z = 0, \quad 0 \leq r \leq 1; \\ &u_r = 0, v = 0 && \text{when} \quad r = 0, \quad z \geq 0. \end{aligned} \tag{5}$$

In addition to these conditions, the continuity of the flow field leads to the restriction that the radial velocity derivative vanishes in the inviscid core and on the edge of the boundary layer, i.e., $u_r = 0$ when $0 \leq r \leq r_\star(= 1-\delta)$ for all z. In view of (5), there exists a singularity when $z = 0$ and $r = 1$. To overcome this difficulty, we assume a very thin boundary layer at the entry.

As in the case of Bingham fluid [3], it may be noted that in addition to the velocity boundary layer, there exists a shear stress boundary layer for Herschel-Bulkley fluid flow. At the shear stress boundary layer, the shear stress becomes zero, and it is thicker than the velocity boundary layer.

The fully developed flow of Herschel-Bulkley fluids has been discussed by Al-Fariss and Pinder [4]. They have found that H_n and the dimensionless plug core radius H_y in the fully developed region are connected by the relation

$$H_n = H_y/K^n, \tag{6}$$

where

$$K = (1 - H_y)^{\frac{n+1}{n}}\left(\frac{n}{n+1} - \frac{2n^2(1-H_y)}{(n+1)(2n+1)} + \frac{2n^3(1-H_y)^2}{(n+1)(2n+1)(3n+1)}\right).$$

Thus, for given n and H_n, eqn (6) determines the corresponding plug core radius H_y in the fully developed state.

Integrating the governing differential equations (3) and (4) throughout the cross-section of the pipe respectively, we obtain their integral forms of the principles of conservation of mass and momentum.

Eqn (3) leads to the overall continuity equation

$$\int_0^1 urdr = \frac{1}{2}, \tag{7}$$

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and eqn (4) becomes

$$\frac{\partial}{\partial z}\int_0^1 u^2 r dr + \frac{1}{4}\frac{dp}{dz} - 4\left[\left(\left|\frac{\partial u}{\partial r}\right|^{n-1} + H_n\left|\frac{\partial u}{\partial r}\right|^{-1}\right)\frac{\partial u}{\partial r}\right]_{r=1} = 0. \qquad (8)$$

For Herschel-Bulkley fluids, following Das [2], we make use of the following axial velocity distribution:

$$u = \begin{cases} U_c, & \text{for} \quad 0 \le r \le 1-\delta, \\ U_c\left[1 - \left(1 - \frac{1-r}{\delta}\right)^{(n+1)/n}\right], & \text{for} \quad 1-\delta \le r \le 1. \end{cases} \qquad (9)$$

In view of this, eqn (7) leads to

$$U_c = \frac{1}{g(\delta)}, \qquad (10)$$

where $g(\delta) = 1 + c_1\delta + c_2\delta^2$ with $c_1 = -\frac{2n}{2n+1}, c_2 = \frac{2n^2}{(2n+1)(3n+1)}$.

Combining eqns (8) and (9), we obtain

$$\frac{f(\delta)}{g^3(\delta)}\frac{d\delta}{dz} = \frac{1}{4}\frac{dp}{dz} + 4\left[\left(\frac{n+1}{n\delta g(\delta)}\right)^n + 4H_n\right], \qquad (11)$$

where $f(\delta)$ is given by

$$f(\delta) = 2d_2c_2\delta^3 + 3d_1c_2\delta^2 - d_3\delta - d_4,$$

with

$$d_1 = -\frac{n(4n+3)}{(2n+1)(3n+2)}, \qquad d_2 = \frac{n^2(9n+7)}{2(2n+1)(3n+1)(3n+2)},$$

$$d_3 = -\frac{n^2(10n+7)}{(2n+1)^2(3n+2)}, \qquad d_4 = \frac{n}{3n+2}.$$

Eqn (11) can be rearranged to get

$$\frac{d\delta}{dz} = \frac{g^3(\delta)}{f(\delta)}\left[\frac{1}{4}\frac{dp}{dz} + 4\left(\frac{n+1}{n\delta g(\delta)}\right)^n + 4H_n\right]. \qquad (12)$$

Fargie & Martin determined dp/dz by applying the momentum equation to the pipe wall. They found that the normalized inverse pressure gradient shows an inflection point. To avoid this they expressed dp/dz in a simple parabolic form (in δ) satisfying appropriate boundary conditions. In the following, in §2.1, we obtain the pressure using the procedure parallel to that of Fargie & Martin, and in §2.2 we obtain dp/dz by applying momentum equation to the pipe wall.

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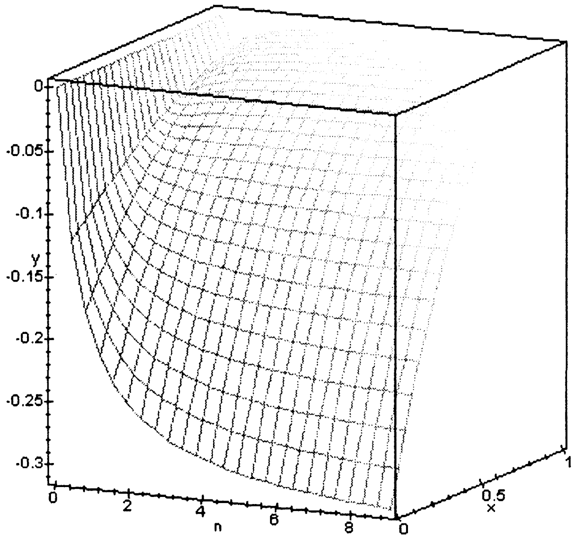

Figure 2: $y(x, n) = f(x)$, $0 \le n \le 9, 0 \le x \le 1$.

2.1 Solution 1

Since with increasing z, δ increases monotonically from zero to the yield value $H_\delta = 1 - H_y$ in the fully developed state, $d\delta/dz$ must correspondingly decrease to zero from an initially large positive value. Because $f(\delta)$ is non-positive(see figure 2), from eqn (12) we obtain

$$\frac{dp}{dz} \le -16 \left[\left(\frac{n+1}{n\delta g(\delta)} \right)^n + H_n \right], \tag{13}$$

where the equality holds only for $\delta = H_\delta$.

Applying eqn (4) to the wall, we obtain

$$\frac{dp}{dz} = 8 \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\left| \frac{\partial u}{\partial r} \right|^{n-1} + H_n \left| \frac{\partial u}{\partial r} \right|^{-1} \right) \frac{\partial u}{\partial r} \right] \right\}_{r=1}. \tag{14}$$

Combining this with eqn (9), we arrive at

$$\frac{dp}{dz} = -8 \left[U_c^n \delta^{-n} \left(\frac{n+1}{n} \right)^n \frac{1+\delta}{\delta} + H_n \right]. \tag{15}$$

Following Fargie and Martin [1], we try to obtain a replacement for dp/dz. In order to avoid the infinite value of dp/dz at $\delta = 0$, we consider its

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reciprocal. The assumed quadratic form for this, in addition to providing correct values at the entry plane and in the fully developed state, should also have the correct gradient when the flow is fully developed. Thus

$$\frac{dp}{dz} = \frac{1}{a\delta^2 + b\delta}, \tag{16}$$

with

$$a = \frac{l_2}{H_\delta} - \frac{1}{l_1 H_\delta^2}, \quad b = \frac{2}{l_1 H_\delta} - l_2, \tag{17}$$

where

$$l_1 = -8\left[\left(\frac{n+1}{nH_\delta g(H_\delta)}\right)^n \frac{1+H_\delta}{H_\delta} + H_n\right], \tag{18}$$

$$l_2 = \left(\frac{n+1}{n}\right)^n \frac{8}{l_1^2 H_\delta^{n+1} g^n(H_\delta)}\left[-\frac{n(1+H_\delta)(c_1 + 2c_2 H_\delta)}{g(H_\delta)} - \frac{(n+1)(1+H_\delta)}{H_\delta} + 1\right]. \tag{19}$$

Combining eqns (16) and (12), we obtain

$$\frac{dz}{d\delta} = \frac{f(\delta)}{g^3(\delta)\left[\frac{1}{4(a\delta^2+b\delta)} + 4\left(\frac{n+1}{n\delta g(\delta)}\right)^n + 4H_n\right]}. \tag{20}$$

Combining eqns (16) and (20), we find that $dp/d\delta$ in the developing region can be expressed as

$$\frac{dp}{d\delta} = \frac{f(\delta)}{(a\delta^2 + b\delta)g^3(\delta)\left[\frac{1}{4(a\delta^2+b\delta)} + 4\left(\frac{n+1}{n\delta g(\delta)}\right)^n + 4H_n\right]}. \tag{21}$$

In the following, we refer this as MDI solution.

Mashelkar[5] has obtained the power-law fluid flow development in a circular pipe by using Fargie & Martin's approach. He infers that the results are valid for $n \leq 1$. Motivated by this observation, here we wish to examine if dp/dz as given by eqn (16) is suitable for all n. If so, it must satisfy the requirement (13), that is

$$a\delta^2 + b\delta \geq -\frac{[n\delta g(\delta)]^n}{16\left\{(n+1)^n + H_n\,[n\delta g(\delta)]^n\right\}}, \quad 0 \leq \delta \leq H_\delta. \tag{22}$$

In figure 3(a), the left ($\lambda_1(\delta)$) and right ($\lambda_2(\delta)$) of the inequality (22) for an H_n(corresponding to $H_\delta = 0.75$) and $n = 0.75, 1.0, 1.25$ are plotted. It seems that $\lambda_1(\delta)$ is always greater than $\lambda_2(\delta)$ in the specified region. But when we magnify the couple of curves for $n = 1.25$ in figure 3(b), we observe

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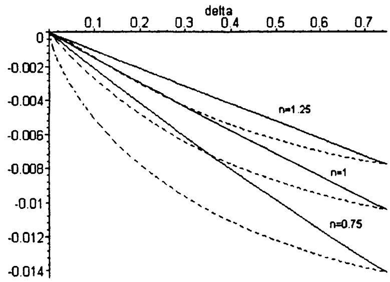
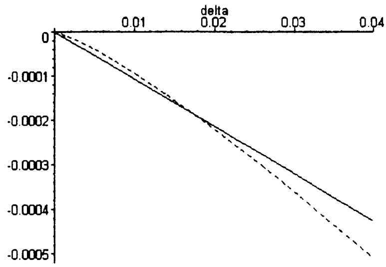


(a) $n = 0.75, 1.0$ and 1.25



(b) $n = 1.25$.

Figure 3: Variation of $\lambda_1(\delta)$(——), $\lambda_2(\delta)$ (- - -) as functions of δ.

that there is a small region near the point $\delta = 0$ where $\lambda_1(\delta) < \lambda_2(\delta)$. Similar observation is made when $n > 1$ [6]. Thus, the solution presented above appears to be applicable only for $n \leq 1$, and its applicability for $n > 1$ is likely to be doubtful.

2.2 Solution 2

When dp/dz is given by (15) the flow can be obtained from the following:

$$\frac{dz}{d\delta} = \frac{f(\delta)}{g^3(\delta)\left[2\left(\frac{n+1}{n\delta g(\delta)}\right)^n \frac{\delta-1}{\delta} + 2H_n\right]} \tag{23}$$

and

$$\frac{dp}{d\delta} = \frac{f(\delta)}{(a\delta^2 + b\delta)g^3(\delta)\left[2\left(\frac{n+1}{n\delta g(\delta)}\right)^n \frac{\delta-1}{\delta} + 2H_n\right]}. \tag{24}$$

This solution will be referred to as ORI solution.

3 Results

The boundary layer development and pressure drop in the entry region can be obtained by solving the relevant differential equations as obtained in §2.1 and §2.2 satisfying $z = 0$, $p = 0$ when $\delta = 0.00001$ by Simpson one-third rule. All our computations use process of automatic halving the step size, which continues until the data obtained from two successive step sizes differ by 10^{-8}. Figures 4 and 5 display the boundary layer development and the pressure drop in the entry region for $n = 0.75$ and 1.25 respectively for $H_n = 2.0$.

Let us define the flow to be fully developed when the axial velocity attains 99% of its fully developed value. Thus, flow is considered fully

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developed for δ given by

$$\frac{1}{1+c_1\delta+c_2\delta^2}=\frac{99}{100}\frac{1}{1+c_1(1-H_y)+c_2(1-H_y)^2}. \tag{25}$$

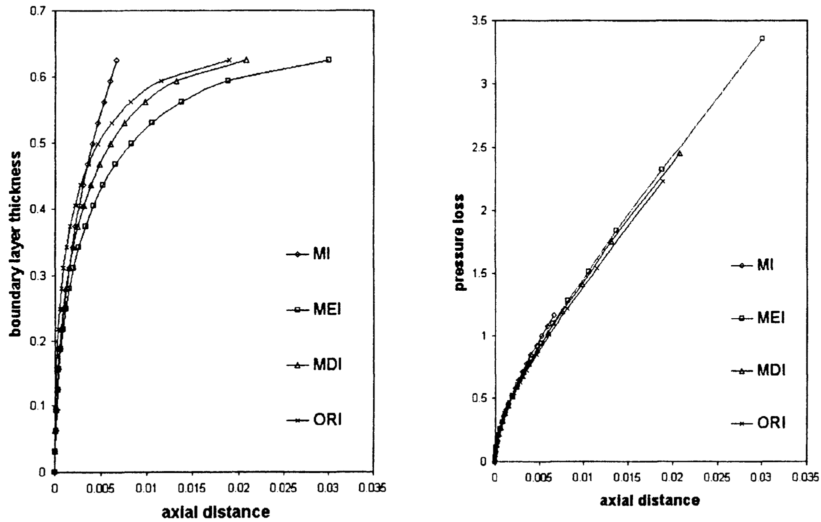

Figure 4: Boundary layer development and pressure drop in the entry region for $n = 0.75$, $H_n = 2.0$.

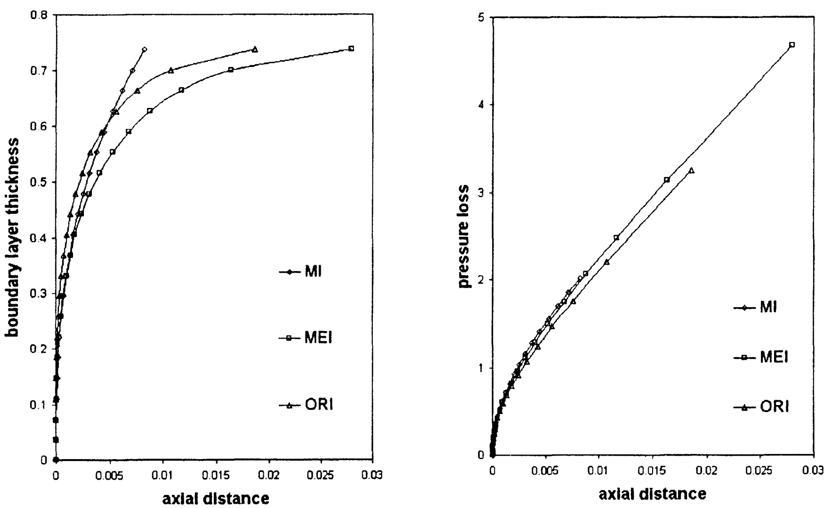

Figure 5: Boundary layer development and pressure drop in the entry region for $n = 1.25$, $H_n = 2.0$.

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The values of z and p computed for this δ give us the length of the entry region L_{99} and the pressure drop P_{99} there.

Das [2] has presented momentum integral(MI) and momentum energy integral(MEI) solutions of the problem under study.

For values of $n \leq 1$ and $H_n = 0.0, 0.5, 1.0, 2.0$ and 5.0, the MI and MEI solutions [2] predict the slowest and fastest rates of growth of δ, perhaps except near the entry. Near the entry, the ORI solution predicts the fastest rate of growth, but the growth becomes slower as fluid flows downstream. Finally we observe that L_{99} as predicted by the ORI solution lies between the values predicted by the MI and MEI solutions. The MEI solution predicts the largest value of L_{99}.

Pressure drop predictions appear to be less sensitive to a particular method of solution. For all the considered values of n and H_n, pressure drop at a given location as predicted by the MI, MEI and ORI solutions are in decreasing order. For $n = 0.75$, pressure drop prediction of the MDI solution lies between the predictions of the MEI and ORI solutions, whereas for $n = 1$, the largest drop in pressure is predicted by the MDI solution except possibly for $H_n = 5.0$.

Figure 6 displays the entry length and pressure drop predicted by the MDI(- - -) and ORI(—) solutions for various of n and H_n. Curves (1)-(5) are respectively for $H_n = 0.0, 0.5, 1.0, 2.0$ and 5.0. As expected, in view of the observations about the boundary layer development, for all n and H_n, entry length and pressure drop predictions lie between the MI and MEI solutions [2]. This has been illustrated in figure 7 for $H_n = 2.0$.

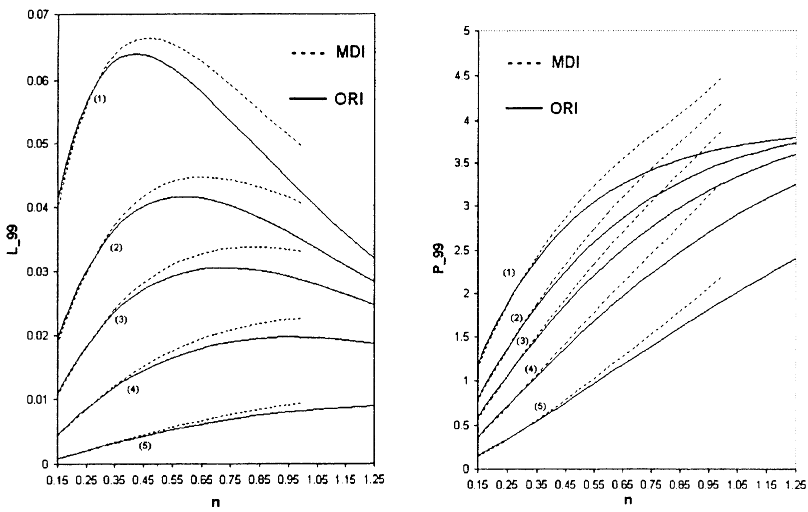

Figure 6: Entry length and pressure drop.

Curves (1)-(5) respectively correspond to $H_n = 0.0, 0.5, 1.0, 2.0$ and 5.0.

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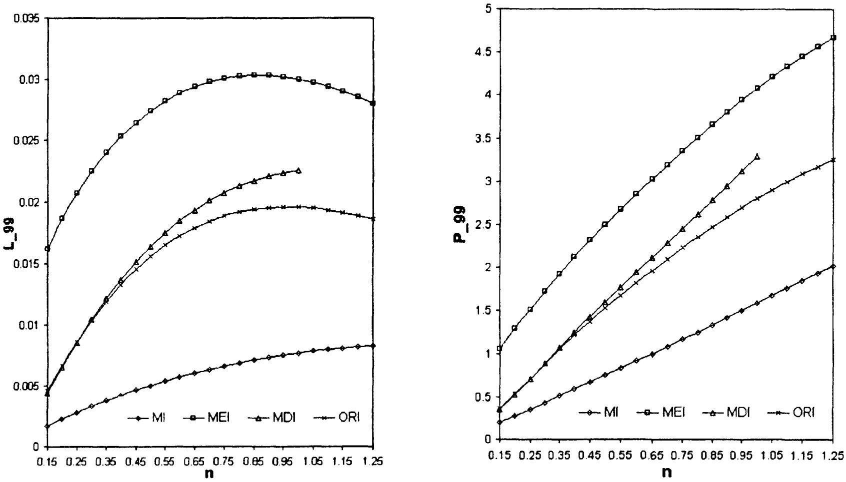

Figure 7: Entry length and pressure drop for $H_n = 2.0$.

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