Aust. J. Phys., 1996, **49**, 79–102

Landau Fermi Liquid Theory*

D. Neilson

School of Physics, University of New South Wales,
Sydney, NSW 2052, Australia.
Present address: Scuola Normale Superiore,
Piazza dei Cavalieri 7, 56126 Pisa, Italy.

Abstract

This article reviews the Landau theory of interacting Fermi liquids such as mobile electrons in solids or helium-3. It starts with Landau's original formulation which takes advantage of the existence of a Fermi surface to map the strongly interacting single-particle excitations near the Fermi surface into a system of weakly interacting quasiparticle excitations. The theory relates microscopic parameters for the quasiparticle energies and scattering strengths to experimental observables. The resulting low lying collective modes of the system, such as zero sound in helium-3, are then discussed. Next the rigorous microscopic basis of the theory is presented. Finally there is an outline of a recent modification of the theory which may resolve some of the puzzles about the nature of the electron states in materials exhibiting high transition temperature superconductivity.

1. Quasiparticles and Landau Parameters

1.1 Quasiparticles in interacting systems of fermions

The Landau theory of Fermi liquids (Landau 1957) replaces the complexities of a strongly interacting system of fermions by a system of weakly interacting quasiparticles lying in states near the Fermi surface. The proximity of the Fermi surface blocks most of the interactions between the low lying quasiparticle excitations which makes them long lived and approximate eigenstates of the system. The theory is particularly suited to transport properties since for interacting Fermi systems these properties are mostly determined by excitations close to the Fermi surface (Baym and Pethick 1991).

While Landau theory has some of the appearance of a phenomenological theory it has a rigorous microscopic basis. Parameters appearing in the theory are specified in terms of microscopic scattering amplitudes of particles sitting on the Fermi surface. At the same time one of the theory's strengths is that the parameters are related to experimentally measurable quantities.

A primary assumption in the theory is that near the Fermi surface there exists a one-to-one correspondence between the physical particles of the system and the long-lived quasiparticle excitations. In the adiabatic approximation we start

* Refereed paper based on a contribution to the fourth Gordon Godfrey Workshop on Atomic and Electron Fluids, held at the University of New South Wales, Sydney, in September 1994.

0004-9506/96/010079$10.00

with a system of non-interacting fermions at time t approaching minus infinity. As t increases we slowly switch on the interactions between the particles at such a rate that by $t = 0$ the interactions have reached full strength. It is assumed that as this happens there continues to be a one-to-one correspondence between the low lying excited states of the non-interacting system and the states of the new interacting system. The excited single-particle states in the interacting system are occupied by quasiparticles. These are not strictly eigenstates of the interacting system but when they are sufficiently close to the Fermi surface they can closely approximate true excited eigenstates for long periods of time. While the interactions between low lying quasiparticles are weak, they cannot be totally neglected and they are treated within a self-consistent field description.

The assumption of a one-to-one mapping for the excited states cannot always hold true for systems of interacting fermions. For example if bound states should develop as the interaction is slowly switched on then the quasiparticle picture breaks down since a bound state is a coherent superposition of many of the non-interacting states.

Let us start with N non-interacting fermions of spin $\frac{1}{2}$ in a volume Ω. The Fermi surface is a sphere of radius equal to the Fermi momentum k_F given by $k_F^3/3\pi^2 = N/\Omega$. At zero temperature all non-interacting single-particle states $|k\sigma\rangle$ with momentum $k \leq k_F$ and spin $\sigma = \pm\frac{1}{2}$ are occupied with occupation number $n_0(k\sigma) = 1$ and all states for $k > k_F$ are empty with $n_0(k\sigma) = 0$.

As we switch on the interaction the quasiparticles continue to obey Fermi statistics and the occupation number of the quasiparticle state $|k\sigma\rangle$ for the ground state remains the step function $n_0(k\sigma)$. Since the quasiparticles are only long lived close to the Fermi surface it is desirable to describe excitations in terms of changes in the occupation numbers, $\delta n(k\sigma) = n(k\sigma) - n_0(k\sigma)$ (Nozières 1964). For low lying excitations the $\delta n(k\sigma)$ will only be non-zero close to the Fermi surface and

$$\delta n(k\sigma) \begin{cases} > 0 & k \gtrsim k_F \\ < 0 & k \lesssim k_F \\ = 0 & |k| \gg k_F \end{cases} . \tag{1}$$

The difference in energies between the excited and ground states can be written in terms of $\delta n(k\sigma)$,

$$\delta E[\delta n(k\sigma)] = \sum_{k\sigma} \epsilon_{k\sigma} \delta n(k\sigma) . \tag{2}$$

This appears identical in form to the expression for the non-interacting system but here the quasiparticle energies $\epsilon_{k\sigma}$ themselves depend on the $\delta n(k\sigma)$,

$$\epsilon_{k\sigma}[\delta n(k\sigma)] = \frac{\delta E[\delta n(k\sigma)]}{\delta n(k\sigma)} , \tag{3}$$

so the excitation energy δE is a nonlinear functional of the $\delta n(k\sigma)$. We denote the energy of a quasiparticle when there are no other excited quasiparticles present by $\epsilon^0_{k\sigma}$. Then we get

$$\delta E = \sum_{k\sigma} \epsilon^0_{k\sigma} \delta n(k\sigma) + \tfrac{1}{2} \sum_{k\sigma, k'\sigma'} f(k\sigma, k'\sigma') \delta n(k\sigma) \delta n(k'\sigma') , \tag{4}$$

where

$$f(k\sigma, k'\sigma') = \frac{\delta^2 E}{\delta n_{k\sigma} \delta n_{k'\sigma'}} \tag{5}$$

measures the interaction between quasiparticles

$$\epsilon_{k\sigma}[\delta n_{k\sigma}] = \epsilon^0_{k\sigma} + \sum_{k'\sigma'} f(k\sigma, k'\sigma') \delta n(k'\sigma') . \tag{6}$$

In the absence of magnetic fields the system is isotropic and $f(k\sigma, k'\sigma')$ can depend only on the relative orientation of the spins $(\sigma.\sigma')$. We can make this dependence explicit by writing

$$f(k\sigma, k'\sigma') = f_s(k, k') + (4\sigma.\sigma') f_a(k, k') , \tag{7}$$

where the functions $f_s(k, k')$ and $f_a(k, k')$ are spin independent.

1.2 The Landau parameters

For quasiparticles lying very close to the Fermi surface the dependence of $f(k\sigma, k'\sigma')$ on the magnitudes of k and k' is not important and $f_s(k, k')$ and $f_a(k, k')$ are functions only of the angle $\zeta = \cos^{-1}(k.k'/k_F^2)$. We expand on the Legendre functions basis,

$$f_s(k, k') = \sum_\ell f_{s\ell} P_\ell(\cos\zeta) , \qquad f_a(k, k') = \sum_\ell f_{a\ell} P_\ell(\cos\zeta) , \tag{8}$$

where $f_{s\ell}$ and $f_{a\ell}$ are the *Landau parameters*. By factoring out the density of states at the Fermi surface, $N_F = (\Omega m^\star k_F)/\pi^2$, where $m^\star$ is the effective mass, the Landau parameters can be expressed in dimensionless form,

$$F_{s\ell} = N_F f_{s\ell}, \qquad F_{a\ell} = N_F f_{a\ell} . \tag{9}$$

Implicit in Landau theory is the hope that the series in (8) converges rapidly with increasing ℓ. The leading parameters can be related to experimentally measurable quantities

$$c^2 = \frac{k_F^2}{3\, m m^\star} [1 + F_{s0}] , \tag{10}$$

$$\frac{1}{m} = \frac{1}{m^\star} + \frac{F_{s1}}{m^\star} , \tag{11}$$

$$\frac{1}{\chi_M} = \frac{\pi^2}{\mu_B^2 m^\star k_F}\left[1 + F_{a0}\right] , \tag{12}$$

where c is the low frequency velocity of ordinary compressive sound waves, χ_M is the magnetic susceptibility and μ_B is the Bohr magneton. We set Planck's constant $\hbar = 1$ throughout. We now derive (10) to (12) (Nozières 1964; Negele and Orland 1988).

1.2.1 F_{so}

For F_{s0} we start with the definitions of the chemical potential, $\mu = (\partial E_0/\partial N)$, and the compressibility of the system,

$$\frac{1}{\chi} = -\Omega\frac{\partial P}{\partial \Omega} = \Omega\frac{\partial^2 E_0}{\partial \Omega^2} , \tag{13}$$

where E_0 is the ground state energy and P the pressure. Combining these gives

$$\frac{1}{\chi} = \frac{N^2}{\Omega}\frac{\mathrm{d}\mu}{\mathrm{d}N}\bigg|_{\Omega} . \tag{14}$$

Using (14) and the relation between compressibility and the low frequency velocity of sound, $c^2 = \Omega/(N\chi m)$, we obtain

$$c^2 = \frac{N}{m}\frac{\mathrm{d}\mu}{\mathrm{d}N} . \tag{15}$$

The chemical potential $\mu = \epsilon_{k_F\sigma}[n_{k\sigma}]$ depends on N both directly because of the change in Fermi momentum,

$$\delta k_F = [\pi^2/(\Omega k_F^2)]\delta N , \tag{16}$$

and also because of changes in the occupation numbers $\delta n_{k\sigma}$. The total change in μ is

$$\delta\mu = \frac{\delta\epsilon_{k\sigma}}{\delta k}\delta k_F + \sum_{k'\sigma'} f(k\sigma, k'\sigma')\delta n(k'\sigma') , \tag{17}$$

where $\delta\epsilon_{k\sigma}/\delta k = \nabla_k\epsilon_{k\sigma}$ is the group velocity v_k of a quasiparticle in state $k\sigma$. For an isotropic system k and v_k are parallel and we can write

$$v_k = \nabla_k\epsilon_{k\sigma} = \frac{k}{m^\star} . \tag{18}$$

Since $\delta n_{k'\sigma'}$ is only appreciable near the Fermi surface we can replace the sum over k' in (17) by an integral over the angle ζ between k and k',

$$\frac{\mathrm{d}\mu}{\mathrm{d}k_F} = \frac{k_F}{m^\star} + \sum_{\sigma'} \int \mathrm{d}(\cos\zeta)\, \frac{\Omega k_F^2}{4\pi^2} f(k\sigma, k'\sigma')\,, \tag{19}$$

or using (16),

$$\frac{\mathrm{d}\mu}{\mathrm{d}N} = \frac{\pi^2}{\Omega k_F m^\star} + \sum_{\sigma'} \int \mathrm{d}(\cos\zeta)\, \frac{f(k\sigma, k'\sigma')}{4}\,. \tag{20}$$

Equation (10) then follows using (15) and the orthogonality of the Legendre polynomials. The parameter F_{s0} is thus determined from experimental measurements of $m^\star$ and c^2.

1.2.2 F_{s1}

For the parameter F_{s1} we consider the current J_k associated with the movement of a quasiparticle having velocity $v_k = \nabla_k \epsilon_{k\sigma}$. The relationship between v_k and J_k is made complicated by the fact that when a quasiparticle moves forward in the medium the net current is reduced by a compensating backflow of other quasiparticles. These move to fill in the space vacated behind the quasiparticle as it propagates forward. This effect is particularly pronounced in a medium of helium atoms with their large hard cores. For electrons which lack a hard core the effect is much smaller.

To determine the effect of backflow on the total current we perform a Galilean transformation on the Hamiltonian for the system $\mathcal{H} = \sum_i (p_i^2/2m) + V$ in the centre of mass frame to another frame that is moving with velocity q/m relative to the centre of mass. Here p_i is the momentum operator for the physical particle i and V represents the interactions between the particles. Since V depends only on the relative separation of the particles it is not affected by the transformation so the Hamiltonian in the moving frame is

$$\mathcal{H}_q = \mathcal{H} - q.\sum_{i=1}^{N} \frac{p_i}{m} + N\frac{q^2}{2m}\,. \tag{21}$$

Writing the energy for the state $|\phi\rangle$ in the moving frame as $E_q \equiv \langle\phi|\mathcal{H}_q|\phi\rangle$ then

$$-\lim_{q\to 0} \nabla_q E_q = \langle\phi|\sum_{i=1}^{N} \frac{p_i}{m}|\phi\rangle\,. \tag{22}$$

Since p_i/m is the velocity operator for the physical particle the right-hand side of (22) is the total current for the state $|\phi\rangle$.

In the ground state the current must vanish because of reflection symmetry so the current associated with the quasiparticle excitation $k\sigma$ is simply

$$J_k = -\lim_{q\to 0} \nabla_q \epsilon_{k\sigma}\,. \tag{23}$$

Under the Galilean transformation the quasiparticle energy $\epsilon_{k\sigma}$ changes because the state $k\sigma$ becomes $(k-q)\sigma$ leading to an energy change of $-q.\nabla_k\epsilon_{k\sigma}$ and also because the occupation numbers $n(k\sigma)$ change (see equation 4). The $n(k\sigma)$ change because states which had been on the Fermi surface in the centre of mass frame are shifted away from the surface. In the direction of q these states now lie above the Fermi surface and the $\delta n(k\sigma)$ will be negative while states in the direction of $(-q)$ will lie below the surface giving a positive $\delta n(k\sigma)$. States in directions transverse to q will to lowest order be unaffected. Aligning angles so that $\zeta=0$ is along the direction of q then the changes $\delta n(k\sigma)$ are

$$\delta n(k\sigma) = -(q\cos\zeta)\delta(|k|-k_F)\,. \tag{24}$$

The total energy shift is

$$\delta\epsilon_{k\sigma} = -q.\nabla_k\epsilon_{k\sigma} + \sum_{k'\sigma'} f(k\sigma,k'\sigma')[-(q\cos\zeta)\delta(|k'|-k_F)]\,, \tag{25}$$

and using (23)

$$J_k = \nabla_k\epsilon_{k\sigma} + \sum_{k'\sigma'} f(k\sigma,k'\sigma')(\cos\zeta)\delta(|k'|-k_F)\,. \tag{26}$$

For a translationally invariant system the total current is a constant of motion and commutes with the interaction term V in the Hamiltonian. The current for the quasiparticle $k\sigma$ must thus be the same in the interacting system as in the non-interacting one where it is

$$J_k = k/m\,, \tag{27}$$

with m the bare mass. Combining (26) and (27) and using (18) gives

$$\frac{1}{m} = \frac{1}{m^\star} + \frac{\Omega k_F}{4\pi^2}\sum_{\sigma'}\int \mathrm{d}(\cos\zeta)\; f(k\sigma,k'\sigma')\cos\zeta\,, \tag{28}$$

from which (11) follows. Thus the effective mass $m^\star$ determines F_{s1}.

1.2.3 F_{ao}

For the parameter F_{a0} we consider the shift in the quasiparticle energy $\epsilon_{k\sigma}$ caused by a magnetic field H. When H is switched on $\epsilon_{k\sigma}$ changes because of the shift in the single-particle energies $-2\mu_B\sigma H$ and because of the changes in the occupation numbers. The total change in $\epsilon_{k\sigma}$ is

$$\delta\epsilon_{k\sigma} = -2\mu_B\sigma H + \sum_{k'\sigma'} f(k\sigma,k'\sigma')\delta n(k'\sigma')\,. \tag{29}$$

Let us assume that $\delta\epsilon_{k\sigma}$ depends linearly on H with some proportionality constant η so $\delta\epsilon_{k\sigma} = -\eta\sigma H$. The consistency of this can be checked at the end of the calculation. The shift in the Fermi momentum is then given by

$$\delta k_F(\sigma = \pm\tfrac{1}{2}) = \sigma\frac{m^\star}{k_F}\eta H\,. \tag{30}$$

Since the up and down shifts in k_F are symmetric the chemical potential and the average of the k_F for the two spins will not change. A total of $4\pi k_F^2\delta k_F$ additional quasiparticle states with spins parallel to H will be shifted from above the Fermi surface to below it and an equal number of states with spins anti-parallel to H will move up out of the Fermi surface. For parallel spins $\delta n(k\sigma) = +1$ and for anti-parallel spins $\delta n(k\sigma) = -1$. Equation (29) then becomes

$$\begin{aligned}\delta\epsilon_{k\sigma} &= -2\mu_B\sigma H + \sigma\frac{\Omega m^\star k_F}{4\pi^2}\eta H\sum_{\sigma'}(4\sigma.\sigma')\int \mathrm{d}(\cos\zeta)f(k\sigma,k'\sigma') \\ &= -2\mu_B\sigma H + \sigma\frac{\Omega m^\star k_F}{\pi^2}\eta H f_{a0}\,,\end{aligned} \tag{31}$$

using (7) and (8). Equation (31) is consistent with the assumption that $\delta\epsilon_{k\sigma}$ is proportional to H with the constant η given by

$$\eta = 2\mu_B - \eta F_{a0} = \frac{2\mu_B}{1+F_{a0}}\,. \tag{32}$$

The magnetic susceptibility of a system is $\chi_M = (1/\Omega)(M/H)$ where the magnetic moment M is determined by the shift in occupation numbers

$$M = \sum_{k\sigma} 2\mu_B\sigma\delta n(k\sigma)\,, \tag{33}$$

and so using (30)

$$\begin{aligned}M &= \Omega\frac{m^\star k_F}{2\pi^2}\mu_B\eta H \\ &= \Omega\frac{m^\star k_F}{\pi^2}\frac{\mu_B^2}{1+F_{a0}}H\,,\end{aligned} \tag{34}$$

and from this (12) follows giving F_{a0} in terms of χ_M and $m^\star$.

2. Inhomogeneous Excited States on a Macroscopic Scale

2.1 The Boltzmann equation for quasiparticles

We have assumed until now that the system is spatially uniform in density but provided inhomogeneous variations change slowly over distances comparable to the spacing between particles we can generalise (4) and write (Nozières 1964)

$$\delta E = \sum_{k\sigma} \int \mathrm{d}^3 r \; \epsilon^0_{k\sigma} \delta n(k\sigma, r) + \tfrac{1}{2} \sum_{k\sigma k'\sigma'} \int \mathrm{d}^3 r \int \mathrm{d}^3 r' f(k\sigma r, k'\sigma' r') \delta n(k\sigma, r) \delta n(k'\sigma', r') \,. \tag{35}$$

For helium atoms where the interactions are finite in range we can assume that interactions act only over distances for which $\delta n(k\sigma, r)$ is almost independent of r. Equation (35) then becomes

$$\delta E = \sum_{k\sigma} \int \mathrm{d}^3 r \; \epsilon^0_{k\sigma} \delta n(k\sigma, r) + \tfrac{1}{2} \sum_{k\sigma k'\sigma'} \int \mathrm{d}^3 r f(k\sigma, k'\sigma') \delta n(k\sigma, r) \delta n(k'\sigma', r) \,, \tag{36}$$

where we take $f(k\sigma, k'\sigma') = \int \mathrm{d}^3 r' f(k\sigma r, k'\sigma' r')$.

Let us further assume that the inhomogeneous variations $\delta n(k\sigma, r) = [n(k\sigma, r) - n_0(k\sigma)]$ are small compared with the mean value $n_0(k\sigma)$ that is, $\delta n(k\sigma, r)/n_0(k\sigma) \ll 1$. Since in the absence of an external driving force the $n_0(k\sigma)$ is independent of r and t the linearised Boltzmann equation of motion for $\delta n(k\sigma, r)$ is (Nozières 1964)

$$\frac{\partial}{\partial t} \delta n(k\sigma, r) + \nabla_r \delta n(k\sigma, r) . \nabla_k \epsilon^0_{k\sigma} - \nabla_k n_0(k\sigma) . \nabla_r \epsilon_{k\sigma}(r) = \left(\frac{\partial}{\partial t} \delta n(k\sigma, r) \right)_{\text{collision}} \,. \tag{37}$$

Since $\delta n(k\sigma, r)$ is only appreciable near the Fermi surface (37) describes the time development of a system of quasiparticles which are long-lived and weakly interacting. The system resembles a dilute gas and for many applications the collisional term on the right-hand side can be neglected. In (37) there is a term $\nabla_r \epsilon_{k\sigma}(r)$ which is not present in the classical Boltzmann equation for a dilute gas. This term is associated with the spatial variations of the quasiparticle energy,

$$\epsilon_{k\sigma}(r) = \epsilon^0_{k\sigma} + \sum_{k'\sigma'} f(k\sigma, k'\sigma') \delta n(k'\sigma', r) \,, \tag{38}$$

so that

$$\nabla_r \epsilon_{k\sigma}(r) = \sum_{k'\sigma'} f(k\sigma, k'\sigma') \nabla_r \delta n(k'\sigma', r) \,. \tag{39}$$

Then

$$\frac{\partial}{\partial t}\delta n(k\sigma, r) + \nabla_r \delta n(k\sigma, r).v_k$$

$$+\delta(\epsilon_{k\sigma} - \mu)v_k.\sum_{k'\sigma'} f(k\sigma, k'\sigma')\nabla_r \delta n(k'\sigma', r) = \left(\frac{\partial}{\partial t}\delta n(k\sigma, r)\right)_{\text{collision}}, \quad (40)$$

where we have used (18), and $\nabla_k n_0(k\sigma) = -v_k \delta(\epsilon_{k\sigma} - \mu)$.

For electrons (36) cannot immediately be used because Coulomb interactions are long range. However we can still use it if we first replace the long-range part of the interaction between points r and r' by an equivalent electrostatic interaction. This acts between the average charge densities at r and r',

$$V_{\mathrm{H}}(r - r') = \tfrac{1}{2}\int \mathrm{d}^3 r \int \mathrm{d}^3 r' \sum_{k\sigma} \delta n(k\sigma, r) \sum_{k'\sigma'} \delta n(k'\sigma', r') \frac{e^2}{|r - r'|}. \quad (41)$$

We can account for $V_{\mathrm{H}}(r)$ by introducing a Hartree electric field $\mathcal{E}_{\mathrm{H}}(r)$ which obeys the Poisson equation, $\nabla.\mathcal{E}_{\mathrm{H}}(r) = 4\pi e \sum_{k\sigma} \delta n(k\sigma, r)$. With $\mathcal{E}_{\mathrm{H}}(r)$ treated as an external field the remaining interactions between electrons are of finite range and (36) can be used.

We consider two applications of the Boltzmann equation, the first collective oscillations in a neutral system in the absence of an external driving term and the second the response of electrons to an external electric field.

2.2 Collective mode in the neutral system

We search for periodic solutions to the Boltzmann equation for an uncharged liquid in the absence of an external field. These are of the form

$$\delta n(k\sigma, r) = \delta n(k\sigma)\mathrm{e}^{i(q.r - \omega t)}. \quad (42)$$

The periodic perturbations to which the Landau theory can be applied are macroscopic and satisfy the conditions $q \ll k_F$ and $\omega \ll \mu$. We will assume that ω is also much greater than the collisional frequency ν, so that we can neglect the collisional term in (40). For sufficiently low temperatures this condition will be satisfied since the collisional frequency vanishes with temperature. Equation (40) is then homogeneous and we have

$$(q.v_k - \omega)\delta n(k\sigma) + q.v_k \delta(\epsilon_{k\sigma} - \mu) \sum_{k'\sigma'} f(k\sigma, k'\sigma')\delta n(k'\sigma') = 0. \quad (43)$$

Solutions to (43) only exist for discrete values of ω/q corresponding to free oscillations of the medium with phase velocity $v_\phi = \omega/q$. These oscillations form the collective modes of the system.

Replacing $\delta n(k\sigma)$ by a function $u(k\sigma)$ which gives the displacement of the Fermi surface in the k direction,

$$\delta n(k\sigma) = -\frac{\partial n_0}{\partial |k|} u(k\sigma) = \delta(\epsilon_{k\sigma} - \mu)|v_k| u(k\sigma)\,, \tag{44}$$

(43) becomes

$$(q.v_k - \omega)u(k\sigma) + q.v_k \sum_{k'\sigma'} f(k\sigma, k'\sigma')\delta(\epsilon_{k'\sigma'} - \mu)u(k'\sigma') = 0\,. \tag{45}$$

Expressing the phase velocity $s = v_\phi / v_F$ in units of the Fermi velocity $v_F = k_F/m^\star$, and with a slight change of notation we get

$$(\cos\theta - s)u(\theta,\phi,\sigma) + \cos\theta \sum_{\sigma'} \int \{\mathrm{d}\phi' \mathrm{d}(\cos\theta')\} \frac{m^\star k_F}{(2\pi)^3} f(\zeta\sigma\sigma') u(\theta',\phi',\sigma') = 0\,, \tag{46}$$

where (θ,ϕ) and (θ',ϕ') give the directions of k and k', and ζ is as usual the angle between k and k'.

When $u(\theta,\phi,\sigma)$ is independent of spin (46) has the solution

$$u(\theta,\phi) = \frac{\cos\theta}{s - \cos\theta} \int \{\mathrm{d}\phi' \mathrm{d}(\cos\theta')\} \frac{F_s(\zeta)}{4\pi} u(\theta',\phi')\,. \tag{47}$$

If the phase velocity eigenvalue $s \lesssim 1$ the collective mode can only be damped by excitations of two or more particles and the mode will be a sharp resonance. For $s < 1$ the velocity will be matched by some of the quasiparticle excitations and the collective mode will be strongly damped.

Approximating $F_s(\zeta)$ by F_{s0} the eigenvalues of (47) are given by solutions of

$$\frac{s}{2}\log\frac{s+1}{s-1} = 1 + \frac{1}{F_{s0}}\,. \tag{48}$$

Provided $F_{s0} > 0$ a solution to (48) always exists with $s \gtrsim 1$. For small positive F_{s0} the phase velocity approaches unity as $s = 1 + 2\exp(-2/F_{s0})$ and for large F_{s0} it increases as $s = \sqrt{F_{s0}/3}$.

The eigenfunction $u(\theta,\phi)$ has the form

$$u(\theta,\phi) = \frac{\cos\theta}{s - \cos\theta} \times 4\pi C\,, \tag{49}$$

where $C = F_{s0} \int \{\mathrm{d}\phi' \mathrm{d}(\cos\theta')\} u(\theta',\phi')$. For $s > 1$ $u(\theta,\phi)$ has a maximum in the direction of q where $\theta = 0$, for $\theta = \pm\pi/2$ the $u(\theta,\phi)$ is zero and in the $(-q)$ direction it is a negative minimum, but with a magnitude less than in the q direction. Thus $u(\theta,\phi)$ distorts the Fermi surface into an oval shape with the elongated end pointing in the direction of propagation. For s approaching unity the distortion becomes a nodule on the Fermi surface localised in the q direction and all the quasiparticle excitations in the collective mode will have the same velocity equal to s. To distinguish this mode from ordinary sound it is called zero sound. Zero sound is a quite different mode from the ordinary compressive sound waves which form the low frequency collective mode for $\omega \ll \nu$. In ordinary sound the whole Fermi surface is uniformly displaced as the density oscillates so the shape of the Fermi surface does not change. In zero sound the Fermi surface displacement $u(\theta, \phi)$ is highly asymmetric and points towards the forward direction.

2.3 External field acting on a charged system

If an external electric field $\mathcal{E}(r,t)$ acts on a system of electrons of charge e then the total driving force comes from the sum of $\mathcal{E}(r,t)$ and the Hartree field $\mathcal{E}_{\mathrm{H}}(r,t)$,

$$\mathcal{F}(r,t) = e\{\mathcal{E}(r,t) + \mathcal{E}_{\mathrm{H}}(r,t)\}\,. \tag{50}$$

We again look for solutions of the Boltzmann equation for a particular wave number q and frequency ω,

$$\mathcal{E}(r,t) = \mathcal{E}\mathrm{e}^{i(q.r-\omega t)}, \qquad \delta n(k\sigma, r) = \delta n(k\sigma)\mathrm{e}^{i(q.r-\omega t)}\,, \tag{51}$$

with ω in the range $\nu \ll \omega \ll \mu$ so that the collisional term can be neglected. Setting the collisional term in (40) equal to zero and adding the driving term associated with $\mathcal{F}(t)$ we get

$$(q.v_k - \omega)\delta n(k\sigma) + q.v_k\ \delta(\epsilon_{k\sigma} - \mu)\sum_{k'\sigma'} f(k\sigma, k'\sigma')\delta n(k'\sigma') + i\mathcal{F}.v_k\ \delta(\epsilon_{k\sigma} - \mu) = 0\,. \tag{52}$$

The quasiparticles excited by the external field will result in a current

$$I(r,t) = e\sum_{k\sigma} \delta n(k\sigma) J_k \mathrm{e}^{i(q.r-\omega t)}\,, \tag{53}$$

where J_k is the current of the quasiparticle $k\sigma$.

Solution of (52) can be mathematically involved and we choose for illustration an elementary example. Taking a spatially uniform external field we can set $q = 0$ and (52) has the solution

$$\delta n(k\sigma) = \frac{i\mathcal{F}.v_k}{\omega}\delta(\epsilon_{k\sigma} - \mu)\,. \tag{54}$$

Combining (53) and (54) and taking $J_k = k/m$

$$I(t) = \frac{ie}{\omega}\mathcal{F}.\sum_{k\sigma} v_k\ \delta(\epsilon_{k\sigma} - \mu)\left(\frac{k}{m}\right)\mathrm{e}^{-i\omega t}$$

$$= \frac{ie}{\omega}\mathcal{F}.\sum_{k\sigma}\delta(k - k_F)\left(\frac{k}{m}\right)\mathrm{e}^{-i\omega t}$$

$$= \frac{ie\mathcal{F}}{m\omega}N\mathrm{e}^{-i\omega t}\,. \tag{55}$$

Recalling that the conductivity tensor $\sigma_{\alpha\beta}$ is the ratio of current I to the driving force $\mathcal{F}$ we get the limiting long wavelength expression

$$\sigma_{\alpha\beta}(\omega) = \frac{iNe^2}{m\omega}\delta_{\alpha\beta}\,, \tag{56}$$

a result which can be obtained directly from translational invariance arguments.

3. Microscopic Basis of the Theory

3.1 Single-particle Green's functions

We now look at the microscopic theory on which the Landau Fermi liquid theory is based (Abrikosov 1963; Nozières 1964; Brown 1972; Jones and March 1973; Negele and Orland 1988; Bedell 1994). We assume a uniform system and recall that the single-particle Green's function $G(k\sigma, t)$ which describes the propagation of a bare particle when $k > k_F$ or a hole when $k < k_F$ over a time interval t is

$$G(k\sigma, t) = i\langle\Psi_0^N|\mathcal{T}\left\{a_{k\sigma}(t)a^\dagger_{k\sigma}(0)\right\}|\Psi_0^N\rangle\,, \tag{57}$$

with $|\Psi_0^N\rangle$ the exact N-particle interacting ground state of the system. The operator $\mathcal{T}$ time orders the annihilation and creation operators $a_{k\sigma}(t)$ and $a^\dagger_{k\sigma}(0)$.

Because of the time-ordering operator, Green's function has a discontinuity at time $t = 0$,

$$G(k\sigma, t = 0^+) - G(k\sigma, t = 0^-) = i\,, \tag{58}$$

while for $t = 0^\pm$

$$G(k\sigma, t = 0^+) = i(1 - m_k),$$

$$G(k\sigma, t = 0^-) = -im_k\,. \tag{59}$$

Here $m_k = \langle\Psi_0^N|a^\dagger_{k\sigma}(0)a_{k\sigma}(0)|\Psi_0^N\rangle$ is the distribution of *bare* particle states $k\sigma$ in the interacting ground state. The discontinuity in $G(k\sigma, t)$ does not depend on m_k.

From equation (59) for $t = 0^-$ we have

$$m_k = \frac{i}{2\pi} \oint_C \mathrm{d}\omega G(k\sigma,\omega)\,, \tag{60}$$

where the contour C is closed in the upper half of the complex ω plane.

In the Heisenberg picture (57) is

$$G(k\sigma,t) = \begin{cases} i\,\langle\Psi_0^N|\mathrm{e}^{i\mathcal{H}t}a_{k\sigma}\mathrm{e}^{-i\mathcal{H}t}a^\dagger_{k\sigma}|\Psi_0^N\rangle & t>0 \\ -i\,\langle\Psi_0^N|a^\dagger_{k\sigma}\mathrm{e}^{i\mathcal{H}t}a_{k\sigma}\mathrm{e}^{-i\mathcal{H}t}|\Psi_0^N\rangle & t<0 \end{cases}, \tag{61}$$

with a time independent $|\Psi_0^N\rangle$. Introducing a complete set of *exact* eigenstates of the full Hamiltonian, $\mathcal{H}|\Psi_n^{N\pm1}\rangle = E_n^{N\pm1}|\Psi_n^{N\pm1}\rangle$ for systems with $N\pm1$ particles we can express (61) in the Lehmann representation as

$$G(k\sigma,t) = \begin{cases} i\sum_n |\langle\Psi_n^{N+1}|a^\dagger_{k\sigma}|\Psi_0^N\rangle|^2\mathrm{e}^{-i\omega_{n0}t} & t>0 \\ -i\sum_n |\langle\Psi_n^{N-1}|a_{k\sigma}|\Psi_0^N\rangle|^2\mathrm{e}^{i\omega_{n0}t} & t<0, \end{cases}, \tag{62}$$

with excitation energies $\omega_{n0} = |E_n^{N\pm1} - E_0^N|$. Let us introduce $\xi_{n0} = \omega_{n0} \mp \mu$ as the excitation energy relative to the chemical potential $\mu = |E_0^{N\pm1} - E_0^N|$. This is the excitation energy for a system with a fixed number of particles.

Introducing the real positive spectral density functions

$$A_+(k\sigma,\omega) = \sum_n |\Psi_n^{N+1}|a^\dagger_{k\sigma}|\Psi_0^N|^2\delta(\omega-\xi_{n0}),$$
$$A_-(k\sigma,\omega) = \sum_n |\Psi_n^{N-1}|a_{k\sigma}|\Psi_0^N|^2\delta(\omega-\xi_{n0})\,, \tag{63}$$

(62) can be written in the form of a spectral expansion

$$G(k\sigma,t) = \begin{cases} i\mathrm{e}^{-i\mu t}\int_0^\infty \mathrm{d}\omega A_+(k\sigma,\omega)\mathrm{e}^{-i\omega t} & t>0 \\ -i\mathrm{e}^{-i\mu t}\int_0^\infty \mathrm{d}\omega A_-(k\sigma,\omega)\mathrm{e}^{i\omega t} & t<0, \end{cases}, \tag{64}$$

and hence

$$G(k\sigma,\omega) = \int_0^\infty \mathrm{d}\omega' \left\{ \frac{A_+(k\sigma,\omega')}{\omega'-\omega+\mu-i0^+} + \frac{A_-(k\sigma,\omega')}{\omega'+\omega-\mu-i0^+} \right\}. \tag{65}$$

3.2 Single-particle excited states

The excited state $a^\dagger_{k\sigma}(0)|\Psi_0^N\rangle$ consists of the N-particle ground state to which an additional bare particle $k\sigma$ has been added at time $t=0$. The $a^\dagger_{k\sigma}(0)|\Psi_0^N\rangle$ is not an eigenstate of the Hamiltonian, being some superposition of eigenstates of different energies with the distribution given in (64). Here $A_+(k\sigma,\omega)$ is the probability density at energy $\omega+\mu$ for the state $a^\dagger_{k\sigma}(0)|\Psi_0^N\rangle$ and $A_-(k\sigma,\omega)$

the probability density at $(-\omega+\mu)$ for $\Psi_0^N|a_{k\sigma}(0)\rangle$. If $a_{k\sigma}^\dagger(0)|\Psi_0^N\rangle$ were an exact eigenstate with energy ξ_k then $A_+(k\sigma,\omega)$ would simply be the delta function $\delta(\omega-\xi_k)$.

Even though $a_{k\sigma}^\dagger(0)|\Psi_0^N\rangle$ is not itself a quasiparticle state it may initially contain a quasiparticle at energy ξ_k. If the quasiparticle decays with a lifetime $1/\Gamma_k$ then there will be a peak in $A_+(k\sigma,\omega)$ which is centred on $\omega=\xi_k>0$ and with a half-width of Γ_k. The area under the peak will be a spectral strength z_k. The remaining contributions from states of other energies will make up a smoothly varying background with a spectral strength of $(1-m_k-z_k)$ (see equation 59). The smaller z_k is in the range $0\le z_k\le 1$, the greater will be the tendency of the single-particle excited states to mix with states involving excitations of two or more particles. For $z_k=1$ there would be no mixing and the single-particle states would be pure eigenstates. For $z_k=0$ the mixing with excitations of two or more particles would be complete leaving behind no detectable quasiparticle component. We concentrate on the quasiparticle case with $k>k_F$ but the argument for the quasihole proceeds in the same way. A quasihole $k\sigma$ leads to a peak in $A_-(k\sigma,\omega)$ at $\xi_k<0$ and the background spectral strength would be $(-m_k-z_k)$.

To study the properties of the quasiparticle state we need to isolate it by filtering the background out of $a_{k\sigma}^\dagger(0)|\Psi_0^N\rangle$. With this aim let us look at the propagator for the state $a_{k\sigma}^\dagger(0)|\Psi_0^N\rangle$. Choosing $t>0$ we distort the contour of integration in (64) from the positive real axis into the lower half of the complex ω plane. We take it first down the negative imaginary axis for some finite interval $0\le\Im m\ \omega\le-\alpha$ and then we run it parallel to the real axis out to $\Re e\ \omega$ approaches infinity

$$\begin{aligned}-iG(k\sigma,t) &= \mathrm{e}^{-i\mu t}\\ &\quad\times\left(\int_0^{-i\alpha}\mathrm{d}\omega A_+(k\sigma,\omega)\mathrm{e}^{-i\omega t}\right.\\ &\quad\left.+\,\mathrm{e}^{-\alpha t}\int_0^{\infty}\mathrm{d}\omega A_+(k\sigma,\omega-i\alpha)\mathrm{e}^{-i\omega t}\right)+\mathcal{R}\,,\end{aligned}\tag{66}$$

where $\mathcal{R}$ is the residue contribution from any singularities in $A_+(k\sigma,\omega)$ that the contour passed over as we pushed it into the lower half plane. For a given time t let us choose the value of the constant α such that $\alpha t\gg 1$. Then the $\mathrm{e}^{-\alpha t}$ factor in (66) is negligible and we can drop the second integration.

A quasiparticle state contained in $a_{k\sigma}^\dagger(0)|\Psi_0^N\rangle$ will lead to a pole in $A_+(k\sigma,\omega)$ in the lower half of the complex ω plane. With the peak parameters we introduced above the pole will be at $\omega=\xi_k-i\Gamma_k$ with a residue of $z_k/2\pi i$. The longer the lifetime of the quasiparticle the closer the pole will be to the real axis. If $\Gamma_k<\alpha$ then the residue of the pole will be picked up in (66) and

$$-iG(k\sigma,t)=\mathrm{e}^{-i\mu t}\int_0^{-i\alpha}\mathrm{d}\omega A_+(k\sigma,\omega)\mathrm{e}^{-i\omega t}+z_k\mathrm{e}^{-i\xi_k t}\mathrm{e}^{-\Gamma_k t}\,.\tag{67}$$

For large t such that $\alpha t \gg 1$ the $\exp(-i\omega t)$ factor in (67) oscillates rapidly with frequency causing destructive interference and the contribution from the integral in (67) will be small. Provided t remains less than $1/\Gamma_k$ then $\alpha \gg \Gamma_k$ and we will pick up the residue of the pole giving

$$-iG(k\sigma, t) = z_k e^{-i\xi_k t} e^{-\Gamma_k t}\,. \tag{68}$$

When t becomes so large that $t \gg 1/\Gamma_k$ the constant α can be chosen smaller than Γ_k leaving no contribution $\mathcal{R}$ from the pole. By this time the quasiparticle excitation has decayed completely away and the entire spectral strength of $G(k\sigma, t)$ is located in the smooth background.

3.3 Quasiparticles and self-energies

In perturbation expansions the effect of interactions on a single-particle Green's function $G(k\sigma, \omega)$ can be completely absorbed into the self-energy correction $\Sigma(k\sigma, \omega)$. The Dyson equation gives the relation between $G(k\sigma, \omega)$ for the interacting system and the non-interacting $G_0(k\sigma, \omega) = [(k^2/2m) - \omega - i0^{\pm}]^{-1}$ in terms of $\Sigma(k\sigma, \omega)$,

$$G(k\sigma, \omega) = G_0(k\sigma, \omega) + G_0(k\sigma, \omega)\Sigma(k\sigma, \omega)G(k\sigma, \omega)\,. \tag{69}$$

We now show that the quasiparticle energy ξ_k, its lifetime $1/\Gamma_k$ and its spectral strength z_k can be related to $\Sigma(k\sigma, \omega)$ (Brown 1972; Negele and Orland 1988).

The solution of (69) is

$$\begin{aligned} G(k\sigma, \omega) &= \frac{G_0(k\sigma, \omega)}{1 - \Sigma(k\sigma, \omega)G_0(k\sigma, \omega)} \\ &= \frac{1}{(k^2/2m) - \omega - \mu - \Sigma(k\sigma, \omega)}\,. \end{aligned} \tag{70}$$

If $G(k\sigma, \omega)$ contains a quasiparticle pole its energy will be at $\xi_k = (k^2/2m) - \Re e\ \Sigma(k\sigma, \xi_k)$ and its half-width will be $\Gamma_k = -\Im m\ \Sigma(k\sigma, \xi_k)$.

We can show in a finite perturbation expansion that $\Im m\ \Sigma(k\sigma, \omega)$ will vanish as ω^2 and that it will change sign at the Fermi surface as follows. Near the Fermi surface the dominant decay channel for a quasiparticle is into two quasiparticles, call them p_a and p_b, and a quasihole h. Denoting by $|V_{\max}|$ a finite upper bound on the strength of the interaction within the phase space available for scattering we can then write an upper limit for $\Im m\ \Sigma(k\sigma, \omega)$,

$$\Im m\ \Sigma(k\sigma, \omega) \leq \sum_{p_a p_b h} V_{\max}^2 2\pi\delta(\omega - \xi_{p_a} - \xi_{p_b} + \xi_h)\,, \tag{71}$$

where $\xi_{p_a} > 0$, $\xi_{p_b} > 0$ and $\xi_h < 0$ are the particle and hole energies of the final states. Energy conservation requires that ξ_{p_a}, ξ_{p_b} and $|\xi_h|$ are all less than ω. Then we have

$$\Im\Sigma(k\sigma,\omega) \leq V_{\max}^2{}^2\pi \int_0^\infty \mathrm{d}\xi_{p1} \int_0^\infty \mathrm{d}\xi_{p2} \int_0^{-\infty} \mathrm{d}\xi_{h1}$$

$$\times N(\xi_{p1})N(\xi_{p2})N(\xi_{h1})\delta(\omega - \xi_{p1} - \xi_{p2} + \xi_{h1})$$

$$= V_{\max}^2 2\pi \int_0^\omega \mathrm{d}\xi_{p1} \int_0^\omega \mathrm{d}\xi_{p2} N(\xi_{p1})N(\xi_{p2})N(\xi_{p1} + \xi_{p2} - \omega)\,. \tag{72}$$

Introducing $N_{\max}$ as an upper bound on the density of $N(\xi)$ in the range $0 \leq \xi \leq \omega$ we get

$$\Im m\ \Sigma(k\sigma,\omega) \leq V_{\max}^2 2\pi N_{\max}^3 \omega^2\,. \tag{73}$$

Thus, provided the interaction strength and the density of states remain finite in the vicinity of the Fermi surface, $\Im m\ \Sigma(k\sigma,\omega)$ is bounded by a constant times ω^2 and the quasiparticle lifetime diverges at the Fermi surface as ω goes to zero. A similiar argument for a quasihole produces an inequality on the value for $\Im m\ \Sigma(k\sigma,\omega)$ just below the Fermi surface but with a change of sign so that we have

$$\Im m\ \Sigma(k\sigma,\omega) \sim \mathrm{sign}(\omega)\omega^2\,. \tag{74}$$

$\Im m\ \Sigma(k\sigma,\omega)$ passes through zero at the Fermi surface and from (70) this implies that the pole in $G(k\sigma,\omega)$ crosses the real ω axis as ω passes through the Fermi surface.

The spectral strength z_k of the quasiparticle peak is determined by the residue of the pole in (70). Near the Fermi surface

$$z_k = \left.\frac{1}{1 + \partial\Sigma/\partial\omega}\right|_{\omega=\xi_k}\,. \tag{75}$$

Since $\Re e\ \Sigma(k\sigma,\omega)$ changes from a negative to a positive sign when ω passes through the Fermi energy its gradient must be positive there. Provided $\Re e\ \Sigma(k\sigma,\omega)$ is continuous (75) gives the inequality $0 < z_{k_F} \leq 1$.

3.4 Quasiparticle propagator

We have noted that the state $a_{k\sigma}^\dagger(0)|\Psi_0^N\rangle$ differs from a quasiparticle excitation because of the spectral contributions from the background of excitations with two or more particles. We had to filter out this background in order to isolate the quasiparticle state. Refering to (64), we want to confine the frequency integration to a small interval centred on the quasiparticle peak in $A_+(k\sigma,\omega)$,

$$A_+(k\sigma,\omega) = \left(\frac{z_k}{\pi}\right)\frac{\Gamma_k}{\Gamma_k^2+(\xi_k-\omega)^2}\,. \tag{76}$$

Suppose we can construct a state $|\psi_{k\sigma}\rangle$ that approximates a quasiparticle $k\sigma$ added to the N-particle ground state $|\Psi_0^N\rangle$ (Nozières 1964; Jones and March 1973). We can then introduce a creation operator $q_{k\sigma}^\dagger$ which adds just the quasiparticle $k\sigma$ to the ground state, $|\psi_{k\sigma}\rangle = q_{k\sigma}^\dagger|\Psi_0^N\rangle$. From the discussion in the preceding section the propagator for the quasiparticle fluctuation from the ground state should for $t>0$ be of the form,

$$\langle\Psi_0^N|q_{k\sigma}(t)q_{k\sigma}^\dagger(0)|\Psi_0^N\rangle = i\mathrm{e}^{-i\mu t}\int_0^\infty \mathrm{d}\omega A_+(k\sigma,\omega)\mathrm{e}^{-i\omega t}\left[\frac{\alpha^2}{\alpha^2+(\xi_k-\omega)^2}\right], \tag{77}$$

where we have introduced the function $\alpha^2/[\alpha^2+(\xi_k-\omega)^2]$ as a filter to pick out frequencies centred on $\omega=\xi_k$ with a passband width of α. This filter differentiates $q_{k\sigma}^\dagger$ from $a_{k\sigma}^\dagger$ (see equation 64). By keeping α as small as possible we eliminate most of the background. In the limit α goes to zero the filter would become the delta function $\delta(\xi_k-\omega)$ but we cannot actually take this limit since $A_+(k\sigma,\omega)$ has a half-width of Γ_k. We must keep the passband width $\alpha>\Gamma_k$ in order to encompass the entire peak.

We construct the states $|\psi_{k\sigma}\rangle$ and $\langle\psi_{k\sigma}|$ in such a way that we recover (77). We define $|\psi_{k\sigma}\rangle = q_{k\sigma}^\dagger|\Psi_0^N\rangle$ as

$$|\psi_{k\sigma}\rangle = \frac{\alpha}{\sqrt{z_k}}\int_{-\infty}^0 \mathrm{d}t'\mathrm{e}^{-i\xi_k t'}\mathrm{e}^{\alpha t'}a_{k\sigma}^\dagger(t')|\Psi_0^N\rangle, \tag{78}$$

and $\langle\psi_{k\sigma}| = \langle\psi_0^N|q_{k\sigma}$ as

$$\langle\psi_{k\sigma}| = \frac{\alpha}{\sqrt{z_k}}\int_0^\infty \mathrm{d}t''\langle\Psi_0^N|a_{k\sigma}(t'')\mathrm{e}^{i\xi_k t''}\mathrm{e}^{-\alpha t''}\,. \tag{79}$$

The factor $\alpha/\sqrt{z_k}$ normalises the states. In the Heisenberg picture $q_{k\sigma}(t) = \exp(i\mathcal{H}t)q_{k\sigma}\exp(-i\mathcal{H}t)$, and the propagator is

$$\langle\Psi_0^N|q_{k\sigma}(t)q_{k\sigma}^\dagger(0)|\Psi_0^N\rangle = \frac{\alpha^2}{z_k}\int_0^\infty \mathrm{d}t''\mathrm{e}^{i\xi_k t''}\mathrm{e}^{-\alpha t''}\int_{-\infty}^0 \mathrm{d}t'\mathrm{e}^{-i\xi_k t'}\mathrm{e}^{\alpha t'}$$
$$\times\,\langle\Psi_0^N|a_{k\sigma}(t''+t)a_{k\sigma}^\dagger(t')|\Psi_0^N\rangle\,. \tag{80}$$

Since we have taken care to construct $t''>t'$ the ground state expectation value on the right-hand side of (80) is Green's function $G(k\sigma,t''+t-t')$ for $t>0$.

Using equation (57) we can carry out the time integrations recovering the form we want (equation 77). If the peak part of $A_+(k\sigma,\omega)$ transmitted by the frequency filter is given by (76) then substituting this we get

$$\langle\Psi_0^N|q_{k\sigma}(t)q_{k\sigma}^\dagger(0)|\Psi_0^N\rangle i\mathrm{e}^{-i\mu t}\int_0^\infty \mathrm{d}\omega\frac{z_k}{\pi}\frac{\Gamma_k}{\Gamma_k^2+(\xi_k-\omega)^2}\mathrm{e}^{-i\omega t}\times\frac{1}{z_k}\frac{\alpha^2}{\alpha^2+(\xi_k-\omega)^2}$$
$$=\mathrm{e}^{-i\xi_k t}\mathrm{e}^{-\Gamma_k t}\,. \tag{81}$$

For times $t < 1/\Gamma_k$ the state $q_{k\sigma}^\dagger(t)|\Psi_0^N\rangle$ thus acts as an eigenstate of the interacting system.

3.5 Excited state Green's function

Let us now calculate the Green's function propagator for a bare particle in the quasiparticle excited state $|\psi_{k\sigma}\rangle$

$$G_{k\sigma}(k'\sigma',t) = i\langle\psi_{k\sigma}|\mathcal{T}\left\{a_{k\sigma}(t)a_{k\sigma}^\dagger(0)\right\}|\psi_{k\sigma}\rangle\,. \tag{82}$$

With the definitions for $\langle\psi_{k\sigma}|$ and $|\psi_{k\sigma}\rangle$ given by (78) and (79) $G_{k\sigma}(k'\sigma',t)$ is the ground state expectation value of four single-particle creation and annihilation operators. It closely resembles the two-particle Green's function for the ground state $|\Psi_0^N\rangle$,

$$K(k\sigma,t'';k'\sigma',t;k'\sigma',0;k\sigma,t') = \langle\Psi_0^N|\mathcal{T}\left\{a_{k\sigma}(t'')a_{k'\sigma'}(t)a_{k'\sigma'}^\dagger(0)a_{k\sigma}^\dagger(t')\right\}|\Psi_0^N\rangle\,, \tag{83}$$

but it is not identical since in (82) only two of the four creation and annihilation operators are acted upon by the time ordering operator $\mathcal{T}$. This can be remedied by shifting time scales in the definitions (78) and (79). Let us redefine

$$|\psi_k\rangle = \frac{\alpha}{\sqrt{z_k}}\int_{-\infty}^0 \mathrm{d}t'\mathrm{e}^{-i\xi_k(t'+\tau')}\mathrm{e}^{\alpha t'}a_{k\sigma}^\dagger(t'+\tau')|\Psi_0^N\rangle,$$

$$\langle\psi_k| = \frac{\alpha}{\sqrt{z_k}}\int_0^\infty \mathrm{d}t''\langle\Psi_0^N|a_{k\sigma}(t''+\tau'')\mathrm{e}^{i\xi_k(t''+\tau'')}\mathrm{e}^{-\alpha t''}\,, \tag{84}$$

where τ' and τ'' are constants. This does not affect the construction of the frequency filter in equation (77) and so (84) is an equally acceptable definition. Choosing the values of the constants $\tau' \ll 0$ and $\tau'' \gg t$ so as to bracket the time interval from 0 to t then, with the help of the convergence factors $\exp(-\alpha t)$, the four operators in (82) are by construction correctly time ordered and

$$G_{k\sigma}(k'\sigma',t) = \frac{i\alpha^2}{z_k}\int_{-\infty}^0 \mathrm{d}t'\int_0^\infty \mathrm{d}t''\mathrm{e}^{-i\xi_k(t''+\tau''-t'-\tau')}\mathrm{e}^{\alpha(t'-t'')}$$
$$\times K(k\sigma,t''+\tau'';k'\sigma',t;k'\sigma',0;k\sigma,t'+\tau')\,. \tag{85}$$

Equation (85) provides the connection between Green's function for the excited state $|\psi_k\rangle$ and the two-particle Green's function for the ground state $|\Psi_0^N\rangle$. Separating out the non-interacting part of $K(k\sigma, t''+\tau''; k'\sigma', t; k'\sigma', 0; k\sigma, t'+\tau')$ which represents the free propagation of the two particles from the interacting part $\Delta K(k\sigma, t''+\tau''; k'\sigma', t; k'\sigma', 0; k\sigma, t'+\tau')$ we write

$$K(k\sigma, t''+\tau''; k'\sigma', t; k'\sigma', 0; k\sigma, t'+\tau')$$

$$= -G(k\sigma, t''+\tau''-t'-\tau')G(k'\sigma', t)$$

$$+ G(k\sigma, t''+\tau'')G(k\sigma, t-t'-\tau')\delta_{kk',\sigma\sigma'}$$

$$+ \Delta K(k\sigma, t''+\tau''; k'\sigma', t; k'\sigma', 0; k\sigma, t'+\tau') . \tag{86}$$

The second term on the right-hand side is the exchange term in which we interchange the two incoming free particle propagators when they are in identical states.

In the interaction term $\Delta K(k\sigma, t''+\tau''; k'\sigma', t; k'\sigma', 0; k\sigma, t'+\tau')$ the two particles must be within range of the interaction, so in (86) this term will be smaller by a factor of $1/\Omega$ compared with the non-interacting part where both particles can propagate independently throughout the volume Ω. Thus to leading order (85) is

$$G_{k\sigma}(k'\sigma', t) = \frac{i\alpha^2}{z_k}\int_{-\infty}^{0} \mathrm{d}t' \int_0^\infty \mathrm{d}t'' \mathrm{e}^{-i\xi_k(t''+\tau''-t'-\tau')}\mathrm{e}^{\alpha(t'-t'')}$$

$$\times \left[-G(k\sigma, t''+\tau''-t'-\tau')G(k'\sigma', t)\right.$$

$$\left.+ G(k\sigma, t''+\tau'')G(k\sigma, t-\tau'-t')\delta_{kk',\sigma\sigma'}\right] . \tag{87}$$

The first term on the right hand side is simply $G(k'\sigma', t)$ because the quasiparticle wave functions are normalised

$$\frac{i\alpha^2}{z_k}\int_{-\infty}^{0} \mathrm{d}t' \int_0^\infty \mathrm{d}t'' \mathrm{e}^{-i\xi_k(t''+\tau''-t'-\tau')}\mathrm{e}^{\alpha(t'-t'')}$$

$$\times G(k\sigma, t''+\tau''-t'-\tau')G(k'\sigma', t)$$

$$= \langle\psi_k|\psi_k\rangle \times \; G(k'\sigma', t)$$

$$= G(k'\sigma', t) , \tag{88}$$

so for $k\sigma \neq k'\sigma'$ $G_{k\sigma}(k'\sigma', t)$ and $G(k'\sigma', t)$ are equal to leading order. For $k\sigma = k'\sigma'$ the exchange term in (87) also contributes. Because of the $\exp(-\alpha t)$

convergence factors we can replace both the single-particle Green's functions under the integrations by their quasiparticle parts,

$$G(k\sigma, t) = \begin{cases} iz_k e^{-i\xi_k t} e^{-\Gamma_k t} & t > 0 \\ 0 & t < 0, \end{cases} \tag{89}$$

so that equation (87) becomes

$$G_{k\sigma}(k'\sigma', t) = G(k'\sigma', t) - iz_k e^{-i\xi_k t} e^{-\Gamma_k t} \delta_{kk', \sigma\sigma'} \tag{90}$$

Thus to leading order the $G_{k\sigma}(k'\sigma', t)$ and $G(k'\sigma', t)$ differ only by the additional contribution from the quasiparticle pole for $k\sigma = k'\sigma'$. With $k\sigma = k'\sigma'$ (89) and (90) give for times $|t| < 1/\Gamma_k$,

$$G_{k\sigma}(k\sigma, t) = \begin{cases} 0 & t > 0 \\ -iz_k e^{-i\xi_k t} e^{-\Gamma_k t} & t < 0, \end{cases} \tag{91}$$

and so $G_{k\sigma}(k\sigma, \omega)$ has a pole in the upper half of the complex ω plane while $G(k\sigma, \omega)$ had its pole in the lower half plane. This means that the excitation $k\sigma$ is a quasiparticle for $G(k\sigma, \omega)$ but a quasihole for $G_{k\sigma}(k\sigma, \omega)$. The effect comes from the fermion nature of quasiparticles which requires that, if there is already an excitation $k\sigma$ out of the ground state, then a further excitation $k\sigma$ can only be a quasihole which takes the system from $|\psi_k\rangle$ and returns it to the ground state.

From (91) the change in the bare particle distribution function when the quasiparticle $k\sigma$ is added is

$$\begin{aligned} \Delta m_{k\sigma} &= i[G_{k\sigma}(k\sigma, t = 0^-) - G(k\sigma, t = 0^-)] \\ &= z_{k\sigma} \, . \end{aligned} \tag{92}$$

Thus $z_{k\sigma}$ represents the fraction of the bare particle $k\sigma$ which is contained in the quasiparticle $k\sigma$. By analogy with (60) we write

$$G_{k\sigma}(k\sigma, t = 0^-) = \frac{1}{2\pi} \oint_C \mathrm{d}\omega G_{k\sigma}(k\sigma, \omega) \, , \tag{93}$$

closing the contour C in the upper half of the complex ω plane. When k passes down through k_F and the excitations change from quasiparticles to quasiholes the pole in $G_{k\sigma}(k\sigma, \omega)$ crosses the real axis. When this happens the pole is no longer within the contour and the value of $m_{k\sigma}$ will drop by a discontinuous amount equal to z_{k_F}.

3.6 Microscopic expression for the quasiparticle interaction energy

To leading order we have established that $G_{k\sigma}(k'\sigma', t)$ and $G(k'\sigma', t)$ are equal for $k\sigma \neq k'\sigma'$ (equation 90). To obtain a microscopic interpretation of the quasiparticle interaction energy (Nozières 1964; Jones and March 1973) we must retain the next higher order correction terms. In (89) the addition of the quasiparticle $k\sigma$ with $k\sigma \neq k'\sigma'$ changes $z_{k'}$ and $\xi_{k'}$ in $G(k'\sigma', t)$. The change in energy $\xi_{k'}$ as a result of the addition of the quasiparticle $k\sigma$ is by definition $f(k'\sigma', k\sigma)$ (see equation 6). Denoting the change in $z_{k'}$ as $\delta_k(z_{k'})$, the difference between the Green's functions for $k\sigma \neq k'\sigma'$ is

$$G_{k\sigma}(k'\sigma', t) - G(k'\sigma', t) = i[z_{k'} + \delta_k(z_{k'})]\mathrm{e}^{-i(\xi_{k'}+f(k\sigma,k'\sigma'))t}\mathrm{e}^{-\Gamma_{k'}t} - iz_{k'}\mathrm{e}^{-i\xi_{k'}t}\mathrm{e}^{-\Gamma_{k'}t}. \tag{94}$$

Since both $f(k\sigma, k'\sigma')$ and $\delta_k(z_{k'})$ are of order $1/\Omega$ the difference to leading order is

$$G_{k\sigma}(k'\sigma', t) - G(k'\sigma', t) = [z_{k'}f(k\sigma, k'\sigma')t + i\delta_k(z_{k'})]\mathrm{e}^{-i\xi_{k'}t}\mathrm{e}^{-\Gamma_{k'}t}. \tag{95}$$

For large t we can neglect the $\delta_k(z_{k'})$ term and solve for $f(k\sigma, k'\sigma')$,

$$f(k\sigma, k'\sigma') = \frac{1}{z_{k'}t}\left[G_{k\sigma}(k'\sigma', t) - G(k'\sigma', t)\right]\mathrm{e}^{i\xi_{k'}t}\mathrm{e}^{\Gamma_{k'}t}. \tag{96}$$

Using equations (85) to (88) we see that contributions to equation (96) come only from the interaction part of the two-particle Green's function, so that

$$f(k\sigma, k'\sigma') = \frac{i\alpha^2}{z_k z_{k'}t}\int_{-\infty}^{0}\mathrm{d}t'\int_{0}^{\infty}\mathrm{d}t''\mathrm{e}^{-i\xi_k(t''+\tau''-t'-\tau')}\mathrm{e}^{\alpha(t'-t'')}\mathrm{e}^{i\xi_{k'}t}\mathrm{e}^{\Gamma_{k'}t} \times \Delta K[k\sigma, (t''+\tau''); k'\sigma', t; k'\sigma', 0; k\sigma, (t'+\tau')]. \tag{97}$$

Equation (97) provides the formal microscopic expression relating $f(k\sigma, k'\sigma')$ to the two-body Green's function for the ground state.

Since ΔK contains only the interacting part of the two-particle Green's function we can directly relate $f(k\sigma, k'\sigma')$ to the irreducible two-particle scattering amplitude γ, which is defined schematically by

$$\Delta K = GG\gamma GG. \tag{98}$$

Using the $\exp(-\alpha t)$ convergence factors in (97) to replace the external propagators by their quasiparticle parts we obtain,

$$\begin{aligned}
&\Delta K[k\sigma, (t''+\tau''); k'\sigma', t; k'\sigma', 0; k\sigma, (t'+\tau')] \\
&\equiv \int \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4[G(k\sigma, t''+\tau''-t_1)G(k'\sigma', t-t_2)] \qquad (99)\\
&\times\gamma(k\sigma t_1, k'\sigma' t_2, k'\sigma' t_3, k\sigma t_4) \times [G(k'\sigma', -t_3)G(k\sigma, t'+\tau'-t_4)] \\
&\approx z_k^2 z_{k'}^2 \int \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4[\mathrm{e}^{-i\xi_k(t''+\tau''-t_1+t'+\tau'-t_4)}\mathrm{e}^{-i\xi_{k'}(t-t_2-t_3)}] \\
&\quad\times\int \mathrm{d}\omega_1\mathrm{d}\omega_2\mathrm{d}\omega_3\mathrm{d}\omega_4\mathrm{e}^{-i\omega_1 t_1}\mathrm{e}^{-i\omega_2 t_2}\mathrm{e}^{-i\omega_3 t_3}\mathrm{e}^{-i\omega_4 t_4} \\
&\times\Gamma(k\sigma\omega_1, k'\sigma'\omega_2, k'\sigma'\omega_3, k\sigma\omega_4)\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta_{k+k';k'+k}. \qquad (100)
\end{aligned}$$

For the vertex function Γ the energy and momentum conservation delta functions shared by ΔK and γ have been made explicit,

$$\gamma(k\sigma\omega_1, k'\sigma'\omega_2, k'\sigma'\omega_3, k\sigma\omega_4)$$
$$= \Gamma(k\sigma\omega_1, k'\sigma'\omega_2, k'\sigma'\omega_3, k\sigma\omega_4) \times [\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)\delta_{k+k';k'+k}] . \quad (101)$$

Substituting (100) into (97) we get

$$f(k\sigma, k'\sigma') = 2\pi i z_k z_{k'} \Gamma(k\sigma\xi_k, k'\sigma'\xi_{k'}, k'\sigma'\xi_{k'}, k\sigma\xi_k) . \quad (102)$$

Thus the quasiparticle energy $f(k\sigma, k'\sigma')$ is proportional to $\Gamma(k\sigma\xi_k, k'\sigma'\xi_{k'}, k'\sigma'\xi_{k'}, k\sigma\xi_k)$, the irreducible forward scattering amplitude of two quasiparticles on the Fermi surface with the four external propagators removed.

4. Marginal Fermi Liquid Theory

The copper-oxide based metals exhibiting high transition temperatures to a superconducting state have in their normal state some peculiar non-Landau Fermi liquid-like properties. In an effort to understand these from a phenomenological point of view a theory has been proposed lying between Landau Fermi liquid theory and theories for systems of localised excitations. Marginal Fermi liquid theory (Varma 1991) retains a Fermi surface but the lifetimes of the low lying excited states do not diverge as the energy ω goes to zero. Instead when ω gets very small the thermal energy T replaces the Fermi energy as the applicable energy scale and the quasiparticle lifetimes remain finite for non-zero temperatures.

The polarisability in the Landau Fermi liquid theory is

$$\chi(\mathbf{q}, \omega) = \frac{\chi^0(\mathbf{q}, \omega)}{1 + f(\mathbf{k}, \mathbf{k}')\chi^0(\mathbf{q}, \omega)} . \quad (103)$$

In the limit of small $\mathbf{q}$ the Lindhard function $\chi^0(\mathbf{q}, \omega)$ for the quasiparticles is

$$\Re e\ \chi^0(\mathbf{q}, \omega)|_{\lim_{\mathbf{q}\to 0}} = z_{k_F}^2 \frac{m^\star k_F}{2\pi^2}\left[-1 + \frac{\omega}{v_F q}\log\left|\frac{v_F q + \omega}{v_F q - \omega}\right|\right] ,$$

$$\Im m\ \chi^0(\mathbf{q}, \omega)|_{\lim_{\mathbf{q}\to 0}} = \begin{cases} 0 & \omega > q v_F \\ -z_{k_F}^2 \dfrac{\pi N_F}{2}\left(\dfrac{\omega}{q v_F}\right) & \omega < q v_F . \end{cases} \quad (104)$$

Replacing the quasiparticle interaction $f(k, k')$ in (103) by the first Landau parameter f_0^s we obtain

$$\Im m\ \chi(\mathbf{q}, \omega) = \begin{cases} 0 & \omega > q v_F \\ -\dfrac{z_{k_F}^2 \pi N_F}{(2 + z_{k_F}^2 F_0^s)^2} \times \left(\dfrac{\omega}{q v_F}\right) & \omega < q v_F . \end{cases} \quad (105)$$

For systems with excitations that are spatially localised there is little or no $\mathbf{q}$ dependence in $\chi(\mathbf{q},\omega)$ and the low energy scale is controlled by the external temperature rather than by any intrinsic scale implied by the Hamiltonian,

$$\Im m\ \chi(\mathbf{q},\omega) \sim \frac{\omega}{T}. \tag{106}$$

In marginal Fermi liquid theory the proposed form for the imaginary part of the polarisability at small ω is

$$\Im m\ \chi(\mathbf{q},\omega) \sim \begin{cases} -N_F\frac{\omega}{T} & \omega \ll T \\ -N_F & T < \omega \ll \omega_c, \end{cases}. \tag{107}$$

where ω_c is some cut-off energy, the exact value of which is not important for small ω.

The corresponding real and imaginary parts of the self-energy are

$$\Re e\ \Sigma(\omega) = \lambda\omega \log\left[\frac{\max(|\omega|,T)}{\omega_c}\right],$$

$$\Im m\ \Sigma(\omega) = \mathrm{sign}(\omega)\lambda\frac{\pi}{2} \times \max(|\omega|,T)\,, \tag{108}$$

where λ is a coupling constant. Here $\Im m\ \Sigma(\omega)$ changes sign at $\omega = 0$ so that quasiparticles still change to quasiholes at the Fermi surface. However, since for non-zero temperatures the change in sign is discontinuous with a jump of $\lambda\pi T$ the lifetimes remain finite at the Fermi surface. This result can be contrasted with the Landau Fermi liquid theory expression (74) which gives inverse lifetimes that vanish quadratically with ω near the Fermi surface.

Equation (108) can be used to determine a spectral strength for the quasiparticles,

$$z_k = \frac{1}{1-\lambda\log\{\max(\epsilon k, T/\omega_e\}}. \tag{109}$$

At zero temperature the spectral weight vanishes logarithmically as k approaches the Fermi surface and ξ_k goes to zero, but at non-zero temperatures z_k is bounded from below and z_{k_F} remains finite. The very existence of quasiparticles depends on the temperature making them truly marginal.

Acknowledgments

I thank M. P. Das for his insightful comments on the marginal Fermi liquid theory and F. Bassani for the hospitality and facilities of the Scuola Normale Superiore where this article was written.

References

Abrikosov, A. A., Gorkov, L. P., and Dzyaloshinski, I. E. (1963). 'Methods of Quantum Field Theory in Statistical Physics', Chap. 4 (Prentice Hall: New Jersey).

Baym, G., and Pethick, C. J. (1991). 'Landau Fermi Liquid Theory, Concepts and Applications' (Wiley: New York).

Bedell, K. S. (1994). *In* 'Modern Perspectives on Many Body Physics' (Eds M. P. Das and J. Mahanty) (World Scientific: Singapore).
Brown, G. E. (1972). 'Many Body Problems' (North Holland: Amsterdam).
Jones, W., and March, N. H. (1973). 'Theoretical Solid State Physics', Vol. 1, Chap. 6 (Wiley–Interscience: London).
Landau, L. D. (1957). *Sov. Phys. JETP* **3**, 920; *ibid.* **5**, 101 (1957).
Negele, J., and Orland, H. (1988). 'Quantum Many Particle Systems', Chap. 6 (Addison–Wesley: New York).
Nozières, P. (1964). 'Theory of Interacting Fermi Systems' (Benjamin: New York).
Varma, C. M. (1991). *J. Phys. Chem. Solids* **52**, 1349.

Manuscript received 1 August, accepted 29 August 1995