Landau Level Ground–State Degeneracy, and Its Relevance for a General Quantization Procedure

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ABSTRACT

The quantum dynamics of a two-dimensional charged spin 1/2 particle is studied for general, symmetry–free curved surfaces and general, nonuniform magnetic fields that are, when different from zero, orthogonal to the defining two surface. Although higher Landau levels generally lose their degeneracy under such general conditions, the lowest Landau level, the ground state, remains degenerate. Previous discussions of this problem have had less generality and/or used supersymmetry, or else have appealed to very general mathematical theorems from differential geometry. In contrast our discussion relies on simple and standard quantum mechanical concepts.

The mathematical similarity of the physical problem at hand and that of a phase-space path integral quantization scheme of a general classical system is emphasized. Adopting this analogy in the general case leads to a general quantization procedure that is invariant under general coordinate transformations – completely unlike any of the conventional quantization prescriptions – and therefore generalizes the concept of quantization to new and hitherto inaccesible situations.

In a complementary fashion , the so-obtained picture of general quantization helps to derive useful semiclassical formulas for the Hall current in the case of a filling factor equal to one for a general surface and magnetic field.

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1. INTRODUCTION AND SUMMARY

For nonrelativistic electrons endowed with their usual spin magnetic moment (i.e., $g_B = 2$) motion in a two-dimensional plane perpendicular to a homogeneous magnetic field has a number of interesting properties. Without taking the spin contribution into account the energy levels of a free particle split into the degenerate Landau levels endowed with the sequence of energy eigenvalues $E_n = (n + \frac{1}{2})\hbar\omega_c, n = 0, 1, 2, ...$, where $\omega_c = eB/mc$. When the spin is included each level splits with half the states rising in energy and the other half falling in energy. Thanks to a proper magnetic moment ($g_B = 2$) those levels that rise exactly overlap with those levels that fall from the next higher Landau level leading to combined energy values given by $E_n = n\hbar\omega_c, n = 0, 1, 2, ...$. While all levels but the lowest contain spin up and spin down states the lowest level consists only of spin down states and has exactly zero energy for any value of ω_c. It is common to regard the level degeneracy as due to translational symmetry, and for all but the lowest Landau level this viewpoint is correct. For the lowest Landau level, however, an additional symmetry applies that preserves the degeneracy even under circumstances where the degeneracy of the higher levels is lifted. As we shall see the circumstances for which degeneracy of the ground state remains are exceptionally broad including cases where the magnetic field is not uniform in strengh as well as cases where a (non)uniform magnetic field is everywhere orthogonal to a two surface that does not have constant curvature. A surface of constant curvature such as the plane (zero curvature) or the sphere (positive curvature) is necessary to have degeneracy of the higher levels, but a generally symmetry-less surface – loosely referred to as a "potato", as may arise by deforming a sphere – even in the presence of a nonuniform magnetic field, maintains degeneracy of the lowest Landau level.

The existence of a degenerate ground state for electrons moving in the presence of nonuniform magnetic field everywhere perpendicular to a (compactified) plane has been known for some time[1]; a compactified plane arises due to periodic boundary conditions, or, effectively, when the magnetic field vanishes outside some compact region. These properties have been demonstrated using methods of supersymmetric quantum mechanics applied to underlying plane surfaces[2]. Recently, the degeneracy of the ground state has been extended to cases of a nonuniform magnetic field everywhere perpendicular to a general, compact, symmetry-free underlying surface[3]. The methods entailed in this proof used contemporary techniques in differential geometry. In this paper we demonstrate that straightforward techniques of nonrelativistic quantum mechanics are sufficient for this more general situation as well.

1.1 Euclidean Path Integral

Although our method of proof will involve partial differential equations, we wish to present our basic results in the form of path integrals. The purpose behind this form of presentation is twofold: on one hand, path integrals involve a functional formulation that is manifestly close in formal appearence to the underlying classical theory; and, on the other hand, the ultimate expressions may be given a form that makes manifest their covariance under coordinate transformations. This feature will be of considerable interest when attention is turned to a mathematical analog system, namely, that of a phase- space path integral for a general classical Hamiltonian which is at once rigorous in its formulation, and, simultaneously, covariant under general coordinate transformations. However, more on the analog system later (see Sec.1.4).

Consider, initially, a charged spin-1/2 particle moving on a two-dimensional plane and subject to a uniform magnetic field perpendicular to the plane. We assume also that the spin is polarized along the magnetic field. The Euclidean space path integral for the propagator is given, in a convenient gauge, by the formal expression ($\hbar = 1, g_B = 2$)

$$\mathcal{N} \int \exp\Big\{ i \int (m\omega_c y\dot{x})dt \ - \ \frac{1}{2}\int [m(\dot{x}^2 + \dot{y}^2) - \omega_c]dt \Big\} \Pi dxdy \ . \tag{1.1}$$

The propagator represents the matrix element

$$< x'', y''|e^{-HT}|x', y' > \ , \tag{1.2}$$

where H has a spectrum given by $n\omega_c, n = 0, 1, 2, ...$ Let us next take the limit $\omega_c \to \infty$; theoretically we can do so by letting $m \to 0$, while empirically such a limit is approached by choosing large magnetic fields. The result of such a limit is the matrix elements of a *projection operator*,

$$\lim_{m\to 0} < x'', y''|e^{-HT}|x', y' > = < x'', y''|\Pi|x', y' > \ . \tag{1.3}$$

In the present case the explicit form is easily worked out, and one finds that

$$\begin{aligned} < x'', y''|\Pi|x', y' > = \ & (eB/2\pi c)\exp\{\frac{i}{2}(eB/c)(y'' + y')(x'' - x') \\ & -\frac{1}{4}(eB/c)[(y'' - y')^2 + (x'' - x')^2]\} \ . \end{aligned} \tag{1.4}$$

It is readily verified that this expression represents the integral kernel of a projection operator. The rank of the projection operator Π – which equals the degeneracy of the lowest Landau level – is given in turn by $\int < x, y|\Pi|x, y > dxdy$, which diverges in the present case.

It is also useful to consider the matrix elements of the projection operator somewhat more abstractly. To this end we introduce the notation $\mathcal{K}(x'', y''; x', y')$ instead of $< x'', y''|\Pi|x', y' >$, and observe that for $\mathcal{K}$ to represent a projection operator it is necessary and sufficient that $\mathcal{K}^*(x'', y''; x', y') = \mathcal{K}(x', y'; x'', y'')$ and $\mathcal{K}(x''', y'''; x', y') = \int \mathcal{K}(x''', y'''; x'', y'')\mathcal{K}(x'', y''; x'y')dx''dy''$. When these conditions are satisfied then the rank of the so-determined projection operator is given by $\int \mathcal{K}(x, y; x, y)dxdy$.

Let us generalize our physical situation so that the electron moves in the presence of a local potential $V(x, y)$ as well as the uniform magnetic field. However, we do not Euclideanize the potential, only the kinetic term, so the expression of interest is represented by the formal path integral

$$\mathcal{N} \int \exp\Big\{i \int [m\omega_c y\dot{x} - V(x, y)]dt \ - \ \frac{1}{2} \int [m(\dot{x}^2 + \dot{y}^2) - \omega_c]dt\Big\}\Pi dxdy \ . \qquad (1.5)$$

In the limit that $m \to 0$ we still expect that the Hilbert space collapses to the lowest Landau level, but in general the result is no longer a projection operator. Instead, there is a dynamical evolution generated by the hermitian Hamiltonian which is determined by an integral kernel that is given by[4]

$$\mathcal{H}(x'', y''; x', y') = \int \mathcal{K}(x'', y''; x, y)V(x, y)\mathcal{K}(x, y; x', y')dxdy \ . \qquad (1.6)$$

In words, the Hamiltonian is given by the two-sided projection of the potential V onto the lowest Landau level. We denote the ultimate limit (a unitary propagator for the lowest Landau level at zero mass or high magnetic field limit) by

$$K(x'', y'', t''; x', y', t') \ = < x'', y''|e^{-i\mathcal{H}T}|x', y' > =$$

$$\lim_{m\to 0} \mathcal{N} \int \exp\Big\{i \int [m\omega_c y\dot{x} - V(x, y)]dt \ - \ \frac{1}{2} \int [m(\dot{x}^2 + \dot{y}^2) - \omega_c]dt\Big\}\Pi dxdy \ , \qquad (1.7)$$

where $T = t'' - t' > 0$. Additionally, it follows that

$$\lim_{t''\to t'} K(x'',y'',t'';x',y',t') \;=\; \mathcal{K}(x'',y'';x',y') \; . \tag{1.8}$$

Why have we chosen to rotate only the kinetic energy and not the potential energy to imaginary time? The answer lies in our desire to obtain a genuine Wiener measure on (x,y) path space so as to put the path integral expression for a unitary time evolution in the projected Hilbert subspace on a sound mathematical foundation. In particular, we note that

$$K(x'',y'',t'';x',y',t') =$$

$$\lim_{m\to 0}(2\pi c/eB)\int \exp\{i(eB/c)\int y dx \;-\; i\int V(x,y)dt\}\exp\{\int(\omega_c/2)\}d\mu_W(x,y) \; , \tag{1.9}$$

where μ_W denotes a pinned Wiener measure as commonly appears in the Feynman-Kac formula. The expression $\int y dx$ is to be interpreted as a (Stratonovich) stochastic integral, in which case this path integral expression for K is without any ambiguity and rigorously defined for each $m > 0$; convergence as $m \to 0$ is assured for a wide class of potentials. [4] As a well-defined integral one may also consider its rigorous reformulation under coordinate transformations. Under such transformations the phase factor transforms under the rules of the ordinary calculus in spite of the fact that the functions involved are Brownian and not classical (e.g., C^1) in character; these transformation properties are the result of the Stratonovich (mid-point) prescription. The Brownian motion itself transforms as one might expect: as initially formulated the two-dimensional, planar Brownian motion was described by Cartesian coordinates; after the transformation the same two-dimensional, planar Brownian motion should be described, in general, by curvilinear coordinates.

1.2 General Field and Surface

With the foregoing elementary and familiar problem as background we turn our attention to present analogous results in more general circumstances. For present purposes we introduce intrinsic coordinates x^1 and x^2 lying in the surface, which as usual, may be described by a Riemanian metric $ds^2 = g_{ab}(x)dx^a dx^b$. The surface may be compact or noncompact and may have an arbitrary genus (number of handles), although for the most part we restrict attention to a simply connected manifold. In addition, we assume there is a magnetic field present that is described by a vector potential $A_b(x)$ in the standard way, $B_{ab}(x) = \partial_a A_b(x) - \partial_b A_a(x)$. As an antisymmetric tensor in two dimensions it is clear that $B_{ab}(x) = \epsilon_{ab}\lambda(x)$, where ϵ_{ab} is the Levi-Civita tensor density and $\lambda(x)$ is a scalar density. Without loss of generality, we shall always orient the surface so that the total magnetic flux is nonnegative.

The path integral that represents the desired generalization of the ones given earlier reads

$$\lim_{m\to 0} \mathcal{N} \int \exp\{i(e/c)\int A_b(x)dx^b \;-\; i\int V(x)dt\}$$

$$\times \exp\{-\frac{1}{2}m\int g_{ab}(x)\dot{x}^a\dot{x}^b dt \;+\; \frac{e}{2mc}\int s^{ab}(x)B_{ab}(x)dt\}\Pi\sqrt{g(x)}dx^1dx^2 \;, \tag{1.10}$$

where the spin tensor $s^{ab} = \sqrt{g}\epsilon^{ab}/2$. The structure of this expression has been chosen with several issues in mind. The terms in the exponent, except the one containing $V(x)$, plus the form of the integration measure describe the Euclidean propagator of a charged spin-1/2 particle moving on the curved surface in the presence of a magnetic field everywhere orthogonal to the surface. In particular, the final term in the exponent represents a generalization of the term $\int(\omega_c/2)dt$ and describes the interaction of the polarized spin-1/2 with the magnetic field ($g_B = 2$). For this form of interaction the degeneracy of the lowest Landau level is not destroyed by a nonuniform field and/or a curved geometry. Moreover the energy of the lowest Landau level remains equal to zero. These facts lie at the heart of what is proved in the following Section. As a consequence, when $V \equiv 0$ and in the limit $m \to 0$ the path integral (1.10) leads to an integral kernel for a projection operator on a degenerate lowest Landau level, while for $V \neq 0$ a unitary evolution on the corresponding Hilbert subspace is obtained.

There are two kinds of transformations of the formal path integral of interest. By construction the expression is invariant under coordinate transformations, $x \to \bar{x} = \bar{x}(x)$, assuming that the indicated quantities transform like tensors of the appropriate kind. A second kind of transformation involves a change of gauge of the vector potential, $A_b(x) \to A_b(x) + \partial_b \Lambda(x)$. The only consequence of such a transformation is the appearence of a total derivative leading to a phase factor of the form $\exp\{i(e/c)[\Lambda(x'') - \Lambda(x')]\}$. Such a factor only affects the local phase of the wave function, a modification without physical content. Of course, transformations that combine both gauge and coordinate changes are important as well as we shall see in the next subsection.

1.3 Reinterpretation in Phase Space

It is often useful to take the mathematical formulation appropriate to one physical situation and reinterpret it in an entirely different physical situation. Hamiltonian mechanics for particles and ray optics provides just one example of the utility of such a reinterpretation. Quantization of two-dimensional particles in a magnetic field and a phase-space path integral quantization of a particle, as we now shall see, provides yet another example.

We return, first of all, to the case of a particle moving on the plane in the presence of a uniform magnetic field and an auxiliary potential V. For present purposes let us introduce new variables, viz.,

$$q = \sqrt{eB/\Omega c}\, x \quad , \; p = \sqrt{eB\Omega/c}\, y \; ,$$

$$\nu = eB/mc = \omega_c \; , \; h(p,q) = V(x,y) \; . \tag{1.11}$$

In terms of these variables the former path integral, Eq.(1.5), assumes the form

$$\lim_{\nu\to\infty} \mathcal{N} \int \exp\{i \int [p\dot{q} - h(p,q)]dt \; - \; \frac{1}{2\nu} \int (\Omega^{-1}\dot{p}^2 + \Omega\dot{q}^2 - \nu^2)dt\} \Pi dpdq \; . \tag{1.12}$$

Apart from the limit and the ν-dependent factor in the integrand the expression in question resembles a formal phase-space path integral. The additional factor may be interpreted as a regularizing factor, more specifically as a continuous-time regularization, for in the limit $\nu \to \infty$, the factor in question formally becomes unity. To gain insights into the consequences of such a regularization we first specialize to the case $h = 0$, and define

$$\begin{aligned} \mathcal{K}(p'',q'';p',q') \; &= \; \lim_{\nu\to\infty} \mathcal{N} \int \exp\{i \int p\dot{q}dt \; - \; \frac{1}{2\nu} \int (\Omega^{-1}\dot{p}^2 + \Omega\dot{q}^2 - \nu^2)dt\} \Pi dpdq \\ &= \; \exp\{\frac{i}{2}(p''+p')(q''-q') - \frac{1}{4}[\Omega^{-1}(p''-p')^2 + \Omega(q''-q')^2]\} \; , \end{aligned} \tag{1.13}$$

as follows from (1.4), with the proviso that we have rescaled the integration measure to absorb the prefactor, namely, $(eB/2\pi c)dxdy = dpdq/2\pi$. It readily follows that

$$\mathcal{K}(p''',q''';p',q') = \int \mathcal{K}(p''',q''';p'',q'')\mathcal{K}(p'',q'';p',q')dp''dq''/2\pi \tag{1.14}$$

and $\mathcal{K}^*(p'',q'';p',q') = \mathcal{K}(p',q';p'',q'')$; therefore $\mathcal{K}$ represents a projection operator, but a projection onto what? Just as in the planar motion in a magnetic field, the projection operator projects onto the relevant Hilbert space for the subsequent quantum mechanics.

In the present case $\mathcal{K}$ denotes a projection operator on $L^2(\mathbf{R}^2, dpdq/2\pi)$ onto the relevant functional Hilbert space for the problem at hand. Nevertheless the integral kernel for the projection operator is, at first sight, unfamiliar in its quantum mechanical meaning. Insight into that meaning is gained by first observing that $\mathcal{K}$ is a positive definite function, i.e., satisfies

$$\sum \alpha_j^* \alpha_k \mathcal{K}(p_j, q_j; p_k, q_k) \;=\; \int |\sum \alpha_k \mathcal{K}(p, q; p_k, q_k)|^2 dpdq/2\pi \;\geq 0 \;, \tag{1.15}$$

in virtue of the properties of $\mathcal{K}$ previously given. As a consequence the (Gel'fand, Naimark, Segal) GNS Theorem[5] asserts that there exists a representation of $\mathcal{K}$ as the inner product of two Hilbert space vectors that is unique up to unitary equivalence; namely, there exists an abstract Hilbert space $\mathbf{H}$ and vectors $|p, q> \in \mathbf{H}$, for all $(p, q) \in \mathbf{R}^2$, such that $\mathcal{K}(p'', q''; p', q') = < p'', q''|p', q' >$ for all argument pairs. In special cases – such as the one presently under consideration – these vectors are generated by a transitively acting group (or a group up to factor) on a fixed fiducial vector, but this situation is far more the exception than the rule.

In the present case the appropriate states are given by

$$|p, q> = \; e^{-iqP} e^{ipQ} |\Omega> \tag{1.16}$$

for all $(p, q) \in \mathbf{R}^2$, where Q and P denote irreducible self-adjoint Heisenberg operators and $|\Omega>$ is a normalized vector that satisfies $(\Omega Q + iP)|\Omega> = 0$. In terms of the Schrödinger representation it follows that

$$< p'', q''|p', q' > = < \Omega| e^{-ip''Q} e^{i(q''-q')P} e^{ip'Q} |\Omega >$$

$$= \; \sqrt{\Omega/\pi} \int \exp\{-\Omega x^2/2 - ip''x + ip'(x + q'' - q')\} \exp\{-\Omega(x + q'' - q')^2/2\} dx$$

$$= \exp\{\frac{i}{2}(p''+p')(q''-q') - \frac{1}{4}[\Omega^{-1}(p''-p')^2 + \Omega(q''-q')^2]\}$$

$$\equiv \mathcal{K}(p'',q'';p',q') \ . \tag{1.17}$$

The GNS Theorem then effectively asserts the unique association of the Weyl group and the Heisenberg operators with this particular kernel. Of course, the states $|p,q>$ in question are just the familiar canonical coherent states[6], which in $\mathbf{H}$ admit a resolution of unity in the form

$$\mathbf{1} \ = \ \int |p,q><p,q|dpdq/2\pi \ . \tag{1.18}$$

These states provide a representation basis for an arbitrary vector $|\psi> \in \mathbf{H}$, given by $\psi(p,q) \equiv <p,q|\psi>$, with an inner product given by $\|\psi\|^2 \ \equiv \ \int |\psi(p,q)|^2 dpdq/2\pi \ = \ <\psi|\psi>$. Finally, the propagator that arises when $h(p,q) \neq 0$ is just the coherent-state matrix element of the evolution operator, namely

$$<p'',q''|e^{-i\mathcal{H}T}|p',q'>$$

$$= \lim_{\nu\to\infty} \mathcal{N} \int \exp\{i\int[p\dot{q} - h(p,q)]dt \ - \ \frac{1}{2\nu}\int(\Omega^{-1}\dot{p}^2 + \Omega\dot{q}^2 - \nu^2)dt\}\Pi dpdq$$

$$\equiv \ K(p'',q'',t'';p',q',t') \ . \tag{1.19}$$

In this expression

$$\mathcal{H}(p'',q'';p',q') \equiv <p'',q''|\mathcal{H}|p',q'> = \int <p'',q''|p,q> h(p,q) <p,q|p',q'> dpdq/2\pi \ , \tag{1.20}$$

or abstractly

$$\mathcal{H} \ = \ \int h(p,q)|p,q><p,q|dpdq/2\pi \ , \tag{1.21}$$

which relates the Hamiltonian operator $\mathcal{H}$ and its c-number representative $h(p,q)$.

Let us interpret the integral in (1.19) as one involving a Wiener measure and a Stratonovich stochastic integral. In that case it becomes appropriate to discuss coordinate transformations. In particular consider a change of canonical coordinates $\bar{p} = \bar{p}(p, q)$, $\bar{q} = \bar{q}(p, q)$ for which $pdq = \bar{p}d\bar{q} + dF(\bar{q}, q)$. This equation which holds for classical (C^1) functions holds for Brownian paths as well. In light of the discussion in the previous subsection, we have chosen to link a gauge transformation with a suitable coordinate transformation so as to preserve the form of the classical action (and of the associated classical equations of motion). With the Wiener measure reinterpreted as planar Brownian motion expressed in curvilinear coordinates, an expression such as (1.19) transforms covariantly under a canonical change of coordinates.

Recapitulation

In this subsection we have reinterpreted the mathematics appropriate to a charged spin-1/2 particle moving in a two-dimensional plane in the presence of a uniform magnetic field and an auxiliary potential as a phase-space, path- integral quantization procedure. Admittedly the reinterpreted expression has the form of a phase-space path integral apart from the unusual ν-dependent factor in the integrand. This factor has apparently introduced a metric into phase space for the purpose of quantization where none seems to be present in alternative quantization procedures, e.g., the standard Schrödinger prescription. However, we assert that a metric is implicitly used in Schrödinger quantization when one recognizes that the Schrödinger rules of quantization work correctly only in certain coordinates, namely Cartesian coordinates. [7] A flat metric appears when it is recognized that global Cartesian coordinates exist only in a globally flat space. The role of the ν-dependent factor is, of course, to regularize the formal integral which then may be reinterpreted as a well-defined Brownian motion integral.

Based on an analogous reinterpretation of the motion of charged spin-1/2 particles in a magnetic field we shall, in the next subsection, propose a quantization scheme for phase spaces endowed with general symplectic forms and general and unrelated metric structures. In so doing we will encounter an unexpected surprise related to the quantization of such systems, namely, each path does *not* contribute to the path integral with equal weight in the general case.

1.4 Quantization of General Systems

As was the case in the previous subsection we initiate our discusion with the kinematics. Let the phase-space variables be denoted by $\xi = (\xi^1, \xi^2)$, set $(e/c)A_a = a_a, (e/c)B_{ab} = b_{ab}$, $m = 1/\nu$, in which case attention focusses on

$$\lim_{\nu\to\infty} \mathcal{N} \int \exp\{i \int a_b(\xi) d\xi^b\}$$

$$\times \exp\{-\frac{1}{2\nu} \int g_{ab}(\xi)\dot{\xi}^a\dot{\xi}^b dt \ + \ \frac{\nu}{2} \int s^{ab}(\xi) b_{ab}(\xi) dt\} \Pi \sqrt{g(\xi)} d\xi^1 d\xi^2 \ . \tag{1.22}$$

In the next Section we shall prove that the kernel defined by this expression corresponds to a projection operator on a nontrivial subspace of the Hilbert space $L^2(\Gamma, \sqrt{g} d\xi^1 d\xi^2)$, ($\Gamma$ denotes the phase-space manifold). This subspace will be identified with the Hilbert space $\mathbf{H}$ of the quantum system. As a consequence the kernel satisfies

$$\mathcal{K}(\xi''; \xi') \ = \ \int \mathcal{K}(\xi''; \xi)\mathcal{K}(\xi; \xi') \sqrt{g} d\xi^1 d\xi^2 \ ,$$

$$\mathcal{K}^*(\xi''; \xi') \ = \ \mathcal{K}(\xi'; \xi'') \ . \tag{1.23}$$

We shall analyse phase-spaces with an $\mathbf{R}^2$ topology and derive the formula for the dimension of $\mathbf{H}$, and even a local expression for the semiclassical density of quantum states. The case of a compact Riemanian phase-space manifold will be also discussed and illustrated by examples. For the latter case the compactibility condition

$$\frac{1}{2}\int b_{ab}(\xi)d\xi^a\wedge d\xi^b \;=\; 2\pi n, \qquad n=1,2,3,... \tag{1.24}$$

should hold; the dimension of $\mathbf{H}$ is then finite and given by

$$D \;=\; n \,+\, 1 \,-\, g\;, \tag{1.25}$$

where g denotes the number of handles on the surface.

From the viewpoint of classical mechanics $a_b(\xi)d\xi^b$ denotes the one form whose exterior derivative

$$da_b(\xi)d\xi^b \;=\; \partial_a a_b(\xi)d\xi^a\wedge d\xi^b \;=\; \frac{1}{2}b_{ab}(\xi)d\xi^a\wedge d\xi^b \tag{1.26}$$

denotes the symplectic two form on the manifold. In simple cases, namely canonical coordinates, the one form is just pdq and the symplectic form then is $dp\wedge dq$. The symplectic form is, in this simple case, the same volume element that appears in the formal path integral measure, namely $\Pi dpdq$. It is noteworthy in the general case that the volume element required in the path integral measure is not (proportional to) the symplectic form volume element. In the general case the volume element $\sqrt{g(\xi)}d\xi^1 d\xi^2$ appears in the path integral measure while the symplectic form is given by $\frac{1}{2}b_{ab}(\xi)d\xi^a\wedge d\xi^b$. This fact flies in the face of conventional wisdom that in a path integral "all paths enter with equal weight". Of course, the additional weighting factor $\exp[(\nu/2)\int s^{ab}b_{ab}(\xi)dt]$ belies this conventional wisdom as well.

Conventionally, the phrase "symplectic form" is reserved to a nondegenerate skew-symmetric matrix, ω_{ab}, such that $\omega_{ab}\omega^{bc}=-\delta_a^c$. In this paper, however, we refer loosely to the skew-symmetric matrix b_{ab} as a symplectic form even if it may be degenerate in some regions and even when it is not degenerate it may fail to be a square root of unity in the sense noted above. Our justification for this terminology arises from the fact that b_{ab} are the coefficients in the exterior derivative of the one form $a_b(\xi)d\xi^b$ that figures in the action functional for the system at hand[8].

The kernel $\mathcal{K}(\xi'';\xi')$ is a positive-definite functional, and as such, according to the GNS Theorem, may be represented as the inner product of (not necessarily normalized) vectors $|\xi>\equiv|\xi^1,\xi^2>$ in an abstract Hilbert space $\mathbf{H}$, namely,

$$< \xi''|\xi' > = \mathcal{K}(\xi'';\xi') \ . \tag{1.27}$$

These vectors are continuously labelled and, in virtue of the projection property of $\mathcal{K}$, they admit a resolution of unity in $\mathbf{H}$ according to

$$\mathbf{1} \ = \int |\xi >< \xi| \sqrt{g} d\xi^1 d\xi^2 \ . \tag{1.28}$$

These are just the properties that make the vectors $\{|\xi>\}$ into a set of coherent states. It must be emphasized, however, that in the general case there is no few-parameter unitary representation of a group (or a group up to the factor) that generates all the states $|\xi>$ as unitary transformations of a fixed fiducial vector. However convenient such a group may be there is, in the general case, no symmetry of the phase-space manifold that would support the existence of such a transitively acting group. The difference in viewpoint regarding quantization advocated here could not be greater than the conventional quantization viewpoint in which one promotes several of the classical phase-space variables to self-adjoint operators appropriate to some low-dimensional closed Lie algebra. These two quantization procedures coincide for a limited number of cases, but will surely lead to different results in the general case. The existence of the physical analog of the quantum Hall effect speaks to the validity of the alternative quantization scheme advocated in this subsection in the general case.

The introduction of a nonvanishing Hamiltonian and nontrivial dynamics proceeds as in the elementary case. The propagator is given by

$$K(\xi'',t'';\xi',t') \;=\; \lim_{\nu\to\infty}\mathcal{N}\int \exp\{i\int[a_b(\xi)\dot{\xi}^b - h(\xi)]dt\}$$

$$\times\exp\{-\frac{1}{2\nu}\int g_{ab}(\xi)\dot{\xi}^a\dot{\xi}^b dt \;+\; \frac{\nu}{2}\int s^{ab}(\xi)b_{ab}(\xi)dt\}\Pi\sqrt{g(\xi)}d\xi^1 d\xi^2$$

$$\equiv <\xi''|e^{-i\mathcal{H}T}|\xi'> \quad . \tag{1.29}$$

Here $\mathcal{H}$ and h are related by

$$\mathcal{H} \;=\; \int h(\xi)|\xi><\xi|\sqrt{g}d\xi^1 d\xi^2 \ . \tag{1.30}$$

To ensure that a unitary evolution exists it is sufficient for $\mathcal{H}$ to be essentially self-adjoint on the finite linear span of the coherent states.

With the final formulas we have achieved our goal of presenting a manifestly coordinate invariant quantization procedure appropriate to a general symplectic form and geometry of the underlying two manifold. One should mention that the present approach to quantization has been extended to Kähler manifolds of an arbitrary even dimension,[9] and for flat phase-spaces Wiener measure in (1.9) may be replaced by a probabilistic measure for a general Poisson process.[10]

2. THE STRUCTURE OF THE LOWEST LANDAU LEVEL

Consider an electron moving on an arbitrary smooth two-dimensional surface Γ as described in Sec. 1.2. The path integral expression (1.10) with a fixed value of the mass parameter m and with $V(x) \equiv 0$ yields the integral kernel of the operator $\exp\{-H[A,g]T\}$ where ($\hbar = 1$)

$$H[A,g] \;=\; -\frac{1}{2m}\Big[\frac{1}{\sqrt{g}}(\partial_a + i(e/c)A_a)g^{ab}\sqrt{g}(\partial_b + i(e/c)A_b)\Big] \;-\; \frac{e}{2mc\sqrt{g}}B_{12} \; . \qquad (2.1)$$

It follows from (2.1) that the limit $m \to 0$ in the path integral (1.10) for $V \equiv 0$ is equivalent to taking the following operator limit (in the sense of matrix elements)

$$\Pi \;=\; \lim_{T\to\infty} \exp\{-H[A,g]T\} \; . \qquad (2.2)$$

The limit operator Π exists and is a nontrivial projection operator in the Hilbert space $L^2(\Gamma, \sqrt{g}dx^1dx^2)$ if and only if : $H[A,g] \geq 0$ and there exists a nontrivial subspace $\mathbf{H}$ of normalizable eigenvectors ϕ satisfying

$$H[A,g]\phi \;=\; 0 \; . \qquad (2.3)$$

In the following we shall construct the solutions of Eq. (2.3) for a manifold Γ admiting a global parametrization ($\mathbf{R}^2$ topology) generalizing the Aharonov-Casher[1] approach to a flat surface with an arbitrary magnetic field, and then we shall briefly discuss two examples of compact manifolds. Before doing this we should take advantage of the fact that for any two-dimensional surface one can always choose a (local) coordinate system, say u and v, $(u,v) \in \mathbf{R}^2$, such that the metric becomes conformally flat i.e.[11]

$$ds^2 \;=\; e^{2w(u,v)}\big(du^2 \;+\; dv^2\big) \; . \qquad (2.4)$$

In this special coordinate system the matrix elements of the Hamiltonian $H[A,g]$ are given by the following expression

$$
\begin{aligned}
<\psi|H[A,g]|\phi> = & -\frac{1}{2m}\int \psi^* e^{-2w}\Big\{\big[\partial_u + i(e/c)A_u)\big]^2\phi + \big[\partial_v + i(e/c)A_v)\big]^2\phi\Big\} e^{2w} dudv \\
& - \frac{e}{2mc}\int \psi^*(\partial_u A_v - \partial_v A_u)\phi \, dudv \\
& \equiv \frac{1}{2m}\int dudv (\mathcal{D}\psi)^* \mathcal{D}\phi
\end{aligned}
\tag{2.5}
$$

where

$$
\mathcal{D}\phi \;=\; \big[(\partial_u - i\partial_v) + i(e/c)(A_u - iA_v)\big]\phi \; . \tag{2.6}
$$

From (2.5) and (2.6) it follows that $H[A,g] \geq 0$ indeed and that the ground states (with polarized spin) are all solutions of the following equation

$$
\big[(\partial_u - i\partial_v) + i(e/c)(A_u - iA_v)\big]\phi \;=\; 0 \; . \tag{2.7}
$$

Obviously the relevant solutions must be square integrable with respect to the measure $\sqrt{g}dx^1dx^2$ and must satisfy the topological constraints in the case of compact manifold Γ. Equation (2.7) gives us control on the singularities of ϕ. Indeed, in the neighborhood of any point there always exists a local non-singular solution, say ρ, which does not vanish.[13] Any other solution ϕ can be expressed in terms of ρ as $\phi(u,v) \;=\; f(u-iv)\rho(u,v)$, f being a holomorphic function. Hence, any singularity (or zero) of ϕ is a singularity (zero) of a holomorphic function. We conclude from this that a square integrable solution is supposed to be smooth. This has implications on the topological restrictions. Mathematically, Eq. (2.7) defines a holomorphic bundle and ϕ is a global section. There are known strong mathematical methods which give us the dimension of the space of solutions to (2.7) in the compact case in terms of topological invariants: the flux of the magnetic field and the Euler characteristic of the surface. We shall illustrate them in Sec. 2.2. On the other hand, in Sec. 2.1 we show that even in a non-compact, topologically flat, case the magnetic flux and the integral of a Gauss curvature – provided that they are finite – determine the dimension of **H**.

2.1 Surface with R^2 Topology

We assume now that there exists a global coordinate system (u, v) satisfying (2.4). Then it follows from Eq.(2.7) that the subspace **H** of the ground states is spanned by the linearly independent functions

$$\phi_k(u,v) \;=\; (u-iv)^k e^{-F(u,v)} e^{iG(u,v)} \tag{2.8}$$

with $k = 0, 1, 2..., N(= D-1) \leq \infty$, and real functions F, G satisfying the equations

$$(\partial_u^2 + \partial_v^2)F(u,v) \;=\; (e/c)(\partial_u A_v - \partial_v A_u)\ , \tag{2.9}$$

$$(\partial_u^2 + \partial_v^2)G(u,v) \;=\; (e/c)(\partial_u A_u + \partial_v A_v)\ . \tag{2.10}$$

The condition of square integrability of ϕ_k demands that the function $(u^2+v^2)^k \exp[-2F(u,v) + 2w(u,v)]$ should decay at least as $(u^2+v^2)^{-(1+\epsilon)}$ for $|u|, |v| \to \infty$ with $\epsilon > 0$. Suppose now that the following integrals are finite ($\Phi \geq 0$)

$$\Phi \;=\; (e/c)\int(\partial_u A_v - \partial_v A_u)dudv\ , \tag{2.11}$$

$$\Psi \;=\; -\int(\partial_u^2 + \partial_v^2)w(u,v)dudv \tag{2.12}$$

The solution of (2.9) can be written as

$$F(u,v) \;=\; \frac{e}{4\pi c}\int du'dv'\{[\partial_u A_v(u',v') - \partial_v A_u(u',v')]\ln[(u-u')^2 + (v-v')^2]\}\ . \tag{2.13}$$

For large $|u|^2+|v|^2$ we obtain the following estimation, using (2.9), (2.11),(2.12) and (2.13),

$$|\phi_k(u,v)|^2 \exp[-2w(u,v)] \sim (u^2+v^2)^{k-(\Phi+\Psi)/2\pi} \ . \tag{2.15}$$

Hence to attain square integrability k must satisfy the inequality

$$k \ < \ \frac{1}{2\pi}(\Phi+\Psi) \ - \ 1 \ . \tag{2.16}$$

The expressions for Φ and Ψ can be easily transformed into a geometric, coordinate independent form

$$\Phi \ = \ \frac{e}{2c}\int(\partial_a A_b(x) - \partial_b A_a(x))dx^a \wedge dx^b \ , \tag{2.17}$$

$$\Psi \ = \ \frac{1}{2}\int R(x)\sqrt{g(x)}dx^1 dx^2 \tag{2.18}$$

where R is the scalar curvature given by the Riemann tensor of g

$$R \ = \ R^{\alpha\beta}{}_{\alpha\beta} = -2e^{-2w}(\partial_u^2+\partial_v^2)w \ . \tag{2.19}$$

We emphasise however, that in this case Ψ and Φ are not topological invariants. Finally, from (2.16) our expression for the dimension of the lowest Landau level reads

$$D \ = \ \textit{largest integer less than}\Big[\frac{1}{2\pi}\Phi \ + \ \frac{1}{2\pi}\Psi \ - \ 1\Big] \ . \tag{2.20}$$

Clearly for infinite Φ and/or Ψ the subspace $\mathbf{H}$ is infinite dimensional. However,even in this case the following formula for the semiclassical density of electronic states on the surface Γ (with a unidirectional magnetic field)is valid as can be seen from Eqs. (2.17),(2.18) and (2.20)

$$dN(x) \ = \ \frac{e}{4\pi c}(\partial_a A_b(x) \ - \ \partial_b A_a(x))dx^a \wedge dx^b \ + \ \frac{1}{4\pi}R(x)\sqrt{g(x)}dx^1 dx^2 \ . \tag{2.21}$$

2.2 Compact surfaces

The case of a compact two-dimensional manifold Γ with an arbitrary genus $g = 0, 1, 2, ...$, can be discussed using geometrical methods. First of all the vector potential A_b and the coordinates at which the metric tensor g_{ab} takes the form (2.4) are defined only locally and subject to a suitable gauge/coordinate transformation from a one to another local domain. The (normalized) integrals $\frac{1}{2\pi}\Phi$, the magnetic charge, and $\frac{1}{2\pi}\Psi$, the Euler characteristic, are now topological invariants and can take only integer values, namely

$$\frac{1}{2\pi}\Phi \;=\; n, \quad n = 0, 1, 2, ... \tag{2.22}$$

$$\frac{1}{2\pi}\Psi \;=\; 2(1-g), \quad g = 0, 1, 2, ... \;. \tag{2.23}$$

The condition (2.22) is the famous Dirac condition on the monopole while the condition (2.23) is the Gauss-Bonnet Theorem.[11] As mentioned in Section 1.4 the Riemann-Roch-Hirzebruch-Atiyah-Singer (see for example Ref.12) index theorem gives the dimension D of the lowest Landau level as

$$D \;=\; n + (1-g) \;=\; \big[\frac{1}{2\pi}(\Phi + \Psi) - 1\big] + g \tag{2.24}$$

if $n \;>\; 2-2g$ or $g = 0$ and when $n \;\leq\; 0$, necessarily $D \;=\; 0$. Note that Eq. (2.24) extends the formula (2.20) to compact manifolds. Here, again, in the semiclassical limit ($n \gg 1+g$, unidirectional magnetic field) the local expression (2.21) for the density of states remains valid. The manifest expressions for the wave functions which span $\mathbf{H}$ in the case of compact Γ are obtained as the solutions of Eq. (2.7) which satisfy the topological constraints. For the sake of illustration we present three particular examples.

Example 1. Potato

We consider here Eq. (2.7) on a 2-surface Γ which is topologically equivalent to a sphere. The genus $g = 0$, now, and we know from the classification of Riemann surfaces that Γ is conformal to a sphere equipped with the natural metric. The coordinates (u, v) cannot be extended to the entire surface Γ. However, in this case, there exist 'spherical' coordinates (θ, α) such that the scalar product $ds^2 = g_{\mu\nu} dx^\mu dx^\nu, (\mu, \nu = \theta, \alpha)$ takes on the following appearance

$$ds^2 = e^{2w}(d\theta^2 + sin^2\theta d\alpha^2) \ . \tag{2.25}$$

Let A_μ be a vector potential carrying the magnetic charge $\Phi = 2\pi n$. According to (2.24), the number of linearly independent solutions of (2.7) is

$$D \ = \ n + 1 \ . \tag{2.26}$$

We shall derive them below, but first here is an outline of our strategy. We write A as

$$A_\mu \ = \ n\tilde{A}_\mu + a_\mu \tag{2.27}$$

where $\tilde{A}$ is a vector potential of the uniform magnetic field corresponding to a magnetic charge $n_0 = 1$. Next, we solve Eq. (2.7) with $\tilde{A}_\mu$ and a_μ, respectively, substituted for A_μ. In the first case we find two linearly independent solutions, $\tilde{\psi}_{(1)}$ and $\tilde{\psi}_{(2)}$, and in the second case a single solution denoted by ϕ'. This is consistent with (2.26). Finally, we define wave functions $\phi_{(1)}, ..., \phi_{(n+1)}$ by

$$\phi_{(i+1)} \ = \ \phi'(\tilde{\psi}_{(1)})^i(\tilde{\psi}_{(2)})^{n+1-i} \ . \tag{2.28}$$

Every $\phi_{(i)}$ is a solution to (2.7) with the vector potential (2.27). It is also easy enough to see (details below) that the $\phi_{(i)}$-s are linearly independent, hence they form a basis of the solutions. More specifically, to express the vector potential A we divide Γ onto two hemispheres and on each of them fix a gauge (if $n > 0$ then there is no global gauge on Γ). Then A and an associated wave function ϕ may be written as

$$(A_\mu,\ \phi) = \begin{cases} (A^+_\mu,\ \phi^+) & \text{if } \theta \leq \frac{\pi}{2} + \epsilon; \\ (A^-_\mu,\ \phi^-) & \text{if } \theta \geq \frac{\pi}{2} - \epsilon \end{cases} \tag{2.29}$$

where $A^\pm_\mu$ and $\phi^\pm$ are well defined on the hemispheres, and on the intersection of the two hemispheres we glue them by a gauge transformation

$$A^+ = A^- + nd\alpha, \quad \phi^+ = e^{-in\alpha}\phi^- \ . \tag{2.30}$$

Through n in the exponent, the gauge transformation contains the information about the magnetic charge. For the uniform magnetic field we choose a vector potential

$$\tilde{A}^\pm = \frac{1}{2}(\pm 1 + cos\theta)d\phi \ . \tag{2.31}$$

The solutions corresponding to $\tilde{A}$ have the following form

$$\begin{aligned} \phi_{(1)} &= \begin{cases} cos\frac{1}{2}\theta & , \\ e^{i\alpha}cos\frac{1}{2}\theta & ; \end{cases} \\ \phi_{(2)} &= \begin{cases} e^{-i\alpha}sin\frac{1}{2}\theta & , \\ sin\frac{1}{2}\theta & . \end{cases} \end{aligned} \tag{2.32}$$

On the other hand, the term a_μ in (2.27) is a globally defined covariant vector field. It follows from the fact that Γ is simply connected, that a_μ can be decomposed into the form

$$a_\mu = \partial_\mu G + \epsilon_\mu{}^\nu \partial_\nu b \tag{2.33}$$

with G and b being real functions on Γ. The solution of (2.7) corresponding to a_μ is

$$\phi' = e^{-(b+iG)} \ . \tag{2.34}$$

We have learned from this example that for a simply connected surface it is enough to find ground states for a uniform magnetic field which has the flux 2π and for all the magnetic fields of zero flux. Then, ground states for an arbitrary magnetic field are generated algebraically from the previous ones.

Example 2. Donut

We consider here a surface topologically equivalent to a torus. This means that the genus $g = 1$, and the dimension of the space of solutions to (2.7) given by (2.24) becomes $D = n$. Geometry of the surface is, up to a pointwise dependent rescaling, equivalent to the geometry of the quotient: the plane R^2 equipped with the flat metric $du^2 + dv^2$ divided by the group of translations generated by two vectors

$$X \;=\; (2\pi, 0), \qquad V \;=\; (u_0, v_0), \quad v_0 \;>\; 0\;. \tag{2.35}$$

The topological conditions which have to be satisfied by a wave function ϕ of a particle interacting with a vector potential A_μ, which has the topological charge n, take the form of certain periodicity conditions. They can be written as

$$\begin{aligned}
A_\mu(u+2\pi, v)dx^\mu \;&=\; A_\mu(u,v)dx^\mu + \frac{2\pi n}{v_0}dv, \\
\phi(u+2\pi, v) \;&=\; \exp\big(-\frac{2\pi n i}{v_0}v\big)\phi(u,v), \\
A_\mu(u+u_0, v+v_0)dx^\mu \;&=\; A_\mu(u,v)dx^\mu, \\
\phi(u+u_0, v+v_0) \;&=\; \phi(u,v),
\end{aligned} \tag{2.36}$$

where $(x^\mu) = (u, v)$. To solve Eq. (2.7) we shall use the same trick as in the previous example. We decompose A into the sum of a vector potential of a uniform magnetic field which carries the topological charge and the rest. However, in the case of a sphere, magnetic field determined uniquely a gauge class of corresponding vector potentials. Now, this one to one correspondence does not hold. The ambiguity consists of magnetic 'vacua' given by constant vector potentials of the form

$$A'_u = \beta_u, \quad A'_v = \beta_v \ , \tag{2.37}$$

βs being constant numbers. In other words, we write A as

$$A_\mu \quad = \quad n\tilde{A}_\mu + a_\mu, \tag{2.38}$$

where for $\tilde{A}$ we can choose

$$\tilde{A}_\mu dx^\mu \quad = \quad \frac{v_0 u - u_0 v}{v_0^2} dv \ ; \tag{2.39}$$

but unlike in (2.33), the Hodge decomposition of a_μ reads

$$a_\mu \quad = \quad \beta_\mu + \partial_\mu G + \epsilon_\mu{}^\nu \partial_\nu b, \tag{2.40}$$

where β_u and β_v are real constants. After the substitution of (2.38) and (2.40) into (2.7), the second and the third term of a_μ [see the RHS of (2.40)] can be eliminated from Eq (2.7) in the same way as in Example 1, i.e., by introducing ψ such that

$$\phi \quad = \quad \psi e^{-(b+iG)} \ . \tag{2.41}$$

In that way, we are left with the following equation

$$(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v} \ + \ n\frac{v_0 u - u_0 v}{2v_0^2} \ + \ \beta)\psi = 0. \tag{2.42}$$

The general solution to (2.42) which satisfies the first periodicity condition, i.e., with respect to the translations generated by the vector $(2\pi, 0)$, can be expressed as

$$\psi \ = l(\bar{z}) \ \exp\left(n\frac{iu_0 v^2}{2v_0^2} + n\frac{v^2 - 2iuv}{2v_0} + 2i\beta v\right) \ , \tag{2.43}$$

where $z := u + iv$, $\beta := \beta_u + i\beta_v$ and the function l is periodic with respect to the vector $(2\pi, 0)$,

$$l(\bar{z}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\bar{z}} \ . \tag{2.44}$$

Applying the second periodicity condition, that with respect to $z \to z + \beta$, we obtain the condition

$$a_{k+n} = a_k \mathrm{exp}\Big[-i(k + \frac{n}{2})(u_0 - iv_0) + 2iv_0\beta\Big] \ . \tag{2.45}$$

Hence, we can fix n arbitrary values for $a_0, ..., a_{n-1}$ and determine by (2.45) all other a_k. It is easy to see that (2.45) guaranties that the obtained sum which gives l converges for every z since $v_0 > 0$.

Summarising this example, we could see above the mechanism which determines the number of independent (polarized spin) ground states as determined by the magnetic flux.

Example 3. Arbitrary surface but topologically trivial magnetic field

Mathematically, Eq (2.7) defines a holomorhic bundle, and a global solution forms a holomorphic section. However, in the previous examples we did not necessarily have to apply the theory of holomorphic bundles. We could just explicitly derive the solutions. On the other hand, if the surface has higher genus then straightforward computations would be very complicated and we only have the formula (2.22). Now, we would like to concentrate on the case when the magnetic flux vanishes, i.e., when $n = 0$. We shall present now how the mathematics works for this example. Suppose a wave function ϕ is a solution to (2.7). We shall see that there are no other solutions linearly independent of ϕ. Indeed, suppose that ϕ' also solves (2.7) with the same vector potential A. Then necessarily

$$\phi' = f(\bar{z})\phi \tag{2.46}$$

where $f(\bar{z})$ is an anti-holomorphic function of z. The only (anti)holomorphic functions on a compact surface are constant functions. However, if ϕ has a zero in some point then perhaps f can have a pole which is compensated by ϕ. Therefore, we have to study zeros of ϕ, and here mathematics gives us a precise answer. First, as we mentioned before every zero of ϕ is a zero of a holomorphic function. Hence, any such zero is of the kind $(\bar{z}-\bar{z}_0)^p$. Second, we have a formula which expresses the magnetic flux by the zeros of ϕ and their orders; this expression reads

$$2\pi n \quad = \quad \Phi = 2\pi \sum p_i \tag{2.47}$$

summed over all zeros. But in our case $\Phi = 0$ it follows that every order $p_i \quad = \quad 0$. Hence, ϕ *cannot vanish at any point.*

Summarising, we have seen that, if $n = 0$ there are two possibilities: either there exists exactly one solution or none. A solution exists if and only if A can be written in the Landau gauge in the form

$$A_a(x) = \sqrt{g(x)}\epsilon_a{}^c(x)\partial_c b(x) \tag{2.48}$$

with b being a global real function on the surface.

2.3 Application to Quantization of General Systems

The results of the previous sections have immediate application to the problem of quantization of a general system discussed in Section 1.4. Namely treating now the two dimensional surface Γ as a phase space of a certain physical system we obtain the representation of the Hilbert space $\mathbf{H}$ of the quantized system. $\mathbf{H}$ is identified with subspace of the Hilbert space $L^2(\Gamma, \sqrt{g}d\xi^1 d\xi^2)$ which contains functions satisfying the polarization condition

$$\Big\{\big[\frac{1}{\sqrt{g}}(\partial_a + ia_a)g^{ab}\sqrt{g}(\partial_b + ia_b)\big] \ + \ s^{ab}b_{ab}\Big\}\phi \ = \ 0 \ . \tag{2.49}$$

In the special coordinate system the solutions of (2.49) are given by the solution of Eq.(2.7) with $(e/c)A_b$, $(e/c)B_{ab}$ replaced by a_b, b_{ab} and in different topological cases we proceed as in Sections 2.1, 2.2. Having found the solutions ϕ_k of Eq.(2.7), which span **H**, we may construct the reproducing kernel $\mathcal{K}$ as

$$\mathcal{K}(\xi'';\xi') \ = \ \sum \beta_{kl}\phi_k(\xi'')\phi_l^*(\xi') \ , \tag{2.50}$$

where β_{kl} are coefficients of a matrix inverse to the Gramm matrix with coefficients $\alpha_{kl} = \int \phi_k^*(\xi)\phi_l(\xi)\sqrt{g(\xi)}d\xi^1 d\xi^2$, and then complete the quantization scheme presented in Section 1.4.

2.4 Quantum Hall Current

The motion of an electron in a general magnetic field and on an arbitrary surface Γ as discussed in Section 1.2 is described by the propagator (1.10). However, according to the reinterpretation given in Sections 1.3 and 1.4 this propagator may be treated as a quantum propagator for the classical system with a phase-space Γ and an action functional

$$\mathcal{A} \ = \ \int \big[\frac{e}{c}A_b(x)\dot{x}^b - V(x)\big]dt \ . \tag{2.51}$$

The corresponding Euler-Lagrange equations read

$$\frac{e}{c}B_{ab}(x)\dot{x}^b \ = \ \partial_a V(x) \ . \tag{2.52}$$

Consider now two points x' and x'' on Γ connected by a curve C. The total electric current J_C which flows through the curve C for the case of fully occupied first Landau level may be calculated using the following semiclassical arguments. Let us treat the electrons as a fluid with the local surface density given by Eq.(2.21) and the local velocity $\dot{x}^a$ which satisfies Hamiltonian equation (2.52). First from Eq. (2.21)we obtain

$$\begin{aligned} dN(x) \; &= \; \frac{e}{4\pi c}(\partial_a A_b(x) \; - \; \partial_b A_a(x))dx^a \wedge dx^b \; + \; \frac{1}{4\pi}R(x)\sqrt{g(x)}dx^1 dx^2 \\ &= \Big[\frac{e}{4\pi c}(\partial_a A_b(x) \; - \; \partial_b A_a(x)) \; + \; \frac{1}{8\pi}R(x)\sqrt{g(x)}\epsilon_{ab}\Big]dx^a \wedge dx^b \; . \end{aligned} \tag{2.53}$$

Then using Eqs.(2.52),(2.53) and the fact that $B_{ab} = (\sqrt{B_{rs}B^{rs}g/2})\epsilon_{ab}$ we have

$$\begin{aligned} J_C \; &= \; \frac{e^2}{2\pi c}\int_C B_{ab}\dot{x}^a dx^b + \frac{1}{4\pi}\int_C R(x)\sqrt{g(x)}\epsilon_{ab}\dot{x}^a dx^b \\ &= \; \frac{e}{2\pi c}[V(x'') - V(x')] + \frac{1}{4\pi}\int_C \frac{R(x)}{\sqrt{B_{ab}B^{ab}/2}}dV(x) \end{aligned} \tag{2.54}$$

The first term on the RHS of Eq.(2.54) gives the standard expression for the quantum Hall current with the filling factor equal to one[14] while the second one is a geometric correction due to the curvature. For a flat surface and uniform magnetic field the standard expression is verified experimentally with an amazing accuracy. We expect also that the generalized formula (2.54) is applicable far beyond the semiclassical limit for physically interesting cases (here semiclassical regime corresponds to the case where the typical magnetic length $[eB/c]^{-1/2}$ is much smaller then the other relevant length scales). This wider applicability is due to the fact that all quantum corrections which can be derived from the expansion in path integral (1.10) around the classical trajectory effectively cancel in the integral along the curve C as long as the external potential $V(x)$ varies very slowly at the ends x', x''. One should notice that Eq. (2.54) makes sense for $B_{ab} \neq 0$, which is in agreement with the semiclassical picture. For an application of the above results to the description of an anomalous Hall current due to the anomalous magnetic moment of an electron in the case of flat surface but general magnetic field see Ref.15.

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