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Landau type theorem for Orlicz spaces

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Introduction

In 1907 Edmund Landau [9] proved the reverse Hölder inequality, i.e., if $(x_n y_n)_{n=1}^{\infty} \in l^1$ for all sequences $(x_n) \in l^p (1 < p < \infty)$, then $(y_n) \in l^q$, where q is the conjugate exponent of p. In his original proof he used the following Dini theorem: if $a_n \geqq 0$ and $\sum_1^\infty a_n = \infty$ then $\sum_1^\infty (a_n/s_n) = \infty$ and $\sum_1^\infty (a_n/s_n^{1+\varepsilon}) < \infty$ for any $\varepsilon > 0$, where s_n is the n-th partial sum. Now, if there were $\sum_1^\infty |y_n|^q = \infty$, then by the Dini theorem with $x_n = |y_n|^{q-1} \Big/ \sum_{k=1}^{n} |y_k|^q$ we would have $(x_n y_n) \notin l^1$ but $(x_n) \in l^p$.

Today everybody is able to prove the above result at once using the uniform boundedness principle or the form of linear continuous functionals on l^p. Landau's theorem belongs to the group of representation theorems of the Köthe dual E^x (=associate space in the other terminology) of a Banach function space E. The case $E = L^M(\mu)$ (i.e., E is an Orlicz space) was thoroughly investigated and it is well-known that $E^x = L^{M^*}(\mu)$ (see for example [14]), where M^* denotes the conjugate function of M (=complementary function in the sense of Young). This fact was originally proved (for convex functions M) by Birnbaum and Orlicz [3]. The purpose of the paper is to present a direct short and elementary proof of Birnbaum's and Orlicz's result and extend it to Orlicz spaces over atomless or counting measures and generated by finite-valued (not necessarily convex) functions. At the end of the paper one can find an example of an Orlicz space $L^M(\mu)$ over purely atomic measure μ and generated by non-convex function M such that its Köthe dual is not isomorphic to any space of the form $L^{M^*}(\nu)$.

We refer readers interested in Orlicz spaces to [8, 11, 14].

Let (S, Σ, μ) be a σ-finite measure space. Moreover, assume that $M: [0, \infty) \to [0, \infty]$ is a non-decreasing function and $M(0) = 0$.

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If $L^0 = L^0(S, \Sigma, \mu)$ denotes the space of Σ-measurable functions (with the usual identification of functions equal μ-almost everywhere) then

$$L^M(\mu) = \{f \in L^0 : m_M(rf) = \int_S M(r|f(s)|)\, d\mu < \infty \text{ for some } r > 0\}$$

is a linear subspace of the space L^0 (called the *Orlicz space*) and the functional

$$\|f\| = \inf\{a > 0 : m_M(f/a) \leqq a\}$$

is a monotone group semi-norm on the space $L^M(\mu)$, i.e., $\|0\| = 0$, $\|f+g\| \leqq \|f\| + \|g\|$ and $|f| \leqq |g|$ μ-almost everywhere implies $\|f\| \leqq \|g\|$. It is clear that the topology determined by $\|\cdot\|$ is Hausdorff iff M is not identically zero. Moreover, $\|\cdot\|$ is an F-norm on $L^M(\mu)$ iff M is continuous at zero or $M(r) = \infty$ for $r > 0$. In papers devoted to Orlicz spaces authors usually assume that M is left continuous, continuous at zero, non-decreasing and $M(0) = 0$. If M is convex continuous at zero and $M \not\equiv 0$, then the topology determined by $\|\cdot\|$ is equivalent to norm topology generated by $|||f||| = \inf\{a > 0 : m_M(f/a) \leqq 1\}$. The F-norm $\|\cdot\|$ is always complete.

We will often write L^M instead of $L^M(\mu)$ and l^M when μ is the counting measure on subsets of natural numbers, i.e., l^M is an Orlicz sequence space.

Let us distinguish the ideal $L^M_A \subset L^M$ consisting of elements with order continuous norm, i.e., $L^M_A = \{f \in L^M : |f| \geqq f_n \downarrow 0 \text{ implies } \|f_n\| \to 0\}$. If (S_n) is a sequence of atoms of finite measure μ such that μ restricted to measurable subsets of $S \setminus \bigcup_{n=1}^{\infty} S_n$ is atomless, then

$$L^M_A = \left\{f \in L^M : \operatorname{supp} f \subset S \setminus \bigcup_n^{\infty} S_n \text{ and } m_M(rf) < \infty \text{ for all } r\text{'s}\right\}$$
$$\oplus \overline{\operatorname{span}}\{1_{S_n} : n \in N\},$$

where 1_{S_n} is the characteristic function of the set S_n.

For a function M we define the *conjugate function* M^* of M by

$$M^*(r) = \sup\{sr - M(s) : s \geqq 0\},$$

(we put $x - \infty = -\infty$ for every real number x). The function M^* is always convex and left continuous. The notation M^{**} means the function $M^{**} = (M^*)^*$. After simple computations we obtain a useful inequality

$$(*) \qquad M^{**}(M^*(u)/u) \leqq M^*(u)$$

(we understand $M^{**}(\infty)$ as ∞). We always have $M^{**} \leqq M$ and $M^* = M^{***}$.

Since we also consider Orlicz spaces $L^M(\mu)$ which are not Banach spaces let us quote the exact definition of the *Köthe dual* $(L^M(\mu))^\times = (L^M)^\times$ of $L^M(\mu)$:

$$(L^M)^\times = \{g \in L^0(S, \Sigma, \mu) : f \cdot g \in L^1(\mu) \text{ for all } f \in L^M(\mu)\}.$$

The space $(L^M)^\times$ is always a Banach space with respect to the norm

$$\|g\| = \sup\{\|f \cdot g\|_{L^1} : m_M(f) \leqq 1\}.$$

Main results

Theorem 1 *If a function* $M \not\equiv 0$ *is convex and left continuous on* $(0, \infty)$ *then* $(L^M)^x = L^{M^*}$.

Proof. It is sufficient to prove $(L^M)^x \subset L^{M^*}$ because the reverse inclusion is a consequence of the inequality $rs \leqq M(r) + M^*(s)$.

Suppose $g \in L^0$ is such that $f \cdot g \in L^1$ for all $f \in L^M$. Therefore g determines a continuous linear functional G on L^M by the equality $G(f) = \int_S f \cdot g \, d\mu$. Put $c = \|G\|$.

Let us consider two cases:

1. $M^*(r) < \infty$ for all r's.

Define

$$g_0(s) = \begin{cases} \dfrac{c+1}{|g(s)|} M^*\left(\dfrac{|g(s)|}{c+1}\right) & \text{for } g(s) \neq 0, \\ 0 & \text{for } g(s) = 0. \end{cases}$$

There is no loss of generality to assume $M(u)$ is finite at some point $u > 0$ (if $M(r) = \infty$ for $r > 0$ then $(L^M)^x \subset LM^* = L^0$ and we are done). It is clear that the support of L^M equals S and so we can choose a sequence (S_n) of measurable sets with the following properties: $S_n \subset S_{n+1}$, $S = \bigcup_{n=1}^{\infty} S_n$, $0 < \mu(S_n) < \infty$ and $g_0 \cdot 1_{S_n} \in L^M$ (see [7, p. 136, Corollary 1]).

Since M is convex and left continuous we have $M = M^{**}$ in virtue of the Fenchel-Moreau theorem ([2, p. 86, Theorem 1.4] or [6, p. 186, Theorem 1]) and therefore using inequality $(*)$ we obtain

$$m_M(g_0 1_{S_n}) \leqq m_{M^*}\left(\frac{|g(\cdot)|}{c+1}\right) = \int_{S_n} g_0(s) \, |g(s)| \, (c+1)^{-1} \, d\mu < \infty$$

because $g_0 1_{S_n} \in L^M$. We claim $m_M(g_0) < \infty$. If $m_M(g_0)$ were infinite then $b = m_M(g_0 1_{S_n}) > 2$ for some n in virtue of $m_M(g_0 1_{S_n}) \uparrow m_M(g_0)$. Moreover, the convexity of M implies $b^{-1} |||g_0 1_{S_n}||| \leqq 1$ and thus

$$\begin{aligned} b = m_M(g_0 1_{S_n}) &\leqq \int_{S_n} g_0(s) \, |g(s)| \, (c+1)^{-1} \, d\mu \\ &= b(c+1)^{-1} \int_{S_n} (b^{-1} g_0(s)) \, |g(s)| \, d\mu \leqq bc(c+1)^{-1} < b \end{aligned}$$

and we have got a contradiction. Therefore, it has to be $m_M(g_0) < \infty$ which implies $g \in L^{M^*}$.

2. There exists $r_0 > 0$ such that $M^*(r) < \infty$ for $r < r_0$ and $M^*(r) = \infty$ for $r > r_0$. We claim $g \in L^\infty$. Indeed, let $r > r_0$ be fixed. We have

$$\infty = \sup\{sr - M(s) : s \geqq 0\} = \sup\{sr - M(s) : s > n\}$$

and so there is a sequence (s_n) increasing to infinity with $rs_n - M(s_n) > 0$, i.e., $M(s_n)/s_n < r$. For a number $s > 0$ choose an index n_0 such that $s < s_{n_0}$. Since the function $M(x)/x$ is non-decreasing we obtain $M(s)/s \leqq M(s_{n_0})/s_{n_0} < r$. Finally $M(s) \leqq rs$. Hence $L^1 \subset L^M$ and this inclusion implies $g \in L^\infty$. Putting

$$g_0(s) = \begin{cases} \dfrac{d}{|g(s)|} M^*\left(\dfrac{|g(s)|}{d}\right) & \text{for } g(s) \neq 0 \\ 0 & \text{for } g(s) = 0, \end{cases}$$

where $d = \max(c+1, (\|g\|_\infty + 1) r^{-1})$, and repeating the arguments used in part 1 we will also obtain $g \in L^{M^*}$.

Remark 1 Let us note that the method used in the first part of the proof is similar to that in [10, Theorem 4].

Theorem 2 *If* $M \not\equiv 0$ *is finite-valued, left continuous, continuous at zero, non-decreasing and* μ *is atomless, then* $(L^M)^x = L^{M^*}$.

Proof. We have only to prove $(L^M)^x \subset L^{M^*}$. The inequality $M^{**} \leqq M$ implies $L^M \subset L^{M^{**}}$. Let us consider two possibilities:

1. There exists a positive number r with $M^{**}(r) > 0$.

Let $g \in (L^M)^x$ and let G be the functional determined by $|g|$, i.e., $G(f) = \int_S f|g|\, d\mu$. This functional restricted to L_A^M remains continuous with respect to the topology τ^{**} induced from $L_A^{M^{**}}$([4, Theorem 2]). The support of L_A^M equals S, and so every positive function $f \in L^{M^{**}}$ is the supremum of some increasing sequence (f_n) of positive functions from L_A^M. Since (f_n) is τ^{**}-bounded and G is τ^{**}-continuous we have $\int_S f|g|\, d\mu = \sup_n \int_S f_n|g|\, d\mu < \infty$, i.e., $f \cdot g \in L^1$ for all $f \in L^{M^{**}}$. Therefore $g \in L^{M^{***}}$ in virtue of Theorem 1 and we are done because $L^{M^{***}} = L^{M^*}$.

2. $M^{**} \equiv 0$.

In this situation $M^*(r) = \infty$ for $r > 0$, i.e., $L^{M^*} = \{0\}$. Moreover, $\liminf_{r \to \infty} M(r)/r = 0$. The last equality gives $(L^M)^x = \{0\}$ ([4, Corollary 1]). Finally $(L^M)^x = L^{M^*}$.

Remark 2 The assumption that μ is atomless is essential. If $M(r) = r^p (0 < p < 1)$ then $M^*(r) = \infty$ for $r > 0$ and the Orlicz sequence space l^{M^*} is trivial, but $(l^p)^x = l^\infty$.

Now we will pay attention to the case of Orlicz sequence spaces l^M. Let us recall that functions M and N are *equivalent at zero* if there exist positive constants a, b, c, d, x such that

$$aM(bu) \leqq N(u) \leqq cM(du) \quad \text{for } u \in [0, x].$$

Since properties of a space l^M are determined by the behaviour of M in a neighborhood of zero we can assume that M is *finite-valued.*

Theorem 3 *If* M *has all properties listed in Theorem 2, then* $(l^M)^x = l^{M_\infty^*}$, *where* $M_\infty(u) = M(u)$ *for* $0 \leqq u \leqq 1$ *and* $M_\infty(u) = \infty$ *for* $u > 1$.

(The notation M_∞^* means the conjugate function of M_∞ – *not* the function $(M^*)_\infty$).

Proof. Let us consider two cases:

1. $M(u) = 0$ iff $u = 0$.

Denote by $\hat{M}$ the convex minorant of the function M in the interval $[0, 1]$, i.e., $\hat{M}(t) \leqq M(t)$ for all $t \in [0,1]$ and if N is a convex function on $[0,1]$ and it satisfies $N(t) \leqq M(t)$ for $t \in [0,1]$ then $N(t) \leqq \hat{M}(t)$ for all $t \in [0,1]$. Putting $\hat{M}(u) = \infty$ for $u > 1$ we obtain that $\hat{M}$ is convex on the half line $[0, \infty)$ and $\hat{M} \leqq M_\infty$. Thus $\hat{M} \leqq M_\infty^{**}$ and $l^M = lM_\infty \subset l^{M_\infty^{**}} \subset l^{\hat{M}}$.

Let (y_n) be a sequence of positive numbers belonging to $(l^M)^x$. The functional G defined on l_A^M by the formula $G((x_n)) = \sum_1^\infty x_n y_n$ is continuous with respect to the topology induced from $l_A^{\hat{M}}$ (see [5, Theorem 5.1]), and so it will remain continuous with respect to the topology induced from $l_A^{M_\infty^{**}}$ because this topology is stronger.

If $0 < x = (x(k)) \in l^{M_\infty^{**}}$, then there exists a sequence (x_n) of elements from l_A^M such that $0 \leqq x_n \uparrow x$. Since the sequence (x_n) is bounded in the topology induced from $l_A^{M_\infty^{**}}$, then $c = \sup G(x_n) < \infty$. Using the famous Fatou lemma we will obtain $c = \sum_1^\infty x(k)\, y(k)$. Therefore $(y_n) \in (l^{M_\infty^{**}})^x = l^{M_\infty^{***}} = l^{M_\infty^*}$ (the first equality follows from Theorem 1). In other words $(l^M)^x \subset l^{M_\infty^*}$.

On the other hand the inclusion $l^{M_\infty} \subset l^{M_\infty^{**}}$ gives $l^{M_\infty^*} = (l^{M_\infty^{**}})^x \subset (l^{M_\infty})^x \subset (l^M)^x$.

Finally $(l^M)^x = l^{M_\infty^*}$.

2. $M([0, t]) = \{0\}$ for some $t > 0$.

It is obvious that $l^M = l^\infty$, and so $(l^M)^x = l^1$. The proof will be finished if we show $l^{M_\infty^*} = l^1$.

Suppose first $t \geqq 1$. Under this assumption we have $M_\infty^*(u) = \sup\{uv - M(v): 0 \leqq v \leqq 1\} = u$, i.e., $l^{M_\infty^*} = l^1$.

If $t < 1$ then $M_\infty^*(u) \geqq \sup\{uv - M(v): 0 \leqq v \leqq t\} = tu$. The convexity of the function M_∞^* implies $M_\infty^*(u) \leqq M_\infty^*(1)\, u \leqq u$ for $u \in [0, 1]$. Therefore M_∞^* is equivalent at zero to the function $N(u) = u$ and thus $l^{M_\infty^{**}} = l^1$ and we are done.

Remark 3 If $M(1) > 0$ and $\liminf_{u \to \infty} M(u)/u > 0$, then $M_\infty^*(u) = M^*(u)$ for u belonging to some neighborhood of zero and therefore $l^{M_\infty^*} = l^{M^*}$.

Indeed, supposing $\liminf_{u \to \infty} M(u)/u > 0$ we find numbers $a > 0$, $v_0 > 1$ such that $M(v)/v > a$ for all $v \geqq v_0$. Taking $v \in (1, v_0)$ we have $M(v)/v \geqq M(1)/v_0 > 0$, and so $\inf\{M(v)/v:\ v > 1\} \geqq \min(a, M(1)/v_0))$. The following inequalities are valid for $u \in [0, \min(a, M(1)/v_0))$:

$$
\begin{aligned}
M^*(u) &= \sup\{uv - M(v): v \geqq 0\} = \sup\{uv - M(v): 0 \leqq v \leqq 1\} \\
&= \sup\{uv - M_\infty(v): 0 \leqq v \leqq 1\} = \sup\{uv - M_\infty(v): v \geqq 0\} = M_\infty^*(u).
\end{aligned}
$$

The assumption $M(1) > 0$ is essential: if $N(u) = 0$ for $0 \leqq u \leqq 2$ and $N(u) = u^2 - 4$ for $u > 2$, then $N^*(u) \geqq \sup\{uv - N(v):\ 0 \leqq v \leqq 2\} = 2u$ and $N_\infty^*(u) = u$, and so the functions N^* and N_∞^* are different, althogh they are equivalent at zero.

Remark 4 If $\liminf_{u \to \infty} M(u)/u = 0$ then $M^*(u) = \infty$ for $u > 0$.

We can find a sequence (u_n) tending to infinity such that $(M(u_n)/u_n)$ converges to zero. Therefore, fixing $r > 0$ and taking sufficiently large n we will obtain $M^*(r) \geqq u_n(r - (M(u_n)/u_n)) \geqq 2^{-1} r u_n$, i.e., $M^*(r) = \infty$.

Remark 5 If $\liminf M(u)/u=0$ (in particular, if M is concave) then $l^{M^*_\infty}=l^\infty$. Indeed, the inequality $\liminf M(u)/u>0$ guarantees the existence of positive numbers c and u_0 such that $M(u)/u>c$ for $0<u\leqq u_0$. Denoting $c_0=\inf\{M(v): u_0\leqq v\leqq 1\}$ we obtain $M^*_\infty(u)=0$ for $u<\min(c, c_0)$. Moreover M^*_∞ is not identically equal to zero. If there were $M^*_\infty(u)=0$ for $u>0$, then we would have $uv\leqq M(v)$ for all $u>0$ and all $v\in[0,1]$. In particular, $u\leqq M(1)$ for all u and we have got a contradiction because we consider finite-valued functions only. Finally, the facts that M^*_∞ vanish in some neighborhood of zero but M^*_∞ is not identically zero imply $l^{M^*_\infty}=l^\infty$.

Remark 6 Theorem 3 (under slightly stronger assumptions about M) was also proved (as a corollary of more general results) by M. Nowak (see [12, Theorem 3.3]) but his proof is more complicated.

Let us note assumptions about measures in Theorems 2 and 3 are important. There exists an Orlicz space $L^M(\mu)$ over a purely atomic measure μ with M non-convex such that the Köthe dual $(L^M(\mu))^x$ is not isomorphic to the space $L^{M*}(v)$ for any measure v.

Theorem 4 *Let M have all properties listed in Theorem 2 and suppose M satisfies the following conditions: $M(u)=0$ iff $u=0$, $\limsup_{u\to\infty} M(2u)/u=\infty$ and $M^{**}(u)=u$. There exists a purely atomic measure μ such that $(L^M(\mu))^x$ is not isomorphic to the space $L^{M^*}(v)$ for any measure v.*

Proof. Using [13, Theorem 1] we can choose a sequence (a_n) of positive numbers such that defining the measure μ on subsets of natural numbers by the formula $\mu(A)=\sum_{n\in A} a_n$ we will have that $L^M(\mu)$ is order isomorphic to l^∞. Therefore $(L^M(\mu))^x$ and l^1 are order isomorphic. If the spaces $(L^M(\mu))^x$ and $L^{M^*}(v)$ were isomorphic for some measure v, then $LM^*(v)$ would be order isomorphic to l^1 on account of [1, Theorem 5]. Thus by Theorem 1 the space $L^{M^{**}}(v)=(L^{M^*}(v))^x$ would be order isomorphic to l^∞, but this is impossible because $M^{**}(u)=u$ implies $L^{M^{**}}(v)=L^1(v)$.

Example of the function M satisfying the assumptions of Theorem 4. Let

$$a_n=\begin{cases}2^{n(n-1)/2} & \text{for odd } n\text{'s}\\ 2^{(n-2)(n-1)/2}+1 & \text{for even } n\text{'s}\end{cases}$$

$$b_n=\begin{cases}2^{n(n-1)/2}+1 & \text{for odd } n\text{'s}\\ 2^{(n+1)/2} & \text{for even } n\text{'s}.\end{cases}$$

Define a function M by the formula

$$M(u)=\begin{cases}u & \text{if } 0\leqq u\leqq 1\\ 2^{n(n+1)/2} & \text{if } a_n<u\leqq b_n.\end{cases}$$

It is easy to see that M is left continuous, continuous at zero, non-decreasing and M vanish only at zero. Putting $u_n=b_{2n-1}$ we get $2u_n\in(a_{2n}\, b_{2n}]$ and

$$M(2u_n)/M(u_n)=2^{2n}\to\infty, \text{i.e., } \limsup_{u\to\infty} M(2u)/u=\infty.$$

Suppose that N is a convex function defined on $[0, \infty)$ satisfying the inequality $N(u) \leqq M(u)$ for all u's. If we show that $N(u) \leqq u$, then the function $\tilde{M}(u)=u$ will be the convex minorant of M on account of the inequality $u \leqq M(u)$. Therefore $M^{**}(u)=u$ because M^{**} is the greatest convex function on $[0, \infty)$ not exceeding M.

Assume first $u \geqq 2$. Since $u \in [b_{2k-1}, b_{2k+1}]$ for some k then $u=cb_{2k-1}$ $(1-c)\, b_{2k+1}$ for some $c \in [0, 1]$. Thus

$$N(u) \leqq cN(b_{2k-1})+(1-c)\, N(b_{2k+1}) \leqq cM(b_{2k-1})+(1-c)\, M(b_{2k+1})$$
$$=cb_{2k-1}+(1-c)\, b_{2k+1}=u.$$

If $1<u \leqq 2$ then repeating the above arguments with $1=b_{2k-1}$ and $2=b_{2k+1}$ we obtain $N(u) \leqq u$. Finally, we have $N(u) \leqq M(u)=u$ for $0 \leqq u \leqq 1$. Thus $N(u) \leqq u$ on the half line and we are done.

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