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Laplace transform in spaces of ultradistributions

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Abstract. The Laplace transform in Komatsu ultradistributions is considered. Also, conditions are given under which an analytic function is a Laplace transformation of an ultradistribution.

0. Introduction

The Laplace transform of distributions was defined and studied by Schwartz, [12]. Later, Carmichael and Pilipović in [1] (see also [2]), considered the Laplace transform in Σ'_α of Beurling-Gevrey tempered ultradistributions and obtained some results concerning the so-called tempered convolution. In particular, they gave a characterization of the space of Laplace transforms of elements from Σ'_α supported by an acute closed cone in $\mathbb{R}^d$. Komatsu has given a great contribution to the investigations of the Laplace transform in ultradistribution and hyperfunction spaces considering them over appropriate domains, see [7] and references therein (see also [14]). Michalik in [9] and Lee and Kim in [8] have adapted the space of ultradistribution and Fourier hyperfunctions to the definition of the Laplace transform, following ideas of Komatsu. Our approach is different. We develop the theory within the space of already constructed ultradistributions of Beurling and Roumieu type. The ideas in the proofs of the two main theorems (theorem 2.1 and theorem 2.5) are similar to those in [13] in the case of Schwartz distributions. In these theorems are characterized ultradistributions defined on the whole $\mathbb{R}^d$ through the estimates of their Laplace transforms. This is the main point of our investigations contrary to other authors who investigated generalized functions supported by cones. We consider a restricted class of ultradistributions assuming conditions $(M.1)$, $(M.2)$ and $(M.3)$ (for example, cases $M_p = p!^s$, $s > 1$) in order to obtain fine representations through the analysis of the corresponding class of subexponentially bounded entire functions. With weaker conditions, $(M.3)'$ instead of $(M.3)$, or even in the case of quasianalyticity, we can obtain different, technically more complicate, structural representations.

1. Preliminaries

The sets of natural, integer, positive integer, real and complex numbers are denoted by $\mathbb{N}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}, \mathbb{C}$. We use the symbols for $x \in \mathbb{R}^d$: $\langle x\rangle = (1 + |x|^2)^{1/2}$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_d}$, $D_j^{\alpha_j} = i^{-1}\partial^{\alpha_j}/\partial x^{\alpha_j}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$. If $z \in \mathbb{C}^d$, by z^2 we will denote $z_1^2 + ... + z_d^2$. Note that, if $x \in \mathbb{R}^d$, $x^2 = |x|^2$.

Following [4], we denote by M_p a sequence of positive numbers $M_0 = 1$ so that:

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$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{Z}_+$;
$(M.2)$ $M_p \leq c_0 H^p \min_{0\leq q\leq p}\{M_{p-q}M_q\}$, $p, q \in \mathbb{N}$, for some $c_0, H \geq 1$;
$(M.3)$ $\sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq c_0 q \frac{M_q}{M_{q+1}}$, $q \in \mathbb{Z}_+$,

although in some assertions we could assume the weaker ones $(M.2)'$ and $(M.3)'$ (see [4]). For a multi-index $\alpha \in \mathbb{N}^d$, M_α will mean $M_{|\alpha|}$, $|\alpha| = \alpha_1 + ... + \alpha_d$. Recall, $m_p = M_p/M_{p-1}$, $p \in \mathbb{Z}_+$ and the associated function for the sequence M_p is defined by

$$M(\rho) = \sup_{p\in\mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

It is non-negative, continuous, monotonically increasing function, which vanishes for sufficiently small $\rho > 0$ and increases more rapidly then $\ln \rho^p$ when ρ tends to infinity, for any $p \in \mathbb{N}$.

Let $U \subseteq \mathbb{R}^d$ be an open set and $K \subset\subset U$ (we will use always this notation for a compact subset of an open set). Then $\mathcal{E}^{\{M_p\},h}(K)$ is the space of all $\varphi \in C^\infty(U)$ which satisfy $\sup_{\alpha\in\mathbb{N}^d} \sup_{x\in K} \frac{|D^\alpha \varphi(x)|}{h^\alpha M_\alpha} < \infty$ and $\mathcal{D}_K^{\{M_p\},h}$ is the space of all $\varphi \in C^\infty\left(\mathbb{R}^d\right)$ with supports in K, which satisfy $\sup_{\alpha\in\mathbb{N}^d} \sup_{x\in K} \frac{|D^\alpha \varphi(x)|}{h^\alpha M_\alpha} < \infty$;

$$\mathcal{E}^{(M_p)}(U) = \varprojlim_{K\subset\subset U} \varprojlim_{h\to 0} \mathcal{E}^{\{M_p\},h}(K), \quad \mathcal{E}^{\{M_p\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}(K),$$

$$\mathcal{D}_K^{(M_p)} = \varprojlim_{h\to 0} \mathcal{D}_K^{\{M_p\},h}, \quad \mathcal{D}^{(M_p)}(U) = \varinjlim_{K\subset\subset U} \mathcal{D}_K^{(M_p)},$$
$$\mathcal{D}_K^{\{M_p\}} = \varinjlim_{h\to\infty} \mathcal{D}_K^{\{M_p\},h}, \quad \mathcal{D}^{\{M_p\}}(U) = \varinjlim_{K\subset\subset U} \mathcal{D}_K^{\{M_p\}}.$$

The spaces of ultradistributions and ultradistributions with compact support of Beurling and Roumieu type are defined as the strong duals of $\mathcal{D}^{(M_p)}(U)$ and $\mathcal{E}^{(M_p)}(U)$, resp. $\mathcal{D}^{\{M_p\}}(U)$ and $\mathcal{E}^{\{M_p\}}(U)$. For the properties of these spaces, we refer to [4], [5] and [6]. In the future we will not emphasize the set U when $U = \mathbb{R}^d$. Also, the common notation for the symbols (M_p) and $\{M_p\}$ will be *.

If $f \in L^1$, then its Fourier transform is defined by $(\mathcal{F} f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx$, $\xi \in \mathbb{R}^d$.

By $\mathfrak{R}$ is denoted a set of positive sequences which monotonically increases to infinity. For $(r_p) \in \mathfrak{R}$, consider the sequence $N_0 = 1$, $N_p = M_p \prod_{j=1}^p r_j$, $p \in \mathbb{Z}_+$. One easily sees that this sequence satisfies $(M.1)$ and $(M.3)'$ and its associated function will be denoted by $N_{r_p}(\rho)$, i.e. $N_{r_p}(\rho) = \sup_{p\in\mathbb{N}} \log_+ \frac{\rho^p}{M_p \prod_{j=1}^p r_j}$, $\rho > 0$. Note, for given r_p and every $k > 0$ there is $\rho_0 > 0$ such that $N_{r_p}(\rho) \leq M(k\rho)$, for $\rho > \rho_0$.

It is said that $P(\xi) = \sum_{\alpha\in\mathbb{N}^d} c_\alpha \xi^\alpha$, $\xi \in \mathbb{R}^d$, is an ultrapolynomial of the class (M_p), resp. $\{M_p\}$, whenever the coefficients c_α satisfy the estimate $|c_\alpha| \leq CL^\alpha/M_\alpha$, $\alpha \in \mathbb{N}^d$ for some $L > 0$ and $C > 0$, resp. for every $L > 0$ and some $C_L > 0$. The corresponding operator $P(D) = \sum_\alpha c_\alpha D^\alpha$ is an ultradifferential operator of the class (M_p), resp. $\{M_p\}$ and they act continuously on $\mathcal{E}^{(M_p)}(U)$ and $\mathcal{D}^{(M_p)}(U)$, resp. $\mathcal{E}^{\{M_p\}}(U)$ and $\mathcal{D}^{\{M_p\}}(U)$ and the corresponding spaces of ultradistributions.

We denote by $\mathcal{S}_2^{M_p,m}\left(\mathbb{R}^d\right)$, $m > 0$, the space of all smooth functions φ which satisfy

$$\sigma_{m,2}(\varphi) := \left(\sum_{\alpha,\beta\in\mathbb{N}^d} \int_{\mathbb{R}^d} \left| \frac{m^{|\alpha|+|\beta|} \langle x\rangle^{|\alpha|} D^\beta \varphi(x)}{M_\alpha M_\beta} \right|^2 dx \right)^{1/2} < \infty, \tag{1}$$

supplied with the topology induced by the norm $\sigma_{m,2}$. The spaces $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ of tempered ultradistributions of Beurling and Roumieu type respectively, are defined as the strong duals of the spaces

$\mathcal{S}^{(M_p)} = \varprojlim_{m\to\infty} \mathcal{S}_2^{M_p,m}\left(\mathbb{R}^d\right)$ and $\mathcal{S}^{\{M_p\}} = \varinjlim_{m\to 0} \mathcal{S}_2^{M_p,m}\left(\mathbb{R}^d\right)$, respectively. All the good properties of $\mathcal{S}^*$ and its strong dual follow from the equivalence of the sequence of norms $\sigma_{m,2}$, $m > 0$, with each of the following sequences of norms (see [2], [10]):

(*a*) $\sigma_{m,p}$, $m > 0$; $p \in [1, \infty]$ is fixed;

(*b*) $s_{m,p}$, $m > 0$; $p \in [1, \infty]$ is fixed, where $s_{m,p}(\varphi) := \sum_{\alpha,\beta\in\mathbb{N}^d} \frac{m^{|\alpha|+|\beta|}\|| \cdot |^{\beta} D^{\alpha}\varphi(\cdot)\|_{L^p}}{M_\alpha M_\beta}$;

(*c*) s_m, $m > 0$, where $s_m(\varphi) := \sup_{\alpha\in\mathbb{N}^d} \frac{m^{|\alpha|}\|D^{\alpha}\varphi(\cdot)e^{M(m|\cdot|)}\|_{L_\infty}}{M_\alpha}$.

If we denote by $\mathcal{S}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$ the space of all infinitely differentiable functions on $\mathbb{R}^d$ for which the norm $\sigma_{m,\infty}$ is finite (obviously it is a Banach space), then $\mathcal{S}^{(M_p)}\left(\mathbb{R}^d\right) = \varprojlim_{m\to\infty} \mathcal{S}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$ and $\mathcal{S}^{\{M_p\}}\left(\mathbb{R}^d\right) = \varinjlim_{m\to 0} \mathcal{S}_\infty^{M_p,m}\left(\mathbb{R}^d\right)$. Also, for $m_2 > m_1$, the inclusion $\mathcal{S}_\infty^{M_p,m_2}\left(\mathbb{R}^d\right) \longrightarrow \mathcal{S}_\infty^{M_p,m_1}\left(\mathbb{R}^d\right)$ is a compact mapping. In [11] and [2] it is proved that $\mathcal{S}^{\{M_p\}} = \varprojlim_{r_i,s_j\in\mathfrak{R}} \mathcal{S}^{M_p}_{(r_p),(s_q)}$, where $\mathcal{S}^{M_p}_{(r_p),(s_q)} = \left\{\varphi \in C^\infty\left(\mathbb{R}^d\right) | \gamma_{(r_p),(s_q)}(\varphi) < \infty\right\}$ and $\gamma_{(r_p),(s_q)}(\varphi) = \sup_{\alpha,\beta\in\mathbb{N}^d} \frac{\left\|\langle x\rangle^{|\beta|}D^\alpha\varphi(x)\right\|_{L^2}}{\left(\prod_{p=1}^{|\alpha|} r_p\right) M_\alpha \left(\prod_{q=1}^{|\beta|} s_q\right) M_\beta}$. Also, the Fourier transform is a topological automorphism of $\mathcal{S}^*$ and of $\mathcal{S}'^*$.

2. Laplace transform

For a set $B \subseteq \mathbb{R}^d$ denote by $\operatorname{ch} B$ the convex hull of B.

Theorem 2.1. *Let B be a connected open set in $\mathbb{R}^d_\xi$ and $T \in \mathcal{D}'^*(\mathbb{R}^d_x)$ be such that, for all $\xi \in B$, $e^{-x\xi}T(x) \in \mathcal{S}'^*(\mathbb{R}^d_x)$. Then the Fourier transform $\mathcal{F}_{x\to\eta}\left(e^{-x\xi}T(x)\right)$ is an analytic function of $\zeta = \xi + i\eta$ for $\xi \in \operatorname{ch} B$, $\eta \in \mathbb{R}^d$. Furthermore, it satisfies the following estimates:*

for every $K \subset\subset \operatorname{ch} B$ there exist $k > 0$ and $C > 0$, resp. for every $k > 0$ there exists $C > 0$, such that

$$|\mathcal{F}_{x\to\eta}(e^{-x\xi}T(x))(\xi + i\eta)| \leq Ce^{M(k|\eta|)},\ \forall \xi \in K, \forall \eta \in \mathbb{R}^d. \tag{2}$$

Proof. Let K be a fixed compact subset of $\operatorname{ch} B$. There exists $0 < \varepsilon < 1/4$ and $\xi^{(1)}, ..., \xi^{(l)} \in B$ such that the convex hull Π of the set $\{\xi^{(1)}, ..., \xi^{(l)}\}$ contains the closed 4ε neighborhood of K (obviously $\Pi \subset\subset \operatorname{ch} B$). We shell prove that the set

$$\left\{S \in \mathcal{D}'^* | S(x) = T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}}, \xi \in K\right\} \tag{3}$$

is bounded in $\mathcal{S}'^*$. Note that by the condition in the theorem $T(x)e^{-x\xi} \in \mathcal{S}'^*$ and $e^{\varepsilon\sqrt{1+|x|^2}}$ is the restriction on the real axis of the function $e^{\varepsilon\sqrt{1+z^2}}$ that is analytic and single valued on the strip $\mathbb{R}^d + i\{y \in \mathbb{R}^d | |y| < 1/4\}$, and hence $e^{\varepsilon\sqrt{1+|x|^2}}$ is in $\mathcal{E}^*$. Note that

$$T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}} = \sum_{k=1}^{l} e^{\varepsilon\sqrt{1+|x|^2}} a(x,\xi)T(x)e^{-x\xi^{(k)}}, \tag{4}$$

where $a(x,\xi) = e^{-x\xi}\left(\sum_{k=1}^{l} e^{-x\xi^{(k)}}\right)^{-1}$. The function $a(x,\xi)$ satisfies the following conditions:

i) $0 < a(x,\xi) \leq 1$, $(x,\xi) \in \mathbb{R}^d \times \Pi$;

ii) $e^{\varepsilon'\sqrt{1+|x|^2}} a(x,\xi) \leq e^{\varepsilon'}$, $(x,\xi) \in \mathbb{R}^d \times K$, and $\forall \varepsilon' \leq 4\varepsilon$;

iii) $a(x,\xi) \in C^{\infty}\left(\mathbb{R}^{2d}\right)$.

iii) it's obvious. To prove *i*), take $\xi \in \Pi$. Then there exist $t_1, ..., t_l \geq 0$ such that $\xi = \sum_{k=1}^{l} t_k \xi^{(k)}$ and $\sum_{k=1}^{l} t_k = 1$. Then, by the weighted arithmetic mean-geometric mean inequality, we have

$$e^{-x\xi} = \prod_{k=1}^{l} e^{-xt_k\xi^{(k)}} \leq \sum_{k=1}^{l} t_k e^{-x\xi^{(k)}} \leq \sum_{k=1}^{l} e^{-x\xi^{(k)}},$$

from where it follows *i*). For the prove of *ii*), note that, for $(x,\xi) \in \mathbb{R}^d \times K$,

$$e^{\varepsilon' \sqrt{1+|x|^2}} a(x,\xi) \leq e^{\varepsilon' + \varepsilon'|x|} a(x,\xi) = e^{\varepsilon'} \max_{|t|\leq \varepsilon'} e^{-tx} a(x,\xi) = e^{\varepsilon'} \max_{|t|\leq\varepsilon'} a(x,\xi+t) \leq e^{\varepsilon'},$$

where the last inequality follows from *i*).

Now we will estimate the derivatives of $a(x,\xi)$. Let $s = \max_{\xi\in\Pi} |\xi|$. Then $a(z,\xi)$ is an analytic function of $z = x+iy$ on the strip $\mathbb{R}^d + i\{y \in \mathbb{R}^d \| y|s < \pi/4\}$, for every fixed $\xi \in \Pi$, because

$$\left|\sum_{k=1}^{l} e^{-z\xi^{(k)}}\right|^2 = \left|\sum_{k=1}^{l} e^{-x\xi^{(k)}} e^{-iy\xi^{(k)}}\right|^2 \geq \left(\sum_{k=1}^{l} e^{-x\xi^{(k)}} \cos y\xi^{(k)}\right)^2 \geq \left(\sum_{k=1}^{l} e^{-x\xi^{(k)}} \frac{\sqrt{2}}{2}\right)^2,$$

and hence

$$\left|\sum_{k=1}^{l} e^{-z\xi^{(k)}}\right| \geq \frac{\sqrt{2}}{2} \sum_{k=1}^{l} e^{-x\xi^{(k)}} > 0, \tag{5}$$

Take $0 < r < 1/\sqrt{d}$ so small such that $rs\sqrt{d} < \pi/4$. Then, from Cauchy integral formula, we have

$$|\partial_z^{\alpha} a(x,\xi)| \leq \frac{\alpha!}{r^{|\alpha|}} \sup_{|w_1-x_1|\leq r,...,|w_d-x_d|\leq r} \left|\frac{e^{-w\xi}}{\sum_{k=1}^{l} e^{-w\xi^{(k)}}}\right|.$$

If we use the inequality (5), we get (we put $w = u + iv$)

$$\begin{aligned} \left|\frac{e^{-(u+iv)\xi}}{\sum_{k=1}^{l} e^{-(u+iv)\xi^{(k)}}}\right| &\leq \frac{\sqrt{2}e^{-u\xi}}{\sum_{k=1}^{l} e^{-u\xi^{(k)}}} = \frac{\sqrt{2}e^{-x\xi}e^{-(u-x)\xi}}{\sum_{k=1}^{l} e^{-x\xi^{(k)}} e^{-(u-x)\xi^{(k)}}} \\ &\leq \frac{\sqrt{2}e^{-x\xi}e^{|u-x||\xi|}}{\sum_{k=1}^{l} e^{-x\xi^{(k)}} e^{-|u-x||\xi^{(k)}|}} \leq \frac{\sqrt{2}e^{-x\xi}e^{rs\sqrt{d}}}{\sum_{k=1}^{l} e^{-x\xi^{(k)}} e^{-rs\sqrt{d}}} = \sqrt{2}e^{2rs\sqrt{d}} a(x,\xi). \end{aligned}$$

So, we obtain the estimate

$$\left|\partial_x^{\alpha} a(x,\xi)\right| \leq \sqrt{2}e^{2s} \frac{\alpha!}{r^{|\alpha|}} a(x,\xi). \tag{6}$$

Note that, by the previous estimate and the property *ii*) of $a(x,\xi)$, it follows that $a(x,\xi) \in \mathcal{S}^*$ for every $\xi \in K$ and the set $\{a(x,\xi) | \xi \in K\}$ is a bounded set in $\mathcal{S}^*$. We will estimate the derivatives of $e^{\varepsilon\sqrt{1+|x|^2}}$. The function $e^{\varepsilon\sqrt{1+z^2}}$ is analytic on the strip $\mathbb{R}^d + i\{y \in \mathbb{R}^d \| y| < 1/4\}$, where we take the principal branch of the square root which is single valued and analytic on $\mathbb{C} \backslash (-\infty, 0]$. If we take $r < 1/(8d)$, from the Cauchy integral formula, we get the estimate $\left|\partial_z^{\alpha} e^{\varepsilon\sqrt{1+|x|^2}}\right| \leq \frac{\alpha!}{r^{|\alpha|}} \sup_{|w_1-x_1|\leq r,...,|w_d-x_d|\leq r} \left|e^{\varepsilon\sqrt{1+w^2}}\right|$. Put $w = u + iv$ and estimate as follows

$$\begin{aligned} \left|e^{\varepsilon\sqrt{1+w^2}}\right| &= e^{\mathrm{Re}\left(\varepsilon\sqrt{1+w^2}\right)} \leq e^{\left|\varepsilon\sqrt{1+w^2}\right|} \leq e^{\varepsilon\sqrt[4]{(1+|u|^2-|v|^2)^2+4(uv)^2}} \leq e^{\varepsilon\sqrt{1+|u|^2-|v|^2+2|uv|}} \\ &\leq e^{\varepsilon\sqrt{1+2|u|^2}} \leq e^{\varepsilon\sqrt{1+4|u-x|^2+4|x|^2}} \leq e^{\varepsilon\sqrt{1+1+4|x|^2}} \leq e^{2\varepsilon\sqrt{1+|x|^2}}. \end{aligned}$$

Hence

$$\left|\partial_x^{\alpha} e^{\varepsilon\sqrt{1+|x|^2}}\right| \leq \frac{\alpha!}{r^{|\alpha|}} e^{2\varepsilon\sqrt{1+|x|^2}}. \tag{7}$$

If we take r small enough we can make the previous estimates for the derivatives of $a(x,\xi)$ and $e^{\varepsilon\sqrt{1+|x|^2}}$ to hold for the same r. Now we obtain

$$\begin{aligned}\left|D_x^{\alpha}\left(e^{\varepsilon\sqrt{1+|x|^2}} a(x,\xi)\right)\right| &\leq \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{(\alpha-\beta)!}{r^{|\alpha-\beta|}} e^{2\varepsilon\sqrt{1+|x|^2}}\cdot\sqrt{2}e^{2s}\frac{\beta!}{r^{|\beta|}}a(x,\xi)\\ &\leq \sqrt{2}e^{2s}\frac{\alpha!}{r^{|\alpha|}}2^{|\alpha|}e^{2\varepsilon\sqrt{1+|x|^2}}a(x,\xi).\end{aligned}$$

Using the property *ii*) of the function $a(x,\xi)$, we get

$$\left|D_x^{\alpha}\left(e^{\varepsilon\sqrt{1+|x|^2}} a(x,\xi)\right)\right| \leq \sqrt{2}e^{2s}\frac{\alpha!2^{|\alpha|}}{r^{|\alpha|}}e^{2\varepsilon\sqrt{1+|x|^2}}a(x,\xi) \leq \sqrt{2}e^{2s+2\varepsilon}\frac{\alpha!2^{|\alpha|}}{r^{|\alpha|}},\ \forall\xi\in K. \tag{8}$$

By this estimate and proposition 7 of [3] one has $e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)$ is a multiplier for $\mathcal{S}'^*$. Because of (4), (3) is a subset of $\mathcal{S}'^*$. Now to prove that (3) is bounded in $\mathcal{S}'^*$. We will give the prove only in the $\{M_p\}$ case, the (M_p) case is similar. Let $\psi\in\mathcal{S}^{\{M_p\}}$. There exists $h>0$ such that $\psi\in\mathcal{S}_{\infty}^{M_p,h}$. Note that

$$\left\langle e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)T(x)e^{-x\xi^{(k)}},\psi(x)\right\rangle = \left\langle T(x)e^{-x\xi^{(k)}}, e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)\psi(x)\right\rangle,\ \forall k\in\{1,...,l\},\forall\xi\in K.$$

Choose $m\leq h/4$. By (8), we have

$$\begin{aligned}&\frac{m^{|\alpha|+|\beta|}\langle x\rangle^{\beta}\left|D^{\alpha}\left(e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)\psi(x)\right)\right|}{M_{\alpha}M_{\beta}}\\ &\leq m^{|\alpha|+|\beta|}\langle x\rangle^{\beta}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{\sqrt{2}e^{2s+2\varepsilon}(\alpha-\gamma)!2^{|\alpha-\gamma|}|D^{\gamma}\psi(x)|}{r^{|\alpha-\gamma|}M_{\alpha}M_{\beta}}\\ &\leq C_1\sigma_{h,\infty}(\psi)\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{h^{|\alpha|+|\beta|}(\alpha-\gamma)!2^{|\alpha-\gamma|}}{4^{|\alpha|+|\beta|}r^{|\alpha-\gamma|}M_{\alpha-\gamma}h^{|\gamma|+|\beta|}} \leq C_1\sigma_{h,\infty}(\psi)\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{h^{|\alpha|-|\gamma|}(\alpha-\gamma)!}{2^{|\alpha|}r^{|\alpha-\gamma|}M_{\alpha-\gamma}}\\ &\leq C\sigma_{h,\infty}(\psi),\ \forall\xi\in K.\end{aligned}$$

Hence $e^{\varepsilon\sqrt{1+|x|^2}}a(x,\xi)T(x)e^{-x\xi^{(k)}}$, $\xi\in K$, is bounded in $\mathcal{S}'^{\{M_p\}}$. Buy (4), the set (3) is bounded in $\mathcal{S}'^{\{M_p\}}$.

We will prove that $e^{-\varepsilon\sqrt{1+|x|^2}}\in\mathcal{S}^*$. In order to do that we will estimate the derivatives of $e^{-\varepsilon\sqrt{1+|x|^2}}$ with the Cauchy integral formula (similarly as for $e^{\varepsilon\sqrt{1+|x|^2}}$). We obtain

$$\left|\partial_z^{\alpha}e^{-\varepsilon\sqrt{1+|x|^2}}\right| \leq \frac{\alpha!}{r^{|\alpha|}}\sup_{|w_1-x_1|\leq r,...,|w_d-x_d|\leq r}\left|e^{-\varepsilon\sqrt{1+w^2}}\right|,$$

where, $0<r<1/(8d)$. Let $w=u+iv$. Then, if we put $\rho=\sqrt{(1+|u|^2-|v|^2)^2+4(uv)^2}$, $\cos\theta=\dfrac{1+|u|^2-|v|^2}{\sqrt{(1+|u|^2-|v|^2)^2+4(uv)^2}}$, $\sin\theta=\dfrac{2uv}{\sqrt{(1+|u|^2-|v|^2)^2+4(uv)^2}}$ (where $\theta\in(-\pi,\pi)$), we have that $\theta\in(-\pi/2,\pi/2)$ (because $\cos\theta>0$ and $\theta\in(-\pi,\pi)$) and

$$\operatorname{Re}\sqrt{1+|u|^2-|v|^2+2iuv} = \operatorname{Re}\sqrt{\rho(\cos\theta+i\sin\theta)} = \operatorname{Re}\sqrt{\rho}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) = \sqrt{\rho}\cos\frac{\theta}{2}\geq\frac{\sqrt{\rho}}{2},$$

where the second equality holds because we take the principal branch of $\sqrt{z}$. Because $r < 1/(8d)$, we get

$$\begin{aligned}\left|e^{-\varepsilon\sqrt{1+w^2}}\right| &= e^{\mathrm{Re}\left(-\varepsilon\sqrt{1+w^2}\right)} \leq e^{-\frac{\varepsilon}{2}\sqrt[4]{(1+|u|^2-|v|^2)^2+4(uv)^2}} \leq e^{-\frac{\varepsilon}{2}\sqrt{1+|u|^2-|v|^2}}\\ &\leq e^{-\frac{\varepsilon}{2}\sqrt{1+\frac{|x|^2}{2}-|u-x|^2-|v|^2}} \leq e^{-\frac{\varepsilon}{4}\sqrt{1+|x|^2}}.\end{aligned}$$

Hence, we obtain

$$\left|\partial_x^\alpha e^{-\varepsilon\sqrt{1+|x|^2}}\right| \leq \frac{\alpha!}{r^{|\alpha|}} e^{-\frac{\varepsilon}{4}\sqrt{1+|x|^2}}. \tag{9}$$

From this, it easily follows that $e^{-\varepsilon\sqrt{1+|x|^2}} \in \mathcal{S}^*$. So $e^{-x\xi}T(x) \in \mathcal{S}'^*\left(\mathbb{R}^d_x\right)$, for $\xi \in K$, because $e^{-x\xi}T(x) = T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}}e^{-\varepsilon\sqrt{1+|x|^2}}$ and we proved that $T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}} \in \mathcal{S}'^*\left(\mathbb{R}^d_x\right)$, for $\xi \in K$.

Put $f(\xi + i\eta) = \mathcal{F}_{x\to\eta}(e^{-x\xi}T(x))$. We will prove that f is an analytic function on $\mathrm{ch}\, B + i\mathbb{R}^d$. Let U be an arbitrary bounded open subset of $\mathrm{ch}\, B$ such that $K = \overline{U} \subset\subset \mathrm{ch}\, B$. For $\psi \in \mathcal{S}^*$ and $\xi \in U$, we have

$$\begin{aligned}\langle f(\xi+i\eta), \psi(\eta)\rangle &= \left\langle \mathcal{F}_{x\to\eta}\left(e^{-x\xi}T(x)\right), \psi(\eta)\right\rangle = \left\langle e^{-x\xi}T(x), \mathcal{F}(\psi)(x)\right\rangle\\ &= \left\langle e^{-x\xi}T(x), \int_{\mathbb{R}^d} e^{-ix\eta}\psi(\eta)d\eta\right\rangle = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x), e^{-\varepsilon\sqrt{1+|x|^2}}\int_{\mathbb{R}^d} e^{-ix\eta}\psi(\eta)d\eta\right\rangle\\ &= \left\langle\left(e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x)\right)\otimes 1_\eta, e^{-\varepsilon\sqrt{1+|x|^2}}e^{-ix\eta}\psi(\eta)\right\rangle\\ &= \int_{\mathbb{R}^d}\left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x)e^{-ix\eta}, e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle\psi(\eta)d\eta.\end{aligned}$$

Hence

$$f(\xi+i\eta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x)e^{-ix\eta}, e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle. \tag{10}$$

First we will prove that $f \in C^\infty\left(U \times \mathbb{R}^d_\eta\right)$. We will prove the differentiability only in ξ_1 and in the $\{M_p\}$ case. The existence of the rest of the derivatives is proved in analogous way and the (M_p) case is treated similarly. Let $\xi^{(0)} = \left(\xi_1^{(0)}, ..., \xi_d^{(0)}\right) = \left(\xi_1^{(0)}, \xi'\right) \in U$, $\xi = \left(\xi_1^{(0)} + \xi_1, \xi_2^{(0)}, ..., \xi_d^{(0)}\right) = \left(\xi_1^{(0)} + \xi_1, \xi'\right)$, $x = (x_1, ..., x_d) = (x_1, x')$. Let $0 < |\xi_1| < \delta < \varepsilon < 1$ such that the ball with radius δ and center in $\xi^{(0)}$ is contained in U. Then, by using (4) and (10), we obtain

$$\begin{aligned}&\frac{f(\xi+i\eta) - f(\xi^{(0)}+i\eta)}{\xi_1} - \left\langle e^{\varepsilon\sqrt{1+|x|^2}}(-x_1)e^{-x\xi^{(0)}}T(x)e^{-ix\eta}, e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle\\ &= \sum_{k=1}^{l}\left\langle e^{-ix\eta}e^{-x\xi^{(k)}}T(x)e^{\varepsilon\sqrt{1+|x|^2}}\left(\frac{a(x,\xi) - a\left(x,\xi^{(0)}\right)}{\xi_1} + x_1 a\left(x,\xi^{(0)}\right)\right), e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle.\end{aligned}$$

It is enough to prove that, for every $\psi \in \mathcal{S}^{\{M_p\}}$, $e^{\varepsilon\sqrt{1+|x|^2}}\left(\frac{a(x,\xi) - a\left(x,\xi^{(0)}\right)}{\xi_1} + x_1 a\left(x,\xi^{(0)}\right)\right)\psi(x) \longrightarrow 0$, when $\xi_1 \longrightarrow 0$, in $\mathcal{S}^{\{M_p\}}$. First note that

$$e^{\varepsilon\sqrt{1+|x|^2}}\left(\frac{a(x,\xi) - a\left(x,\xi^{(0)}\right)}{\xi_1} + x_1 a\left(x,\xi^{(0)}\right)\right) = e^{\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\left(\frac{e^{-x_1\xi_1} - 1}{\xi_1} + x_1\right).$$

Now, we get

$$\frac{e^{-x_1\xi_1} - 1}{\xi_1} + x_1 = \frac{1}{\xi_1}\sum_{n=1}^{\infty}\frac{(-1)^n x_1^n \xi_1^n}{n!} + x_1 = \sum_{n=2}^{\infty}\frac{(-1)^n x_1^n \xi_1^{n-1}}{n!}.$$

So, for $j \in \mathbb{N}$, $j \geq 2$ and $0 < |\xi_1| < \delta < \varepsilon < 1$, we have

$$\begin{aligned}\left|D^j_{x_1}\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\right| &= \left|D^j_{x_1}\left(\sum_{n=2}^{\infty}\frac{(-1)^n x_1^n \xi_1^{n-1}}{n!}\right)\right| = \left|\sum_{n=j}^{\infty}\frac{(-1)^n n! x_1^{n-j}\xi_1^{n-1}}{(n-j)!n!}\right|\\ &\leq |\xi_1|\sum_{n=j}^{\infty}\frac{|x_1|^{n-j}|\xi_1|^{n-2}}{(n-j)!} \leq |\xi_1|\sum_{n=j}^{\infty}\frac{|x_1|^{n-j}|\xi_1|^{n-j}}{(n-j)!} \leq \delta e^{|x_1|\delta}.\end{aligned}$$

Using similar technic, we obtain the estimates

$$\left|D_{x_1}\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\right| \leq \delta|x_1|e^{|x_1|\delta} \text{ and } \left|\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\right| \leq \delta|x_1|^2 e^{|x_1|\delta}.$$

So, in all cases, we have $\left|D^j_{x_1}\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\right| \leq \delta\langle x_1\rangle^2 e^{|x_1|\delta}$. By using (8), we get (for simpler notation we write j for the d-tuple $(j,0,...,0)$)

$$\begin{aligned}&\left|D^\alpha\left(e^{\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\psi(x)\right)\right|\\ &= \left|\sum_{\beta\leq\alpha}\sum_{j\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{j}D^{\beta-j}\left(e^{\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\right)D^j\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)D^{\alpha-\beta}\psi(x)\right|\\ &\leq \sum_{\beta\leq\alpha}\sum_{j\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{j}\sqrt{2}e^{2s}\frac{(\beta-j)!2^{|\beta-j|}}{r^{|\beta-j|}}e^{2\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\delta\langle x_1\rangle^2 e^{|x_1|\delta}|D^{\alpha-\beta}\psi(x)|\\ &\leq C\delta\langle x_1\rangle^2\sum_{\beta\leq\alpha}\sum_{j\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{j}\left(\frac{2}{r}\right)^{|\beta-j|}(\beta-j)!|D^{\alpha-\beta}\psi(x)|,\end{aligned}$$

where we used the inequality $e^{2\varepsilon\sqrt{1+|x|^2}}a(x,\xi^{(0)})e^{|x_1|\delta} \leq e^{3\varepsilon\sqrt{1+|x|^2}}a(x,\xi^{(0)}) \leq e^{3\varepsilon}$, which follows from the property *ii)* of $a(x,\xi)$. Because $\psi \in \mathcal{S}^{\{M_p\}}$, there exists $m > 0$ such that $\psi \in \mathcal{S}^{M_p,m}_\infty$. Choose h such that $h < m/4$, $h < 1/4$ and $hH < m$. We get

$$\begin{aligned}&\frac{h^{|\alpha|+|\beta|}\langle x\rangle^\beta\left|D^\alpha\left(e^{\varepsilon\sqrt{1+|x|^2}}a\left(x,\xi^{(0)}\right)\left(\frac{e^{-x_1\xi_1}-1}{\xi_1}+x_1\right)\psi(x)\right)\right|}{M_\alpha M_\beta}\\ &\leq C\delta\sum_{\gamma\leq\alpha}\sum_{j\leq\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{j}\left(\frac{2}{r}\right)^{|\gamma-j|}(\gamma-j)!\frac{\langle x_1\rangle^2\langle x\rangle^{|\beta|}h^{|\alpha|+|\beta|}|D^{\alpha-\gamma}\psi(x)|}{M_{\alpha-\gamma}M_{\gamma-j}M_jM_\beta}\\ &\leq C_1\delta\sum_{\gamma\leq\alpha}\sum_{j\leq\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{j}\left(\frac{2}{r}\right)^{|\gamma-j|}(\gamma-j)!\frac{\langle x\rangle^{|\beta|+2}h^{|\alpha|+|\beta|}H^{|\beta|+2}|D^{\alpha-\gamma}\psi(x)|}{M_{\alpha-\gamma}M_{\gamma-j}M_jM_{\beta+2}}\\ &\leq C_2\delta\sigma_{m,\infty}(\psi)\sum_{\gamma\leq\alpha}\sum_{j\leq\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{j}\left(\frac{2}{r}\right)^{|\gamma-j|}(\gamma-j)!\frac{h^{|\alpha|+|\beta|}H^{|\beta|}}{m^{|\alpha|-|\gamma|}m^{|\beta|+2}M_{\gamma-j}M_j}\\ &\leq C_3\delta\sigma_{m,\infty}(\psi)\sum_{\gamma\leq\alpha}\sum_{j\leq\gamma}\binom{\alpha}{\gamma}\binom{\gamma}{j}\left(\frac{2}{r}\right)^{|\gamma-j|}\left(\frac{h}{m}\right)^{|\alpha|-|\gamma|}\left(\frac{hH}{m}\right)^{|\beta|}\frac{h^{|\gamma|}(\gamma-j)!}{M_{\gamma-j}M_j} \leq C_0\delta\sigma_{m,\infty}(\psi),\end{aligned}$$

where we use $(M.2)$ and the fact $\frac{k^p p!}{M_p} \to 0$, when $p \to \infty$. Now, from this it follows that

$$e^{\varepsilon\sqrt{1+|x|^2}}\left(\frac{a(x,\xi)-a\left(x,\xi^{(0)}\right)}{\xi_1}+x_1 a\left(x,\xi^{(0)}\right)\right)\psi(x) \longrightarrow 0,\ \xi_1 \longrightarrow 0$$

in $\mathcal{S}^{\{M_p\}}$ and by the above remarks, the differentiability of $f(\xi+i\eta)$ on $U\times\mathbb{R}^d_\eta$ follows. Also, from the previous, we can conclude that $\partial^\alpha_\xi f(\xi+i\eta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}(-x)^\alpha e^{-x\xi}T(x)e^{-ix\eta}, e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle$ and similarly $\partial^\alpha_\eta f(\xi+i\eta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}(-ix)^\alpha e^{-x\xi}T(x)e^{-ix\eta}, e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle$. From this and the arbitrariness of U, the analyticity of $f(\xi+i\eta)$ follows because it satisfies the Cauchy-Riemann equations. So, for $\zeta=\xi+i\eta$, we get

$$f(\zeta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\zeta}T(x), e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle \tag{11}$$

and $\partial^\alpha_\zeta f(\zeta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}(-x)^\alpha e^{-x\zeta}T(x), e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle$, for $\zeta\in U+i\mathbb{R}^d_\eta$, for each fixed U (ε depends on U).

Now we will prove the estimates (2) for $f(\xi+i\eta)$. Let $K\subset\subset \operatorname{ch} B$ be arbitrary but fixed. First we will consider the (M_p) case. We know that $\mathcal{S}^{(M_p)}$ is a (FS) - space and $\mathcal{S}^{(M_p)} = \varprojlim_{h\to\infty}\mathcal{S}^{M_p,h}_\infty$. If we denote the closure of $\mathcal{S}^{(M_p)}$ in $\mathcal{S}^{M_p,h}_\infty$ by $\widetilde{\mathcal{S}}^{M_p,h}_\infty$ then $\mathcal{S}^{(M_p)} = \varprojlim_{h\to\infty}\widetilde{\mathcal{S}}^{M_p,h}_\infty$ and the projective limit is reduced. Then $\mathcal{S}'^{(M_p)} = \varinjlim_{h\to\infty}\widetilde{\mathcal{S}}'^{M_p,h}_\infty$ which is injective inductive limit with compact maps (because the projective limit is with compact maps). Because we proved that the set $\left\{S\in\mathcal{D}'^*|S(x)=T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}}, \xi\in K\right\}$ is bounded in $\mathcal{S}'^{(M_p)}$, it follows that there exists $h>0$ such that $\left\{S\in\mathcal{D}'^*|S(x)=T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}}, \xi\in K\right\}\subseteq\widetilde{\mathcal{S}}'^{M_p,h}_\infty$ and it's bounded there. By (9), we have the estimate

$$\begin{aligned}
\frac{h^{|\alpha|+|\beta|}\langle x\rangle^\beta\left|D^\alpha_x\left(e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right)\right|}{M_\alpha M_\beta} &\leq \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\frac{(2h)^{|\alpha|-|\gamma|}(2h)^{|\gamma|}h^{|\beta|}\langle x\rangle^\beta|\eta|^\gamma(\alpha-\gamma)!e^{-\frac{\varepsilon}{4}\sqrt{1+|x|^2}}}{2^{|\alpha|}r^{|\alpha-\gamma|}M_{\alpha-\gamma}M_\gamma M_\beta}\\
&\leq C_1\frac{1}{2^{|\alpha|}}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\left(\frac{2h}{r}\right)^{|\alpha|-|\gamma|}\frac{(\alpha-\gamma)!e^{M(h\langle x\rangle)}e^{M(2h|\eta|)}e^{-\frac{\varepsilon}{4}\langle x\rangle}}{M_{\alpha-\gamma}}\\
&\leq C'e^{M(2h|\eta|)},
\end{aligned}$$

where we use that $e^{M(h\langle x\rangle)}e^{-\frac{\varepsilon}{4}\langle x\rangle}$ is bounded and $\frac{k^p p!}{M_p}\to 0$ when $p\to\infty$. Then, for $\xi\in K$ and $\eta\in\mathbb{R}^d$,

$$|f(\xi+i\eta)| = \left|\left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x), e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle\right| \leq C\left\|e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right\|_{\widetilde{\mathcal{S}}^{M_p,h}_\infty} \leq \tilde{C}e^{M(2h|\eta|)}.$$

Now we will consider the $\{M_p\}$ case. $\mathcal{S}^{\{M_p\}}$ is a (DFS) - space and $\mathcal{S}^{\{M_p\}} = \varinjlim_{h\to 0}\mathcal{S}^{M_p,h}_\infty$, where the inductive limit is injective with compact maps. Let $h>0$ be fixed. For shorter notation, denote by F the set $\left\{S\in\mathcal{D}'^*|S(x)=T(x)e^{-x\xi+\varepsilon\sqrt{1+|x|^2}}, \xi\in K\right\}$ and by J the inclusion $\mathcal{S}^{M_p,h}_\infty\longrightarrow\mathcal{S}^{\{M_p\}}$. Because we already proved that F is a bounded subset of $\mathcal{S}'^{\{M_p\}}$, its image under ${}^tJ$ (the transposed mapping of J) is a bounded subset of $\mathcal{S}'^{M_p,h}_\infty$. By the above calculations we see that $e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}$ is in $\mathcal{S}^{M_p,m}_\infty$, for every $m>0$. Hence, for $\xi\in K$

and $\eta \in \mathbb{R}^d$, we have

$$\begin{aligned} |f(\xi+i\eta)| &= \left|\left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x), e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle\right| = \left|\left\langle {}^tJ\left(e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\xi}T(x)\right), e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle\right| \\ &\leq C'_h\left\|e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right\|_{\mathcal{S}^{M_p,h}_\infty} \leq C_h e^{M(2h|\eta|)}, \end{aligned}$$

where we used the above estimate for $\dfrac{h^{|\alpha|+|\beta|}\langle x\rangle^\beta\left|D^\alpha\left(e^{-ix\eta}e^{-\varepsilon\sqrt{1+|x|^2}}\right)\right|}{M_\alpha M_\beta}$. □

Remark 2.2. *If, for $S \in \mathcal{D}'^*$, the conditions of the theorem are fulfilled, we call $\mathcal{F}_{x\to\eta}\left(e^{-x\xi}S(x)\right)$ the Laplace transform of S and denote it by $\mathcal{L}(S)$. Moreover, by (11), $\mathcal{L}(S)(\zeta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\zeta}S(x), e^{-\varepsilon\sqrt{1+|x|^2}}\right\rangle$, for $\zeta \in U + i\mathbb{R}^d_\eta$, where $\overline{U} \subset\subset \operatorname{ch} B$ and ε depends on U.*

Note that, if for $S \in \mathcal{D}'^*$ the conditions of the theorem are fulfilled for $B = \mathbb{R}^d$, then the choice of ε can be made uniform for all $K \subset\subset \mathbb{R}^d$.

For the next theorem we need the following technical results.

Lemma 2.3. *Let $(k_p) \in \mathfrak{R}$. There exists $(k'_p) \in \mathfrak{R}$ such that $k'_p \leq k_p$ and $\prod_{j=1}^{p+q} k'_j \leq 2^{p+q}\prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_j$, for all $p, q \in \mathbb{Z}_+$.*

Proof. Define $k'_1 = k_1$ and inductively $k'_j = \min\left\{k_j, \dfrac{j}{j-1}k'_{j-1}\right\}$, for $j \geq 2$, $j \in \mathbb{N}$. Obviously $k'_j \leq k_j$ and one easily checks that (k'_j) is monotonically increasing. To prove that k'_j tends to infinity, suppose the contrary. Then, because (k'_j) is a monotonically increasing sequence of positive numbers, it follows that it is bounded by some $C > 0$. Because $(k_j) \in \mathfrak{R}$, there exists j_0, such that, for all $j \geq j_0$, $j \in \mathbb{N}$, $k_j \geq 2C$. So, for all $j \geq j_0 + 1$, $k'_j = \dfrac{j}{j-1}k'_{j-1}$. We get that $k'_j = \dfrac{j}{j_0}k'_{j_0} \to \infty$, when $j \longrightarrow \infty$, which is a contradiction. Hence $(k'_j) \in \mathfrak{R}$. Note that, for all $p, j \in \mathbb{Z}_+$, we have $k'_{p+j} \leq \dfrac{p+j}{j}k'_j$. Hence $\prod_{j=1}^{p+q} k'_j = \prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_{p+j} \leq \prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} \dfrac{p+j}{j}k'_j = \dfrac{(p+q)!}{p!q!}\prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_j \leq 2^{p+q}\prod_{j=1}^{p} k'_j \cdot \prod_{j=1}^{q} k'_j$. □

We will construct certain class of ultrapolynomials similar to those in [4], (see (10.9)′ in [4]), which will have the added beneficence of not having zeroes in a strip containing the real axis.

Let $c > 0$ be fixed. Let $k > 0$, $l > 0$ and $(k_p) \in \mathfrak{R}$, $(l_p) \in \mathfrak{R}$ be arbitrary but fixed. Choose $q \in \mathbb{Z}_+$ such that $\dfrac{c\sqrt{d}}{lm_p} < \dfrac{1}{2}$, for all $p \in \mathbb{N}$, $p \geq q$ in the (M_p) case and $\dfrac{c\sqrt{d}}{l_pm_p} < \dfrac{1}{2}$, for all $p \in \mathbb{N}$, $p \geq q$ in the $\{M_p\}$ case. Consider the entire functions

$$P_l(w) = \prod_{j=q}^{\infty}\left(1 + \frac{w^2}{l^2m_j^2}\right),\ w \in \mathbb{C}^d \tag{12}$$

in the (M_p) case, resp.

$$P_{l_p}(w) = \prod_{j=q}^{\infty}\left(1 + \frac{w^2}{l_j^2m_j^2}\right),\ w \in \mathbb{C}^d \tag{13}$$

in the $\{M_p\}$ case. It is easily checked that the entire function $P_l(w_1, 0, ..., 0)$, resp. $P_{l_p}(w_1, 0, ..., 0)$, of one variable satisfies the condition c) of proposition 4.6 of [4]. Hence, $P_l(w)$, resp. $P_{l_p}(w)$, satisfies the equivalent conditions a) and b) of proposition 4.5 of [4]. Hence, there exist $L > 0$ and $C' > 0$, resp. for every $L > 0$ there exists $C' > 0$, such that $|P_l(w)| \leq C'e^{M(L|w|)}$, resp. $|P_{l_p}(w)| \leq C'e^{M(L|w|)}$, for all $w \in \mathbb{C}^d$ and $P_l(D)$, resp. $P_{l_p}(D)$, are ultradifferential operators of (M_p), resp. $\{M_p\}$, type. It is easy to check that $P_l(w)$ and $P_{l_p}(w)$ don't have zeroes in $W = \mathbb{R}^d + i\{v \in \mathbb{R}^d \| v_j| \leq c,\ j = 1, ..., d\}$. For $w = u + iv \in W$, $|u| \geq 2c\sqrt{d}$, we have $\left|w^2\right| \geq \frac{|w|^2}{4}$ and $\left|1 + \frac{w^2}{l_j^2 m_j^2}\right| \geq 1$, for $j \geq q$. We estimate as follows

$$\begin{aligned}|P_{l_p}(w)| &= \left|\prod_{j=q}^{\infty}\left(1 + \frac{w^2}{l_j^2 m_j^2}\right)\right| = \sup_p \prod_{j=q}^{p}\left|1 + \frac{w^2}{l_j^2 m_j^2}\right| \geq \sup_p \prod_{j=q}^{p} \frac{\left|w^2\right|}{l_j^2 m_j^2} \geq \sup_p \prod_{j=q}^{p} \frac{|w|^2}{4 l_j^2 m_j^2}\\ &= \frac{\prod_{j=1}^{q-1} 4l_j^2}{|w|^{2q-2}}\left(\sup_p \frac{|w|^p M_{q-1}}{M_p \prod_{j=1}^{p} 2l_j}\right)^2 = C_0'\left(\frac{M_{q-1}\prod_{j=1}^{q-1} k_j}{|w|^{q-1}}\right)^2 e^{2N_{2l_p}(|w|)} \geq C_0' \frac{e^{N_{2l_p}(|w|)}}{e^{2N_{k_p}(|w|)}},\end{aligned}$$

where we put $C_0' = \prod_{j=1}^{q-1} \frac{4l_j^2}{k_j^2}$ and $l_p = l$ and $k_p = k$ in the (M_p) case. For $w \in W$, because $P_l(w)$, resp. $P_{l_p}(w)$, doesn't have zeroes in W, we get that there exist $C_0 > 0$ such that

$$|P_l(w)| \geq C_0 e^{-2M(|w|/k)} e^{M(|w|/(2l))}, \text{ resp. } |P_{l_p}(w)| \geq C_0 e^{-2N_{k_p}(|w|)} e^{N_{2l_p}(|w|)},\ w \in W. \tag{14}$$

Now, by using Cauchy integral formula, we can estimate the derivatives of $1/P_l(x)$, resp. $1/P_{l_p}(\xi)$. We will introduce some notations to make the calculations less cumbersome. For $r > 0$, denote by $B_r(a)$ the polydisc with center at a and radii r, i.e. $\{z \in \mathbb{C}^d \| z_j - a_j| < r,\ j = 1, 2, ..., d\}$ and by $T_r(a)$ the corresponding polytorus $\{z \in \mathbb{C}^d \| z_j - a_j| = r,\ j = 1, 2, ..., d\}$. We will do it for the $\{M_p\}$ case, for the (M_p) case it is similar. We already know that on W, $1/P_{l_p}(w)$ is analytic function (P_{l_p} doesn't have zeroes in W). Hence

$$\left|\partial_w^\alpha \frac{1}{P_{l_p}(x)}\right| \leq \frac{\alpha!}{r^{|\alpha|}} \cdot \left\|\frac{1}{P_{l_p}(z)}\right\|_{L^\infty(T_r(x))} \leq \frac{\alpha!}{C_0 r^{|\alpha|}} \cdot \left\|\frac{e^{2N_{k_p}(|z|)}}{e^{N_{2l_p}(|z|)}}\right\|_{L^\infty(T_r(x))},$$

for arbitrary but fixed $r \leq c$ (so $\overline{B_r(x)} \subseteq W$). For $x \in \mathbb{R}^d \backslash B_{2r\sqrt{d}}(0)$, there exists $j \in \{1, ..., d\}$ such that $|x_j| \geq 2r\sqrt{d}$. Then, on $T_r(x)$, $|z| \geq |x| - |z - x| = |x| - r\sqrt{d} \geq |x|/2$, i.e. $e^{N_{2l_p}(|z|)} \geq e^{N_{2l_p}(|x|/2)} = e^{N_{4l_p}(|x|)}$. Moreover, for such x, we have

$$e^{2N_{k_p}(|z|)} \leq e^{2N_{k_p}(|x| + r\sqrt{d})} \leq 4e^{2N_{k_p}(2r\sqrt{d})} e^{2N_{k_p}(2|x|)} = C_1 e^{2N_{k_p}(2|x|)},$$

where in the last inequality we used that $e^{M(\lambda+\nu)} \leq 2e^{M(2\lambda)}e^{M(2\nu)}$, for $\lambda \geq 0$, $\nu \geq 0$. So, we obtain $\left|\partial_w^\alpha \frac{1}{P_{l_p}(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} \frac{e^{2N_{k_p}(2|x|)}}{e^{N_{4l_p}(|x|)}}$. For x in $B_{2r\sqrt{d}}(0)$, $\left\|e^{2N_{k_p}(|z|)} e^{-N_{2l_p}(|z|)}\right\|_{L^\infty(T_r(x))}$ is bounded, so we can conclude that the above inequality holds, possible with another constant C. Analogously, we can prove that, for the (M_p) case, $\left|\partial_w^\alpha \frac{1}{P_l(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} \frac{e^{2M(2|x|/k)}}{e^{M(|x|/(4l))}}$. This is important, because, if $k > 0$ is fixed, resp. $(k_p) \in \mathfrak{R}$ is fixed, then we can find $l > 0$, resp. $(l_p) \in \mathfrak{R}$, such that $e^{2M(2|x|/k)} e^{-M(|x|/(4l))} \leq C'' e^{-M(|x|/k)}$, resp. $e^{2N_{k_p}(2|x|)} e^{-N_{4l_p}(|x|)} \leq C'' e^{-N_{k_p}(|x|)}$, for some $C'' > 0$. This inequality trivially follows from proposition 3.6 of [4] in the (M_p) case. To prove the inequality in the $\{M_p\}$ case, first note that $e^{2N_{k_p}(2|x|)} e^{N_{k_p}(|x|)} \leq e^{3N_{k_p/2}(|x|)}$. By lemma 2.3, there exists $(k_p') \in \mathfrak{R}$ such that $k_p' \leq k_p/2$ and $\prod_{j=1}^{p+q} k_j' \leq 2^{p+q} \prod_{j=1}^{p} k_j' \cdot \prod_{j=1}^{q} k_j'$, for all $p, q \in \mathbb{Z}_+$. So $e^{3N_{k_p/2}(|x|)} \leq e^{3N_{k_p'}(|x|)}$. If we put

$N_0 = 1$ and $N_p = M_p \prod_{j=1}^{p} k'_j$, for $p \in \mathbb{Z}_+$, then, by the properties of (k'_p), it follows that N_p satisfies $(M.1)$, $(M.2)$ and $(M.3)'$ where the constant H in $(M.2)$ for this sequence is equal to $2H$. Moreover, note that $N(\lambda) = N_{k'_p}(\lambda)$, for all $\lambda \geq 0$. We can now use proposition 3.6 of [4] for $N(|x|)$ (i.e. for $N_{k'_p}(|x|)$) and obtain $e^{3N_{k'_p}(|x|)} \leq c'' e^{N_{k'_p}(4H^2|x|)} = c'' e^{N_{k'_p/(4H^2)}(|x|)}$, for some $c'' > 0$. Now take l_p such that $4l_p = k'_p/(4H^2)$, $p \in \mathbb{Z}_+$ and the desired inequality follows. So, we obtain

$$\left|\partial_x^\alpha \frac{1}{P_l(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-M(|x|/k)}, \text{ resp. } \left|\partial_x^\alpha \frac{1}{P_{l_p}(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-N_{k_p}(|x|)}, \ x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d,$$

where C depends on k and l, resp. (k_p) and (l_p), and M_p; $r \leq c$ arbitrary but fixed. Moreover, from the above observation and (14), we obtain

$$|P_l(w)| \geq \tilde{C} e^{M(|w|/k)}, \text{ resp. } |P_{l_p}(w)| \geq \tilde{C} e^{N_{k_p}(|w|)}, \ w \in W, \tag{15}$$

for some $\tilde{C} > 0$.

Lemma 2.4. *let $g : [0,\infty) \longrightarrow [0,\infty)$ be an increasing function that satisfies the following estimate:*
for every $L > 0$ there exists $C > 0$ such that $g(\rho) \leq M(L\rho) + \ln C$.
Then there exists subordinate function $\epsilon(\rho)$ such that $g(\rho) \leq M(\epsilon(\rho)) + \ln C'$, for some constant $C' > 1$.

For the definition of subordinate function see [4].

Proof. If $g(\rho)$ is bounded then the claim of the lemma is trivial (we can take C' large enough such that the inequality will hold for arbitrary subordinate function). Assume that g is not bounded. We can easily find continuous strictly increasing function $f : [0,\infty) \longrightarrow [0,\infty)$ which majorizes g such that for every $L > 0$ there exists $C > 0$ such that $f(\rho) \leq M(L\rho) + \ln C$. Hence, there exists $\rho_1 > 0$ such that $f(\rho) > 0$ for $\rho \geq \rho_1$. There exists $\rho_0 > 0$ such that $M(\rho) = 0$ for $\rho \leq \rho_0$ and $M(\rho) > 0$ for $\rho > \rho_0$. Because $M(\rho)$ is continuous and strictly increasing on the interval $[\rho_0, \infty)$ and $\lim_{\rho\to\infty} M(\rho) = \infty$, M is bijection from $[\rho_0,\infty)$ to $[0,\infty)$ with continuous and strictly increasing inverse $M^{-1} : [0,\infty) \longrightarrow [\rho_0,\infty)$. Define $\epsilon(\rho)$ on $[\rho_1,\infty)$ in the following way $\epsilon(\rho) = M^{-1}(f(\rho))$ and define it linearly on $[0,\rho_1)$ such that it will be continuous on $[0,\infty)$ and $\epsilon(0) = 0$. Then $\epsilon(\rho)$ is strictly increasing and continuous on $[0,\infty)$. Moreover, for $\rho \in [\rho_1,\infty)$, it satisfies $f(\rho) = M(\epsilon(\rho))$. Hence, there exists $C' > 1$ such that $f(\rho) \leq M(\epsilon(\rho)) + \ln C'$, for $\rho \geq 0$. It remains to prove that $\epsilon(\rho)/\rho \longrightarrow 0$ when $\rho \longrightarrow \infty$. Assume the contrary. Then, there exist $L > 0$ and a strictly increasing sequence ρ_j which tends to infinity when $j \longrightarrow \infty$, such that $\epsilon(\rho_j) \geq 2L\rho_j$, i.e. $f(\rho_j) \geq M(2L\rho_j)$. For this L, by the condition for f, choose $C > 1$ such that $f(\rho) \leq M(L\rho) + \ln C$. Then we have $M(2L\rho_j) \leq M(L\rho_j) + \ln C$, which contradicts the fact that $e^{M(\rho)}$ increases faster then ρ^p for any p. One can obtain this contradiction by using equality (3.11) of [4]. □

Theorem 2.5. *Let B be a connected open set in $\mathbb{R}^d_\xi$ and f an analytic function on $B + i\mathbb{R}^d_\eta$. Let f satisfies the condition:*
for every compact subset K of B there exist $C > 0$ and $k > 0$, resp. for every $k > 0$ there exists $C > 0$, such that

$$|f(\xi + i\eta)| \leq C e^{M(k|\eta|)}, \ \forall \xi \in K, \forall \eta \in \mathbb{R}^d. \tag{16}$$

Then, there exists $S \in \mathcal{D}'^*(\mathbb{R}^d_x)$ such that $e^{-x\xi}S(x) \in \mathcal{S}'^*(\mathbb{R}^d_x)$, for all $\xi \in B$ and

$$\mathcal{L}(S)(\xi + i\eta) = \mathcal{F}_{x\to\eta}\left(e^{-x\xi}S(x)\right)(\xi + i\eta) = f(\xi + i\eta), \ \xi \in B, \ \eta \in \mathbb{R}^d. \tag{17}$$

Proof. Because of (16), for every fixed $\xi \in B$, $f_\xi = f(\xi + i\eta) \in \mathcal{S}'^*(\mathbb{R}^d_\eta)$. Put $T_\xi(x) = \mathcal{F}^{-1}_{\eta\to x}\left(f_\xi(\eta)\right)(x) \in \mathcal{S}'^*(\mathbb{R}^d_x)$ and $S_\xi(x) = e^{x\xi}T_\xi(x) \in \mathcal{D}'^*(\mathbb{R}^d_x)$. We will show that S_ξ does not depend on $\xi \in B$. Let U be an arbitrary, but fixed, bounded connected open subset of B, such that $K = \overline{U} \subset\subset B$.

Let $c > 2$ be such that $|\xi_j| \leq c/2$, for $\xi = (\xi_1, ..., \xi_d) \in K$. In the (M_p) case, choose $s > 0$ such that

$\int_{\mathbb{R}^d} e^{M(k|\eta|)}e^{-M(\frac{s}{2}|\eta|)}d\eta < \infty$ and $e^{2M(k|\eta|)} \leq \tilde{c}e^{M(\frac{s}{2}|\eta|)}$, for some constant $\tilde{c} > 0$. For the $\{M_p\}$ case, by the conditions in the theorem, for every $k > 0$ there exists $C > 0$, such that $\ln_+ |f(\xi + i\eta)| \leq M(k|\eta|) + \ln C$ for all $\xi \in K$ and $\eta \in \mathbb{R}^d$. The same estimate holds for the nonnegative increasing function

$$g(\rho) = \sup_{|\eta|\leq\rho} \sup_{\xi\in K} \ln_+ |f(\xi + i\eta)|.$$

If we use lemma 2.4 for this function we get that there exists subordinate function $\epsilon(\rho)$ and a constant $C > 1$ such that $g(\rho) \leq M(\epsilon(\rho)) + \ln C$. From this we have that $\ln_+ |f(\xi + i\eta)| \leq g(|\eta|) \leq M(\epsilon(|\eta|)) + \ln C$, i.e.

$$|f(\xi + i\eta)| \leq Ce^{M(\epsilon(|\eta|))},\ \forall\xi \in K, \forall\eta \in \mathbb{R}^d, \tag{18}$$

for some $C > 1$. By lemma 3.12 of [4], there exists another sequence $\tilde{N}_p$, which satisfies $(M.1)$, such that $\tilde{N}(\rho) \geq M(\epsilon(\rho))$ and $k'_p = \tilde{n}_p/m_p \longrightarrow \infty$ when $p \longrightarrow \infty$. Take $(k_p) \in \mathfrak{R}$ such that $k_p \leq k'_p$, $p \in \mathbb{Z}_+$. Then

$$e^{N_{k_p}(\rho)} = \sup_p \frac{\rho^p}{M_p \prod_{j=1}^p k_j} \geq \sup_p \frac{\rho^p}{M_p \prod_{j=1}^p k'_j} = e^{\tilde{N}(\rho)} \geq e^{M(\epsilon(\rho))}.$$

Hence, from (18), it follows that $|f(\xi + i\eta)| \leq Ce^{N_{k_p}(|\eta|)}$, for all $\xi \in K$ and $\eta \in \mathbb{R}^d$. Choose $(s_p) \in \mathfrak{R}$ such that $\int_{\mathbb{R}^d} e^{N_{k_p}(|\eta|)}e^{-N_{2s_p}(|\eta|)}d\eta < \infty$ and $e^{2N_{k_p}(|\eta|)} \leq \tilde{c}e^{N_{2s_p}(|\eta|)}$, for some $\tilde{c} > 0$.

Now, for the chosen c and s, resp. (s_p), by the discussion before the theorem, we can find $l > 0$, resp. $(l_p) \in \mathfrak{R}$, and entire functions $P_l(w)$ as in (12), resp. $P_{l_p}(w)$ as in (13), such that they don't have zeroes in $W = \mathbb{R}^d + i\{v \in \mathbb{R}^d | |v_j| \leq c,\ j = 1, ..., d\}$ and the following estimates hold

$$\left|\partial_x^\alpha \frac{1}{P_l(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-M(s|x|)},\ \text{resp.}\ \left|\partial_x^\alpha \frac{1}{P_{l_p}(x)}\right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-N_{s_p}(|x|)},\ x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d,$$

where C depends on s and l, resp. (s_p) and (l_p), and M_p; $r \leq c$ is arbitrary but fixed. For shorter notation, we will denote $P_l(w)$ and $P_{l_p}(w)$ by $P(w)$ in both cases. Define the entire functions $P_\xi(w) = P(w - i\xi) = \prod_{j=q}^{\infty}\left(1 + \frac{(w - i\xi)^2}{l^2 m_j^2}\right)$ in the (M_p) case, resp. $P_\xi(w) = P(w - i\xi) = \prod_{j=q}^{\infty}\left(1 + \frac{(w - i\xi)^2}{l_j^2 m_j^2}\right)$ in the $\{M_p\}$ case. As we noted in the construction of the entire functions $P(w)$ (the discussion before the theorem), $P(w)$ satisfies the equivalent conditions a) and b) of proposition 4.5 of [4]. Hence, there exist $L > 0$ and $C' > 0$, resp. for every $L > 0$ there exists $C' > 0$, such that $|P(w)| \leq C'e^{M(L|w|)}$, $w \in \mathbb{C}^d$ and $P(D)$ are ultradifferential operators of (M_p), resp. $\{M_p\}$, type. So, we obtain

$$|P_\xi(w)| = |P(w - i\xi)| \leq C'e^{M(L|w-i\xi|)} \leq C''e^{M(2L|w|)},\ w \in \mathbb{C}^d,$$

because $\xi = (\xi_1, ..., \xi_d)$ is such that $|\xi_j| \leq c/2$, for $j = 1, ..., d$. Hence, by proposition 4.5 of [4], $P_\xi(D)$ is an ultradifferential operator of class (M_p), resp. of class $\{M_p\}$, for every $\xi = (\xi_1, ..., \xi_d)$ such that $|\xi_j| \leq c/2$, $j = 1, ..., d$. Moreover, by the properties of $P(w)$, it follows that $P_\xi(w)$ is an entire function that doesn't have zeroes in $\mathbb{R}^d + i\{v \in \mathbb{R}^d | |v_j| \leq c/2,\ j = 1, ..., d\}$ for all $\xi \in K$. So, by using the Cauchy integral formula to estimate the derivatives, one obtains that $P_\xi(\eta)$ and $1/P_\xi(\eta)$ are multipliers for $\mathcal{S}'^*(\mathbb{R}^d_\eta)$. Also, by (15), we have $|P_\xi(\eta)| = |P(\eta - i\xi)| \geq \tilde{C}e^{M(s|\eta-i\xi|)} \geq \tilde{C}'e^{M(\frac{s}{2}|\eta|)}$, for all $\xi \in K$ and $\eta \in \mathbb{R}^d$ in the (M_p) case and similarly, $|P_\xi(\eta)| = |P(\eta - i\xi)| \geq \tilde{C}e^{N_{s_p}(|\eta-i\xi|)} \geq \tilde{C}'e^{N_{2s_p}(|\eta|)}$, for all $\xi \in K$ and $\eta \in \mathbb{R}^d$, in the $\{M_p\}$ case. For $\xi \in B$, put $f_\xi(\eta) = f(\xi + i\eta)$. Then $f_\xi(\eta)/P_\xi(\eta) \in L^1\left(\mathbb{R}^d_\eta\right) \cap \mathcal{E}^*\left(\mathbb{R}^d_\eta\right)$, for all $\xi \in K$. Observe that

$$e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}\left(f_\xi(\eta)\right)(x) = e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)P_\xi(\eta)}{P_\xi(\eta)}\right)(x) = e^{x\xi}P_\xi(D_x)\left(\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)}{P_\xi(\eta)}\right)(x)\right),$$

i.e.

$$S_\xi(x) = e^{x\xi} P_\xi(D_x)\left(\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)}{P_\xi(\eta)}\right)(x)\right). \tag{19}$$

Let $P(w) = \sum_\alpha c_\alpha w^\alpha$. For simpler notation, put $R(\eta) = f_\xi(\eta)/P_\xi(\eta)$ and calculate as follows

$$\begin{aligned} P(D_x)\left(e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}(R)(x)\right) &= \sum_\alpha c_\alpha \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-i\xi)^\beta e^{x\xi} D_x^{\alpha-\beta}\mathcal{F}^{-1}_{\eta\to x}(R)(x)\\ &= e^{x\xi}\sum_\alpha c_\alpha \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-i\xi)^\beta D_x^{\alpha-\beta}\mathcal{F}^{-1}_{\eta\to x}(R)(x). \end{aligned}$$

Note that

$$\begin{aligned} &\sum_\alpha c_\alpha \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-i\xi)^\beta D_x^{\alpha-\beta}\mathcal{F}^{-1}_{\eta\to x}(R)(x)\\ &= \mathcal{F}^{-1}_{\eta\to x}\left(\sum_\alpha c_\alpha \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(-i\xi)^\beta \eta^{\alpha-\beta}R(\eta)\right)(x) = \mathcal{F}^{-1}_{\eta\to x}\left(\sum_\alpha c_\alpha(\eta-i\xi)^\alpha R(\eta)\right)(x)\\ &= \mathcal{F}^{-1}_{\eta\to x}\left(P(\eta-i\xi)R(\eta)\right)(x) = \mathcal{F}^{-1}_{\eta\to x}\left(P_\xi(\eta)R(\eta)\right)(x) = P_\xi(D_x)\mathcal{F}^{-1}_{\eta\to x}(R)(x). \end{aligned}$$

From this and (19), we get $S_\xi(x) = P(D_x)\left(e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)}{P_\xi(\eta)}\right)(x)\right)$. Now, for $w = \eta - i\xi$, we have

$$e^{x\xi}\mathcal{F}^{-1}_{\eta\to x}\left(\frac{f_\xi(\eta)}{P_\xi(\eta)}\right)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{f(\xi+i\eta)e^{(\xi+i\eta)x}}{P(\eta-i\xi)}d\eta = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d-i\xi}\frac{f(iw)e^{iwx}}{P(w)}dw.$$

The function $\frac{f(iw)e^{iwx}}{P(w)}$ is analytic for $iw \in U+i\mathbb{R}^d$, i.e. $w\in\mathbb{R}^d - iU$ (because $P(w)$ is analytic in the last set and doesn't have zeroes there). Using the growth estimates for f and P, from the theorem of Cauchy-Poincaré, it follows that the last integral doesn't depend on $\xi \in U$. From this and the arbitrariness of U it follows that $S_\xi(x)$ doesn't depend on $\xi \in B$. We will denote this by $S(x)$. Now, by the observations in the beginning, it follows that $\mathcal{F}_{x\to\eta}\left(e^{-x\xi}S(x)\right) = f_\xi$ as ultradistributions in η for every fixed $\xi\in B$. By theorem 2.1, it follows that $\mathcal{F}_{x\to\eta}\left(e^{-x\xi}S(x)\right)$ is analytic function for $\zeta = \xi + i\eta \in B + i\mathbb{R}^d$, hence the equality (17) holds pointwise. □

Remark 2.6. *If f is an analytic function on $O = B + i\mathbb{R}^d_\eta$ and satisfies the conditions of the previous theorem then, by this theorem and theorem 2.1, it follows that f is analytic on* $\operatorname{ch} B + i\mathbb{R}^d_\eta$ *and satisfies the estimates (2) for every* $K \subset\subset \operatorname{ch} B$.

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