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Laplace transforms related to excursions of a one-dimensional diffusion

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Various known expressions in terms of hyperbolic functions for the Laplace transforms of random times related to one-dimensional Brownian motion are derived in a unified way by excursion theory and extended to one-dimensional diffusions.

Keywords: Brownion motion; hyperbolic function; local time

1. Introduction

It is well known that the Laplace transforms of many random times derived from a one-dimensional Brownian motion (BM) admit simple expressions in terms of hyperbolic functions. This paper offers a unified approach to these results, and presents their generalizations for a one-dimensional diffusion, using Itô's (1971) excursion theory. See Jeanblanc *et al*. (1997) for a survey of related results involving the Feynman–Kac formula for the distribution of an additive functional of BM, and see Borodin and Salminen (1996) for a vast array of fomulae for the distribution of functionals of a one-dimensional diffusion.

Section 2 presents the basic univariate formulae in a table, together with commentary and proofs. Section 3 shows how the univariate formulae can be combined with independence results from excursion theory to obtain various multivariate Laplace transforms. In the case of BM, these results have been applied to process control and stockmarket prices by Taylor (1975), and to the asymptotic distribution of windings of planar BM by Pitman and Yor (1986).

2. Univariate transforms

Let I be a subinterval of the real line. Let $(P^x, x \in I)$ govern $X = (X_t, t \geqslant 0)$ as a non-singular diffusion on I. For background and precise definitions, see Itô and McKean (1965), Rogers and Williams (1987) or Borodin and Salminen (1996). Assume for simplicity that X is recurrent. Let $0, x \in I$ with $0 \leqslant x$. Let $\lambda \geqslant 0$. In each row of Table 1, the left-hand entry is the P^x expectation of some functional of the diffusion path, mostly for $x = 0$. The middle entry gives a general expression for this expectation in terms of three *basic functions*: $g_\lambda(x, 0)$, the λ-potential density with respect to the speed measure; $s(x)$, the scale function; $\phi_\lambda(x)$, the P^0 Laplace transform of $T_x = \inf\{t: X_t = x\}$. These basic functions are interpreted probabilistically by Rows 1, 3 and 5 of Table 1. Analytic expressions for these functions, in terms of the semigroup or generator of X, are standard.

Explicit formulae for the basic functions are known for many diffusions, including Bessel and Ornstein–Uhlenbeck processes (Borodin and Salminen 1996). In particular, the third column of the table gives formulae derived from the second column in the case when X is a reflecting Brownian motion (RBM) on $I = [0, \infty)$, in terms of hyperbolic functions of θx, where $\theta = (2\lambda)^{1/2}$. All the formulae in the third column were obtained by Knight (1969), who also inverted most of these transforms. For X a standard BM on $I = (-\infty, \infty)$ the formulae in Table 1 apply for all $x \geqslant 0$ with

$$g_\lambda(x, 0) = \theta^{-1}\mathrm{e}^{-\theta x}, \qquad s(x) = 2x, \qquad \phi_\lambda(x) = \mathrm{e}^{-\theta x},$$

where $\theta = (2\lambda)^{1/2}$. Then $s_\lambda(x) = \theta^{-1}(1 - \mathrm{e}^{-2\theta x})$.

The following commentary introduces the notation of Table 1, line by line, and indicates proofs of the formulae by application of Itô's excursion theory.

Table 1. Some basic Laplace transforms

Row	Probabilistic quantity	General expression for $0 \leqslant x$	Expression for RBM with $\theta = (2\lambda)^{1/2}$
1	$P^x(L_{W_\lambda})$	$g_\lambda(x, 0)$	$\dfrac{\exp(-\theta x)}{\theta}$
2	$P^0(\mathrm{e}^{-\lambda\tau_\ell})$	$\exp\left(\dfrac{-\ell}{g_\lambda(0, 0)}\right)$	$\exp(-\ell\theta)$
3	$P^0(L_{T_x})$	$s(x)$	x
4	$P^0(M_{\tau_\ell} \leqslant x)$	$\exp\left(\dfrac{-\ell}{s(x)}\right)$	$\exp\left(\dfrac{-\ell}{x}\right)$
5	$P^0(\mathrm{e}^{-\lambda T_x})$	$\phi_\lambda(x)$	$\dfrac{1}{\cosh(\theta x)}$
6	$P^0(L_{T_x \wedge W_\lambda})$	$s_\lambda(x) = g_\lambda(0, 0) - \phi_\lambda(x) g_\lambda(x, 0)$	$\dfrac{\tanh(\theta x)}{\theta}$
7	$P^0\{\mathrm{e}^{-\lambda\tau_\ell} 1(M_{\tau_\ell} \leqslant x)\}$	$\exp\left(\dfrac{-\ell}{s_\lambda(x)}\right)$	$\exp\{-\ell\theta\coth(\theta x)\}$
8	$P^0(\mathrm{e}^{-\lambda G_x})$	$\dfrac{s_\lambda(x)}{s(x)}$	$\dfrac{\tanh(\theta x)}{\theta x}$
9	$P^0\{\mathrm{e}^{-\lambda(T_x - G_x)})$	$\dfrac{\phi_\lambda(x) s(x)}{s_\lambda(x)}$	$\dfrac{\theta x}{\sinh(\theta x)}$

2.1. Row 1

Let $L = (L_t,\ t \geq 0)$ be a local time process of X at 0, and let W_λ be exponentially distributed with rate λ, independent of X. This row identifies the potential density probabilistically as

$$g_\lambda(x, 0) = P^x(L_{W_\lambda}) = P^x \int_0^\infty \lambda\, e^{-\lambda t} L_t\, dt = P^x \int_0^\infty e^{-\lambda t}\, dL_t = c \int_0^\infty e^{-\lambda t} p(t, x, 0)\, dt,$$

where $p(t, x, y) = p(t, y, x) = P^x(X_t \in dy)/(m\, dy)$ is the jointly continuous transition density of X relative to the speed measure m, and c is a constant depending on the normalization of local time and conventions regarding constant factors in the definition of the scale function and speed measure of X. To be precise, it is supposed that $L_t = L_t^0$ where the process of local times $(L_t^y,\ t \geq 0,\ y \in I)$ is such that

$$\int_0^t f(X_s)\, ds = c \int_I f(y) L_t^y\, m\, dy$$

almost surely for every non-negative Borel function f. In the third column, for X a RBM, say $X = |B|$ where B is a BM, and also for $X = B$, we take L to be the occupation density of B at 0 relative to Lebesgue measure. Then Lévy's equivalence holds; L_t and $|B_t|$ have the same P^0 distribution.

2.2. Row 2

Let $(\tau_\ell,\ \ell \geq 0)$ be the inverse of L. The general expression for the Laplace transform of τ_ℓ is well known for L, the local time process of X at 0 for any recurrent point 0 of a strong Markov process X. This formula follows immediately from the probabilistic definition of $g_\lambda(0, 0)$ in Row 1, by Itô's excursion theory. Let P^0 govern a Poisson point process N on $(0, \infty)$ with rate λ, independent of X, and mark each excursion of X away from 0 by the times of points of N during the excursion, if any. Then, as explained by Greenwood and Pitman (1980) and Rogers and Williams (1987, Section VI.53), one obtains a homogeneous Poisson point process of marked excursions on the local time scale. (In the case when X spends positive Lebesgue time at 0, this process must also count marks between excursions.) Let W_λ be the time of the first point of N. Then L_{W_λ} is the time of the first marked excursion on the local time scale; so L_{W_λ} has exponential distribution with rate $1/P(L_{W_\lambda}) = 1/g_\lambda(0, 0)$. Thus

$$P^0(e^{-\lambda \tau_\ell}) = P^0(W_\lambda > \tau_\ell) = P^0(L_{W_\lambda} > \ell) = \exp\left(\frac{-\ell}{g_\lambda(0, 0)}\right).$$

Analysis of this formula, together with Krein's theory of strings, allowed Knight (1981) and Kotani and Watanabe (1982) to characterize the Lévy measures of the process of inverse local times $(\tau_\ell,\ \ell \geq 0)$. In particular, these Lévy measures are absolutely continuous with respect to Lebesgue measure on $(0, \infty)$, and the densities are Laplace transforms. See also Section 6 of Pitman (1996).

2.3. Row 3

This row defines $s(x)$ for $x > 0$. Note that $1/s(x)$ is the rate per unit local time of excursions from 0 that reach x. So, by the Poisson character of the excursion process, and the strong Markov property of X, for $0 < x < y$, given that an excursion reaches x, the chance that it reaches y is

$$P^x(T_y < T_0) = \frac{1/s(y)}{1/s(x)} = \frac{s(x)}{s(y)}.$$

That is to say, *the function* $s(x)$ *serves as a scale function for* X *on the interval* $[0, \infty]$, *with* $s(0) = 0$.

2.4. Row 4

Here $M_t = \max_{0 \leqslant s \leqslant t} X_s$. This is implied by Row 3 and the Poisson character of excursions on the local time scale, just as Row 1 implied Row 2.

2.5. Row 5

This row defines $\phi_\lambda(x)$. The evaluation of $\phi_\lambda(x)$ for RBM is made by the following well-known argument: for $\theta = (2\lambda)^{1/2}$, apply the optional sampling theorem to the martingale $\cosh(\theta|B_t|)\exp(-\lambda t)$ which is the average of the two martingales $\exp(\pm\theta B_t - \lambda t)$.

2.6. Row 6

This row defines a new function

$$\begin{aligned} s_\lambda(x) &:= P^0(L_{T_x \wedge W_\lambda}) \\ &= P^0(L_{W_\lambda}) - P^0\{(L_{W_\lambda} - L_{T_x})1(T_x < W_\lambda)\} \\ &= g_\lambda(0, 0) - \phi_\lambda(x) g_\lambda(x, 0) \end{aligned}$$

by application of the strong Markov property of X at time T_x, and the definitions of Rows 1 and 3. Substituting the formulae of Rows 1 and 3 for RBM gives the expression $s_\lambda(x) = \theta^{-1}\tanh(\theta x)$ for RBM.

2.7. Row 7

This is implied by Row 6, just as Row 1 implies Row 2, and Row 3 implies Row 4. In terms of the Poisson point process of marked excursions, Row 6 shows that $1/s_\lambda(x)$ is the rate of excursions that *either reach x or are marked*. The left-hand and middle entries of Row 7 show two different ways of computing the probability of no such excursions up to local time ℓ.

2.8. Row 8

Here G_x is the last zero of X before time T_x. Consider the first excursion that either reaches x or is marked. Compute the probability that this excursion reaches x, first by conditioning on G_x, then from the ratio of Poisson rates $\{1/s(x)\}/\{1/s_\lambda(x)\}$, to see that this probability is given by both the left and central entries of Row 8.

2.9. Row 9

The Poisson character of the excursion process implies that G_x and $T_x - G_x$ are independent (last exit decomposition). So Row 9 follows from Rows 5 and 8. For X a BM or RBM, the result is implicit in Williams' description of the process $(X_{G_x+t}, 0 \leqslant t \leqslant T_x - G_x)$ as a BES(3) process started at 0 and run till it first hits x (Williams 1974, equation (67.2) of Chapter II). More generally, if the upper end point of the basic interval I on which X is defined is b say, Williams' results show that the P^0 distribution of $(X_{G_x+t}, 0 \leqslant t \leqslant T_x - G_x)$ is identical with the $\hat{P}^0$ distribution of $(X_t, 0 \leqslant t \leqslant T_x)$ where the family of diffusion laws $(\hat{P}^x, x \in [0, b))$ conditions X to hit b before 0 (the Doob h transform of X for $h(x) = s(x)$). So Row 9 implies an expression for the $\hat{P}^0$ Laplace transform of T_x:

$$\hat{P}^0\{\exp(-\lambda T_x)\} = \frac{\phi_\lambda(x)s(x)}{s_\lambda(x)}. \tag{1}$$

The generator $\hat{A}$ for this conditioned diffusion is $\hat{A} = s^{-1}As$, where A is the generator of X. Using the standard fact (Itô and McKean 1965, Section 4.6) that $1/\phi_\lambda(x)$ is a solution of $Af = \lambda f$, it is easy enough to check that the inverse of the right side of (1) solves $\hat{A}f = \lambda f$ off 0. A more careful discussion of boundary behaviour is required to make this observation into an analytic proof of (1). See Jeanblanc *et al.* (1997) for related results.

3. Multivariate transforms

As noted by Knight (1969, 1978), for each $x > 0$ the Poisson character of the excursion process implies that $(X_t, 0 \leqslant t \leqslant G_x)$ given $L_{T_x} = \ell$ has the same distribution as $(X_t, 0 \leqslant t \leqslant \tau_\ell)$ given $(M_{\tau_\ell} < x)$. Combined with Rows 4 and 7 of Table 1 this implies that

$$P^0\{\exp(-\lambda G_x)|L_{T_x} = \ell\} = P^0\{\exp(-\lambda\tau_\ell)|M_{\tau_\ell} < x\} = \exp\{-\ell\xi_\lambda(x)\}, \tag{2}$$

where

$$\xi_\lambda(x) := \frac{1}{s_\lambda(x)} - \frac{1}{s(x)} \overset{\text{RBM}}{=} \theta\coth(\theta x) - \frac{1}{x} \tag{3}$$

and the notation $\overset{\text{RBM}}{=}$ means equality in the case when X is RBM, with $\theta = (2\lambda)^{1/2}$. This result for RBM is due to Knight (1969). Combine (2) and (3) with the fact that L_{T_x} has exponential distribution with rate $1/s(x)$ to obtain

$$P^0\{\exp(-\alpha L_{T_x} - \lambda G_x)\} = \frac{s_\lambda(x)}{s(x)\{1+\alpha s_\lambda(x)\}} \stackrel{\text{RBM}}{=} \frac{1}{\theta x \coth(\theta x) + \alpha x}. \tag{4}$$

Using the independence of (L_{T_x}, G_x) and $T_x - G_x$, and Row 9, this implies that

$$P^0\{\exp(-\alpha L_{T_x} - \lambda T_x)\} = \frac{\phi_\lambda(x)}{1+\alpha s_\lambda(x)} \stackrel{\text{RBM}}{=} \frac{\theta}{\theta \cosh(\theta x) + \alpha \sinh(\theta x)}. \tag{5}$$

Williams (1976) obtained this formula for RBM and used it to deduce the closely related formulae of Taylor (1975).

Consider now the occupation times

$$A_t^+ := \int_0^t \mathrm{d}s\, 1(X_s > 0), \qquad A_t^- := \int_0^t \mathrm{d}s\, 1(X_s \leqslant 0). \tag{6}$$

As a preliminary for computation of the P^0 joint Laplace transform of $A^+_{T_x}$, $A^-_{T_x}$ and L_{T_x}, observe from (2) and the independence of positive and negative excursions that there is the identity

$$P^0\{\exp(-\lambda A^+_{\tau_\ell}) M_{\tau_\ell} < x\} P^0\{\exp(-\lambda A^-_{\tau_\ell})\} = \exp\{-\ell \xi_\lambda(x)\}. \tag{7}$$

Now write $A^+_{\tau_\ell} = B_\ell(x) + C_\ell(x)$ where $B_\ell(x)$ is the total length of those positive excursions before time τ_ℓ which fail to reach x, and $C_\ell(x)$ is the total length of those positive excursions before time τ_ℓ which do reach x. By another application of the Poisson character of the excursion process, $B_\ell(x)$ and $C_\ell(x)$ are independent. Since the events $(M_{\tau_\ell} < x)$ and $(C_\ell(x) = 0)$ are identical, and $B_\ell(x) \downarrow 0$ as $x \downarrow 0$, it follows that

$$P^0\{\exp(-\lambda A^+_{\tau_\ell}) | M_{\tau_\ell} < x\} = P^0[\exp\{-\lambda B_\ell(x)\}] \uparrow 1 \text{ as } x \downarrow 0$$

and hence from (7)

$$P^0\{\exp(-\lambda A^-_{\tau_\ell})\} = \exp\{-\ell \xi_\lambda(0+)\}, \tag{8}$$

where

$$\xi_\lambda(0+) := \lim_{x \downarrow 0} \xi_\lambda(x) \stackrel{\text{BM}}{=} \frac{\theta}{2} \tag{9}$$

and $\stackrel{\text{BM}}{=}$ means equality in the case when X is BM, with $\theta = (2\lambda)^{1/2}$. Now (7) gives

$$P^0\{\exp(-\lambda A^+_{\tau_\ell}) | M_{\tau_\ell} < x\} = \exp[-\ell\{\xi_\lambda(x) - \xi_\lambda(0+)\}] \tag{10}$$

and (8), (10) and the independence of positive and negative excursions combine to yield

$$P^0\{\exp(-\lambda A^+_{\tau_\ell} - \mu A^-_{\tau_\ell}) | M_{\tau_\ell} < x\} = \exp[-\ell\{\xi_\lambda(x) - \xi_\lambda(0+) + \xi_\mu(0+)\}]. \tag{11}$$

Conditioning on $L_{T_x} = \ell$, as before, and then integrating out ℓ yields the formula

$$P^0\{\exp(-\alpha L_{T_x} - \lambda A^+_{G_x} - \mu A^-_{G_x})\} = \frac{s_\lambda(x)/s(x)}{1 + \{\alpha + \xi_\mu(0+) - \xi_\lambda(0+)\} s_\lambda(x)}. \tag{12}$$

Since $A^-_{T_x} = A^-_{G_x}$ and $A^+_{T_x} = A^+_{G_x} + (T_x - G_x)$ where $T_x - G_x$ is independent of $(L_{T_x}, A^-_{G_x}, A^+_{G_x})$ by the last exit decomposition, (12) combined with Row 9 of Table 1 yields finally

$$P^0\{\exp(-\alpha L_{T_x} - \lambda A^+_{T_x} - \mu A^-_{T_x})\} = \frac{\phi_\lambda(x)}{1 + \{\alpha + \xi_\mu(0+) - \xi_\lambda(0+)\} s_\lambda(x)} \tag{13}$$

$$\overset{\mathrm{BM}}{=} \frac{\theta}{\theta \cosh(\theta x) + \{2\alpha + (2\mu)^{1/2} \sinh(\theta x)}. \tag{14}$$

Pitman and Yor (1986, Proof of Theorem 4.2) found this formula for BM by martingale calculus and applied it to the asymptotic distribution of windings of a planar BM.

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