LAPLACE TRANSFORMS WHICH ARE NEGATIVE POWERS OF QUADRATIC POLYNOMIALS

G. LETAC AND J. WESOŁOWSKI

ABSTRACT. We find the distributions in $\mathbb{R}^n$ for the independent random variables X and Y such that $\mathbb{E}(X|X+Y) = a(X+Y)$ and $\mathbb{E}(q(X)|X+Y) = bq(X+Y)$ where q runs through the set of all quadratic forms on $\mathbb{R}^n$ orthogonal to a given quadratic form v. The essential part of this class is provided by distributions with Laplace transforms $(1 - 2\langle c, s \rangle + v(s))^{-p}$ that we describe completely, obtaining a generalization of a Gindikin theorem. This leads to the classification of natural exponential families with the variance function of type $\frac{1}{p}m \otimes m - \varphi(m)M_v$, where M_v is the symmetric matrix associated to the quadratic form v and $m \mapsto \varphi(m)$ is a real function. These natural exponential families extend the classical Wishart distributions on Lorentz cones already considered by Jensen, and later on by Faraut and Korányi.

1. INTRODUCTION

As an attempt to extend the celebrated Lukacs' theorem on characterization of the gamma distribution (Lukacs, 1955), Wang (1981) observes the following interesting fact. Its statement uses the entire function

$$(1.1) \qquad f_p(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\Gamma(m+p)},$$

where $p > 0$.

Theorem 1.1. *Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be non-Dirac independent positive random variables valued in $\mathbb{R}^2$ such that there exist real numbers a and b with the following property:*

$$(1.2) \qquad \mathbb{E}(X|X+Y) = a(X+Y), \quad \mathbb{E}(X_i^2|X+Y) = b(X_i+Y_i)^2, \quad i=1,2.$$

Then $0 < a^2 < b < a < 1$ and for $p = \frac{a^2-ab}{b-a^2}$ there exist non-negative numbers a_1, a_2, a_{12} such that $a_1a_2 - a_{12} \geq 0$, and for small enough (s_1, s_2) one has

$$(1.3) \qquad \begin{aligned} \mathbb{E}(e^{s_1X_1+s_2X_2}) &= (1 - a_1s_1 - a_2s_2 + (a_1a_2 - a_{12})s_1s_2)^{-p}, \\ \mathbb{E}(e^{s_1Y_1+s_2Y_2}) &= (1 - a_1s_1 - a_2s_2 + (a_1a_2 - a_{12})s_1s_2)^{-\frac{1-a}{a}p}. \end{aligned}$$

Furthermore if $a_1 a_2 - a_{12} > 0$, the distribution of X is

$$(\theta_1 \theta_2 - 1)^p e^{\theta_1 x_1 + \theta_2 x_2} \frac{1}{\Gamma(p)} (x_1 x_2)^{p-1} f_p(x_1 x_2) \mathbf{1}_{(0,\infty)^2}(x_1, x_2) dx_1 dx_2,$$

where $\theta_1 = \frac{-a_2}{a_1 a_2 - a_{12}}$ and $\theta_2 = \frac{-a_1}{a_1 a_2 - a_{12}}$.

The result is not difficult: just multiply (1.2) by $e^{s_1(X_1+Y_1)+s_2(X_2+Y_2)}$ and take expectations to obtain differential equations for the Laplace transforms of X and Y.

A quasi-automatic way to extend Wang's result is to consider X and Y independent in $\mathbb{R}^n$ such that there exist a and b with

$$\mathbb{E}(X|X+Y) = a(X+Y), \quad \mathbb{E}(X_i^2|X+Y) = b(X_i + Y_i)^2, \quad i = 1, 2, \ldots, n.$$

The same method of proof gives that there exists $p > 0$ and coefficients c_T with $T \subset \{1, \ldots, n\}$ such that the Laplace transform of X has the form

$$(1.4) \qquad \left(\sum_T c_T s^T \right)^{-p}$$

with the convention $s^T = \prod_{i \in T} s_i$. Special cases of these distributions occur in literature (see Bernardoff (2006), Bar-Lev *et al.* (1994), Griffiths (1984), and comment 4 in Section 6 below). Actually the set of acceptable parameters $((c_T)_T; p)$ is not known in general, and in most of the known cases, the density is unreachable.

A better way to extend Wang's bivariate characterization to n dimensions is based on the following reformulation: let us observe that the second part of (1.2) can be rewritten by saying that

$$(1.5) \qquad \mathbb{E}(q(X)|X+Y) = bq(X+Y)$$

for all quadratic forms on $\mathbb{R}^2$ of type $q(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$ that are having a diagonal representative matrix $M_q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. This two dimensional linear space of quadratic forms can be seen as the space of the quadratic forms on $\mathbb{R}^2$ which are orthogonal to the quadratic form $v(x) = 2x_1 x_2$ whose representative matrix is $M_v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Here the space of quadratic forms has the scalar product $\langle q_1, q_2 \rangle = \operatorname{tr} M_{q_1} M_{q_2}$ and orthogonality means $\operatorname{tr} M_{q_1} M_{q_2} = 0$. Similarly, Bobecka and Wesołowski (2004) in their Theorem 3, instead of the second part of (1.2), consider the conditions $\mathbb{E}(X_1^2|X+Y) = b(X_1 + Y_1)^2$ and $\mathbb{E}(X_1 X_2|X+Y) = b(X_1 + Y_1)(X_2 + Y_2)$. These conditions can be expressed by taking q's in (1.5) with the representative matrices of the form $M_q = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & 0 \end{pmatrix}$. These quadratic forms are orthogonal to v where $M_v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, namely $v(x) = x_2^2$. Our extension will be based on the consideration of (1.5) for $n \geq 2$ dimensions by taking all quadratic forms q which are orthogonal to a fixed v as in the previous two cases. The Laplace transforms of the distributions that we get are essentially of the form $(1 - 2\langle c, s \rangle + v(s))^{-p}$. The discussion is unfortunately somewhat obscured by consideration of several cases. But the heart of our method can be seen in formulas (2.13) to (2.15). This is covered in Section 2.

In Section 3 we identify the acceptable triples (c, v, p) for the Laplace transform of the form $(1 - 2\langle c, s\rangle + v(s))^{-p}$. We compute the corresponding probability distributions. This includes a generalization of the Gindikin Theorem (see Gindikin (1975)) about Wishart distributions on symmetric cones for the Lorentz cone (look at the comments after Theorem 3.1). Section 4 is devoted to a complete description by convolution semigroups of the exceptional case isolated by Bobecka and Wesołowski (2004).

After the point of view of regression illustrated by (1.5) (recall that statisticians use the term regression curve for the graph of the function $h(u) = \mathbb{E}(V \,|U = u)$ when (U, V) is a vector random variable), after the second point of view of Laplace transforms which are negative powers of quadratic polynomials, a third one is given by natural exponential families (NEF) on $\mathbb{R}^n$. We consider the NEF such that their variance function V_F has the following property: for some $p > 0$ the matrix

$$V_F(m) - \frac{1}{p}m \otimes m$$

is proportional to a constant symmetric matrix M_v for all m. Surprisingly this condition is more or less equivalent to (1.5). The source of this point of view is the fact that when $M_v = \mathrm{diag}(1, -1, \ldots, -1)$, then

$$V_F(m) = \frac{1}{p}m \otimes m + \frac{1}{4p}(m_1^2 - m_2^2 - \ldots - m_n^2)M_v$$

is the variance function of the NEF of the Wishart distributions on the Lorentz cone with shape parameter $p \geq (n - 2)/2$. This is the subject of Section 5. Section 6 links the present paper with literature, specially on Euclidean Jordan algebras.

2. CHARACTERIZATION BY QUADRATIC REGRESSION

Here is our extension. As usual, $L_X(s) = \mathbb{E}(e^{\langle s, X\rangle})$ denotes the Laplace transform of the random variable X.

Theorem 2.1. *Let $n \geq 2$. Let v be a quadratic form on $\mathbb{R}^n$ of rank r and let Q_v be the space of quadratic forms q on $\mathbb{R}^n$ such that $\mathrm{tr}(M_q M_v) = 0$. Let X and Y be non-Dirac independent random variables on $\mathbb{R}^n$ having exponential moments. Assume that there exists $a \in \mathbb{R}$ such that $\mathbb{E}(X|X + Y) = a(X + Y)$. Then $0 < a < 1$ and $L_Y = (L_X)^{(1-a)/a}$. Furthermore there exists a real number b such that for all $q \in Q_v$,*

$$(2.6) \qquad\qquad \mathbb{E}(q(X)|X + Y) = bq(X + Y)$$

if and only if $0 < a^2 \leq b \leq a < 1$ and

1. *Either $b = a$. In this case $r = 1$. We write $v = \lambda e_1 \otimes e_1$ for some vector e_1 of norm 1 and some real $\lambda \neq 0$. Then $X = e_1 X'$, where X' has an arbitrary distribution on the real line.*
2. *Or $b = a^2$. In this case X has normal distribution in $\mathbb{R}^n$ with arbitrary mean, and covariance proportional to v, where v or $-v$ is semi-positive definite.*
3. *Or $a^2 < b < a$.*
   - *Either $r \geq 2$, and there exist a real number λ and $c \in \mathbb{R}^n$ such that for $p = \frac{a^2 - ab}{b - a^2}$ one has*

$$(2.7) \qquad\qquad \mathbb{E}(e^{\langle s, X\rangle}) = (1 - 2\langle c, s\rangle + \lambda v(s))^{-p}.$$

  – *Or $r = 1$. We write $v = \lambda e_1 \otimes e_1$ for some vector e_1 of norm 1 and some real $\lambda \neq 0$. Then there exists c in $\mathbb{R}^n$ orthogonal to e_1 and a real function f_1 defined on some open interval in $\mathbb{R}$ such that for $X = \langle X, e_1 \rangle e_1 + X'$ one has*

(2.8) $$\mathbb{E}(e^{\langle s, X \rangle}) = \mathbb{E}(e^{s_1 \langle X, e_1 \rangle + \langle s', X' \rangle}) = (f_1(s_1) - \langle c, s' \rangle)^{-p},$$

  where $s = s_1 e_1 + s'$ with $\langle s', e_1 \rangle = 0$.

Proof. Since X and Y have exponential moments there exist two open maximal convex sets $\Theta(X)$ and $\Theta(Y)$ containing 0 on which their Laplace transforms $L_X(s)$ and L_Y are finite. Denote their intersection by Θ. We also write $\kappa_X = \log L_X$ and $\kappa_Y = \log L_Y$. Since $\mathbb{E}(X|X+Y) = a(X+Y)$ holds, let us multiply both sides by $e^{\langle s, X+Y \rangle}$ and take expectations. We get for s in Θ:

$$(1-a)L_X'(s)L_Y(s) = aL_X(s)L_Y'(s)$$

that we rewrite as $(1-a)\kappa_X' = a\kappa_Y'$. Since X and Y are non-Dirac, $a = 0$ and $a = 1$ are impossible. Since κ_X and κ_Y are convex, a and $(1-a)$ have the same sign and $a \in (0,1)$. Since $\kappa_X(0) = \kappa_Y(0) = 0$ we get $L_Y(s) = (L_X(s))^{(1-a)/a}$ on Θ and by the principle of maximal analyticity for Laplace transforms we get $\Theta(X) = \Theta(Y) = \Theta$. For simplicity we now denote $\kappa = \kappa_X$.

We use the following notation: if $P(s) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha s_1^{\alpha_1} \ldots s_n^{\alpha_n}$ is any real polynomial with respect to n variables, we denote by $P(\frac{\partial}{\partial s})$ the differential operator

$$P\left(\frac{\partial}{\partial s}\right) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \left(\frac{\partial}{\partial s_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial s_n}\right)^{\alpha_n}.$$

In particular if f is any C^2 real function of n variables with gradient f' and Hessian f'' and if q is a quadratic form, note that we can write $q\left(\frac{\partial}{\partial s}\right)(f) = \mathrm{tr}\,(M_q f'')$ and $q(f') = \mathrm{tr}\,(M_q(f' \otimes f'))$.

The next equality is again rather standard. For $s \in \Theta$ we multiply both sides of (2.6) by $e^{\langle s, X+Y \rangle}$ and take expectations. We get for all $q \in Q_v$

(2.9) $$\left(1 - \frac{b}{a}\right) q\left(\frac{\partial}{\partial s}\right)\kappa = \left(\frac{b}{a^2} - 1\right) q(\kappa'),$$

since in general we have for any quadratic form q on $\mathbb{R}^n$

$$q\left(\frac{\partial}{\partial s}\right) e^{\kappa(s)} = e^{\kappa(s)}\left(q\left(\frac{\partial}{\partial s}\right)\kappa(s) + q(\kappa'(s))\right)$$

and

$$q\left(\frac{\partial}{\partial s}\right)(fg)(s) = g(s)q\left(\frac{\partial}{\partial s}\right)f(s) + 2q(f'(s), g'(s)) + f(s)q\left(\frac{\partial}{\partial s}\right)g(s)$$

with the notation $q(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$ for the polarized form of q. We now discuss the solutions of (2.9) in the various cases $b = a$, $b = a^2$, $a^2 < b < a$, and we will show after that $b \notin [a^2, a]$ is impossible.

THE CASE $b = a$.

From (2.9) we can claim that $q(\kappa'(s)) = 0$ for all $q \in Q_v$ and for all $s \in \Theta$. Suppose first that $v = 0$. Then Q_v is the set of all quadratic forms: taking q positive definite implies $\kappa'(s) = 0$ and X Dirac, a case that we have excluded. Suppose that $v \neq 0$. Then $q(\kappa'(s)) = 0$ can be reformulated as $\mathrm{tr}\,[M_q(\kappa'(s) \otimes \kappa'(s))] = 0$. Since this is true for any $q \in Q_v$, this implies the existence of a scalar function $f(s)$ such

that $\kappa'(s) \otimes \kappa'(s) = f(s)v$. Since X is non-Dirac, there exists s_0 such that $\kappa'(s_0) \neq 0$ and thus v has rank 1. Without loss of generality, by changing the orthonormal basis into a new basis $(e_1, \ldots, e_n)$ if necessary, we assume that M_v is written by blocks as $M_v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, where the square diagonal blocks are of order 1 and $n - 1$, respectively. Thus

$$\left(\frac{\partial \kappa}{\partial s_1}\right)^2 = f(s), \ \frac{\partial \kappa}{\partial s_j} = 0, \ j = 2, \ldots, n.$$

Thus X is concentrated on the line $\mathbb{R}e_1$. This leads to the result.

THE CASE $b = a^2$.

By (2.9) we get that $q(\frac{\partial}{\partial s})\kappa(s) = 0$ for all s. Furthermore this is true for all $q \in Q_v$. Thus there exists a scalar function $s \mapsto g(s)$ on Θ such that $\kappa''(s) = g(s)v$. We show that g is a constant function. To see this, without loss of generality by changing the orthonormal basis if necessary, we assume that $M_v = (v_{ij})$ is diagonal. Thus we can write for any i, j, l in $\{1, \ldots, n\}$

$$\frac{\partial^3 \kappa}{\partial s_i \partial s_j \partial s_l} = \frac{\partial g}{\partial s_l} v_{ij} = \frac{\partial g}{\partial s_i} v_{lj} = \frac{\partial g}{\partial s_j} v_{il}.$$

Since $n \geq 2$, by choosing $i = j \neq l$ we easily deduce from this that $\frac{\partial g}{\partial s_i} = 0$ for all i. Thus X is Gaussian as indicated, and v or $-v$ has to be semipositive definite.

THE CASE $a^2 < b < a$.

We introduce the positive number $p = \frac{a^2 - ab}{b - a^2}$ such that (2.9) becomes $pq(\frac{\partial}{\partial s})\kappa = q(\kappa')$ which can be rewritten

$$\mathrm{tr}\,(M_q(p\kappa'' - \kappa' \otimes \kappa')) = 0.$$

Since this holds true for all $q \in Q_v$, this is equivalent to the existence of a scalar function g on Θ such that

$$\kappa''(s) - \frac{1}{p}\kappa'(s) \otimes \kappa'(s) = g(s)v.$$

THE SUBCASE $a^2 < b < a$ AND $2 \leq r$.

We assume that $r \geq 2$. We prove first that either g has no zeros, or g is identically 0. If $g(s_0) = 0$ for some $s_0 \in \Theta$, by changing the orthonormal basis if necessary we assume that $\kappa'(s_0)$ is a multiple of e_1, where $e = (e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Thus $\kappa''(s_0)$ is proportional to the block matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, where the diagonal blocks are square matrices of order 1 and $n - 1$, respectively. Since $\kappa''(s_0)$ is the covariance of the probability $e^{\langle s_0, x \rangle - \kappa(s_0)}\mu(dx)$ where $X \sim \mu$, this implies that μ is concentrated on $\mathbb{R}e_1$, thus $\kappa''(s)$ is proportional to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for all s, which implies that $g(s) = 0$ for all $s \in \Theta$. Thus κ is a function of s_1 alone and we get

(2.10)
$$\frac{\partial^2 \kappa}{\partial s_1^2} = \frac{1}{p}\left(\frac{\partial \kappa}{\partial s_1}\right)^2.$$

This easily shows that X_1 is gamma distributed with the shape parameter p. This provides formula (2.7) for the case $\lambda = 0$.

We now assume that g has no zeros on Θ. Thus we denote $f = -1/g$. We now rewrite

$$(2.11) \qquad \kappa''(s) = \frac{1}{p}\kappa'(s) \otimes \kappa'(s) - \frac{1}{f(s)}v$$

as an equality between two symmetric bilinear forms on $\mathbb{R}^n$ with respect to the variables h_1 and h_2 of $\mathbb{R}^n$. We obtain

$$(2.12) \qquad \kappa''(s)(h_1, h_2) = \frac{1}{p}\kappa'(s)(h_1)\kappa'(s)(h_2) - \frac{1}{f(s)}v(h_1, h_2).$$

We now take the differential of both sides of (2.12) and estimate it in the direction h_3 of $\mathbb{R}^n$. For simplification, write $a_i = \kappa'(s)(h_i) = \langle \kappa'(s), h_i \rangle$. We get the equality

(2.13)

$$\kappa'''(s)(h_1, h_2, h_3) = \frac{1}{p}a_1\kappa''(s)(h_2, h_3) + \frac{1}{p}a_2\kappa''(s)(h_1, h_3) + \frac{\langle f'(s), h_3 \rangle}{f(s)^2}v(h_1, h_2)$$

$$(2.14) \qquad = \frac{2}{p^2}a_1 a_2 a_3 - \frac{1}{pf(s)}(a_1 v(h_2, h_3) + a_2 v(h_1, h_3) + a_3 v(h_1, h_2))$$

$$(2.15) \qquad + \left(\frac{a_3}{pf(s)} + \frac{\langle f'(s), h_3 \rangle}{f(s)^2} \right) v(h_1, h_2)$$

(we have transformed (2.13) with the help of (2.12)). Now since (2.13) and (2.14) are trilinear symmetric forms in (h_1, h_2, h_3), this implies that (2.15) is also a trilinear symmetric form.

Since $(h_1, h_2) \mapsto v(h_1, h_2)$ is not of rank 0 or 1, the fact that (2.15) is a trilinear symmetric form implies that the linear form

$$h_3 \mapsto \frac{a_3}{pf(s)} + \frac{\langle f'(s), h_3 \rangle}{f(s)^2}$$

is zero, that is, $f(s)\kappa'(s) + pf'(s) = 0$. Taking the differential of this equality and using (2.12) gives $f''(s) = \frac{1}{p}v$. This implies the existence of constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ such that $f(s) = \alpha + \langle \beta, s \rangle + \frac{1}{2p}v(s)$. Now the equality $f(s)\kappa'(s) + pf'(s) = 0$ gives that $\kappa(s) = -p\log(f(s)/\alpha)$ since $\kappa(0) = 0$ (note that f is never zero on Θ and thus $\alpha = f(0) \neq 0$). Thus there exists $\lambda = \frac{1}{2p\alpha} \neq 0$ and $c = -\frac{\beta}{2\alpha} \in \mathbb{R}^n$ such that $\mathbb{E}(e^{\langle s, X \rangle}) = (1 - 2\langle c, s \rangle + \lambda v(s))^{-p}$. For further use in Section 5, note that

$$(2.16) \qquad f(s) = \frac{\lambda}{2p}(1 - 2\langle c, s \rangle + \lambda v(s)).$$

THE SUBCASE $a^2 < b < a$ AND $r = 1$.

If g is identically zero on Θ we are sent back to (2.10) as in the case $r \geq 2$. In this case, (2.8) holds with $c = 0$ and an affine f_1 which is the same as (2.7) with $\lambda = 0$.

If g is not identically zero, denote

$$\Theta_0 = \{s \in \Theta \; ; \; g(s) \neq 0\}.$$

Since g is analytic, Θ_0 is an open set and we define $f = -1/g$ on Θ_0.

We write $v(s) = \pm\langle w, s \rangle^2$ for some non-zero vector $w \in \mathbb{R}^n$. We take without loss of generality w as the unit vector e_1 (by doing a suitable linear transformation),

and we write $s = s_1 e_1 + \ldots + s_n e_n$ where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$. We rewrite (2.11) suitably expressed in coordinates as

$$(2.17) \qquad \frac{\partial^2 \kappa(s)}{\partial s_1^2} = \frac{1}{p} \left(\frac{\partial \kappa(s)}{\partial s_1} \right)^2 - \frac{1}{f(s)},$$

$$(2.18) \qquad \frac{\partial^2 \kappa(s)}{\partial s_i \partial s_j} = \frac{1}{p} \frac{\partial \kappa(s)}{\partial s_i} \frac{\partial \kappa(s)}{\partial s_j} \text{ if } (i,j) \neq (1,1).$$

First differentiate (2.17) with respect to s_j with $j \neq 1$. In (2.18) take $i = 1$ and differentiate with respect to s_1. Taking the difference between these two equalities makes that the third derivatives disappear. Plugging in the resulting equation formulas (2.17) and (2.18) we get for $j \neq 1$:

$$(2.19) \qquad \frac{1}{p} \frac{\partial \kappa(s)}{\partial s_j} = -\frac{1}{f(s)} \frac{\partial f(s)}{\partial s_j}.$$

Now differentiate (2.19) with respect to s_i, $i = 1, \ldots, n$, and using again (2.18) and (2.19) we get for $(i,j) \neq (1,1)$:

$$(2.20) \qquad \frac{\partial^2 f(s)}{\partial s_i \partial s_j} = 0.$$

Since Θ_0 is not empty, it contains an open cube $\Theta_1 = (u_1, v_1) \times \ldots \times (u_n, v_n)$ and there exists a real function $s_1 \mapsto C_1(s_1)$ on (u_1, v_1) and constants $C_2, \ldots, C_n$ such that for all s in Θ_1 we have

$$(2.21) \qquad f(s) = C_1(s_1) + C_2 s_2 + \ldots + C_n s_n.$$

Furthermore, by definition, f is never zero on Θ_0, and without loss of generality we may assume that $f > 0$ on Θ_1 (if not, we change v into $-v$). Here we have used the fact that Θ_1 is connected.

On the other hand integrating (2.19) leads to the existence of functions $D_j(s)$ on Θ_1 such that D_j does *not* depend on s_j and such that for any $j = 2, \ldots, n$, we have $\kappa(s) = -p \log f(s) + D_j(s)$. All the D_j's are therefore equal to a common function $s_1 \mapsto D(s_1)$ defined on (u_1, v_1), and thus

$$(2.22) \qquad \kappa(s) = -p \log f(s) + D(s_1).$$

We carry (2.22) into (2.18) taken for $(i,j) = (1,2)$. After cancellation with the help of (2.20) we find that D is a constant. Finally writing $c_j = -C_j e^{-D/p}$ for $j = 2, \ldots, n$ and $f_1(s_1) = C_1(s_1) e^{-D/p}$, we get (2.8) on Θ_1. By the principle of maximal analyticity for Laplace transforms, we conclude that (2.8) holds on Θ itself.

THE CASE $b \notin [a^2, a]$.

The last thing to prove is to check that $b \notin [a^2, a]$ is impossible. The algebraic calculations of the case $a^2 < b < a$ are still valid, but with $p < 0$. If $r \geq 2$ we end up with the Laplace transform $(1 - 2\langle c, s \rangle + \lambda v(s))^{-p}$. We discuss the case $\lambda \neq 0$. Since $Q_v = Q_{\lambda v}$, we replace λv by v. Without loss of generality we can also choose the coordinates such that $v(s) = s_1^2 + s_2^2 + \ldots + s_k^2 - s_{k+1}^2 - \ldots - s_r^2$, where $0 \leq k \leq r \leq n$. Then

$$1 - 2\langle c, s \rangle + v(s) = 1 - 2c_1 s_1 - \ldots - 2c_r s_r - \ldots - 2c_n s_n + s_1^2 + s_2^2 + \ldots + s_k^2 - s_{k+1}^2 - \ldots - s_r^2.$$

This shows that for $j > r$ the second derivative of the convex function $s_j \mapsto -p \log(1 - 2\langle c, s \rangle + \lambda v(s))$ is $4pc_j^2(1 - 2\langle c, s \rangle + \lambda v(s))^2$. Since $p < 0$ this implies $c_j = 0$.

Thus without loss of generality we assume $r = n$. Equality (2.11) applied to the case $p < 0$ shows that either $k = 0$ or $k = n$: this comes from the fact that

$$\kappa''(s) + \frac{1}{|p|}\kappa'(s) \otimes \kappa'(s) = \frac{1}{f(s)}v$$

is necessarily positive definite as the sum of a positive definite operator $\kappa''(s)$ and of a semipositive definite one. Write $\epsilon = -1$ for $k = 0$ and $\epsilon = 1$ for $k = n$. Furthermore we see that $1 - 2\langle c, s \rangle + v(s) = a + v(s - \epsilon c)$ where $a = 1 - v(c)$. If $\mu(dx)$ is the probability with Laplace transform $(1 - 2\langle c, s \rangle + v(s))^{-p}$ defined on an open convex set Θ containing 0, consider the possibly unbounded measure $e^{\langle \epsilon c, x \rangle}\mu(dx)$ with Laplace transform $(a + v(s))^{-p}$ defined on $\Theta + \epsilon c$. Renormalization according to a and discussion according to the facts that either v or $-v$ is positive definite and that $a < 0, a = 0, a > 0$ leads us to consider the four cases

$$(1 - s_1^2 - \ldots - s_n^2)^{-p},$$
$$(1 + s_1^2 + \ldots + s_n^2)^{-p},$$
$$(s_1^2 + \ldots + s_n^2)^{-p},$$
$$(-1 + s_1^2 + \ldots + s_n^2)^{-p}.$$

None of these 4 functions $\psi(s)$ can be the Laplace transform of some positive measure on $\mathbb{R}^n$. Because if it is the case, then for a fixed $s_0 \in \mathbb{R}^n \setminus \{0\}$ the function $t \mapsto \psi(s_0 t)$ defined on some interval of the real line would be the Laplace transform of a positive measure on $\mathbb{R}$. But $t \mapsto \log \psi(ts_0)$ is never convex. We finally treat the case $\lambda = 0$ in a similar way. Thus the case $r \geq 2$ is settled.

If $r = 1$ we end up with a Laplace transform $L(s_1)(1 - C(s_1)s_2)^{-p}$ (after some changing of coordinates). Since the logarithm of $s_2 \mapsto L(s_1)(1 - C(s_1)s_2)^{-p}$ is not convex for $p < 0$ the proof of the direct part of Theorem 2.1 is complete. The converse statements are easily obtained by reversing the above process.

3. LAPLACE TRANSFORMS $(1 - 2\langle c, s \rangle + v(s))^{-p}$

In this section we will describe distributions $\mu = \mu_{p,c,v}$ on $\mathbb{R}^n$ whose Laplace transforms on a certain open convex set Θ containing 0 are of the form $(1 - 2\langle c, s \rangle + v(s))^{-p}$, where v is a quadratic form of rank $r \leq n$, $c \in \mathbb{R}^n$ and $p > 0$. These distributions occur in Theorem 2.1 with λv replacing v. However since $Q_v = Q_{\lambda v}$ for $\lambda \neq 0$ we may assume $\lambda = 1$.

Before this, we recall a famous result due to Gindikin (1975): there exists a positive measure ν_p on $\mathbb{R}^n$ such that for $-\theta$ in the open Lorentz cone

$$(3.23) \qquad \Omega = \{x \in \mathbb{R}^n;\ x_1 > (x_2^2 + \ldots + x_n^2)^{1/2}\}$$

one has

$$\int e^{\langle \theta, x \rangle}\nu_p(dx) = (\theta_1^2 - (\theta_2^2 + \ldots + \theta_n^2))^{-p}$$

if and only if $p \geq (n - 2)/2$. Under these circumstances for $p > (n - 2)/2$ the measure ν_p is concentrated on Ω and is equal to

$$(3.24) \qquad \nu_p(dx) = (x_1^2 - x_2^2 - \ldots - x_n^2)^{p - \frac{n}{2}} \frac{dx}{2^{2p - \frac{n}{2}}\Gamma_\Omega(p)}$$

where $\Gamma_\Omega(p) = (2\pi)^{\frac{n-2}{2}}\Gamma(p)\Gamma(p-\frac{n-2}{2})$. For $p = (n-2)/2$ the measure ν_p is the image of the Lebesgue measure on $\mathbb{R}^{n-1}$ by $(x_2,\ldots,x_n) \mapsto ((x_2^2+\ldots+x_n^2)^{1/2}, x_2,\ldots,x_n)$, up to a suitable multiplicative constant and is thus concentrated on the boundary of Ω.

Next, if ρ is a non-negative number and if p is still $\geq (n-2)/2$ we use $\nu_p(dx)$ to build a more general measure $\nu_{p,\rho}$ on the closure $\overline{\Omega}$ of Lorentz cone (3.23). It is defined by

$$\nu_{p,\rho} = \sum_{j=0}^{\infty} p(p+1)\ldots(p+j-1)\frac{\rho^j}{j!}\nu_{p+j}$$

where ν_p is defined in (3.24). By the Newton binomial theorem

$$(1-z)^{-p} = \sum_{j=0}^{\infty} p(p+1)\ldots(p+j-1)\frac{z^j}{j!}$$

correct for $|z| < 1$ and applied to $z = \rho/(\theta_1^2 - (\theta_2^2+\ldots+\theta_n^2))$, the Laplace transform of $\nu_{p,\rho}$ is defined on $-\Omega_\rho$ where

$$(3.25) \qquad \Omega_\rho = \{\theta \in \mathbb{R}^n; \ \theta_1 > (\theta_2^2 + \ldots + \theta_n^2 + \rho)^{1/2}\}$$

and is

$$(\theta_1^2 - \theta_2^2 - \ldots - \theta_n^2 - \rho)^{-p}.$$

Note that $\nu_{p,0} = \nu_p$ and that $\Omega_0 = \Omega$. If $p > (n-2)/2$ the explicit form of ν_p given in (3.24) leads to the relatively explicit form

$$(3.26) \qquad \nu_{p,\rho}(dx) = \frac{[v(x)]^{p-\frac{n}{2}}}{2^{2p-\frac{n}{2}}(2\pi)^{\frac{n-2}{2}}\Gamma_\Omega(p)} f_{p-\frac{n-2}{2}}\left(\frac{\rho v(x)}{4}\right)\mathbf{1}_\Omega(x)dx$$

where the function f_p is defined by (1.1) and where $v(x) = x_1^2 - x_2^2 - \ldots - x_n^2$. If $p = (n-2)/2$ the measure $\nu_{p,\rho}$ contains the singular part $\nu_{\frac{n-2}{2}}$ and an absolutely continuous part. It is

$$(3.27) \qquad \nu_{\frac{n-2}{2},\rho}(dx) = \nu_{\frac{n-2}{2}}(dx) + \frac{\rho}{2^{\frac{n}{2}}(2\pi)^{\frac{n-2}{2}}\Gamma(\frac{n-2}{2})} f_2\left(\frac{\rho v(x)}{4}\right)\mathbf{1}_\Omega(x)dx.$$

We skip the proofs of (3.26) and of (3.27), which are easily done by checking either the densities or the Laplace transforms of both sides.

In order to state the theorem, we introduce a notation. Given a quadratic form v on a Euclidean space E with rank $r \leq \dim E = n$ we associate to v a symmetric endomorphism s_v of E defined by $\langle s_v(x), x \rangle = v(x)$. Its image F has dimension r, and the restriction of s_v is an automorphism of F which has an inverse. Therefore we define a quadratic form v^{-1} on F by $\langle s_v^{-1}(x), x \rangle = v^{-1}(x)$. For instance if $E = \mathbb{R}^n$ has its natural Euclidean structure, a suitable change of orthonormal basis gives $v(x) = \lambda_1 x_1^2 + \ldots + \lambda_k x_k^2 - \lambda_{k+1}x_{k+1}^2 - \ldots - \lambda_r x_r^2$ with $\lambda_j > 0$ and $r \leq n$. Then $F = \mathbb{R}^r \times \{0\}$ and

$$v^{-1}(x) = \frac{x_1^2}{\lambda_1} + \ldots + \frac{x_k^2}{\lambda_k} - \frac{x_{k+1}^2}{\lambda_{k+1}} - \ldots - \frac{x_r^2}{\lambda_r}.$$

If $c = (c_1,\ldots,c_n)$ in this new system of coordinates, then the projection c_F of c on the space F is $c_F = (c_1,\ldots,c_r,0,\ldots,0)$.

Theorem 3.1. *Let $p > 0$, let $c \in \mathbb{R}^n$ and let v be a quadratic form on $\mathbb{R}^n$ with signature $(k, r - k, n - r)$ (that is, having, respectively, k plus signs, $r - k$ minus signs and $n-r$ zeros in the Sylvester decomposition). We write c_F as the orthogonal projection of c on the image F of the symmetric endomorphism s_v, and we write $\rho = v^{-1}(c_F) - 1$. Denote by $\mu_{p,c,v}$ the probability on $\mathbb{R}^n$ with Laplace transform $(1 - 2\langle c, s \rangle + v(s))^{-p}$. Then $\mu_{p,c,v}$ exists if and only if the following circumstances are met:*

1. *$k = 0$. In this case, $-v$ is semipositive definite and $\mu_{p,c,v}$ is the distribution of the random vector*

(3.28)
$$XT^{1/2} + Tc \,,$$

   where X and T are independent, X is centered and normally distributed in $\mathbb{R}^n$ with covariance $-v$ and T is a gamma variable $\gamma_{p,2}$.
2. *$k = 1 = n$, $p > 0$ and $c_1^2 \geq 1$. If $c_1 = \epsilon \cosh\theta$ with $\epsilon = \pm 1$ let $T_1 \sim \gamma_{p,e^\theta}$ and $T_2 \sim \gamma_{p,e^\theta}$ be independent. With this notation $\mu_{p,c,v}$ is the distribution of $\epsilon(T_1 + T_2)$.*
3. *$k = 1 < n$, $(n - 2)/2 \leq p$ and $\rho \geq 0$. Changing the coordinates we may assume that $v(x) = x_1^2 - x_2^2 - \ldots - x_n^2$ and $c_1 > 0$. With this notation*

(3.29)
$$\mu_{p,c,v}(dx) = e^{-c_1 x_1 + c_2 x_2 + \ldots + c_n x_n} \nu_{p,\rho}(dx).$$

Comments. Note that this statement offers a generalization of the Gyndikin Theorem, by showing that if $\rho \geq 0$ the function defined on $-\Omega_\rho$ (see (3.25)) by $(\theta_1^2 - \theta_2^2 - \ldots - \theta_n^2 - \rho)^{-p}$ is the Laplace transform of a positive measure if and only if $p \geq (n - 2)/2$.

To get an intuitive feeling of the probability (3.29) in case 3, observe that for $\rho = 0$ it is the classical Wishart distribution on the closed Lorentz cone $\overline{\Omega}$. Now, for $\rho > 0$, define $R = 1 - (1 + \rho)^{1/3}$, $c_0 = (1 - R)c$ and $v_0 = (1 - R)v$. This choice of R implies $\rho_0 = v_0(c_0) - 1 = 0$. Now consider the random variable (X, J) valued in $\overline{\Omega} \times \mathbb{N}$ such that the distribution of the integer valued random variable J is the negative binomial distribution

$$\left(\delta_0 + \sum_{j=1}^\infty \frac{p(p+1)\ldots(p+j-1)}{j!} R^j \delta_j \right) (1 - R)^p$$

and such that the distribution of X conditioned by $J = j$ is the Wishart distribution on the closed Lorentz cone $\mu_{p+j, c_0, v_0}(dx)$. Then routine calculation of the Laplace transform of X shows that $X \sim \mu_{p,c,v}$.

Proof. Given v with the signature $(k, r - k, n - r)$ we first choose orthonormal coordinates such that $v(s) = \lambda_1 s_1^2 + \ldots + \lambda_k s_k^2 - \lambda_{k+1} s_{k+1}^2 - \ldots - \lambda_r s_r^2$, where $\lambda_j > 0$ for $j = 1, \ldots, r$. If $(X_1, \ldots, X_n)$ is a random variable having Laplace transform $(1 - 2\langle c, s \rangle + v(s))^{-p}$, then $(Y_1, \ldots, Y_n) = (\lambda_1^{-1/2} X_1, \ldots, \lambda_r^{-1/2} X_r, X_{r+1}, \ldots, X_n)$ has Laplace transform $(1 - 2\langle c', s \rangle + v'(s))^{-p}$, with $v'(s) = s_1^2 + \ldots + s_k^2 - s_{k+1}^2 - \ldots - s_r^2$ and with $c'_j = \lambda_1^{-1/2} c_j$ for $j \leq r$ and $c'_j = c_j$ for $j > r$. Note that $v'(c') = v^{-1}(c_F)$. Thus from now on in the remainder of the proof of the present theorem we assume that

$$v(s) = s_1^2 + \ldots + s_k^2 - s_{k+1}^2 - \ldots - s_r^2.$$

Note that the Hessian matrix $v''(s)$ is constant and equal to M_v. Since the convex function $\kappa(s) = -p\log(1 - 2\langle c, s\rangle + v(s))$ has second derivative

$$\kappa''(s) = \frac{p}{(1 - 2\langle c, s\rangle + v(s))^2}[(2c - v'(s))\otimes(2c - v'(s)) - (1 - 2\langle c, s\rangle + v(s))v''(s)],$$

this implies that $k = 0$ or 1. To see this, observe that $k \geq 2$ would imply that the restriction of the above quadratic form to $s_1, \ldots, s_k$ is not positive definite: the sum of a quadratic form containing at least two negative signs in its signature and of a quadratic form of rank one cannot be positive definite.

THE CASE $k = 0$. Writing

$$\frac{1}{(1 - 2\langle c, s\rangle - s_1^2 - \ldots - s_r^2)^p} = \int_0^\infty e^{-t(1 - 2\langle c, s\rangle - s_1^2 - \ldots - s_r^2)} t^{p-1} \frac{dt}{\Gamma(p)},$$

the existence and the probabilistic interpretation of the corresponding distribution μ is as follows: suppose that X is standard normal in $\mathbb{R}^r$ and is considered as a variable in $\mathbb{R}^n$ concentrated on $\mathbb{R}^r \times \{0\}$. If T is a gamma variable $\gamma_{p,2}$ independent of X, then the distribution $\mu_{p,c,v}$ of

$$(3.30) \qquad\qquad XT^{1/2} + cT$$

has the above Laplace transform.

THE CASE $k = n = 1$. First we observe that $k = n = 1$ implies that $c_1^2 \geq 1$. Indeed $c_1^2 < 1$ implies that $1 - 2c_1 s_1 + s_1^2$ has no real root. Thus the Laplace transform is defined on the whole line, but computation shows that its logarithm κ satisfies $\kappa''(c) < 0$. If $c_1^2 = \cosh^2\theta \geq 1$ with $\epsilon = \pm 1$ being the sign of c_1, then

$$(3.31) \qquad\qquad \frac{1}{(1 - 2c_1 s_1 + s_1^2)^p} = \frac{1}{(1 - s_1\epsilon e^\theta)^p}\cdot\frac{1}{(1 - s_1\epsilon e^{-\theta})^p},$$

and $\mu_{p,c,v}$ is clearly the convolution of two gamma distributions γ_{p,e^θ} and $\gamma_{p,e^{-\theta}}$ when $\epsilon = 1$ and symmetric of this one when $\epsilon = -1$.

THE CASE $k = 1$ AND $r = n \geq 2$. The case $k = n = 1$ has shown $c_1^2 \geq 1$. We denote

$$\rho = c_1^2 - c_2^2 - \ldots - c_n^2 - 1.$$

We discuss separately the three cases $\rho = 0$, $\rho > 0$ and $\rho < 0$.

THE SUBCASE $k = 1$, $r = n \geq 2$ AND $\rho = 0$.
    With $\rho = 0$ denote

$$c_1 = \epsilon(1 + c_2^2 + \ldots + c_n^2)^{1/2}$$

with $\epsilon = \pm 1$. Suppose $\epsilon = 1$. Then $(-c_1, c_2, \ldots, c_n)$ is in $-\Omega$ and for $p \geq (n-2)/2$ the measure $e^{-c_1 x_1 + c_2 x_2 + \ldots + c_n x_n}\nu_p(dx)$ is bounded, with Laplace transform

$$((s_1 - c_1)^2 - (s_2 + c_2)^2 - \ldots - (s_n + c_n)^2)^{-p}.$$

This shows that

$$(3.32) \qquad\qquad e^{-c_1 x_1 + c_2 x_2 + \ldots + c_n x_n}\nu_p(dx)$$

is the desired probability. This probability is a Wishart distribution on the cone Ω in the sense of Jordan algebras (see Faraut and Korányi (1994), Jensen (1988), Massam (1994) or Casalis and Letac (1994)). If $\epsilon = -1$ we take its image by

$x \mapsto -x$. If $p < (n-2)/2$ it is easy to use the Gindikin Theorem to see that the probability $\mu_{p,c,v}$ does not exist.

THE SUBCASE $k = 1$, $r = n \geq 2$ AND $\rho > 0$.

Assume first that $p \geq (n-2)/2$. Suppose that $c_1 > 0$. The fact that $\rho > 0$ implies that $(-c_1, c_2, \ldots, c_n)$ is in $-\Omega_\rho$, and the measure on $\overline{\Omega}$ defined by $e^{-c_1 x_1 + c_2 x_2} \nu_{p,\rho}(dx)$ is bounded and has the Laplace transform

$$((s_1 - c_1)^2 - (s_2 + c_2)^2 - \ldots - (s_n + c_n)^2 - \rho)^{-p} = (1 - 2\langle c, s \rangle + v(s))^{-p}.$$

This shows that $e^{-c_1 x_1 + c_2 x_2 + \ldots + c_n x_n} \nu_{p,\rho}(dx)$ is the desired probability $\mu_{p,c,v}$. If $c_1 < 0$ we consider its image by $x \mapsto -x$ for getting $\mu_{p,c,v}$.

Recall that $\rho > 0$. We now assume $p < (n-2)/2$, and we prove that $\mu_{p,c,v}$ does not exist. Suppose that $\mu_{p,c,v}$ exists. For any integer N we have $\mu_{p,c,v}^{*N} = \mu_{Np,c,v}$. Since $\mu_{Np,c,v}$ is concentrated on the closed convex cone $\overline{\Omega}$ for N large enough by the previous analysis, $\mu_{p,c,v}$ is also concentrated on $\overline{\Omega}$.

We now consider two polynomials

$$v(\theta) = \theta_1^2 - \theta_2^2 - \ldots - \theta_n^2, \quad \Delta(\theta) = 1 - 2\langle c, \theta \rangle + v(\theta).$$

Observe that if x and x' are in the cone Ω defined by (3.23), the Schwarz inequality applied to $(x_2, \ldots, x_n)$ and $(x_2', \ldots, x_n')$ shows that $v(x + x') > v(x) + v(x')$. Since

$$\Omega \supset \Omega_\rho = \{x \in \mathbb{R}^n \; ; \; x_1 > 0, \; v(x) > \rho\}$$

the inequality $v(x + x') > v(x) + v(x')$ holds on Ω_ρ. This shows that Ω_ρ is an additive semigroup, that is, $x + x'$ is in Ω_ρ if x and x' are in Ω_ρ. We apply it to $x = -\theta$ and $x' = c$ to obtain that Δ is positive on $-\Omega_\rho$:

$$\Delta(\theta) = 1 - 2\langle c, \theta \rangle + v(\theta) = v(-\theta + c) - v(c) + 1 = v(-\theta + c) - \rho > 0.$$

We now apply the operator $v(\frac{\partial}{\partial \theta})$ to the two members of the following identity:

$$(\Delta(\theta))^{-p} = \int_{\overline{\Omega}} e^{\langle \theta, x \rangle} \mu_{p,c,v}(dx).$$

This identity is true for all θ in $-\Omega_\rho$ since Δ is positive there. We get

$$(3.33) \qquad v\left(\frac{\partial}{\partial \theta}\right)(\Delta(\theta))^{-p} = \int_{\overline{\Omega}} e^{\langle \theta, x \rangle} v(x) \mu_{p,c,v}(dx).$$

Now a thorough and standard computation of the left hand side of (3.33) shows that it is equal to $4(\rho + \Delta(\theta))(\Delta(\theta))^{-p-2} p(p - p_0(\theta))$, where

$$p_0(\theta) = \frac{(n-2)\Delta(\theta) - 2\rho}{2(\Delta(\theta) + \rho)} < \frac{n-2}{2}.$$

Note that $\rho + \Delta(\theta)$ is positive on $-\Omega_\rho$. Furthermore $\rho + \Delta(\theta)$ does not depend on p. Since the right hand side of (3.33) is non-negative the existence of $\mu_{p,c,v}$ would imply $p \geq p_0(\theta)$ for all $\theta \in -\Omega_\rho$. Since Δ is unbounded on $-\Omega_\rho$, this implies that $p \geq (n-2)/2$, which contradicts our assumption.

THE SUBCASE $k = 1$, $2 \leq r = n$ AND $\rho < 0$.

With $\rho < 0$ and $p \geq (n-2)/2$ we now prove that $\mu_{p,c,v}$ does not exist. If not, standard reasoning shows that there would exist a positive measure $\nu_{p,\rho}$ such that either (3.26) or (3.27) hold (according to $p > (n-2)/2$ or $p = (n-2)/2$): for this compute the Laplace transforms of the right hand sides of (3.26) or (3.27). But the

function $x \mapsto f_p(\rho v(x))$ is not positive when $\rho < 0$. To see this we relate f_p to the classical Bessel function J_p by the formula

$$\left(\frac{z}{2}\right)^{p-1} f_p\left(-\frac{1}{4}z^2\right) = J_{p-1}(z)$$

which implies (see Watson (1966)Jensen (1988), Massam (1994) or Casalis and Letac (1994)), chapter XV) that f_p has an infinity of simple zeros on $(-\infty, 0)$ (and no other zeros in the complex plane).

Finally, we show that $k = 1$ and $0 < p < (n-2)/2$ implies that $\mu_{p,c,v}$ does not exist. The statement is clear: since $\rho < 0$, the existence of $\mu_{p,c,v}$ would imply the existence of $\mu_{Np,c,v}$ for any integer N. But we have already seen that $\mu_{p,c,v}$ does not exist for $\rho < 0$ and $p > (n-2)/2$.

THE CASE $k = 1$ AND $1 \leq r < n$.

In this case, if $c_{r+1} = \ldots = c_n = 0$ the problem of the existence of $\mu_{p,c,v}$ is solved by the previous cases $k = 1$, $r = n \geq 2$ or $k = 1 = n$. If $c_i \neq 0$ for some i with $r < i \leq n$, we prove that $\mu_{p,c,v}$ does not exist. To see this, we observe that $(1 - 2c_1 s_1 + s_1^2 - 2c_2 s_2)^{-p}$ is not a Laplace transform of the distribution of a random variable (X_1, X_2). For in this case we have $\mathbb{E}(e^{t(X_1 - X_2 c_1/c_2)}) = (1 + t^2)^{-p}$ which is impossible since $t \mapsto (1 + t^2)^{-p}$ is not a convex function around 0. More generally the following function defined in a neighborhood of zero by

$$(1 - 2c_1 s_1 + s_1^2 - 2c_2 s_2 - s_2^2 - \ldots - s_r^2 - 2c_{r+1} s_{r+1} - \ldots - 2c_n s_n)^{-p}$$

cannot be the Laplace transform of a probability if $(c_{r+1}, \ldots, c_n) \neq 0$. If $c_{r+1} \neq 0$ (say), then inserting $s_i = 0$ for $i \neq 1$ and $i \neq r + 1$ brings us back to the preceding impossible case.

4. LAPLACE TRANSFORMS $(f_1(s_1) - s_2)^{-p}$

Our next task is to study the distributions μ_{p,f_1} in $\mathbb{R}^2$ (essentially) which occur in Theorem 2.1 for $r = 1$ and $a^2 < b < a$, with Laplace transform $(f_1(s_1) - s_2)^{-p}$. Actually, replacing $\mathbb{R}^2$ by $\mathbb{R}^{m+1}$ is rather natural. Note that since $(s_1, s_2) \mapsto (f_1(s_1) - s_2)^{-p}$ is a Laplace transform, then it must be convex, thus f_1 must be concave. Furthermore, the domain of definition of a Laplace transform is convex, thus the interior of the domain of definition of f_1 must be an open convex subset C of $\mathbb{R}^m$ and the interior of the domain of definition of $(s_1, s_2) \mapsto (f_1(s_1) - s_2)^{-p}$ is $\{(s_1, s_2) \in C \times \mathbb{R}; f_1(s_1) - s_2 > 0\}$. In order to avoid difficulties about continuity at the boundary of the Laplace transforms, we prefer to work with the interior of the domain of definition of Laplace transforms rather than with the domain itself. Recall that the Laplace transform of a probability is *not* continuous in general at one of the boundary points: continuity holds only along straight lines containing the boundary point (an example due to Hoffmann-Jorgensen can be found in Barndorff-Nielsen (1978), page 104).

Proposition 4.1. *Let f_1 be a concave function defined on some convex open subset C of $\mathbb{R}^m$ containing 0 in its closure and let $p > 0$. A necessary and sufficient condition for the existence of a probability μ_{p,f_1} on $\mathbb{R}^{m+1}$ with Laplace transform $(f_1(s_1) - s_2)^{-p}$ is the existence of some infinitely divisible distribution $(\nu_t)_{t>0}$ on $\mathbb{R}^m$ such that the Laplace transform of ν_t is $\exp t(1 - f_1(s_1))$. Under these circumstances*

(4.34) $$\mu_{p,f_1}(dx, dt) = \nu_t(dx)\gamma_{p,1}(dt).$$

That means that if $(Y(t))_{t\geq 0}$ is a Lévy process on $\mathbb{R}^m$ governed by $(\nu_t)_{t>0}$ and if T is an independent random variable with distribution $\gamma_{p,1}(dt)$, then μ_{p,f_1} is the distribution in $\mathbb{R}^{m+1}$ of $(Y(T),T)$.

Proof. The part $\Leftarrow$ is standard. In order to prove the converse $\Rightarrow$ we consider a random variable $(X,T) \sim \mu_{p,f_1}(dx,dt)$ in $\mathbb{R}^m \times \mathbb{R}$ such that on the set

$$D = \{(s_1,s_2) \, ; s_1 \in C, \ s_2 < f_1(s_1)\}$$

we have

$$\mathbb{E}(e^{\langle s_1,X\rangle + s_2 T}) = (f_1(s_1) - s_2)^{-p}.$$

Since μ_{p,f_1} is a probability, $(0,0)$ belongs to the closure of D, and therefore $\lim_{s_1 \to 0} f_1(s_1) = 1$ for radial limit. Thus we can claim that for $s_2 < 1$ we have $\mathbb{E}(e^{s_2 T}) = (1 - s_2)^{-p}$. In other words the marginal distribution of T must be $\gamma_{p,1}$, where for $a > 0$ we denote $\gamma_{p,a}(dt) = e^{-at}a^p t^{p-1}\mathbf{1}_{(0,\infty)}(t)\frac{dt}{\Gamma(p)}$. Therefore if we denote by $\nu_t(dx)$ the conditional distribution of X knowing $T = t$, we get $\mu_{p,f_1}(dx,dt) = \nu_t(dx)\gamma_{p,1}(dt)$, and thus on D:

$$\begin{aligned}
(f_1(s_1) - s_2)^{-p} &= \int_{\mathbb{R}^m}\int_0^\infty e^{\langle s_1,x\rangle + s_2 t}\nu_t(dx)\gamma_{p,1}(dt) \\
&= \int_0^\infty e^{s_2 t}\left(\int_{\mathbb{R}^m} e^{\langle s_1,x\rangle}\nu_t(dx)\right)\gamma_{p,1}(dt).
\end{aligned}$$

Since $\lim_{s_1 \to 0} f_1(s_1) = 1$ there is a non-empty convex subset C_1 of C on which the concave function f_1 is positive. For $s_1 \in C_1$ and $s_2 < f_1(s_1)$ we have

$$\left(\frac{f_1(s_1)}{f_1(s_1) - s_2}\right)^p = \int_0^\infty e^{s_2 t}\left(\int_{\mathbb{R}^m} e^{\langle s_1,x\rangle}\nu_t(dx)\right)(f_1(s_1))^p\gamma_{p,1}(dt).$$

Thus for fixed $s_1 \in C_1$ we can claim that

$$\left(\int_{\mathbb{R}^m} e^{\langle s_1,x\rangle}\nu_t(dx)\right)(f_1(s_1))^p\gamma_{p,1}(dt) = \gamma_{p,f_1(s_1)}(dt).$$

Equating the densities we get that for almost all $t > 0$ we have on C_1:

$$\int_{\mathbb{R}^m} e^{\langle s_1,x\rangle}\nu_t(dx) = e^{t(1-f_1(s_1))}.$$

Since the the function on C_1 defined by $e^{t(1-f_1(s_1))}$ is the Laplace transform of a probability ν_t for almost all $t > 0$, one concludes by continuity that it is the Laplace transform of some probability ν_t for all $t > 0$, and $(\nu_t)_{t>0}$ is a convolution semigroup. This ends the proof.

Comments.

1. Note that in the above proof $C_1 \neq C$ in general: for example if $f_1(s_1) = 1 - \|s_1\|^2$, then C_1 is the open unit ball and C is the Euclidean space $\mathbb{R}^m$.
2. There is nothing special in the gamma distribution in the statement of Proposition 4.1. If $\mathbb{E}(e^{\langle s_1,X\rangle + s_2 T})$ has the form $e^{\kappa_0(\kappa_1(s_1) + s_2)}$ and if the distribution α of T is concentrated in $[0,\infty)$, consider the conditional distribution $X|T = t \sim \nu_t(dx)$. Then α almost surely we have

(4.35) $$e^{t(\kappa_1(s_1) - \kappa_1(0))} = \int_{\mathbb{R}^m} e^{\langle s_1,x\rangle}\nu_t(dx).$$

   If (4.35) holds for $t = 1$ (say), then the fact that (4.35) holds α almost surely implies that the Jorgensen set of ν_1 contains the support of α.

In particular, if the support of α contains some interval $[0, a]$, then ν_1 is infinitely divisible. See also comment 5 in Section 6.

3. Let us also mention that in Theorem 2.1, the second part of 3, we have considered $X = \langle X, e_1 \rangle + X'$ where $X' = Tc$ is a multiple of a constant vector c orthogonal to e_1. With the notation of Proposition 4.1, we have $\langle X, e_1 \rangle = Y(T)$.

5. Variance functions $\frac{1}{p} m \otimes m - \varphi(m) M_v$

In this section, we are going to interpret some of the above results in terms of natural exponential families (NEF). Recall some definitions about NEF. Given a positive measure μ on $\mathbb{R}^n$, not necessarily bounded, consider its Laplace transform, namely the function on $\mathbb{R}^n$ valued in $[0, \infty]$ defined by

$$\theta \mapsto L_\mu(\theta) = \int_{\mathbb{R}^n} e^{\langle \theta, x \rangle} \mu(dx) \leq \infty.$$

We denote by $D(\mu)$ the set of $\theta \in \mathbb{R}^n$ such that $L_\mu(\theta) < \infty$. Hölder inequality implies that $D(\mu)$ is a convex set and that $\kappa_\mu = \log L_\mu$ is a convex function on it. One denotes by $\Theta(\mu)$ the interior of $D(\mu)$. For instance, if μ is a probability which has exponential moments (that is, there exists $a > 0$ such that $\int_{\mathbb{R}^n} e^{a(|x_1| + \ldots + |x_n|)} \mu(dx) < \infty$), then $\Theta(\mu)$ is not empty.

Let us assume that $\Theta(\mu)$ is not empty. If furthermore μ is not concentrated on some affine hyperplane, then κ_μ is strictly convex and real-analytic on $\Theta(\mu)$. This implies that its differential κ'_μ (that we consider as taking its values in $\mathbb{R}^n$) is one-to-one. Denote the image of $\Theta(\mu)$ by the function κ'_μ as the open subset $M(\mu) \subset \mathbb{R}^n$. Denote also by $m \mapsto \theta = \psi_\mu(m)$ the inverse of $\theta \mapsto m = \kappa'_\mu(\theta)$. The NEF $F = F(\mu)$ generated by μ is the set of probabilities $F(\mu) = \{P(\theta, \mu); \ \theta \in \Theta(\mu)\}$ where

$$P(\theta, \mu)(dx) = e^{\langle \theta, x \rangle - \kappa_\mu(\theta)} \mu(dx).$$

Note that $F = F(\mu) = F(\mu_1)$ does not imply $\mu = \mu_1$, but only the existence of $(a, b) \in \mathbb{R}^{n+1}$ such that $\mu_1(dx) = e^{\langle a, x \rangle + b} \mu(dx)$. In this case $\Theta(\mu_1) = a + \Theta(\mu)$ but $M(\mu_1) = M(\mu)$, and we would rather denote it by $M(F)$. Note also that any element of F generates F, although the converse is not true: $F = F(\mu)$ does not necessarily imply that $\mu \in F$. The Laplace transform of $P(\theta, \mu)$ is $s \mapsto \frac{L_\mu(\theta + s)}{L_\mu(\theta)}$ and is defined for $s \in \Theta(\mu) - \theta$.

It is easily seen that $m = \kappa'_\mu(\theta) = \int_{\mathbb{R}^n} x P(\theta, \mu)(dx)$. For this reason $M(F)$ is called the domain of the means of F. The Hessian matrix $\kappa''_\mu(\theta)$ is the covariance matrix of $P(\theta, \mu)$. Therefore, for $m \in M(F)$ the symmetric matrix $V_F(m) = \kappa''_\mu(\psi_\mu(m))$ is the covariance matrix of the unique element $P(m, F)$ of F with mean m. The function defined on $M(F)$ by $m \mapsto V_F(m)$ and valued among the symmetric positive definite (n, n) matrices is called the variance function of F. Because of the relation $V(k'_\mu(\theta)) = \kappa''_\mu(\theta)$ the variance function determines k_μ up to constants $(a, b) \in \mathbb{R}^{n+1}$, and thus $(V_F, M(F))$ determines F. Finally, we mention that if we take the image G of an NEF F given by $(V_F, M(F))$ by some affine transformation $x \mapsto A(x) + b$ of $\mathbb{R}^n$ where $b \in \mathbb{R}^n$ and A is an invertible $n \times n$ matrix, then $M_G = AM(F) + b$ and

$$(5.36) \qquad\qquad V_G(m) = A V_F(A^{-1}(m - b)) A^T.$$

In this section we are going to find all NEF's F such that there exists a quadratic form v, a positive number p and a real function φ on $M(F)$ such that $V_F(m) = \frac{1}{p}m \otimes m - \varphi(m)M_v$ on $M(F)$, where M_v is the representative matrix of the quadratic form v. As an example, for $v(x) = x_1^2 - x_2^2 - \ldots - x_n^2$ the family F_p of Wishart distributions with the shape parameter $p \geq (n-2)/2$ on the Lorentz cone $\Omega = \{x \in \mathbb{R}^n \ ; x_1 > 0, v(x) > 0\}$ has this property. If P denotes the quadratic representation of the Jordan algebra associated to Ω in the sense of Faraut and Korányi (1994), page 32, then the variance function is $P(m)/p$. Surprisingly we have on Ω

$$V_{F_p}(m) = \frac{1}{p}P(m) = \frac{1}{p}m \otimes m - \frac{1}{4p}(m_1^2 - m_2^2 - \ldots - m_n^2)\mathrm{Diag}(1, -1, \ldots, -1).$$

Note that from (5.36) the image G of an NEF F with $V_F(m) = m \otimes m/p - \varphi(m)M_v$ by the linear automorphism $x \mapsto Ax$ has a variance function of the same form, namely

$$V_G(m) = \frac{1}{p}m \otimes m - \varphi(A^{-1}m)AM_vA^T.$$

This remark will considerably simplify the presentation of our results, since it allows us by proper choice of A to assume that M_v has the simple form

$$(5.37) \qquad M_v = \begin{bmatrix} I_k & 0 & 0 \\ 0 & -I_{r-k} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 5.1. *Let v be a quadratic form on $\mathbb{R}^n$ with representative matrix M_v having rank r and signature $(k, r-k, n-r)$. Let $p > 0$. We assume $n \geq 2$. Let M be a connected open set of $\mathbb{R}^n$ and φ a real function defined on M such that $m \mapsto \frac{1}{p}m \otimes m - \varphi(m)M_v$ is the variance function V_F of some NEF F whose domain of the means is $M = M(F)$.*

Then φ is real analytic on M, has no zeros and has constant sign. Assuming $\varphi > 0$, then either $k = 1$, $r = n$, or $k = 0$, $r = n$, or $k = 0$, $r = n - 1$ or $k = r = 1$, $n = 2$. Except when $k = r = 1$, $n = 2$ there exist some $c \in \mathbb{R}^n$ and some real $\lambda > 0$ such that the NEF F is generated by a probability $\mu_{p,c,\lambda v}$ with Laplace transform $\Delta(s)^{-p}$ with $\Delta(s) = 1 - 2\langle c, s \rangle + \lambda v(s)$. Every element $P(\theta; \mu_{p,c,\lambda v})$ of F has the form $\mu_{p,c_\theta,\lambda_\theta v}$ with $c_\theta = (c - M_v\theta)/\Delta(\theta)$ and $\lambda_\theta = \lambda/\Delta(\theta)$.

1. *If $k = 1$, $r = n$ the vector c must satisfy $\rho = \frac{v^{-1}(c)}{\lambda} - 1 \geq 0$. The domain of the means is $M(F) = \Omega_\rho$ defined by (3.25). If $\rho > 0$ the proportionality factor $\varphi(m) > 0$ is*

$$\varphi(m) = \frac{p\lambda}{2\rho}\left(\left[1 + \rho\frac{v^{-1}(m)}{p^2\lambda}\right]^{1/2} - 1\right).$$

   If $\rho = 0$ the natural exponential family F is the family of Wishart distributions on the Lorentz cone Ω associated to v, with the shape parameter $p \geq (n-2)/2$. In this case $\varphi(m) = \frac{v^{-1}(m)}{4p}$.

2. *If $k = 0$, $r = n$ the domain of the means is $\mathbb{R}^n$. With the Euclidean structure $\|x\|^2 = -v^{-1}(x)$ we have*

$$\varphi(m) = \frac{p}{\lambda(\|c\|^2 + \lambda)}\left[\left(\lambda^2 + (\|c\|^2 + \lambda)\|m\|^2\right)^{1/2} + \lambda\right].$$

3. If $k = 0$, $r = n-1$ we take for simplicity $v(x) = v^{-1}(x) = -x_1^2 - \ldots - x_{n-1}^2$. Then $c_n \neq 0$ and we have $\varphi(m) = \frac{\lambda m_n}{2c_n}$, and the domain of the means is $\mathbb{R}^{n-1} \times (0, \infty)$ if $c_n > 0$ and $\mathbb{R}^{n-1} \times (-\infty, 0)$ if $c_n < 0$.

4. In the exceptional case $k = r = 1$, $n = 2$ the NEF is generated by a probability measure $\mu_{p,f_1}(dx, dt) = \nu_t(dx)\gamma_{p,1}(dt)$ as defined in Section IV. Denote by F_1 the NEF in $\mathbb{R}$ generated by ν_1, denote by (a,b) the domain of the means of F_1 and denote by V_{F_1} its variance function. Then the domain of the means of F is the cone

$$\{(m_1, m_2) \; ; \; m_2 > 0, \; am_2 < m_1 < bm_2\}$$

and $\varphi(m_1, m_2)$ is $-m_2 V_{F_1}(m_1/m_2)$. Consequently

$$V_F(m_1, m_2) = \begin{bmatrix} \frac{m_1^2}{p} + m_2 V_{F_1}(\frac{m_1}{m_2}) & \frac{m_1 m_2}{p} \\ \frac{m_1 m_2}{p} & \frac{m_2^2}{p} \end{bmatrix}.$$

Proof. The quadratic form v cannot be 0 since the rank of $V_F(m)$ is always $n \geq 2$. Since a variance function is real analytic and since $v \neq 0$, the real analyticity of φ is obvious. If $\varphi(m_0) = 0$ for some $m_0 \in M$, then the rank of $V(m_0)$ is $1 \neq n$, a contradiction. Since M is connected, φ has constant sign. Without loss of generality, we choose in the sequel $\varphi(m) > 0$. If not, we change v into $-v$.

Since $n=$rank $V_F(m) \leq 1 + r$ we get $r \geq n - 1$. If $r = n$

$$V_F(m) = \frac{1}{p} m \otimes m + \begin{bmatrix} -\varphi(m)I_k & 0 \\ 0 & \varphi(m)I_{n-k} \end{bmatrix}.$$

Clearly since $\varphi(m) > 0$, this matrix is positive definite only if $k \leq 1$: just watch the first (k, k) block of the matrix $V_F(m)$.

For $r = n - 1$ we consider the matrix

$$V_F(m) = \frac{1}{p} m \otimes m + \begin{bmatrix} -\varphi(m)I_k & 0 & 0 \\ 0 & \varphi(m)I_{n-1-k} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly as above we see that $k \leq 1$. However, $k = 1$ is excluded by considering the $(2, 2)$ submatrix obtained with the first and last rows and columns.

THE CASE $k = 1$, $r = n$.

Let μ be a generating measure of F and write κ as the log of the Laplace transform of μ defined on the open convex set Θ. Then $s \mapsto \kappa'(s)$ is a bijective mapping from Θ on M. Since φ has no zeros, we denote $f(s) = \frac{1}{\varphi(\kappa'(s))}$. Thus we have

$$\kappa''(s) = \frac{1}{p}\kappa'(s) \otimes \kappa'(s) - \frac{1}{f(s)}v.$$

At this point we can use the discussion made after (2.11) and claim that since $r \geq 2$ there exists $c \in \mathbb{R}^n$ and $\lambda > 0$ such that f has the form (2.16) and $\mu = \mu_{p,c,\lambda v}$. This implies that the domain of the means M is the Lorentz cone Ω and that the domain of the Laplace transform Θ is $-\Omega_\rho$, where $\rho = \frac{v^{-1}(c)}{\lambda} - 1 \geq 0$ (by Theorem 3.1, part 3) and Ω_ρ is defined by (3.25).

The only thing left to do is to compute $\varphi(m)$. We use M_v as the representative matrix of v. As usual, the quadratic form associated to M_v^{-1} is denoted v^{-1}. Since

$\kappa(s) = -p \log \Delta$ with $\Delta = 1 - 2\langle c, s \rangle + \lambda v(s)$, we write $m = \kappa'(s) = \frac{2p}{\Delta}(c - \lambda M_v s)$. Denote by $\psi : M \to \Theta$ the inverse function of κ'. Thus

$$s = \psi(m) = \frac{1}{\lambda} M_v^{-1} c - \frac{\Delta}{2\lambda p} M_v^{-1} m.$$

We carry this in the definition of $\Delta(s)$ for computing Δ:

$$
\begin{aligned}
\Delta &= 1 - \frac{2}{\lambda}\langle c, M_v^{-1} c \rangle + \frac{\Delta}{\lambda p}\langle c, M_v^{-1} m \rangle + \frac{1}{\lambda} v\left(M_v^{-1} c - \frac{\Delta}{2p} M_v^{-1} m \right) \\
&= 1 - \frac{1}{\lambda} v^{-1}(c) + \frac{\Delta^2}{4\lambda p^2} v^{-1}(m) = -\rho + \frac{\Delta^2}{4\lambda p^2} v^{-1}(m).
\end{aligned}
$$

In order to solve this second degree equation in Δ we denote

$$\delta = \left[1 + \rho \frac{v^{-1}(m)}{\lambda p^2} \right]^{1/2}.$$

Introducing $\epsilon = \pm 1$ gives $\Delta = \frac{2\lambda p^2}{v^{-1}(m)}(1 + \epsilon\delta)$, and finally for $\rho > 0$ we get

$$\varphi(m) = \frac{1}{f(\psi(m))} = \frac{\lambda p}{\Delta} = \frac{\lambda p}{2} \frac{\epsilon\delta - 1}{\rho}.$$

In order to decide whether $\epsilon = 1$ or not, we observe that $-\varphi(m)v$ must have at least $n - 1$ positive signs in its signature. Since $k = 1$ and $r = n$, this implies $\varphi(m) > 0$ and $\epsilon = 1$. For $\rho = 0$ an easy discussion shows that $\varphi(m) = \frac{v^{-1}(m)}{4p}$.

THE CASE $k = 0$, $r = n$.

The discussion is rather similar, and changes occur only when finding the sign of ϵ for choosing the correct root of the second degree equation giving Δ. We skip the details.

THE CASE $k = 0$, $r = n - 1$.

For $n = 2$ by the discussion given in the proof of Theorem 2.1, case $a^2 < b < a$, $r = 1$ (the second part where g is not identically zero), we get that $\Delta(s) = 1 - 2c_1 s_1 - \lambda s_1^2 - 2c_2 s_2$. For $n > 2$ we have $\Delta(s) = 1 - 2\langle c, s \rangle - \lambda(s_1^2 + \ldots + s_{n-1}^2)$. Thus in both cases $n = 2$ and $n > 2$ we are sent back to a similar discussion as in the case $k = 1$ and $r = n$. Thus $\varphi(m) = p\lambda/\Delta$. Moreover

$$m_n = \frac{\partial \kappa}{\partial s_n} = \frac{2pc_n}{\Delta}.$$

Combining the two formulas we get $\varphi(m) = \frac{\lambda m_n}{2c_n}$.

THE EXCEPTIONAL CASE $k = r = 1$, $n = 2$.

We have seen in Section 2 that the family is generated by a probability μ_{p,f_1} on $\mathbb{R}^2$ with Laplace transform $(f_1(s_1) - s_2)^{-p}$. By analyticity, this Laplace transform is defined on

$$\Theta = \{(s_1, s_2) \; ; \; s_1 \in I, \; s_2 < f_1(s_1)\},$$

where I is the open domain of the definition of f_1. Since this domain is convex, this implies that I is an open interval and that f_1 is concave. Furthermore $0 \in \overline{I}$

and $\lim_{s_1\to 0} f_1(s_1) = 1$. The cumulant function is $k = -p\log f$ where $f(s_1,s_2) = f_1(s_1) - s_2$. Thus

$$m_1 = -p\frac{f_1'(s_1)}{f}, \quad m_2 = p\frac{1}{f}.$$

This implies that $f_1'(s_1) = -m_1/m_2$. Observe that since k is strictly convex, then $(s_1,s_2) \mapsto f_1(s_1) - s_2$ is strictly concave and thus $s_1 \mapsto f_1(s_1)$ is strictly concave, too. Hence $s_1 \mapsto f_1'(s_1) = -m_1/m_2$ is one to one and s_1 is a function of m_1/m_2. Since the cumulant function of F_1 is $1 - f_1(s_1)$, the domain of the means of F_1 is $(a,b) = -f_1'(I)$. We now check that the map from Θ to $(a,b) \times (0,\infty)$ is bijective: for any $(c,y) \in (a,b)\times(0,\infty)$ there exists a unique $s_1 \in I$ such that $c = -f_1'(s_1)$ and a $s_2 < f(s_1)$ such that $y = \frac{p}{f(s_1)-p}$. Thus the domain of the means is as claimed.

Since $\frac{\partial^2 k}{\partial s_1^2} = \frac{m_1^2}{p} - m_2 f_1''(s_1) = \frac{m_1^2}{p} - \varphi(m_1,m_2)$ and since s_1 is a function of m_1/m_2 we get that $\varphi(m_1,m_2) = m_2\varphi_1(m_1/m_2)$, where φ_1 is defined on (a,b). Since $V_{F_1}(-f_1'(s_1)) = -f_1''(s_1)$ we can claim that $\varphi_1 = -V_{F_1}$, and this ends the proof.

6. COMMENTS AND EXAMPLES

1. EXAMPLES. We concentrate on the case $a^2 < b < a$ and $n = 2$. Up to linear changes we can assume that the possible representative matrix M_v of v is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad -I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

To see this we observe that $(2,0,0)$, $(1,1,0)$, $(0,2,0)$, $(1,0,1)$, and $(0,1,1)$ are the possible signatures $(k, r-k, n-r)$ for a non-zero v, and $n = 2$. However cases $(2,0,0)$ and $(0,2,0)$ coincide (corresponding to $-I_2$) as well as $(1,0,1)$ and $(0,1,1)$ (corresponding to B) since $Q_v = Q_{-v}$: see the comments at the beginning of Section 3. Cases B and W have been covered in Bobecka and Wesołowski (2004) and Wang (1981). The case $-I_2$ is new and is described by (3.30).

2. EXPONENTIAL MOMENTS. Wang's theorem is for positive random variables: they do not necessarily have exponential moments, but they have non-trivial Laplace transforms, and this is equivalent after a trivial change of the probability measures of X and Y. In Theorem 2.1, we do not make a hypothesis about positiveness, but only about exponential moments. One can wonder if the hypothesis could be reduced to the existence of the first two moments. There is apparently serious difficulty in deducing from the analog of (2.9), written in terms of Fourier transforms which holds only for all $q \in Q_v$, the fact that all higher derivatives of the Fourier transform of X do exist. This problem does not exist in the one dimensional cases as considered by Lukacs (1955).

3. SIMPLE QUADRATIC FAMILIES. In a celebrated paper Casalis (1996) has characterized all the natural exponential families on $\mathbb{R}^n$ (called simple quadratic families, or Casalis families) such that their variance functions have the form

$$V_F(m) = am \otimes m + B(m) + C,$$

where $m \mapsto B(m)$ is linear, C is a constant matrix and a is a real number. If $a > 0$, $C = 0$ and $B(m)$ is the product of a constant matrix M_v by a linear form $m \mapsto \varphi(m)$, we get the simple quadratic families covered by our Section 5. If $n \geq 3$, among these $2n+4$ Casalis families, only one has this property, namely her gamma family of type 1 which coincides with case 3 of Theorem 5.1. If $n = 2$ two of the

8 Casalis families appear in the exceptional case 4 of Theorem 5.1 by taking F_1 either as a normal family or a Poisson family.

4. Diagonal variance families. In Bar-Lev *et al.* (1994) all the natural exponential families on $\mathbb{R}^n$ such that their variance functions satisfy

$$\mathrm{Diag}(V_F(m)) = (f_1(m_1), \ldots, f_n(m_n))$$

are characterized and called diagonal variance families. When $v \neq 0$ and M_v has zero diagonal, our natural exponential families are of the diagonal variance type. If $n = 2$ we are in the Wang case. For $n \geq 3$ clearly $k = r$ or $k = 0$ is impossible, since it implies that M_v or $-M_v$ would be a semipositive definite matrix, which is impossible if its diagonal is 0. An example of the case where $k = 1$ and $r = n \geq 3$ is $M_v = I_n - J_n$, where J_n is the matrix of ones. For instance for $n = 3$

$$M_v = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

It is easily seen that the characteristic polynomial of M_v is

$$(-1)^n(\lambda - n + 1)(\lambda + 1)^{n-1},$$

thus the signature is $(1, n-1, 0)$.

This example is rather interesting since Theorem 5.1 provides the value of $\varphi(m)$ (part 1). Here $v(m) = \sum_{j=1}^n m_j^2 - (\sum_{j=1}^n m_j)^2$. There are very few cases of diagonal variance families for $n \geq 3$ such that the off diagonal elements can be computed, and the present calculation offers an example of such a situation. Note that for $n \geq 3$ any signature $(k, r-k, n-r)$ can be reached by a symmetric matrix M_v with zero diagonal when $0 < k < r$.

The diagonal variance families such that the diagonal of the variance function is $(m_1^2/p, \ldots, m_n^2/p)$ have a Laplace transform of the form $P(s)^{-p}$ as in (1.4). Bernardoff (2006) has given the necessary and sufficient conditions of infinite divisibility of such a distribution in $\mathbb{R}^n$ under the extra hypothesis that $c_{\{1,\ldots,n\}} \neq 0$. In our case $P(s)$ is a second degree polynomial, thus for $n \geq 3$ this condition $c_{\{1,\ldots,n\}} \neq 0$ cannot be satisfied. Note that Theorem 3.1 shows that our distributions are not infinitely divisible, since p must be $\geq (n-2)/2$. For $n = 2$ we are sent back again to Wang's theorem, where the distributions are infinitely divisible.

5. Exceptional case. It is easy to see that the NEF generated on $\mathbb{R}^{m+1}$ by the distribution appearing in (4.35) is simply

$$\begin{bmatrix} \frac{V_0(m_2)}{m_2^2}m_1 \otimes m_1 + m_2 V_1(\frac{m_1}{m_2}) & \frac{m_1}{m_2}V_0(m_2) \\ \frac{m_1^T}{m_2}V_0(m_2) & V_0(m_2) \end{bmatrix},$$

where V_0 and V_1 are the variance functions of the NEFs generated by α and ν_1 (assuming that 1 is in the support of α).

6. Quadratic regression for Wishart distributions on Jordan algebras. In Letac and Massam (1998) a characterization of Wishart distributions on the symmetric cone Ω associated to a Euclidean Jordan algebra E based on regression conditions is given: consider the two symmetric operators on E,

$$Q_1(x) = d'x \otimes x + P(x), \quad Q_2(x) = -x \otimes x + P(x),$$

where P is the quadratic representation of E and $2d'$ is the Peirce constant of E. Suppose that X and Y are independent random variables valued in $\overline{\Omega}$ and that

there exist real numbers a, b_1 and b_2 such that $\mathbb{E}(X|X+Y) = a(X+Y)$ and

$$\mathbb{E}(Q_i(X)|X+Y) = b_i Q_i(X+Y)$$

for $i = 1, 2$. Then one proves that X and Y are Wishart distributed with the same scale parameter and with shape parameters p and q. In this case

$$b_1 = \frac{p}{p+q} \times \frac{p+1}{p+q+1}, \ b_2 = \frac{p}{p+q} \times \frac{p-d'}{p+q-d'}.$$

It is interesting to compare this result with the present paper.

If S is any symmetric operator on E, consider the quadratic form on E defined by $x \mapsto q_i(S)(x) = \text{trace } (SQ_i(x))$. Denote by $\mathcal{Q}_i$ the space of all quadratic forms $q_i(S)$ on V. Then a reformulation of the above characterization of the Wishart distributions on V is obtained by saying that $\mathbb{E}(q(X)|X+Y) = b_i q(X+Y)$ for all $q \in \mathcal{Q}_i$ and for $i = 1, 2$ and $\mathbb{E}(X|X+Y) = a(X+Y)$ (here $q(x) = q_i(S)(x)$ for some symmetric operator S on E).

The above example suggests that possibly new interesting distributions on a Euclidean space E could be discovered by choosing independent linear subspaces $\mathcal{Q}_1, \ldots, \mathcal{Q}_l$ of the space of quadratic forms on E such that $\mathbb{E}(Q_i(X)|X+Y) = b_i Q_i(X+Y)$ holds for all $Q_i \in \mathcal{Q}_i$ with $i = 1, \ldots, l$ and some fixed real numbers $b_1, \ldots, b_l$. The paper by Letac and Massam (1998) was therefore devoted to a particular case where $l = 2$. The present paper is devoted to the case $l = 1$ where the only restriction for $\mathcal{Q}_1$ is to be of codimension 1 (while (1.4) was considering the case where $\mathcal{Q}_1$ is the set of q such that M_q is diagonal).

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LABORATOIRE DE STATISTIQUE ET PROBABILITÉS, UNIVERSITÉ PAUL SABATIER, 31062 TOULOUSE, FRANCE

E-mail address: `letac@cict.fr`

WYDZIAŁ MATEMATYKI I NAUK INFORMACYJNYCH, POLITECHNIKA WARSZAWSKA, WARSZAWA, POLAND

E-mail address: `wesolo@mini.pw.edu.pl`