Mathematical Programming 62 (1993) 557–574
North-Holland

Laplacian eigenvalues and the maximum cut problem

C. Delorme
LRI, University Paris-Sud, Orsay, France

S. Poljak*
KAM MFF, Charles University, Prague, Czech Republic

Received 30 August 1990
Revised manuscript received 3 February 1993

We introduce and study an eigenvalue upper bound $\varphi(G)$ on the maximum cut $\mathrm{mc}(G)$ of a weighted graph. The function $\varphi(G)$ has several interesting properties that resemble the behaviour of $\mathrm{mc}(G)$. The following results are presented.

We show that φ is subadditive with respect to amalgam, and additive with respect to disjoint sum and 1-sum. We prove that $\varphi(G)$ is never worse that 1.131 $\mathrm{mc}(G)$ for a planar, or more generally, a weakly bipartite graph with nonnegative edge weights. We give a dual characterization of $\varphi(G)$, and show that $\varphi(G)$ is computable in polynomial time with an arbitrary precision.

Key words: Max-cut, eigenvalues, algorithms.

1. Introduction

Let $G=(V,E)$ be a weighted graph on n vertices with weight w_e on the edge e. For each partition $V=S\cup(V\setminus S)$ of V, we have the corresponding value $\sum_{i\in S,\, j\in V\setminus S} w_{ij}$. The max-cut of G, denoted by $\mathrm{mc}(G)$, is the number defined by

$$\mathrm{mc}(G)=\max_{S\subset V}\sum_{\substack{i\in S\\ j\in V\setminus S}} w_{ij}\,. \tag{1}$$

We introduce and investigate a number $\varphi(G)$, defined for every weighted graph G, which is always an upper bound on the max-cut $\mathrm{mc}(G)$, i.e. we have

$$\mathrm{mc}(G)\leqslant\varphi(G). \tag{2}$$

The number $\varphi(G)$ is defined as

$$\varphi(G)=\min_{\Sigma u_i=0}\tfrac{1}{4}\lambda_{\max}\,(L+\mathrm{diag}(u))n \tag{3}$$

Correspondence to: C. Delorme, LRI, University Paris-Sud, F-91405 Orsay Cedex, France.
*The research has been partially done when the second author visited LRI in September 1989.

where L is the Laplacian matrix of the weighted graph G with n vertices, $U = \text{diag}(u)$ is the diagonal matrix with entries u_i on the diagonal and $\lambda_{\max}$ is the maximum eigenvalue of the matrix $L + U$. The minimum is taken over all vectors $u \in \mathbb{R}^n$ satisfying $\sum u_i = 0$. Section 2 contains the details of the definition and the proof of the inequality (2).

Since the value $\varphi(G)$ is obtained as the minimum of the function $f(u) = \frac{1}{4} n \lambda_{\max}(L + \text{diag}(u))$, it is useful to study some properties of f. In Section 3 we show that f is convex, lipschitzian and has unique minimum.

The main results are presented in Sections 4, 5 and 6. In Section 4 we give a dual characterization of $\varphi(G)$ in terms of an optimality certificate, consisting of a family of eigenvectors. Though this certificate cannot be, in general, used for concrete computation of $\varphi(G)$, it has theoretical applications. The bound $\varphi(G)$ has several interesting properties. In many aspects, its behaviour resembles properties of $\text{mc}(G)$, and — perhaps due to this fact — it also seems to well approximate $\text{mc}(G)$. One of such properties is proved in Section 5. We show that $\varphi(G)$ is subadditive with respect to amalgams, and, moreover, additive with respect to 1-sums and disjoint unions of graphs. The subadditivity allows us to establish that $\varphi(G)$ never exceeds $1.131 \text{ mc}(G)$ for planar, or more generally, weakly bipartite graphs. This result is given in Section 6.

Since the max-cut problem is NP-complete [13], the computational aspects are also studied. The number $\varphi(G)$ need to be rational. However, there is a polynomial time algorithm to compute it with arbitrary prescribed precision. The algorithm is obtained by an application of the ellipsoid method, in the form given by [14]. Section 7 contains the technical details.

The method of optimization over the ''varying diagonal'' (we call it a ''correcting vector'', in order to underline the combinatorial meaning) has already been used in a paper by Donath and Hoffman [10] (see also [5]) to obtain a lower bound on the equipartition problem. Their bound has been later improved by Boppana [6]. Since our eigenvalue bound is quite analogous to, or perhaps even simpler, than those on the equipartition problem, we believe that our contribution is in a deeper study of the combinatorial properties. In some sense, we were rather motivated by the work of Fiedler [11], who formulated an eigenvalue lower bound on the graph connectivity and studied its properties. Our bound $\varphi(G)$ on the max-cut problem improves a previous simpler bound

$$\text{mc}(G) \leqslant \tfrac{1}{4} \lambda_{\max}(L) n$$

given by Mohar and Poljak [18], which corresponds to the choice $u = 0$.

2. Definition of the upper bound $\varphi(G)$

Consider a loopless graph G with vertex set V, each pair $\{i, j\}$ of vertices is given a weight w_{ij}. The number of vertices is denoted by n, and the vertices are labelled with the integers $1, 2, \ldots, n$. An ordinary (unweighted) graph is identified with a weighted graph where the weights are 1 on the edges and 0 on the non-edges.

We associate to a subset S of V the vector X_S with n coordinates $x_v = 1$ if $v \in S$ and $x_v = -1$ if $v \in V \setminus S$.

We introduce the *Laplacian matrix* $L = L_G$ with n rows and n columns. The matrix is symmetric and its entries are $L_{ij} = -w_{ij}$ for $i \neq j$ and $L_{ii} = \sum_{j \neq i} w_{ij}$. If G is an ordinary graph, then the Laplacian can be expressed as $L_G = \mathrm{diag}(d) - A$ where A is the adjacency matrix and d the vector of vertex degrees. We will occasionally also use notation $L(G)$ instead of L_G. It will be always clear from the context whether G is a weighted or unweighted graph.

We recall that the maximum eigenvalue λ of a symmetric matrix M satisfies

$$\lambda X^{\mathrm{T}} X \geqslant X^{\mathrm{T}} M X$$

for every vector X. This inequality is called the *Rayleigh principle* (see for example [17]).

Let $u = u_1, u_2, \ldots, u_n$ be a vector of n reals and $U = \mathrm{diag}(u)$ the diagonal matrix on n rows and n columns with $U_{ii} = u_i$. We will call any vector u with $\sum u_i \geqslant 0$ a *correcting vector* (the entry u_i is associated with vertex i).

Lemma 1. *We have*

$$\mathrm{mc}(G) \leqslant \tfrac{1}{4} n \lambda_{\max}(L + U)$$

for every correcting vector u.

Proof. We observe that the following identity holds:

$$X^{\mathrm{T}} L X = \sum w_{ij}(x_i - x_j)^2$$

for the Laplacian L and every vector X. In particular, with $X = X_S$ corresponding to a subset $S \subset V$, we have

$$X_S^{\mathrm{T}} L X_S = \sum w_{ij}(X_{Si} - X_{Sj})^2 = 4 \sum_{\substack{i \in S \\ j \notin S}} w_{ij}$$

which is 4 times the value of the cut induced by S.

We also have $X_S^{\mathrm{T}} U X_S = \sum u_i (X_{Si})^2 = \sum u_i \geqslant 0$ because $X_{Si} = \pm 1$.

Let S induce a maximum cut. Then

$$4\mathrm{mc}(G) = X_S^{\mathrm{T}} L X_S \leqslant X_S^{\mathrm{T}} (L + U) X_S.$$

Applying the Rayleigh principle to the vector X_S, we obtain that

$$4\mathrm{mc}(G) \leqslant \lambda_{\max}(L + U) X_S^{\mathrm{T}} X_S = n \lambda_{\max}(L + U). \qquad \square$$

Based on Lemma 1, we can introduce the upper bound $\varphi(G)$ which is the main notion studied in the paper. We define

$$\varphi(G) = \min_{u} \tfrac{1}{4} n \lambda_{\max}(L + U),$$

the minimum being taken over all correcting vectors u (we will prove in the next section that the minimum is actually attained for some u, and so that the notation is justified). Such a vector u realizing the minimum will be called an *optimum correcting vector.*

The optimum correcting vector u obviously satisfied $\sum_{1 \leqslant i \leqslant n} u_i = 0$.

We first state two properties of φ, namely that φ is (i) positive-homogeneous, and (ii) monotone.

Lemma 2. (i) *If each weight w_{ij} is multiplied by some positive real k, then φ is also multiplied by k.*

(ii) *If G and G′ have the same vertices and the weight functions w and w′ satisfy $w_{ij} \leqslant w'_{ij}$ for all pairs of vertices, then $\varphi(G) \leqslant \varphi(G')$.*

Proof. The first property is obvious. Let us prove the second one. We note that for every vector u and every vector X the relation

$$X^{\mathrm{T}}(L_{G'} + U)X - X^{\mathrm{T}}(L_G + U)X = X^{\mathrm{T}}(L_{G'} - L_G)X \geqslant 0$$

holds. □

Observe that a trivial upper bound on $\mathrm{mc}(G)$ is the sum of all nonnegative weights of edges. We show that our upper bound $\varphi(G)$ is never worse than this trivial bound.

Theorem 1. *Let G be a weighted graph with weight function w. We have*

$$\varphi(G) \leqslant \sum_{w_e \geqslant 0} w_e .$$

Proof. Set $w'_{ij} = \max(0, w_{ij})$ for all pairs of vertices and let G' be the corresponding weighted graph. The second part of the lemma above says $\varphi(G) \leqslant \varphi(G')$. So we may assume that w is already non-negative. Set $m = \sum w_{ij}$, and consider the correcting vector defined by

$$u_i = \frac{4m}{n} - 2 \sum_j w_{ij} . \tag{4}$$

We have, for every X, the relations

$$X^{\mathrm{T}}(L + U)X = \frac{4m}{n} X^{\mathrm{T}}X - \sum w_{ij}(X_i + X_j)^2 \leqslant \frac{4m}{n} X^{\mathrm{T}}X.$$

Applying this to an eigenvector X associated to the largest eigenvalue $\lambda_{\max}$ of $L + U$, we obtain $\lambda_{\max}(L + U) \leqslant 4m/n$, and hence $\varphi(G) \leqslant m$. □

A simpler proof of Theorem 1 will be proposed in Section 5.

Let us remark that the proof of Theorem 1 relied on the "guess" of a good correcting vector. Indeed an awkward choice of a correcting vector may result in a poor bound. For example, consider the ordinary star $K_{1,n-1}$ with the null correcting vector. Then $\frac{1}{4} n \lambda_{\max}(L(K_{1,n-1})) = \frac{1}{4} n^2$ while $\mathrm{mc}(K_{1,n-1}) = n - 1 = \varphi(K_{1,n-1})$.

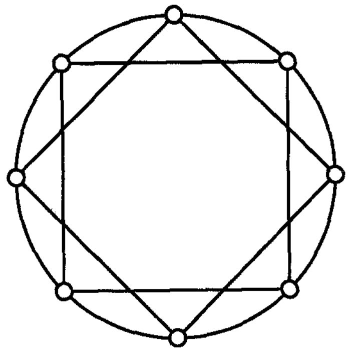

Fig. 1. The exact graph $C_{8,2}$ with mc $= \varphi = 12$.

We will call a weighted graph *exact* if it satisfies $\mathrm{mc}(G) = \varphi(G)$. Clearly, a graph G is exact if and only if some ± 1-vector x belongs to the eigenspace of $\lambda_{\max}(L + \mathrm{diag}(u))$ where u is a correcting vector. Although $\varphi(G)$ provides a good upper bound for $\mathrm{mc}(G)$, exact graphs seem to be relatively rare. Still there are several interesting examples of exact graphs.

An immediate corollary of Theorem 1 is that every weighted bipartite graph with positive weights is exact. Also the ordinary complete graphs with an even number of vertices are exact. (Clearly, $\mathrm{mc}(K_{2k}) = k^2$ and on the other hand, with a null correcting vector, the maximum eigenvalue is $2k$, hence $\varphi \leqslant k^2$.)

An example is also given in Figure 1 (see [18, p. 349]).

Some construction of exact graphs will be given in Section 5. Further examples of exact graphs can be found in [18] and [7]. We prove in [8] that the recognition of exact weighted graphs is NP-complete. The complexity of the unweighted case is an open question.

3. Properties of f

It is advantageous to write the definition of $\varphi(G)$ in the form $\varphi(G) = \min_u f(u)$ where $f(u) = \frac{1}{4} n \lambda_{\max}(L + U)$ and u is constrained by $\sum u_i \geqslant 0$.

In this section we establish some properties of f. It is important to show that f attains a minimum which is unique. Another useful property of f, the convexity, has already been known (see [10, Theorem 1] or [9, Chapter 6]).

Theorem 2. *The function f has the following properties.*

(i) *f is Lipschitzian, that is $|f(u) - f(u')| / |u - u'|$ is bounded by some constant.*

(ii) *f is convex.*

(iii) *f attains its minimum exactly once unless all the weights are null.*

Proof. Let u and u' be two vectors and U and U' the corresponding diagonal matrices.

(i) Without loss of generality, one can assume $f(u) \leqslant f(u')$. Let Y be an eigenvector for the highest eigenvalue of $L+U'$, with $Y^{\mathrm{T}}Y=\frac{1}{4}n$. We have $Y^{\mathrm{T}}(L+U)Y\leqslant f(u)$ and $Y^{\mathrm{T}}(L+U')Y=f(u')$ and thus

$$0\leqslant f(u')-f(u)=\sum_i (u_i'-u_i)y_i^2\leqslant \tfrac{1}{4}\max|u_i'-u_i|n.$$

Thus f is Lipschitzian.

(ii) Let us give a short proof of the convexity of f for convenience. Let t be some real between 0 and 1. Let $u''=(1-t)u+tu'$. Let Y be an eigenvector for the highest eigenvalue of $L+U''$ with $Y^{\mathrm{T}}Y=\frac{1}{4}n$. Then $Y^{\mathrm{T}}(L+U)Y\leqslant f(u)$ and $Y^{\mathrm{T}}(L+U')\leqslant f(u')$. By linear combination we obtain

$$\begin{aligned} f(u'') = Y^{\mathrm{T}}(L+U'')Y &= (1-t)Y^{\mathrm{T}}(L+U)Y+tY^{\mathrm{T}}(L+U')Y \\ &\leqslant (1-t)f(u)+tf(u') \end{aligned}$$

that is the convexity of f.

(iii) Suppose that the minimum is attained at u and u', with $u\neq u'$. By convexity it is also attained at $u''=\frac{1}{2}(u+u')$. Let Y be an eigenvector for the largest eigenvalue of $L+U''$, with $Y^{\mathrm{T}}Y=\frac{1}{4}n$. We then have $Y^{\mathrm{T}}(L+U)Y\leqslant\frac{1}{4}\lambda_{\max}(L+U)n$ and $Y^{\mathrm{T}}(L+U')Y\leqslant \frac{1}{4}\lambda_{\max}(L+U')n$. By averaging, we see that both inequalities are in fact equalities, thus Y is also eigenvector for the highest eigenvalue of $L+U$ and for the highest eigenvalue of $L+U'$ which is the same real. Hence Y is an eigenvector for $U'-U$, with eigenvalue 0. The eigenspace H of $L+U''$ is thus a subspace of the kernel K of $U'-U$. Let $d=\dim K$. We have $d<n$ since $U\neq U'$. Let λ' be the largest eigenvalue of $L+U''$ smaller than λ (it exists since there are non-null entries out of the diagonal of the symmetric matrix $L+U''$). Let us define u^* by $u_i^*=u_i''+(\lambda-\lambda')(d-n)/n$ if the vector with only the ith coordinate non-null is in K and $u_i^*=u_i''+(\lambda-\lambda')d/n$ for the other indices i. Then U^* and U'' have the same trace. Let us compute the quadratic form with matrix $L+U^*$. Every vector Z is decomposed on Z_1 that lies in H and Z_2 orthogonal to H. Then

$$Z^{\mathrm{T}}(L+U'')Z\leqslant\lambda Z_1^{\mathrm{T}}Z_1+\lambda' Z_2^{\mathrm{T}}Z_2$$

and

$$\begin{aligned} Z^{\mathrm{T}}(L+U^*)Z &= Z^{\mathrm{T}}(L+U'')Z+(\lambda-\lambda')\frac{d-n}{n}Z_1^{\mathrm{T}}Z_1 \\ &\quad +(\lambda-\lambda')\frac{d}{n}Z_2^{\mathrm{T}}Z_2\leqslant\frac{d\lambda+(n-d)\lambda'}{n}Z^{\mathrm{T}}Z<\lambda Z^{\mathrm{T}}Z. \end{aligned}$$

Thus we have found a better choice for the system, which contradicts the minimality.

We have proven that the minimum is attained at most once. Let us now prove that it is attained. If $\max|u_i|\to\infty$ with $\sum u_i\geqslant 0$ then $\max u_i\to\infty$ and $\lambda_{\max}(L+U)\geqslant\max_i(u_i+d_i)$, where $d_i=\sum_{j\neq i}w_{ij}$ is the "weighted degree" of the vertex i (see Lemma 4 of Section 7). Now apply the usual compactness argument to the continuous function f that tends to infinity with u on the closed half-space $\sum u_i\geqslant 0$. $\square$

The convexity of f simplifies the possible forms of the optimal correcting vector if the weighted graph has a non trivial group of automorphisms (preserving the weights of the edges).

Corollary 1. *Let u be the optimum correcting vector of a weighted graph G. If vertices i and j belong to the same orbit then $u_i = u_j$. In particular, if G is vertex-transitive, then the optimal correcting vector is the null vector.*

Proof. The proof is a routine use of the convexity and group action. □

We use this corollary to compute φ for various examples (some of them are collected in [7]). As a consequence of this corollary, the previous upper bound of Mohar and Poljak [18] is not improved for the vertex-transitive graphs.

We will use later in this paper some examples:

- Ordinary complete graphs:

$$\mathrm{mc}(K_{2k}) = k^2 = \varphi(K_{2k}),$$

$$\mathrm{mc}(K_{2k+1}) = k(k+1) < \tfrac{1}{4}\ (2k+1)^2 = \varphi(K_{2k+1}).$$

- Ordinary cycles:

$$\mathrm{mc}(C_{2k}) = 2k = \varphi(C_{2k}),$$

$$\mathrm{mc}(C_{2k+1}) = 2k < \left(2 + 2\cos\frac{\pi}{2k+1}\right)\frac{2k+1}{4} = \varphi(C_{2k+1}) \leqslant \frac{25+5\sqrt{5}}{32} \times 2k.$$

The value $(25+5\sqrt{5})/32 = 1.130635\ldots$ comes from the 5-cycle.

4. A characterization of the optimal correcting vector

In this section we give a criterium to check whether a correcting vector u is optimal.

Clearly, in case of exact graphs, it is sufficient to present a subset S which induces a cut of value $\frac{1}{4} n \lambda_{\max}(L_G + \mathrm{diag}(u))$. Then we already know that

- u is the optimum correcting vector;
- S induces the max-cut; and
- the vector X_S with coordinates 1 or -1 is an eigenvector of $L_G + \mathrm{diag}(u)$ for the eigenvalue $\lambda_{\max}(L_G + \mathrm{diag}(u))$.

Thus, we can see that X_S certificates the optimality of u in the case of exact graphs. We will formulate a more general concept of a certificate that can be used for all graphs.

Let u be a correcting vector and $\mathscr{E}$ the eigenspace of $\lambda_{\max}(L_G + \mathrm{diag}(u))$. We say that a finite family $(x(\kappa),\ \kappa\in K)$ from $\mathscr{E}$ is an *optimality certificate* of the correcting vector u if $\sum_{\kappa\in K}((x(\kappa))_i)^2=1$ for every $i=1, \ldots, n$. The notion is justified by Corollary 2 below.

We will use the linear form $\sigma: u\to\sum_i u_i$, the eigenspace $\mathscr{E}$ and the convex cone $\mathscr{C}$ generated by the linear forms $\tau_x: u\to\sum_i x_i^2 u_i$ for all x in the eigenspace $\mathscr{E}$.

Theorem 3. *The correcting term u realizes the minimum if and only if the form σ is in the cone $\mathscr{C}$.*

Proof. If σ is in the cone then, for every increment v with $\sigma(v)\geqslant 0$, we have some $x\in\mathscr{E}$ with $\tau_x(v)\geqslant 0$. Then if we normalize x to $x^{\mathrm{T}}x=\frac{1}{4}n$ we have $f(u+v)\geqslant x^{\mathrm{T}}(L+U+V)x=f(u)+\tau_x(v)\geqslant f(u)$.

If σ is not in the cone $\mathscr{C}$, then some system v separates $\mathscr{C}$ and the form τ_x, that is $\sigma(v)>0$ and $\tau_x(v)<0$ for all $x\neq 0$ in $\mathscr{E}$, and we now prove that the minimum of f does not occur in u.

Let us introduce the spheres $\mathscr{S}_r$ of vectors with $x^{\mathrm{T}}x=r$. These spheres are compact sets and their intersections with $\mathscr{E}$ and the orthogonal subspace $\mathscr{E}^{\perp}$ are also compact. Let $a<0$ be the maximum of $x^{\mathrm{T}}Vx$ on $\mathscr{S}_1\cap\mathscr{E}$, let b be the maximum of $|x^{\mathrm{T}}Vy|$ with $x\in\mathscr{E}\cap\mathscr{S}_1$ and $y\in\mathscr{E}^{\perp}\cap\mathscr{S}_1$ and let c be the maximum of $y^{\mathrm{T}}Vy$ with $y\in\mathscr{E}^{\perp}\cap\mathscr{S}_1$. Let λ and μ be the largest and second largest eigenvalues of $L+U$. Let $x+y$ be a vector in $\mathscr{S}_{n/4}$ such that $x\in\mathscr{E}$ and $y\in\mathscr{E}^{\perp}$.

Then

$$(x+y)^{\mathrm{T}}(L+U+tV)(x+y)-\tfrac{1}{4}\lambda n\leqslant tax^{\mathrm{T}}x+2tb\sqrt{x^{\mathrm{T}}xy^{\mathrm{T}}y}+(tc+\mu-\lambda)y^{\mathrm{T}}y.$$

If $0<t<a(\mu-\lambda)/(b^2-ca)$, the quadratic form appearing on the right hand side, with matrix

$$\begin{bmatrix} ta & tb \\ tb & tc+\mu-\lambda \end{bmatrix},$$

is negative definite. So $f(u+tv)<f(u)$ on the open interval $]0, a(\mu-\lambda)/(b^2-ca)[$. $\square$

Corollary 2. *A vector u is optimum correcting vector if and only if there exists an optimality certificate of u.*

Proof. If $\sigma\in\mathscr{C}$, then $\sigma=\sum_{\kappa\in K}\alpha_\kappa\tau_{x(\kappa)}$ where K is finite and vectors $x(\kappa)$ are in $\mathscr{E}$, and α_k are positive. Clearly, one may assume that the vectors $x(\kappa)$ are distinct and all α_κ are equal to one. Thus, $(x(\kappa),\ \kappa\in K)$ is an optimality certificate. The converse implication is trivial. $\square$

Let us remark that the optimality certificate need not be unique. The existence of an optimality certificate has rather a theoretical importance than a computational use. We will use it in the next section to establish the additivity of φ on 1-sums and 0-sums. It may be noted that every certificate can be replaced by a certificate consisting of at most n vectors of $\mathscr{E}$.

Example. For the ordinary triangle, we already know that $u=0$ is the optimal correcting vector. The eigenvalues are 0 and 3. A certificate is the family of eigenvectors $(1/\sqrt{2})(0, 1, -1)$, $(1/\sqrt{2})(1, 0, -1)$, $(1/\sqrt{2})(1, -1, 0)$.

Corollary 3. *Assume that G is not exact, and let u be the optimum correcting vector. Then the largest eigenvalue $\lambda_{\max}$ of $L+U$ cannot be simple.*

Proof. Assume that $\lambda_{\max}$ is simple, and let X be a corresponding eigenvector; we may assume that X has length $\sqrt{n}$. Then the dimension of the eigenspace $\mathscr{E}$ is 1, and hence also the dimension of the cone $\mathscr{C}$ is 1. Since by Theorem 3 the form $\sigma=(1, 1, \ldots, 1)=(u\mapsto\sum_i u_i)$ belongs to $\mathscr{C}$, it follows that $\mathscr{E}$ contains a vector with coordinates in $\{-1, 1\}$. Hence G is exact, a contradiction. □

The "dual characterization" by an optimality certificate corresponds, in general, to a subgradient characterization of a convex function. However we are not aware of any result that would directly yield our Theorem 3. A very close problem was studied by Overton [19] but his dual characterization has a more complicate form.

5. The function φ and amalgams of graphs

In this section we prove the subadditivity of the bound φ with respect to amalgams. Subadditivity is a crucial property and will be applied later to establish a result on planar graphs. In special cases, namely when the graphs to be amalgamated have 0 or 1 vertex in common, the inequality becomes an equality.

We describe now the *amalgam* of two weighted graphs G' and G''. Let V' and V'' be the vertex sets of G' and G'' (the sets V' and V'' may intersect). Let n', n'' be the number of vertices and let w', w'' be the weight functions of G' and G'' respectively. Then the vertex set of the amalgam G is $V=V'\cup V''$ and its weight function w is defined by

$$w_{ij}=\begin{cases} w'_{ij}+w''_{ij} & \text{if } i \text{ and } j \text{ are both in } V'\cap V'', \\ w'_{ij} & \text{if } i, j \text{ are both in } V' \text{ but not both in } V'', \\ w''_{ij} & \text{if } i, j \text{ are both in } V'' \text{ but not both in } V', \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the Laplacian matrix L of G is the sum of the suitably extended with zeroes Laplacian matrices L' and L'' of G' and G''.

It is easy to see that the max-cut has the following properties.

Lemma 3. *If G is the amalgam of G' and G'', then*

$$\mathrm{mc}(G)\leqslant\mathrm{mc}(G')+\mathrm{mc}(G''). \tag{5}$$

If the amalgamated graphs G' and G'' have at most one common vertex then

$$\mathrm{mc}(G)=\mathrm{mc}(G')+\mathrm{mc}(G''). \qquad \square \tag{6}$$

The following theorem shows that φ behaves in the same way.

Theorem 4. *If G is the amalgam of G' and G'', then*

$$\varphi(G)\leqslant\varphi(G')+\varphi(G''). \tag{7}$$

If the amalgamated graphs G' and G'' have at most one common vertex then

$$\varphi(G)=\varphi(G')+\varphi(G''). \tag{8}$$

Proof. The proof will consist of several steps, starting with some special cases. Let u' and u'' be the optimum correcting vectors, and $(x'(\kappa'),\ \kappa'\in K')$ and $(x''(\kappa''),\ \kappa''\in K'')$ optimality certificates for G' and G'' respectively. (These certificates are defined in Section 5.)

(i) Assume that $V'=V''$. We show that the inequality (7) holds. Consider a (possibly not optimal) correcting vector $u=u'+u''$. Let x be the eigenvector corresponding to the maximum eigenvalue of the matrix $L(G')+\mathrm{diag}(u'+u'')+L(G'')$. Assume $x^{\mathrm{T}}x=\frac{1}{4}|V|$. Using the Rayleigh principle, we have

$$\begin{aligned}\varphi(G)&\leqslant x^{\mathrm{T}}(L(G')+\mathrm{diag}(u'+u'')+L(G'')x\\&\leqslant\lambda_{\max}(L(G')+\mathrm{diag}(u'))x^{\mathrm{T}}x+\lambda_{\max}(L(G'')+\mathrm{diag}(u''))x^{\mathrm{T}}x\\&\leqslant\varphi(G')+\varphi(G'').\end{aligned}$$

(ii) Assume that $V'\cap V''=\emptyset$. We show that the equality (8) holds. Let λ' and λ'' denote the maximum eigenvalue of $L(G')+\mathrm{diag}(u')$ and $L(G'')+\mathrm{diag}(u'')$ respectively. We construct a correcting vector u by

$$u_i=\begin{cases}u_i'+\alpha' & \text{if } i\in V',\\ u_i''+\alpha'' & \text{if } i\in V'',\end{cases} \tag{9}$$

with α' and α'' real numbers such that $n'\alpha'+n''\alpha''=0$ and $\lambda'+\alpha'=\lambda''+\alpha''$. The concatenation of an eigenvector x' for $L(G')+\mathrm{diag}(u')$ and an eigenvector x'' for $L(G'')+\mathrm{diag}(u'')$ is then an eigenvector $x=(x',x'')$ for $L(G)+\mathrm{diag}(u)$ with eigenvalue $\lambda=\lambda'+\alpha'$. It is easy to check that λ is the largest eigenvalue of $L(G)+\mathrm{diag}(u)$ and that $\frac{1}{4}\lambda(n'+n'')=\frac{1}{4}\lambda'n'+\frac{1}{4}\lambda''n''$. Thus we already have proved the inequality (7).

In order to show that the equality (8) holds, we check that the correcting vector u given by formula (9) is already optimum for G. To prove optimality of u, we will construct an optimality certificate $(x_\kappa,\ \kappa\in K)$ for G and u, from the certificates of u' and u''.

Now K is the disjoint union of K' and K''. $x(\kappa)$ is the concatenation $x'(\kappa)$, 0 if $\kappa\in K'$ and the concatenation 0, $x''(\kappa)$ if $\kappa\in K''$. The corresponding sum is the all-one vector. This proves that the chosen u realizes the minimum of f.

(iii) Let $G+v$ be obtained from G by adding an isolated vertex. Then $\varphi(G+v)=\varphi(G)$. This follows immediately from part (ii).

(iv) Let V' and V'' be arbitrary, i.e. they may intersect and need not be identical. We show that the inequality (7) holds. One adds isolated vertices to graphs G' and G'' to obtain the vertex set of G. This does not change the bounds $\varphi(G')$ and $\varphi(G'')$. Then one applies part (i).

This concludes the proof of the first assertion in the theorem.

(v) Let V' and V'' intersect exactly in one vertex. We show that the equality (8) holds. We already know the inequality (7) from part (iv). We will construct the optimum correction vector u and a certificate of optimality.

The proof is similar to that of part (ii). Let $M'=L(G')+\operatorname{diag}(u')$ be the optimally corrected Laplacian of G', n' the number of vertices of G' and $(x'(\kappa'), \kappa'\in K')$ the certificate of the minimum. We also consider the analogous objects for G''.

We introduce the correcting vector u for $L(G)$,

$$u_i=\begin{cases} u_i'+\alpha' & \text{if } i\in V(G')\setminus V(G''),\\ u_i''+\alpha'' & \text{if } i\in V(G'')\setminus V(G'),\\ (1-n')\alpha'+(1-n'')\alpha''+u_i+u_i'' & \text{if } i\in V(G')\cap V(G''),\end{cases}$$

with $\alpha'+\lambda'=\alpha''+\lambda''=(1-n')\alpha'+(1-n'')\alpha''$.

Then it is easily checked that the largest eigenvalue of $L(G)+\operatorname{diag}(u)$ is $(n'\lambda'+n''\lambda'')/(n'+n''-1)$. Moreover the eigenvectors are constituted by "glueing" an eigenvector x' and an eigenvector x'' with the same coordinate on the common vertex i.

From a certificate $\sigma=\sum \tau_{x'(\kappa')}$, it is possible to obtain a new one with the first vector x' satisfying $x_i'=1$ and the other ones satisfying $x_i'=0$. It suffices to replace the matrix X' with n rows and $r=|K'|$ columns representing the r eigenvectors x' by the matrix $X'P$ with P an orthogonal matrix with r rows and columns, the first column of which has coordinates $p_{\kappa 1}=x_i'(\kappa)$. The same modification can be performed on the set of vectors x_i''.

One then builds a certificate for u by glueing the first vectors x' and x'' and completing each other x' (resp. x'') with $n''-1$ zeroes (resp. $n'-1$ zeroes). □

Theorem 4 has several applications. The most important application to planar graphs will appear in Section 6. Here we present as corollaries some constructions of exact graphs.

Corollary 4. *For any weighted graph G, the bound $\varphi(G)$ is at most the sum of the positive weights of its edges.*

Proof. Let G be a graph with weight function w. For every edge e let G_e be the graph consisting only of this edge with weight w_e. Using subadditivity, we have $\varphi(G)\leqslant\sum\varphi(G_e)$. Clearly

$$\varphi(G_e)=\begin{cases} w_e & \text{if } w_e\geqslant 0,\\ 0 & \text{if } w_e<0.\end{cases}\qquad \square$$

Corollary 5. *Any bipartite graph with non-negative weights is exact.*

Proof. Let G be a bipartite graph with a non-negative weight function w. We have $\mathrm{mc}(G) \leqslant \varphi(G) \leqslant \sum w_e$. Since the obvious cut already gives $\mathrm{mc} = \sum w_e$, these three expressions are equal. □

Corollary 6. *If all 2-connected components of G are exact, then G is exact.*

Proof. This can be proven by amalgamating these components and the remaining edges in such an order that the two parts to be amalgamated at each step have only one vertex in common. □

This corollary has the following generalization.

Corollary 7. *Let G and G' be a pair of exact graphs. Assume that the maximum cuts of G and G', respectively, can be realized by partitions which coincide on $V \cap V'$. Then the amalgam G'' of G and G' is exact as well.*

Proof. We have a cut of G'' by combining the compatible partitions of G and G'. This cut has value $\mathrm{mc}(G) + \mathrm{mc}(G') = \varphi(G) + \varphi(G')$. On the other hand we have $\varphi(G'') \leqslant \varphi(G) + \varphi(G') = \mathrm{mc}(G) + \mathrm{mc}(G') \leqslant \mathrm{mc}(G'') \leqslant \varphi(G'')$. □

The *cartesian sum* $(G, k) \times (G', k')$ of weighted graphs G, G' with weight functions w, w' on their edges and k, k' on their vertices is the graph with vertex set $V \times V'$ (and weight $k_i k'_{i'}$ on the vertex (i, i')), and weight function $w_{ij} k_{i'}$ on the edges (i, i'), (j, i') and $k_i w_{i'j'}$ on the edges $(i, i')(i, j')$, the other edges having null weight.

Corollary 8. *If G and G' are exact and the weight functions on vertices k, k' are non-negative, then their cartesian sum is exact.*

Proof. This can be proven by amalgamating the n' copies of G (with weight function $k'_{i'} w$ on $G \times \{i'\}$) and the n copies of G' (with the weight function $k_i w'$ on $\{i\} \times G'$).

If optimal cuts of G and G' are given by S and S' then the cut of $G \times G'$ given by $S \times S' \cup (V \setminus S) \times (V' \setminus S')$ has the wanted value (see [18]).

$$\varphi((G, k) \times (G', k')) \leqslant \sum_i k_i \varphi(G') + \sum_{i'} k'_{i'} \varphi(G)$$

can also be derived from the computation of the eigenvalues of $K \otimes (L' + U') + (L + U) \otimes K'$ where K and K' are the diagonal matrices corresponding to the weight functions k, k' on the vertex sets of G and G'. □

6. Weakly bipartite graphs

It is well known that the max-cut problem is polynomially solvable for planar graphs. We show that our bound $\varphi(G)$ behaves well on planar graphs. The result is formulated in Corollary 6 below.

We need to recall some definitions.

The convex hull of characteristic vectors of bipartite subgraphs of a graph G is called the *bipartite subgraph polytope* and is denoted by $P_{\mathrm{B}}(G)$.

A graph G is called *weakly bipartite* (cf. [15]) if its bipartite subgraph polytope $P_{\mathrm{B}}(G)$ is described by the following system of inequalities:

$$\sum_{e\in c} x_e \leqslant |c|-1 \quad \text{for each odd cycle } c, \tag{10}$$

$$0 \leqslant x_e \leqslant 1 \quad \text{for } e \in E. \tag{11}$$

Theorem 5. *Let G be a weakly bipartite graph with non-negative weights. Then*

$$\varphi(G) \leqslant \frac{25+5\sqrt{5}}{32}\,\mathrm{mc}(G).$$

Proof. The max-cut problem for any graph G with non-negative weights can be written as $\max\{w^{\mathrm{T}}x \mid x \in P_{\mathrm{B}}(G)\}$. Since G is assumed weakly bipartite, it is equivalent to the optimization problem

$$\max \sum_{e\in E} w_e x_e \quad \text{subject to (10) and (11)}. \tag{12}$$

Let C be the set of odd cycles in G. Using the duality of linear programming, there is a collection of non-negative coefficients α_c, $c \in C$ and a collection of non-negative coefficients β_e, $e \in E$ such that

$$\beta_e + \sum_{c \ni e} \alpha_c \geqslant w_e \quad \text{for every edge } e, \tag{13}$$

$$\sum_{e\in E} \beta_e + \sum_{c\in C} \alpha_c(|c|-1) = \mathrm{mc}(G). \tag{14}$$

For an odd cycle $c \subset G$ and a positive coefficient α, let αc be the weighted cycle with all edges bearing weight α. Similarly let βe denote the single edge considered as a graph with the weight β.

Let H be the weighted graph obtained as the amalgam of the weighted graphs $\alpha_c c$, $c \in C$ and $\beta_e e$, $e \in E$. By (13), the weight of each edge of H is at least 1 and G is clearly a subgraph of H. Thus we have $\varphi(G) \leqslant \varphi(H)$ by Lemma 2.

Since H is obtained as the union of weighted subgraphs, using Theorem 4 we get

$$\begin{aligned}\varphi(H) &\leqslant \sum \varphi(\alpha_c c) + \sum \varphi(\beta_e e) \\ &= \sum \alpha_c \varphi(c) + \sum \beta_e \varphi(e) \\ &\leqslant \sum \alpha_c \frac{25+5\sqrt{5}}{32}(|c|-1) + \frac{25+5\sqrt{5}}{32}\sum \beta_e \\ &= \frac{25+5\sqrt{5}}{32}\left(\sum_c \alpha_c(|c|-1) + \sum_e \beta_e\right) = \frac{25+5\sqrt{5}}{32}\,\mathrm{mc}(G)\end{aligned}$$

owing to

$$\varphi(c) \leqslant \frac{25+5\sqrt{5}}{32}(|c|-1)$$

for odd cycles and (14). This proves the theorem. □

Since every planar graph is weakly bipartite due to a result of Barahona ([1], cf. [15]), we have:

Corollary 9. *Every planar graph with nonnegative weights G satisfies*

$$\varphi(G) \leqslant \frac{25+5\sqrt{5}}{32}\,\mathrm{mc}(G). \qquad \square \tag{15}$$

Some other subclasses of weakly bipartite graphs are given in [2] and [12]. However a general characterization of weakly bipartite graphs is not known.

7. Existence of a polynomial algorithm

In this section, we will show that the number $\varphi(G)$ is efficiently computable for a weighted graph G with rational weights. More precisely, we prove that $\varphi(G)$ can be computed with arbitrary prescribed precision $\varepsilon > 0$ by a polynomial time algorithm. The term *polynomial* means that the number of steps of the algorithm is bounded by a polynomial in the number of bits which are necessary to encode the input data, which in our case consist of the integer n, the rational symmetric matrix of weights w_{ij} and a rational precision $\delta > 0$. These notions are precised in [14]. We just recall definitions and a theorem from that book.

If K is a compact subset of $\mathbb{R}^n$ and ε a positive real number, $S(K, \varepsilon)$ is the set of points y such that the closed sphere $S(y, \varepsilon)$ centered at y with radius ε meets K, and $S(K, -\varepsilon)$ is the set of points y such that $S(y, \varepsilon)$ is included in K.

A centered body $(K; n, R, r, a)$ is a n-dimensional set K of points such that $S(a, r) \subset K \subset S(a, R)$.

Rational membership problem RMEM. Given a vector $y \in \mathbb{Q}^n$ and a convex compact set $K \subset \mathbb{R}^n$, decide whether y is in K.

This problem is slightly stronger than the weak membership problem:

Weak membership problem WMEM. Given a vector $y \in \mathbb{Q}^n$, a positive rational ε and a convex compact set $K \subset \mathbb{R}^n$, either

(i) assert y is $S(K, \varepsilon)$, or

(ii) assert y is out of $S(K, -\varepsilon)$.

Weak optimization problem WOPT. Given a vector c in $\mathbb{Q}^n$ and a positive rational number ε, either

(i) find a vector $y \in \mathbb{Q}^n$ such that $y \in S(K, \varepsilon)$ and $c^{\mathrm{T}}x \leqslant c^{\mathrm{T}}y + \varepsilon$ for all $x \in S(K, -\varepsilon)$, or

(ii) assert that $S(K, -\varepsilon)$ is empty.

Theorem 6 (Theorem 4.3.13 of [14]). *There exists an oracle-polynomial time algorithm that solves the following problem*:

Input: *A rational number $\varepsilon > 0$, a centered convex body $(K; n, R, r, a_0)$ given by a weak membership oracle, and a convex function $f: \mathbb{R}^n \to \mathbb{R}$ given by an oracle that, for every $x \in \mathbb{Q}^n$ and $\delta > 0$, returns a rational number t such that $|f(x) - t| \leqslant \delta$.*

Output: *A vector $y \in S(K, \varepsilon)$ such that $f(y) \leqslant f(x) + \varepsilon$ for all $x \in S(K, -\varepsilon)$.* □

Theorem 7. *There exists a polynomial time algorithm which, for a given graph G with rational weights and a rational number $\delta > 0$, computes a rational vector $\bar{u}$ and a rational number $\bar{\lambda}$ such that*

(i) $f(\bar{u}) \leqslant \bar{\lambda}$,

(ii) $\bar{\lambda} \leqslant f(u) + \delta$ *for all* u. □

For simplicity, we will assume that all weights are between -1 and 1. This is not restrictive since φ is positive homogeneous. We also work with a graph on $n > 2$ vertices, since case of $n \leqslant 2$ has no interest.

We need an easy lemma from matrix theory.

Lemma 4. *Each diagonal entry m_{ii} of a real symmetric matrix M is a lower bound for the highest eigenvalue $\lambda_{\max}(M)$ and the highest sum $s_i = m_{ii} + \sum_{j \neq i} |m_{ij}|$ is an upper bound for $\lambda_{\max}(M)$.* □

We now have some lemmas to apply the theory developed in [14].

Let us define the polyhedron $K_0 \subset \mathbb{R}^n$,

$$K_0 \left\{ u \,\middle|\, \sum_i u_i \geqslant 0 \wedge \forall i,\ u_i \leqslant 3(n-1) \right\},$$

and let $\tilde{u}$ be the optimum point, i.e. $f(\tilde{u}) = \varphi(G)$.

Lemma 5. *The optimum point $\tilde{u}$ is in K_0.*

Proof. Each diagonal entry d_i of L is between $1-n$ and $n-1$ and each other entry is between -1 and 1. Hence $\lambda_{\max}(L) \leqslant 2n-2$ and the entries d_i+u_i are less than $2n-2$ at the minimum. Hence at the minimum the terms u_i are at most $3n-3$. □

We choose $B=4n-2$, $a_0=(0, \ldots, 0, 2n)^{\mathrm{T}}$, $r=1$, $R=3(n-1)^2$. Let $K=\{(u, t) \mid \lambda_{\max}(L+U) \leqslant t \leqslant B\}$ (this set is denoted as $G(f, K_0)$ in [14]).

Lemma 6. *We have $S(a_0, r) \subset K \subset S(a_0, R)$ for the L_∞ distance.*

Proof. The first inclusion is a consequence of $\max(u_i)+1-n \leqslant \lambda_{\max}(L) \leqslant \max(u_i)+2n-2$ that gives $\lambda_{\max} \leqslant 2n-1$ if all $|u_i|$'s are at most 1 and $2n+1<4n-2$. The second one comes from the inequalities $u_i \leqslant 3n-3$ and $\sum_i u_i \geqslant 0$ that together give $-3(n-1)^2 \leqslant u_i \leqslant 3(n-1) \leqslant 3(n-1)^2$ and $2n-3(n-1)^2<1-n \leqslant \lambda_{\max}$ and $4n-2<2n+3(n-1)^2$. □

We also notice that $\lambda_{\max}(L+U) \leqslant 4n-4$ if u is in K_0.

Lemma 7. *The rational membership problem RMEM is polynomially solvable for K.*

Proof. Besides the easy tests $u_i \leqslant 3n-3$, $\sum_i u_i \geqslant 0$ and $t \leqslant B$, we have to decide whether $\lambda_{\max}(L+U) \leqslant t$. This is equivalent to check whether $tI-(L+U)$ is positive. This can be done by checking that the principal subdeterminants are non-negative. Thus the problem reduces to computing these n determinants. At last, the computation of the determinant is polynomial on the size of the matrix (see [14, Corollary 1.4.9]). □

In particular, we can solve (WOPT) for the minimization of t subject to $(u, t) \in K$ with $\varepsilon<1$. This gives us a pair $(\bar{u}, \bar{t}) \in S(K, \varepsilon)$ such that $\bar{t} \leqslant t+\varepsilon$ for all $(u, t) \in S(K, -\varepsilon)$.

Let $(u', t') \in K$. Then there exists $(u, t) \in S(K, -\varepsilon)$ with the L_∞-distance of (u, t) and (u', t') at most 2ε. This comes from the choice of K_0, and the slow variations of $\lambda_{\max}$ (we can see that $|\lambda_{\max}(L+U)-\lambda_{\max}(L+U')| \leqslant |u-u'|$ like in Theorem 2).

Hence the true minimum of $\lambda_{\max}$ is between $\bar{t}+\varepsilon$ and $\bar{t}-3\varepsilon$.

Note that f being Lipschitzian gives slightly improved results in comparison with the general theory of [14].

8. Final remarks

A natural question is how well the eigenvalue bound $\varphi(G)$ approximates the actual value of the max-cut $\mathrm{mc}(G)$, and comparison of the eigenvalue bound with other approaches. These questions motivate our further work on this topics.

An important property is that $\varphi(G)$ provides a bound which is asymptotically optimal, because the expected value of the ratio $\varphi(G)/\mathrm{mc}(G)$ tends to 1 for random graphs G with constant edge probabilities.

Let $G_{n,p}$ denote the random graph on n vertices with edge probability p. We have the following result.

Theorem 8. *Let p, $0<p<1$, be fixed. Then*

$$\lim_{n\to\infty} \frac{\varphi(G_{n,p})}{\mathrm{mc}(G_{n,p})}=1.$$

Proof. *Let* e denote the number of edges of $G_{n,p}$ and d the average degree. We have $\mathrm{mc}(G_{n,p}) > \frac{1}{2}e = \frac{1}{4}nd$, since this is true for every graph. In order to estimate $\varphi(G_{n,p})$, we need a result of Juhász [16, Proposition 3]. He proved that, for a random $(0, 1)$-matrix A with fixed density p of 1's, all the eigenvalues λ but the maximum one $\lambda_{\max}$ are of magnitude $|\lambda| \leqslant \mathrm{O}(n^{1/2+\varepsilon})$.

Assume that the correcting vector u is chosen so that the optimized Laplacian matrix $L+\mathrm{diag}(u)$ takes the form $dI-A$ (where d is the average degree). Then $\lambda_{\max}(L+\mathrm{diag}(u)) = d-\lambda_{\max}(A) = d+\mathrm{O}(n^{1/2+\varepsilon})$, and hence

$$\varphi(G_{n,p}) \leqslant \tfrac{1}{4}n(d+\mathrm{O}(n^{1/2+\varepsilon})).$$

We have

$$\lim_{n\to\infty} \frac{\varphi(G_{n,p})}{\mathrm{mc}(G_{n,p})} \leqslant \lim_{n\to\infty} \frac{\frac{1}{4}n(d+\mathrm{O}(n^{1/2+\varepsilon}))}{\frac{1}{4}nd} = 1. \qquad \square$$

The asymptotic optimality of $\varphi(G)$ contrasts with the asymptotic behaviour of the bound from [3, 4] (computed by linear programming), for which the ratio $\varphi(G)/\mathrm{mc}(G)$ tends to $\frac{4}{3}$ on the graphs with fixed edge probabilities, and to 2 on a class of sparse graphs (see [22]). A concrete class of sparse graphs with the latter property are the *Ramanujan graphs* [20]. Moreover, the eigenvalue bound is never worse than 1.131 multiple of the linear programming bound for a nonnegatively weighted graph (see [20]).

On the other hand, the worst case ratio of $\varphi(G)/\mathrm{mc}(G)$ (for G unweighted) is not known. The so far worst known case is with $G=C_5$ where the ratio takes value 1.131.... We conjecture that this value might be true for all graphs. A collection of graph-theoretic examples supporting the conjecture is given in [7].

Computational experiments with the bound $\varphi(G)$ are reported in [21]. The current version of the program deals with instances of sizes up to 20.10^3 vertices and 2.10^6 edges. As a by-product, the code also produces a cut as a lower bound. This cut is typically slightly larger than a cut found by repeated local search in the same amount of time. The typical gap between the upper and lower bound is about 5%.

A version of the code computes the exact value of $\mathrm{mc}(G)$ by the branch and bound strategy, using some theoretical results from [8] for the data initiation in subproblems.

Acknowledgements

We thank the following people with whom we discussed some parts of this paper: A. Neumaier, M. Fukushima, T. Ibaraki, B. Mohar. We also thank two referees for their helpful comments.

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