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LARGE AMPLITUDE FREE VIBRATION ANALYSIS OF TAPERED TIMOSHENKO BEAMS USING COUPLED DISPLACEMENT FIELD METHOD

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Tapered beams are more efficient compared to uniform beams as they provide a better distribution of mass and strength and also meet special functional requirements in many engineering applications. In this paper, the linear and non-linear fundamental frequency parameter values of the tapered Timoshenko beams are evaluated by using the coupled displacement field (CDF) method and closed form expressions are derived in terms of frequency ratio as a function of slenderness ratio, taper ratio and maximum amplitude ratio for hinged-hinged and clamped-clamped beam boundary conditions. The effectiveness of the CDF method is brought out through the solution of the large amplitude free vibrations, in terms of fundamental frequency of tapered Timoshenko beams with axially immovable ends. The results obtained by the present CDF method are validated with the existing literature wherever possible.

Key words: large amplitude vibrations, coupled displacement field method, tapered Timoshenko beams, taper ratio, slenderness ratio, frequency ratio.

1. Introduction

Research on vibrations of beams has been going on for a long period of time. So far, many authors have found different methods to find the free vibration behavior of shear flexible beams. Abrate [1] analyzed the free vibration of non-uniform beams with general shape and arbitrary boundary conditions. Byoung Koo Lee *et al.* [2] studied free vibrations of tapered beams with general boundary condition which involves finding an ordinary differential governing equation of beams which can be solved by numerical methods and the natural frequencies are calculated by combining the Runge Kutta method and the determinant search method. De Rosa *et al.* [3] considered the dynamic behavior of beams with an linearly varying cross-section in which the equation of motion is solved in terms of Bessel functions, and the boundary conditions lead to the frequency equation which is a function of four flexibility coefficients. De Rosa *et al.* [4] calculated the natural vibration frequencies of tapered beams by using the Euler-Bernoulli beam theory in the presence of an arbitrary number of rotationally, axially and elastically flexible constraints and the dynamic analysis is performed by means of the so-called cell discretization method (CDM), according to which the beam is reduced to a set of rigid bars, linked together by elastic sections, where the bending stiffness and the distributed mass of the bars is concentrated. Clementi *et al.* [5] studied the frequency response curves of a non-uniform beam undergoing nonlinear oscillations by using the multiple time scale method in which the axial inertia is neglected, and so the equations of motion are statically condensed on the transversal displacement only.

Firouz-Abadi *et al.* [6] investigated the transverse free vibration of a class of variable-cross-section beams using the Wentzel, Kramers, Brillouin (WKB) approximation in which the governing equation of motion of the Euler–Bernoulli beam including axial force distribution is utilized to obtain a singular differential equation in terms of the natural frequency of vibration and a WKB expansion series is applied to find the solution. Zamorska [7] used Green's function method for the free vibration problem of non uniform

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Bernoulli-Euler beams to find Green's function of the fourth order differential operator, occurring at the beam's equation of motion and proposed the power series method. Jung Woo Lee [8] developed the transfer-matrix method to determine solutions to the free vibration characteristics of a tapered Bernoulli–Euler beam in which the roots of the differential equation are determined by using the Frobenius method to obtain the power series solution for bending vibrations and examined the effect of various taper ratios on the eigen pairs of beams, in which the height of the cross section along the length is linearly reduced.

Raju *et al.* [9] analyzed large amplitude free vibrations of tapered beams using continuum and finite element methods. Mahmoud *et al*. [10] applied the differential transformation method (DTM) for free vibration analysis of beams with uniform and non-uniform cross sections. Meera Saheb *et al.* [11] conducted a free vibration analysis for uniform Timoshenko beams using the coupled displacement field method. Mehmet Cem Ece *et al*. [12] studied the vibrations of an isotropic beam which has a variable cross-section. In [12], the governing equation is reduced to an ordinary differential equation in the spatial coordinate system for a family of cross-section geometries with exponentially varying width. Minmao Liao and Hongzhi Zhong [13] carried out a non-linear vibration analysis by establishing equations of motion for taper Timoshenko beams. Mahmoud Bayat [14] introduced an analytical study on the vibration frequencies of tapered beams using an ancient Chinese method called the Max-Min Approach (MMA) and Homotopy Perturbation Method (HPM) to obtain natural frequency and corresponding displacement of tapered beams. Mohamed Hussien Taha and Samir Abohadima [15] used a mathematical model for vibrations of non-uniform flexural beams by presenting a mathematical model for free vibrations of non-uniform viscoelastic flexural beams. An analytical solution for the fourth order differential equation of beam vibration under appropriate boundary conditions is obtained by factorization and calculated mode shapes and damped natural frequencies for a wide range of beam characteristics. Lewandowski [16] obtained an equation of motion to study non-linear vibrations of beams by employing Hamilton's principle by neglecting inertia forces and applied the Ritz method with continuum solution for determining natural frequencies.

Ramazan *et al.* [17] presented a free vibration analysis of a beam based on the Timoshenko type with different boundary conditions. The solutions are obtained by the method of Lagrange multipliers in which the free vibration problem is posed as a constrained vibration problem. Rossi and Laura [18] determined the natural vibration frequencies of linearly tapered beams subjected to different combinations of edge supports by finite element algorithmic procedures. Si Yuan *et al*. [19] utilized the exact dynamic stiffness method for studying vibration of Bernoulli–Euler members, for the case of flexural free vibration of non-uniform Timoshenko beams with no uniformity of geometric and material properties. Kukla and Zamojska [20] applied Green's function method in the frequency analysis of a beam with a varying cross section. The beam carries an arbitrary number of attached discrete systems. The exact solution to the problem concerns a beam with quadratically a varying cross-section area. Hoseini *et al* [21] used the homotopy analysis method to obtain an accurate analytical solution for fundamental non-linear natural frequency and corresponding displacement of tapered beams. Zhou and Cheung [22] studied the vibration characteristics of tapered beams with a continuously varying rectangular cross-section for a truncated beam and arbitrary positive numbers for a sharp ended beam and obtained the eigen frequency equation by the Rayleigh-Ritz method and the effect of the location convergence is discussed.

The solution for the large amplitude free vibration problems using energy methods involves assuming suitable admissible functions for lateral displacement and the total rotation which leads to two coupled nonlinear differential equations in terms of lateral displacement and the total rotation. This can be achieved with less computational effort by the coupled displacement field method in which lateral displacement and total rotation are coupled through the static equilibrium equation of the shear flexible beam. This method leads to only one undetermined coefficient which can be easily solved using the principle of conservation of total energy (neglecting damping) to solve the problem.

2. Methodology

2.1. Rayleigh-Ritz method

In the Rayleigh-Ritz method the expressions for strain energy and kinetic energy are

$$U = \frac{EI}{2}\int_0^L \left(\frac{d\theta}{dx}\right)^2 dx + \frac{kGA}{2}\int_0^L \left(\frac{dw}{dx}+\theta\right)^2 dx \tag{2.1}$$

$$T = \frac{\rho A \omega_L^2}{2}\int_0^L w^2 dx + \frac{\rho I \omega_L^2}{2}\int_0^L \theta^2 dx \tag{2.2}$$

where E is Young's modulus and I is the area moment of inertia, w is the lateral displacement, θ is the total rotation, k is the shear correction factor, G is the shear modulus, A is the area of cross section, ρ is the mass density of the material of the beam, ω is the radiant frequency, L is the length of the beam and x is the axial coordinate. Suitable admissible functions satisfying mainly the kinetic boundary conditions (sometimes the admissible functions may satisfy some or all the natural boundary conditions and do not violate the variation principles) are assumed for w and θ as

$$w = \sum_{i=1}^{n} a_i f_i(x)\,, \tag{2.3}$$

$$\theta = \sum_{i=1}^{n} b_i f_i{}'(x)\,. \tag{2.4}$$

We consider the equivalence of θ and $\frac{dw}{dx}$ of the beam problem where a_i and b_i are the 2nd undetermined coefficients for the multi-term admissible functions given by the above equations. For the sake of simplicity and clarity, single term admissible functions for θ and w with two undetermined coefficients are chosen and it has been shown that the single term admissible functions with trigonometric functions for various boundary conditions of the beam are found to be accurate for all practical purposes. The Lagrangian (U-T) is minimized with respect to the two undetermined coefficients a and b as

$$\frac{\partial(U-T)}{\partial a} = 0\,, \tag{2.5}$$

$$\frac{\partial(U-T)}{\partial b} = 0\,. \tag{2.6}$$

By solving the above equations, a quadratic equation for the frequency parameter can be obtained in the form of $L\lambda^2 - M\lambda + N = 0$ which has roots of $\frac{-M \pm \sqrt{M^2 - 4LN}}{2L}$ and by solving the above equation the fundamental frequency parameter can be obtained as $\lambda = \frac{\rho A \omega_L^2 L^4}{EI}$ and can be solved to obtain the fundamental frequency parameter of Timoshenko beams for various boundary conditions as a function of the slenderness ratio (β=L/r) and taper ratio, where r is the radius of gyration.

The Rayleigh-Ritz (R-R) method is explained in detail as follows for a tapered Timoshenko hinged-hinged beam boundary condition. The equation for strain energy and kinetic energy for a tapered Timoshenko beam are given as

$$U = \frac{E}{2}\int_0^L I\left(\frac{d\theta}{dx}\right)^2 dx + \frac{kG}{2}\int_0^L A\left(\frac{dw}{dx}+\theta\right)^2 dx\,, \tag{2.7}$$

$$T = \frac{\rho\omega_L{}^2}{2}\int_0^L Aw^2 dx + \frac{\rho\omega_L{}^2}{2}\int_0^L I\theta^2 dx\,, \tag{2.8}$$

$$A = A_0\left(1+\frac{\alpha x}{L}\right), I = I_0\left(1+\frac{\alpha x}{L}\right)^3, \alpha = \frac{\left(h_L - h_0\right)}{h_L} \tag{2.9}$$

where h_L, h_o, are the height of the beam at the left end=0 and the right end =L respectively as given in Fig.1, A_o and I_o are cross sectional area and area moment of inertia at the right side, A is the area at any cross section, I is the moment of inertia at any cross section, α is the taper ratio. The assumed total rotation and transverse displacement for hinged-hinged beam are respectively

$$\theta = b\frac{\pi}{L}\cos\frac{\pi x}{L}\,, \tag{2.10}$$

$$w = a\sin\frac{\pi x}{L}\,. \tag{2.11}$$

Substituting Eqs (2.9), (2.10) and (2.11) in Eq.(2.7) and after simplification, we get

$$U = \frac{b^2 EI_0\pi^2}{2L}\left[\left(0.5+0.0871\alpha^3+0.4241\alpha^2+0.75\alpha\right)\right] + \frac{kGA_0L}{2}\left[\left(b+\frac{a\pi}{L}\right)^2\left(0.5+0.25\alpha\right)\right]. \tag{2.12}$$

Substituting Eqs (2.9),(2.10) and (2.11) in Eq.(2.8) and after simplification, we obtain

$$T = \frac{\rho\omega_L{}^2 a^2 A_0 L}{2}\left(0.5+0.25\alpha\right) + \frac{\rho\omega_L{}^2 b^2 I_0 L}{2}\left(0.5+0.1629\alpha^3+0.5759\alpha^2+0.75\alpha\right), \tag{2.13}$$

$$\frac{\partial(U-T)}{\partial a} = 0\,. \tag{2.14}$$

Applying the Lagrangian and minimizing with respect to a and after simplification the equation becomes as

$$a\left(kGA_0\frac{\pi^2}{L} - \rho\omega_L{}^2 A_0 L\right) + bkGA_0\pi = 0\,, \tag{2.15}$$

$$\frac{\partial(U-T)}{\partial b} = 0 \tag{2.16}$$

Applying the Lagrangian and minimizing with respect to b 'b' and after simplification the equation becomes

$$akGA_0\pi + b\left[\frac{EI_0\pi^2}{L}\frac{\left(0.5+0.0871\alpha^2+0.4241\alpha^3+0.75\alpha\right)}{\left(0.5+0.25\alpha\right)}+\right.$$

$$\left.+kGA_0L-\rho\omega_L{}^2I_0L\frac{\left(0.5+0.1629\alpha^3+0.5759\alpha^2+0.75\alpha\right)}{\left(0.5+0.25\alpha\right)}\right]=0 \tag{2.17}$$

Solving Eqs (2.15) and (2.17) and after simplification the final equation gets the following form

$$L\lambda^2 - M\lambda + N = 0\,, \tag{2.18}$$

which has roots of $\lambda = \dfrac{-M \pm \sqrt{M^2-4LN}}{2L}$

where

$$L=\frac{6.76\varphi}{\beta^6},\qquad M=\left(\frac{21.3841}{\beta^4}\varphi+\frac{66.71834}{\beta^4}\psi+\frac{2.1666}{\beta^2}\right),\qquad N=\frac{210.970}{\beta^2}\psi\,,$$

$$\lambda=\frac{\left(\dfrac{21.3841}{\beta^4}\varphi+\dfrac{66.71834}{\beta^4}\psi+\dfrac{2.1666}{\beta^2}\right)\pm\sqrt{\left(\dfrac{21.3841}{\beta^4}\varphi+\dfrac{66.71834}{\beta^4}\psi+\dfrac{2.1666}{\beta^2}\right)^2-\dfrac{5706.6297}{\beta^8}\psi\varphi}}{\dfrac{13.52}{\beta^6}\varphi} \tag{2.19}$$

where $\psi=\left(\dfrac{0.5+0.0871\alpha^3+0.4241\alpha^2+0.75\alpha}{0.5+0.25\alpha}\right)$ $\varphi=\left(\dfrac{0.5+0.1629\alpha^3+0.5759\alpha^2+0.75\alpha}{0.5+0.25\alpha}\right)$.

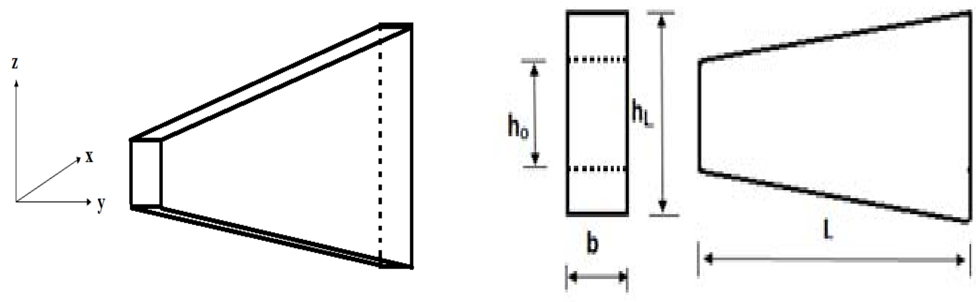

Fig.1. Tapered Timoshenko hinged-hinged beam with linearly varying height (constant width).

2.2. Coupled displacement filed method

Figure1 Tapered Timoshenko hinged-hinged beam with linearly varying height (constant width).

2.2.1. Coupling equation

From the kinematics of the shear flexible beam theory

$$\bar{u} = (x, z) = z(\theta), \tag{2.20}$$

$$\bar{w} = (x, z) = w(x, z) \tag{2.21}$$

where $\bar{u}$ is the axial displacement and $\bar{w}$ is the transverse displacements at any point of the beam, z is the distance of any point from the neutral axis, w is the transverse displacement and θ is the total rotation anywhere on the beam axis and x, z are the independent spatial variables. The axial and shear strains are given by

$$\in_x = z \frac{\partial\theta}{\partial x}, \tag{2.22}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \theta. \tag{2.23}$$

Now, the expressions for the strain energy U and the work done W by the externally applied load are given by

$$U = \frac{EI}{2} \int_O^L \left(\frac{d\theta}{dx}\right)^2 dx + \frac{kGA}{2} \int_O^L \left(\frac{\partial w}{\partial x} + \theta\right)^2 dx, \tag{2.24}$$

$$W = \int_0^L p(x) w(x) dx \tag{2.25}$$

where EI is the flexural rigidity, GA is the shear rigidity, k is the shear correction factor (taken as 5/6 in the present study), $p(x)$ is the static lateral load per unit length acting on the beam, E is Young's modulus, G is the shear modulus, x is the axial coordinate and L is the length of the beam. Applying the principle of minimization of total potential energy, as

$$\delta\,(U\text{-}W) = 0 \tag{2.26}$$

the following equilibrium equations can be obtained

$$kGA \left(\frac{d^2 w}{dx^2} + \frac{d\theta}{dx}\right) + p = 0, \tag{2.27}$$

$$EI \frac{d^2\theta}{dx^2} - kGA \left(\frac{dw}{dx} + \theta\right) = 0 \tag{2.28}$$

where θ is the total rotation, w is the transverse displacement. Equations (2.27) and (2.28) are coupled equations and can be solved for obtaining the solution for the static analysis of the shear deformable beams.

A close observation of Eq.(2.27) shows that it is dependent on the load term 'p' and Eq.(2.28) is independent of the load term 'p'. Hence, Eq.(2.28) is used to couple the total rotation θ and the transverse displacement w, so that the two undetermined coefficients problem (for single term admissible function) becomes a single undetermined coefficient problem and the resulting linear free vibration problem becomes much simpler to solve.

2.3. Evaluation of fundamental frequency parameter using coupled displacement field (CDF) method

The concept of the coupled displacement field method is explained in detail in the following section. In the CDF method, the single term admissible function for the total rotation (θ) is assumed and the function for the transverse displacement (w) is derived using the coupling equation. An admissible function for the total rotation (θ) is assumed for the tapered hinged-hinged beam which satisfies all the applicable boundary conditions and the symmetric condition in the beam domain.

$$\theta = a\frac{\pi}{L}\cos\frac{\pi x}{L}, \tag{2.29}$$

$$\frac{d\theta}{dx} = -a\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L}, \tag{2.30}$$

$$\frac{d^2\theta}{dx^2} = -a\cos\frac{\pi x}{L}\frac{\pi^3}{L^3} \tag{2.31}$$

where is the central lateral displacement of the beam which is also the maximum lateral displacement. Rewriting Eq.(2.28), we get

$$\frac{dw}{dx} = -\theta + \frac{EI}{kGA}\frac{d^2\theta}{dx^2} . \tag{2.32}$$

Substituting Eqs (2.29), (2.31) in Eq.(2.32) and by integrating the above equation, lateral displacement can be obtained as

$$w = -a\left[1 + \left(\frac{\pi}{L}\right)^2 \gamma\right]\sin\pi\varsigma \tag{2.33}$$

where $\quad \gamma = \dfrac{EI}{kGA}, \qquad \varsigma = \dfrac{x}{L}$.

It may be noted here that because of the coupled displacement field concept, the transverse displacement w distribution contains the same undetermined coefficient a as the θ distribution and satisfies all the applicable essential boundary and symmetric conditions.

$$w(0) = w(L) = \left.\frac{dw}{dx}\right|_{x=L/2} = 0 . \tag{2.34}$$

Linear free vibrations of tapered Timoshenko beams can be studied, once the coupled displacement field for the lateral displacement w, for an assumed θ distribution, is evaluated using the principle of conservation of total energy at any instant of time, neglecting damping, which states that $U + T =$ constant. The expression for U and T are already given in Eqs (2.7) and (2.8).

Substituting Eqs (2.9), (2.29), (2.30) and (2.32) in Eqs (2.7) and after simplification, we get

$$U=\frac{a^2 EI_0\pi^2}{2L}\left[(0.5+0.0871\alpha^3+0.4241\alpha^2+0.75\alpha)+3.12\frac{\pi^2}{\beta^2}(0.5+0.25\alpha)\right]. \tag{2.35}$$

Substituting Eqs (2.9), (2.29) and (2.33) in Eq.(2.8) and after simplification, we obtain

$$T=\frac{a^2\rho A_0{\omega_L}^2L^3}{2\pi^2}\left[\left(1+3.12\frac{\pi^2}{\beta^2}\right)^2(0.5+0.25\alpha)+\frac{\pi^2}{\beta^2}\left(0.5+0.1629\alpha^3+0.5759\alpha^2+0.75\alpha\right)\right] \tag{2.36}$$

$$\frac{\partial(U-T)}{\partial a}=0\,,$$

by applying the Lagrangian, for Eqs (2.35), (2.36) the fundamental frequency parameter is obtained and is given as below

$$\lambda=\frac{\rho A\omega_L^2L^4}{EI}=\frac{\pi^4\left[\left(0.5+0.0871\alpha^3+0.4241\alpha^2+0.75\alpha\right)+3.12\frac{\pi^2}{\beta^2}\left(0.5+0.25\alpha\right)\right]}{\left[\left(1+3.12\frac{\pi^2}{\beta^2}\right)^2(0.5+0.25\alpha)+\frac{\pi^2}{\beta^2}\left(0.5+0.1629\alpha^3+0.5759\alpha^2+0.75\alpha\right)\right]}$$

Where λ is the non dimensional fundamental frequency parameter, $\beta=L/r$ (slenderness ratio) and r is the radius of gyration. The same procedure is adopted as discussed in the above section for calculating the fundamental frequency parameter for clamped-clamped tapered Timoshenko beam boundary condition (Tab.1).

3. Large amplitude free vibrations

For an assumed θ distribution, the coupled displacement field for the lateral displacement w is evaluated, after the lateral displacement w is calculated, the large amplitude free vibrations can be studied using the principle of conservation of total energy at any instant of time neglecting damping.

$$U+T+W=\text{constant}. \tag{3.1}$$

Work done due to large amplitudes

$$W=\frac{T_a}{2}\int_0^L\frac{1}{2}\left(\frac{dw}{dx}\right)^2dx \tag{3.2}$$

where w is the transverse displacement obtained from the coupling equation

$$T_a=\frac{E}{2Lr^2}\int_0^L I\left(\frac{dw}{dx}\right)^2dx\,. \tag{3.3}$$

It is to be noted here that w in Eq.(3.3) does not contain shear flexible terms.
T_a is the tension developed in the beam because of large deformations.

W is the work done by the tension developed because of large amplitudes, ρ is the mass density. Where r is the radius of gyration and T_a is evaluated in terms of the amplitude ratio (*a/r*). Substituting the values of w (obtained from the coupled displacement field), i.e., Eq.(2.33) in Eq.(3.2) and solving the work done due to large amplitudes, we get

$$W = \frac{EI_0 \pi^2 a^4}{16 r^2 L} \frac{(2-\alpha)\left(\pi^2\alpha^2 + 3\alpha^2 - 2\pi^2\alpha + 2\pi^2\right)}{8\pi^2} \left(1 + \left(\frac{\pi}{L}\right)^2 \frac{EI}{KGA}\right)^2 . \tag{3.6}$$

Substituting Eqs (2.35), (2.36) and (3.6) in Eq.(3.1) and simplifying.
The following form is obtained

$$\dot{s}^2 + s^2\alpha_1 + s^4\alpha_2 = \text{constant} \tag{3.5}$$

where

$$\alpha_1 = \frac{EI_0\, 2\pi^4}{\rho L^4 A_0} \frac{\left[\left(0.5 + 0.0871\alpha^3 + 0.4241\alpha^2 + 0.75\alpha\right) + \frac{3.12\pi^2}{\beta^2}(0.5 + 0.25\alpha)\right]}{\left[\left(1 + \frac{3.12\pi^2}{\beta^2}\right)^2 (0.5 + 0.25\alpha) + \frac{\pi^2}{\beta^2}\left(0.5 + 0.1629\alpha^3 + 0.5759\alpha^2 + 0.75\alpha\right)\right]} , \tag{3.6}$$

$$\alpha_2 = \frac{3EI_0\pi^2}{35\rho L^4 A_0 r^2} \frac{\left[(2-\alpha)\left(\pi^2\alpha^2 + 3\alpha^2 - 2\pi^2\alpha + 2\pi^2\right)\left(1 + \frac{3.12\pi^2}{\beta^2}\right)^2\right]}{\left[\left(1 + \frac{3.12\pi^2}{\beta^2}\right)^2 (0.5 + 0.25\alpha) + \frac{\pi^2}{\beta^2}\left(0.5 + 0.1629\alpha^3 + 0.5759\alpha^2 + 0.75\alpha\right)\right]} \tag{3.7}$$

The ratio of non linear and linear frequency is expressed as

$$\left(\frac{\omega_{NL}}{\omega_L}\right)^2 = 1 + \frac{3}{2}\left(\frac{\alpha_2}{\alpha_1}\right)\left(\frac{a_m}{r}\right)^2 , \tag{3.8}$$

$$\left(\frac{\omega_{NL}}{\omega_L}\right)^2 = 1 + \frac{3}{70} \frac{(2-\alpha)\left(\pi^2\alpha^2 + 3\alpha^2 - 2\pi^2\alpha + 2\pi^2\right)\left(1 + \left(\frac{3.12\pi^2}{\beta^2}\right)\right)^2}{\pi^2\left[\left(0.5 + 0.0871\alpha^3 + 0.4241\alpha^2 + 0.75\alpha\right) + \frac{3.12\pi^2}{\beta^2}(0.5 + 0.25\alpha)\right]} \left(\frac{a_m}{r}\right)^2 . \tag{3.9}$$

The same procedure is adopted as in the above section for calculating the frequency ratio for clamped-clamped tapered Timoshenko beam boundary condition; respective important expressions are given in Tab.1.

4. Numerical results and discussion

In all computations, Poisson's ratio (ν) and shear correction factor (k) are taken as *0.3* and *5/6* respectively. The concept of the coupled displacement field is used to determine the ratios of non-linear radian frequency ω_{NL} to the linear radian frequency ω_L of tapered Timoshenko beams with the two most practically used hinged-hinged, clamped-clamped beam boundary conditions. The boundary conditions of the beam considered here are hinged-hinged, clamped-clamped with axially immovable ends. Suitable single term trigonometric admissible functions are used to represent the total rotation θ in the coupled displacement field method. The corresponding coupled lateral displacement w is derived using the coupling equation. The numerical results are obtained in terms of $\left(\frac{\omega_{NL}}{\omega_L}\right)$ for various maximum amplitudes, taper parameters and slenderness ratios. To assess the accuracy of the results, the present results obtained from the coupled displacement field method are compared with the existing literature.

Table 1 shows the expressions for the total rotation (θ), derived transverse displacement (w), fundamental frequency parameter (λ) and frequency ratio $\left(\frac{\omega_{NL}}{\omega_L}\right)$ for a clamped-clamped tapered Timoshenko beam. Tables 2 and 3 shows the variation of linear non-dimensional fundamental frequency parameter as a function of the slenderness ratio and taper ratio for hinged-hinged, clamped-clamped beam boundary conditions. To show the effectiveness of the CDF method, the author also solved the tapered Timoshenko hinged-hinged beam boundary condition problem using the famous conventional Rayleigh-Ritz method and the same results are also included in Tab.3 along with the results of other researchers wherever possible. It is observed from Tab.2 and Tab.3 that the non dimensional linear fundamental frequency parameter value increases with an increase in the taper ratio for a given slenderness ratio. It is also found from Tab.2 and Tab.3 that the non dimensional linear fundamental frequency parameter value increases with an increase in the slenderness ratio for a given taper parameter. It is in general observed from Tab.2 and Tab.3 that more frequency values are observed in the case of clamped-clamped beam when compared to the hinged-hinged beam.

Table.1. Expressions for total rotation (θ), derived transverse displacement (w) fundamental frequency parameter (λ) and frequency ratio $\left(\frac{\omega_{NL}}{\omega_L}\right)^2$ for clamped-clamped tapered Timoshenko beams.

Boundary condition	Type of admissible function and assumed total rotation	Expression
clamped-clamped	trigonometric $\theta = a\frac{2\pi}{L}\sin 2\pi\varsigma$	$w = a\left[1+\left(\frac{2\pi}{L}\right)^2\gamma\right]\left[\cos 2\pi\varsigma - 1\right]$
		$\lambda = \frac{16\pi^4}{1.20}\frac{\left[\left(0.5+0.1344\alpha^3+0.5189\alpha^2+0.75\alpha\right)+\frac{123.17}{\beta^2}\left(0.5+0.25\alpha\right)\right]}{\left[\left(1+\frac{123.17}{\beta^2}\right)^2\left(1.5+0.8511\alpha\right)+\frac{39.47}{\beta^2}\left(0.5+0.1156\alpha^3+0.4811\alpha^2+0.75\alpha\right)\right]}$
		$\left(\frac{\omega_{NL}}{\omega_L}\right)^2 = 1+\frac{1}{6106.4752489241871amped}\frac{\left[\left(3-4\pi^2\right)\alpha^3+\left(16\pi^2-6\right)\alpha^2-24\pi^2\alpha+16\pi^2\right]\left(1+\left(\frac{12.48\pi^2}{\beta^2}\right)\right)^2}{\left[\left(0.5+0.1344\alpha^3+0.5189\alpha^2+0.75\alpha\right)+\frac{12.48\pi^2}{\beta^2}\left(0.5+0.25\alpha\right)\right]}\left(\frac{a_m}{r}\right)^2$

Table 2. $\lambda^{1/2}$ values for a tapered Timoshenko hinged-hinged beam.

	Slenderness ratio (β)											
	10			20			23.0951	40		100		
Taper ratio (α)	CDF Method	R-R method	Ref[18]	CDF Method	R-R method	Ref [18]	CDF Method	CDF Method	R-R Method	CDF Method	R-R Method	Ref[18]
0	8.3912	8.6913	8.388	9.4107	9.8398	9.411	9.5180 9.5163^ 9.5163$^{\$}$	9.7470	10.2259	9.8496	10.3445	9.850
0.15	8.8435	8.8404	-	10.0595	10.0547	-	9.96419	10.4695	10.4676	10.5953	10.5950	-
0.1	8.6916	8.9874	8.683	9.8415	9.2696	9.829	10.1899	10.2267	10.7107	10.3446	10.8473	-
0.2	8.9962	9.1322	8.955	10.2789	10.4843	10.228	10.4171	10.7141	10.9549	10.8480	11.1014	-
0.25	9.1496	9.2747	-	10.4996	10.6987	-	10.6458	10.9604	11.2002	11.1024	11.3570	-
0.3	9.3036	9.4148	9.205	10.7214	10.9125	10.610	10.8757	11.2082	11.4463	11.3585	11.6141	-
0.35	9.4580	9.5526	-	10.9443	11.1257	-	11.1068	11.4574	11.6933	11.6161	11.8726	-
0.4	9.6127	9.6880	-	11.1681	11.3382	-	11.3389	11.7079	11.9409	11.8752	12.1323	-
0.45	9.7676	9.8209	-	11.3926	11.5498	-	11.5718	11.9596	12.1891	12.1356	12.3932	-
0.5	9.9225	9.9514	-	11.6178	11.7604	-	11.8055	12.2124	12.4378	12.3973	12.6551	-
0.55	10.0774	10.0795	-	11.8435	11.9700	-	12.0400	12.4661	12.6869	12.6601	12.9181	-
0.6	10.2321	10.2051	-	12.0697	12.1785	-	12.2749	12.7208	12.9362	12.9241	13.1820	-
0.65	10.3866	10.3283	-	12.2962	12.3857	-	12.5104	12.9764	13.1858	13.1891	13.4468	-
0.7	10.5407	10.4491	-	12.5230	12.5917	-	12.7463	13.2327	13.4355	13.4551	13.7123	-
0.75	10.6943	10.5675	-	12.7500	12.7963	-	12.9825	13.4897	13.6853	13.7219	13.9787	-
0.8	10.8475	10.6835	-	12.9771	12.9995	-	13.2190	13.7474	13.9351	13.9897	14.2457	-
0.85	11.0000	10.7971	-	13.2042	13.2013	-	13.4557	14.0057	14.1848	14.2582	14.5133	-
0.9	11.1519	10.9084	-	13.4313	13.4016	-	13.6924	14.2644	14.4345	14.5274	14.7816	-
0.95	11.3030	11.0175	-	13.6583	13.6003	-	13.9293	14.5237	14.6840	14.7974	15.0504	-
1	11.4533	11.3873	-	13.8852	13.4105	-	14.1661	14.7834	14.7470	15.0681	15.8495	-

Table 3. $\lambda^{1/2}$ values for a tapered Timoshenko clamped-clamped beam.

	Slenderness ratio(β)							
	10		20		40	80	100	
Taper Ratio (α)	CDF Method	Ref[18]	CDF Method	Ref[18]	CDF method	CDF method	CDF Method	Ref[18]
0	13.8025	13.8370	18.0930	18.838	21.8857	22.5543	22.6392	22.61
0.1	14.0619	14.0910	18.7304	19.487	22.8503	23.6136	23.7109	-
0.15	14.2010	-	19.0651	-	23.3531	24.1647	24.2683	-
0.2	14.3459	14.3180	19.4092	20.095	23.8682	24.7287	24.8387	-
0.25	14.4964	-	19.7623	-	24.3947	25.3046	25.4210	-
0.3	14.6522	14.5210	20.1240	20.667	24.9316	25.8914	26.0144	-
0.35	14.8130	-	20.4928	-	25.4783	26.4884	26.6180	-
0.4	14.9785	-	20.8690	-	26.0340	27.0948	27.2310	-
0.45	15.1484	-	21.2519	-	26.5980	27.7099	27.8528	-
0.5	15.3226	-	21.6408	-	27.1696	28.3331	28.4827	-
0.55	15.5008	-	22.0355	-	27.7484	28.9638	29.1202	-
0.6	15.6828	-	22.4354	-	28.3338	29.6014	29.7646	-
0.65	15.8683	-	22.8401	-	28.9252	30.2454	30.4156	-
0.7	16.0572	-	23.2493	-	29.5223	30.8955	31.0726	-
0.75	16.2492	-	23.6626	-	30.1246	31.5511	31.7352	-
0.8	16.4441	-	24.0797	-	30.7317	32.2119	32.4030	-
0.85	16.6418	-	24.5002	-	31.3433	32.8776	33.0758	-
0.9	16.8421	-	24.9240	-	31.9591	33.5477	33.7531	-
0.95	17.0448	-	25.3560	-	32.5786	34.2220	34.4346	-
1	17.2498	-	25.7799	-	33.2017	34.9003	35.1201	-

In Tabs 4, 5, 6 and 7 the variation of the frequency ratio $\left(\frac{\omega_{NL}}{\omega_L}\right)$ with the maximum amplitude ratio and taper parameters for different slenderness ratios such as 20 (short beams) and 100 (slender beams). They are given respectively for hinged-hinged, clamped-clamped tapered beam boundary conditions. It is found from Tab.4 and Tab.5 that the frequency ratio is a function of three parameters such as the maximum amplitude ratio, taper parameter and slenderness ratio. It is in general found from Tab.4 and Tab.5 that the frequency ratio increases with an increase of the maximum amplitude ratio for a given taper parameter and slenderness ratio. It is also seen in Tab.4 and Tab.5 that the frequency ratio decreases with an increase of the taper parameter for a given slenderness ratio and amplitude ratio. This is mainly due to the fact that as the taper parameter increases the mass of the beam decreases. A similar frequency ratio variation has been observed for clamped-clamped beam boundary conditions for the slenderness ratio of 20 and 100 and are included in Tab.6 and Tab.7

Table 4. $\frac{\omega_{NL}}{\omega_L}$ for a tapered shear flexible hinged-hinged beam for β=*20.*

	α= *0.25*		α = *0.5*		α = *0.75*		α = *1*	
a_m/r	CDF Method	Ref [13]	CDF Method	Ref [13]	CDF Method	Ref [13]	CDF Method	Ref [13]
0.10	*1.0009*	*1.0009*	*1.0005*	*1.0007*	*1.0003*	*1.0006*	*1.0002*	*1.0005*
0.20	*1.0036*	*1.0037*	*1.0019*	*1.0030*	*1.0011*	*1.0025*	*1.0007*	*1.0021*
0.30	*1.0081*	-	*1.0042*	-	*1.0025*	-	*1.0016*	-
0.40	*1.0144*	*1.0146*	*1.0075*	*1.0119*	*1.0044*	*1.0100*	*1.0028*	*1.0085*
0.50	*1.0224*	-	*1.0118*	-	*1.0069*	-	*1.0044*	-
0.60	*1.0321*	*1.0325*	*1.0169*	*1.0266*	*1.0099*	*1.0224*	*1.0064*	*1.0190*
0.70	*1.0434*	-	*1.0230*	-	*1.0134*	-	*1.0087*	-
0.80	*1.0564*	*1.0570*	*1.0299*	*1.0467*	*1.0175*	*1.0394*	*1.0113*	*1.0336*
0.90	*1.0709*	-	*1.0377*	-	*1.0221*	-	*1.0143*	-
1.00	*1.0868*	*1.0878*	*1.0464*	*1.0721*	*1.0272*	*1.0608*	*1.0177*	*1.0519*
1.10	*1.1042*	-	*1.0559*	-	*1.0328*	-	*1.0213*	-
1.20	*1.1230*	*1.1239*	*1.0662*	*1.1022*	*1.0389*	*1.0864*	*1.0253*	*1.0740*
1.30	*1.1430*	-	*1.0773*	-	*1.0455*	-	*1.0297*	-
1.40	*1.1642*	-	*1.0891*	-	*1.0526*	-	*1.0343*	-
1.50	*1.1865*	*1.1878*	*1.1017*	*1.1552*	*1.0602*	*1.1315*	*1.0393*	*1.1131*
2	*1.3135*		*1.1748*		*1.1047*		*1.0689*	
3	*1.6224*		*1.3622*		*1.2231*		*1.1492*	
4	*1.9752*		*1.5878*		*1.3718*		*1.2531*	
5	*2.3523*		*1.8376*		*1.5420*		*1.3751*	

Table 5. $\frac{\omega_{NL}}{\omega_L}$ for a tapered shear flexible hinged-hinged beam for β=*100*.

	α									
	0.25		*0.4*		*0.5*		*0.75*		*1*	
a_m/r	CDF Method	Ref[13]	CDF Method	Ref[9]	CDF Method	Ref[13]	CDF Method	Ref[13]	CDF Method	Ref[13]
0.10	*1.0008*	*1.0010*	*1.0010*	*1.0010*	*1.0004*	*1.0008*	*1.0003*	*1.0007*	*1.0002*	*1.0006*
0.20	*1.0033*	*1.0040*	*1.0022*	*1.0042*	*1.0017*	*1.0033*	*1.0010*	*1.0028*	*1.0006*	*1.0025*
0.30	*1.0075*	-	*1.0050*	-	*1.0039*	-	*1.0022*	-	*1.0014*	-
0.40	*1.0132*	*1.0158*	*1.0088*	*1.0166*	*1.0069*	*1.0132*	*1.0040*	*1.0113*	*1.0025*	*1.0098*
0.50	*1.0206*	-	*1.0138*	-	*1.0107*	-	*1.0062*	-	*1.0040*	-
0.60	*1.0295*	*1.0353*	*1.0198*	*1.0370*	*1.0154*	*1.0294*	*1.0089*	*1.0252*	*1.0057*	*1.0219*
0.70	*1.0400*	-	*1.0268*	-	*1.0209*	-	*1.0121*	-	*1.0078*	-
0.80	*1.0519*	*1.0619*	*1.0349*	*1.0649*	*1.0272*	*1.0516*	*1.0158*	*1.0444*	*1.0102*	*1.0387*
0.90	*1.0653*	-	*1.0440*	-	*1.0344*	-	*1.0199*	-	*1.0128*	-
1.00	*1.0800*	*1.0950*	*1.0541*	*1.0997*	*1.0423*	*1.0795*	*1.0245*	*1.0685*	*1.0158*	*1.0597*
1.10	*1.0961*	-	*1.0651*	-	*1.0509*	-	*1.0296*	-	*1.0191*	-
1.20	*1.1134*	*1.1344*	*1.0770*	-	*1.0603*	*1.1127*	*1.0351*	*1.0972*	*1.0227*	*1.0849*
1.30	*1.1319*	-	*1.0898*	-	*1.0704*	-	*1.0411*	-	*1.0266*	-
1.40	*1.1516*	-	*1.1035*	-	*1.0813*	-	*1.0475*	-	*1.0308*	-
1.50	*1.1724*	*1.2033*	*1.1180*	-	*1.0928*	*1.1712*	*1.0543*	*1.1479*	*1.0352*	*1.1296*
2	*1.2906*		*1.2017*	*1.3354*	*1.1598*		*1.0948*		*1.0619*	
3	*1.5805*		*1.4140*	*1.6981*	*1.3330*		*1.2027*		*1.1344*	
4	*1.9138*		*1.6663*	-	*1.5430*		*1.3394*		*1.2289*	
5	*2.2717*		*1.9432*	-	*1.7769*		*1.4968*		*1.3406*	

Table 6. $\frac{\omega_{NL}}{\omega_L}$ for a tapered shear flexible clamped-clamped beam for β=*20*.

	α							
	0.25		*0.5*		*0.75*		*1*	
a_m/r	CDF Method	Ref 13]	CDF Method	Ref[13]	CDF Method	Ref 13]	CDF Method	Ref[13]
0.10	*1.0002*	*1.0002*	*1.0001*	*1.0001*	*1.0000*	*1.0001*	*1.0000*	*1.0001*
0.20	*1.0006*	*1.0006*	*1.0003*	*1.0005*	*1.0002*	*1.0004*	*1.0001*	*1.0003*
0.30	*1.0016*	-	*1.0008*	-	*1.0004*	-	*1.0002*	-
0.40	*1.0028*	*1.0025*	*1.0014*	*1.0019*	*1.0007*	*1.0015*	*1.0004*	*1.0012*
0.50	*1.0043*	-	*1.0022*	-	*1.0011*	-	*1.0006*	-
0.60	*1.0062*	*1.0057*	*1.0031*	*1.0043*	*1.0016*	*1.0034*	*1.0009*	*1.0026*
0.70	*1.0084*	-	*1.0043*	-	*1.0023*	-	*1.0012*	-
0.80	*1.0110*	*1.0100*	*1.0056*	*1.0076*	*1.0029*	*1.0060*	*1.0015*	*1.0047*
0.90	*1.0139*	-	*1.0070*	-	*1.0036*	-	*1.0019*	-
1.00	*1.0172*	*1.0156*	*1.0087*	*1.0119*	*1.0045*	*1.0093*	*1.0024*	*1.0073*
1.10	*1.0207*	-	*1.0105*	-	*1.0054*	-	*1.0029*	-
1.20	*1.0246*	*1.0223*	*1.0125*	*1.0170*	*1.0065*	*1.0133*	*1.0034*	*1.0105*
1.30	*1.0288*	-	*1.0146*	-	*1.0076*	-	*1.0040*	-
1.40	*1.0334*	-	*1.0169*	-	*1.0088*	-	*1.0047*	-
1.50	*1.0382*	*1.0344*	*1.0194*	*1.0263*	*1.0100*	*1.0206*	*1.0054*	*1.0164*
2	*1.0670*		*1.0343*		*1.0178*		*1.0095*	
3	*1.1452*		*1.0756*		*1.0397*		*1.0213*	
4	*1.2464*		*1.1308*		*1.0694*		*1.0376*	
5	*1.3657*		*1.1982*		*1.1066*		*1.0582*	

Table 7. $\frac{\omega_{NL}}{\omega_L}$ for a tapered shear flexible clamped-clamped beam for β=*100*.

	α									
	0.25		*0.4*		*0.5*		*0.75*		*1*	
a_m/r	CDF Method	Ref [13]	CDF Method	Ref [9]	CDF Method	Ref[13]	CDF Method	Ref[13]	CDF Method	Ref[13]
0.10	*1.0001*	*1.0002*	*1.0001*	*1.0004*	*1.0001*	*1.0002*	*1.0000*	*1.0002*	*1.0000*	*1.0001*
0.20	*1.0005*	*1.0009*	*1.0003*	*1.0017*	*1.0003*	*1.0008*	*1.0001*	*1.0007*	*1.0001*	*1.0006*
0.30	*1.0011*	-	*1.0007*	-	*1.0006*	-	*1.0003*	-	*1.0001*	-
0.40	*1.0020*	*1.0038*	*1.0013*	*1.0066*	*1.0010*	*1.0031*	*1.0005*	*1.0027*	*1.0003*	*1.0023*
0.50	*1.0032*	-	*1.0021*	-	*1.0015*	-	*1.0008*	-	*1.0004*	-
0.60	*1.0046*	*1.0085*	*1.0030*	*1.0149*	*1.0022*	*1.0070*	*1.0011*	*1.0060*	*1.0006*	*1.0052*
0.70	*1.0062*	-	*1.0040*	-	*1.0030*	-	*1.0015*	-	*1.0008*	-
0.80	*1.0081*	*1.0150*	*1.0052*	*1.0263*	*1.0039*	*1.0124*	*1.0020*	*1.0106*	*1.0010*	*1.0092*
0.90	*1.0103*	-	*1.0066*	-	*1.0050*	-	*1.0025*	-	*1.0013*	-
1.00	*1.0126*	*1.0233*	*1.0082*	*1.0408*	*1.0061*	*1.0193*	*1.0030*	*1.0165*	*1.0016*	*1.0144*
1.10	*1.0153*	-	*1.0099*		*1.0074*	-	*1.0037*	-	*1.0019*	-
1.20	*1.0182*	*1.0334*	*1.0118*		*1.0088*	*1.0278*	*1.0044*	*1.0237*	*1.0023*	*1.0206*
1.30	*1.0213*	-	*1.0138*		*1.0103*	-	*1.0052*	-	*1.0027*	-
1.40	*1.0246*	-	*1.0160*		*1.0120*	-	*1.0060*	-	*1.0031*	-
1.50	*1.0282*	*1.0517*	*1.0183*		*1.0138*	*1.0430*	*1.0069*	*1.0367*	*1.0036*	*1.0320*
2	*1.0497*		*1.0323*	*1.1545*	*1.0243*		*1.0122*		*1.0064*	
3	*1.1087*		*1.0713*	*1.3224*	*1.0540*		*1.0273*		*1.0143*	
4	*1.1863*		*1.1237*		*1.0941*		*1.0481*		*1.0253*	
5	*1.2792*		*1.1876*		*1.1436*		*1.0742*		*1.0392*	

5. Conclusions

The concept of the CDF method applicable to beams presented in this paper is successfully applied to study the large amplitude free vibration behavior of tapered Timoshenko beams with axially immovable ends. The influence of the taper parameter on the linear and non-linear frequency parameter has been studied for two different tapered Timoshenko beam boundary conditions. Accurate closed form expressions for $\left(\frac{\omega_{NL}}{\omega_L}\right)^2$ for the hinged-hinged, clamped-clamped beam boundary conditions are obtained in terms of the maximum amplitude ratio, taper ratio and slenderness ratio for the assumed single term admissible function for the total rotation θ.

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Nomenclature

A – area of cross section
E – Young's modulus

G – shear modulus
H_L – height of beam at right end
H_o – height of beam at left end
I – area moment of inertia
k – shear correction factor
L – length of the beam
r – radius of gyration
T – kinetic energy
U – strain energy
w – transverse displacement
α – taper ratio
β – slenderness ratio
θ – assumed total rotation
ν – Poisson's ratio (0.3)
ω_L – linear frequency
ω_{NL} – nonlinear frequency

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