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# Large Cells in Nonlinear Rayleigh-Bénard Convection

V. L. GERTSBERG and G. I. SIVASHINSKY

*Department of Mathematical Sciences, Tel-Aviv University*
*Ramat-Aviv*

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The bifurcation analysis proposed recently in solving the problem of surface-tension-induced convection in nearly insulated liquid films is carried over to the case of buoyancy-induced convection in a horizontal liquid layer. If the layer is nearly insulated, the longitudinal dimension of the convection cells is significantly greater (near the critical Rayleigh number) than the layer thickness. It turns out that this makes it possible to separate the horizontal and vertical space variables and so to lower the dimensionality of the problem. For example, considering a horizontal layer of liquid enclosed between two rigid plates, one obtains the following two-dimensional equation describing the dynamics of formation of the plane form of convective cells:

$$\Phi_\tau + \nabla^4\Phi + \nabla[(2-(\nabla\Phi)^2)\nabla\Phi] + \alpha\Phi = 0\,.$$

Effects due to the temperature-dependence of viscosity are also discussed.

## § 1. Problem statement and basic equations

This communication is devoted to the derivation of an asymptotic equation describing the dynamics of formation of the plane form of buoyancy-induced convective structure in a nearly insulated liquid layer. The analysis proposed below is largely analogous to the procedure in a recent paper I[1] on surface-tension-driven convection.

It is known (see, e.g., Ref. 2)) that the wavenumber of the disturbance corresponding to the critical Rayleigh number ($Ra_c$), in free convection with nearly insulated boundaries is small. Near $Ra_c$, therefore, the characteristic dimension of the convective cells is significantly greater than the layer thickness. As it turns out, this situation makes it possible to separate the horizontal and vertical space variables (near $Ra_c$) and thus to lower the dimensionality of the problem.

As in I, in the interest of computational convenience, we shall present the arguments for the two-dimensional version of the problem. Generalization of the final result to the three-dimensional case presents no essential difficulty and may be carried out directly on the basis of symmetry considerations.

In suitably selected nondimensional variables, the equation system for the problem of buoyancy-induced convection may be written as follows (see e.g., Ref. 3)).

Heat equation:

$$\frac{\partial\Theta}{\partial t}+\frac{\partial\Psi}{\partial y}\frac{\partial\Theta}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\Theta}{\partial y}+\frac{\partial\Psi}{\partial x}=\nabla^2\Theta\,. \tag{1·1}$$

Equation for stream function:*)

$$\mathrm{Ra}\frac{\partial\Theta}{\partial x}=\nabla^4\Psi\,. \tag{1·2}$$

Equations (1·1) and (1·2) are considered in the strip

$$-\infty<x<\infty\,,\quad 0\leq y\leq 1\,.$$

For the case of a liquid layer between two rigid plates, the boundary conditions at $y=0, 1$ are:

$$\Psi=0\,,\quad \Psi_y=0\,,\quad \Theta_y=b\Theta\,,\qquad (y=0) \tag{1·3}$$

$$\Psi=0\,,\quad \Psi_y=0\,,\quad \Theta_y=-b\Theta\,.\qquad (y=1) \tag{1·4}$$

The following notation has been used in Eqs. (1·1)~(1·4): $\Theta$; nondimensional temperature disturbance in units of difference between the temperature of the bottom ($T_-$) and that of the upper boundary ($T_+$) in the absence of convection. $\Psi$; nondimensional stream function, in units of the conductivity $\varkappa$ of the liquid. $x$, $y$, $t$; nondimensional space and time coordinates, in units of the thickness $d$ of the liquid layer and the time interval $d^2/\varkappa$, respectively. $\mathrm{Pr}=\nu/\varkappa$; Prandtl number, $\mathrm{Ra}=\sigma g(T_- - T_+)d^3/\nu\varkappa$; Rayleigh number, $\sigma$; volume expansion coefficient, $g$; gravitational acceleration, $\nu$; kinematic viscosity. $b$; nondimensional coefficient of heat-exchange between liquid and boundaries (Biot number). To simplify matters, we are assuming that the Biot numbers at the upper and lower boundaries are equal.

## § 2. Asymptotic analysis

Let the Rayleigh number Ra be close to the critical value $\mathrm{Ra}_c$ corresponding to the onset of convective flow

$$\mathrm{Ra}=\mathrm{Ra}_c(1+\varepsilon)\,.\qquad (\varepsilon\ll 1) \tag{2·1}$$

Assuming that the heat loss is small

$$b=\varepsilon^2\beta\,, \tag{2·2}$$

we introduce the following scaled space and time variables:

$$\zeta=x\sqrt{\varepsilon}\,,\qquad \eta=y\,,\qquad \tau=\varepsilon^2 t\,. \tag{2·3}$$

This choice of scalings is dictated by the results of the approximate linear sta-

*) To simplify matters, we assume henceforth that the Prandtl number Pr is infinity.

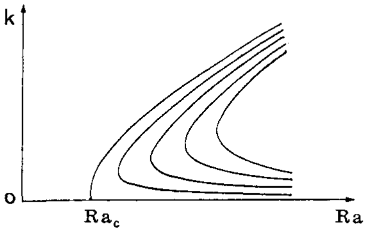


Fig. 1. Marginal stability curves at various Biot numbers ($\partial \mathrm{Ra}(K, b)/\partial b > 0$).

bility analysis (see the Appendix). Figure 1 shows the position of the neutral stability curves ($\Omega=0$) in the ($K$, Ra)-plane. Here $\Omega$ is the rate of instability parameter, $K$ the wavelength of the disturbance. At small $\mathrm{Ra}-\mathrm{Ra}_c$ one obtains in the instability region $K \sim \sqrt{\mathrm{Ra}-\mathrm{Ra}_c}$, $\Omega \sim b \sim (\mathrm{Ra}-\mathrm{Ra}_c)^2$. Thus, it seems reasonable to expect that nonlinear convection should be described by scalings (2·3).

It then follows from Eq. (1·2) that

$$\Psi \sim \sqrt{\varepsilon}\,\Theta \quad \text{for} \quad \varepsilon \ll 1\,. \tag{2·4}$$

On the basis of (2·3) and (2·4), we introduce new scaled variables $\theta$, $\psi$ for the disturbances of temperature and stream function:

$$\Theta(x, y, t, \varepsilon) = \theta(\zeta, \eta, \tau, \varepsilon),$$
$$\Psi(x, y, t, \varepsilon) = \sqrt{\varepsilon}\,\psi(\zeta, \eta, \tau, \varepsilon). \tag{2·5}$$

In terms of the new variables, problem (1·1)~(1·4) becomes

$$\varepsilon^2 \frac{\partial\theta}{\partial\tau} + \varepsilon \frac{\partial\psi}{\partial\eta}\frac{\partial\theta}{\partial\zeta} - \varepsilon \frac{\partial\psi}{\partial\zeta}\frac{\partial\theta}{\partial\eta} + \varepsilon \frac{\partial\psi}{\partial\zeta} = \varepsilon \frac{\partial^2\theta}{\partial\zeta^2} + \frac{\partial^2\theta}{\partial\eta^2}\,, \tag{2·6}$$

$$\mathrm{Ra}_c(1+\varepsilon)\frac{\partial\theta}{\partial\zeta} = \varepsilon^2 \frac{\partial^4\psi}{\partial\zeta^4} + 2\varepsilon \frac{\partial^4\psi}{\partial\zeta^2\partial\eta^2} + \frac{\partial^4\psi}{\partial\eta^4}\,, \tag{2·7}$$

$$\psi=0\,, \quad \psi_\eta=0\,, \quad \theta_\eta=\varepsilon^2\beta\theta\,, \qquad (\eta=0) \tag{2·8}$$

$$\psi=0\,, \quad \psi_\eta=0\,, \quad \theta_\eta=-\varepsilon^2\beta\theta\,. \qquad (\eta=1) \tag{2·9}$$

For the subsequent arguments it is useful to integrate Eq. (2·6) over the interval $0 \le \eta \le 1$. Noting the boundary conditions (2·8), (2·9) and the identity

$$\psi_\eta\theta_\zeta - \psi_\zeta\theta_\eta \equiv (\psi\theta)_{\zeta\eta} - (\psi_\zeta\theta)_\eta - (\psi\theta_\eta)_\zeta\,, \tag{2·10}$$

we can write the resulting integral relation as follows:

$$\varepsilon\frac{\partial}{\partial\tau}\int_0^1 \theta d\eta - \frac{\partial}{\partial\zeta}\int_0^1 \psi\theta_\eta d\eta + \frac{\partial}{\partial\zeta}\int_0^1 \psi d\eta = \frac{\partial^2}{\partial\zeta^2}\int_0^1 \theta d\eta - \beta\varepsilon(\theta/_{\eta=0} + \theta/_{\eta=1}). \tag{2·11}$$

A solution of problem (2·6)~(2·9) will be sought as an asymptotic expansion:

$$\psi = \psi^0 + \varepsilon\psi^1 + \varepsilon^2\psi^2 + \cdots, \qquad \theta = \theta^0 + \varepsilon\theta^1 + \varepsilon^2\theta^2 + \cdots\,. \tag{2·12}$$

The problem in zeroth approximation is

$$\theta^0_{\eta\eta}=0\,,\qquad \psi^0_{\eta\eta\eta\eta}=\mathrm{Ra}_c\theta_\zeta{}^0\,,\tag{2·13}$$

$$\psi^0=0\,,\qquad \psi_\eta{}^0=0\,,\qquad \theta_\eta{}^0=0\,,\qquad (\eta=0)\tag{2·14}$$

$$\psi^0=0\,,\qquad \psi_\eta{}^0=0\,,\qquad \theta_\eta{}^0=0\,.\qquad (\eta=1)\tag{2·15}$$

Hence

$$\theta^0=F(\zeta,\tau),\qquad \psi^0=\frac{1}{24}\mathrm{Ra}_cF_\zeta(\eta^4-2\eta^3+\eta^2).\tag{2·16}$$

The integral relationship (2·11) for the zeroth approximation is*)

$$\frac{\partial}{\partial\zeta}\int_0^1\psi^0\,d\eta-\frac{\partial}{\partial\zeta}\int_0^1\psi^0\theta_\eta{}^0\,d\eta=\frac{\partial^2}{\partial\zeta^2}\int_0^1\theta^0\,d\eta\,.\tag{2·17}$$

Hence, using (2·16), we have

$$(\mathrm{Ra}_c-720)F_{\zeta\zeta}=0\,,\qquad \text{i.e.,}\quad \mathrm{Ra}_c=720\,.\tag{2·18}$$

To determine the as yet unknown function $F(\zeta,\tau)$, we proceed to the next approximation. The equation system and boundary conditions are as follows:

$$\psi_\eta{}^0\theta_\zeta{}^0-\psi_\zeta{}^0\theta_\eta{}^0+\psi_\zeta{}^0=\theta^0_{\zeta\zeta}+\theta^1_{\eta\eta}\,,\tag{2·19}$$

$$\mathrm{Ra}_c\theta_\zeta{}^0+\mathrm{Ra}_c\theta_\zeta{}^1=2\psi^0_{\zeta\zeta\eta\eta}+\psi^1_{\eta\eta\eta\eta}\,,\tag{2·20}$$

$$\psi^1=0\,,\qquad \psi_\eta{}^1=0\,,\qquad \theta_\eta{}^1=0\,,\qquad (\eta=0)\tag{2·21}$$

$$\psi^1=0\,,\qquad \psi_\eta{}^1=0\,,\qquad \theta_\eta{}^1=0\,.\qquad (\eta=1)\tag{2·22}$$

Hence, in view of (2·16), (2·18) we have

$$\theta^1=G(\zeta,\tau)+F_\zeta{}^2(6\eta^5-15\eta^4+10\eta^3)+\frac{1}{2}F_{\zeta\zeta}(2\eta^6-6\eta^5+5\eta^4-\eta^2),\tag{2·23}$$

$$\begin{aligned}\psi^1=&\frac{10}{7}F_\zeta F_{\zeta\zeta}(2\eta^9-9\eta^8+12\eta^7-20\eta^3+15\eta^2)\\&+\frac{1}{14}F_{\zeta\zeta\zeta}(2\eta^{10}-10\eta^9+15\eta^8-42\eta^6+84\eta^5-70\eta^4+20\eta^3+\eta^2)\\&+30F_\zeta(\eta^4-2\eta^3+\eta^2)+30G_\zeta(\eta^4-2\eta^3+\eta^2).\end{aligned}\tag{2·24}$$

For the first approximation, the integral relationship (2·11) gives

*) The integral relationship (2·17) is clearly just the solvability condition for the problem in the next (i.e., the first) approximation.

$$\frac{\partial}{\partial \tau}\int_0^1 \theta^0 d\eta - \frac{\partial}{\partial \eta}\int_0^1 (\psi^0 \theta_\eta{}^1 + \psi^1 \theta_\eta{}^0) d\eta + \frac{\partial}{\partial \zeta}\int_0^1 \psi^1 d\eta$$
$$= \frac{\partial^2}{\partial \zeta^2}\int_0^1 \theta^1 d\eta - \beta(\theta^0/_{\eta=0} + \theta^0/_{\eta=1}). \qquad (2\cdot 25)$$

Inserting the expressions found above for $\theta^0$, $\theta^1$, $\psi^0$, $\psi^1$ into this equation, we obtain the required equation for $F(\zeta, \tau)$:

$$\frac{\partial F}{\partial \tau} + \frac{17}{462}\frac{\partial^4 F}{\partial \zeta^4} + \frac{\partial}{\partial \zeta}\left[\left(1 - \frac{10}{7}\left(\frac{\partial F}{\partial \zeta}\right)^2\right)\frac{\partial F}{\partial \zeta}\right] + 2\beta F = 0\,. \qquad (2\cdot 26)$$

Unlike Marangoni's convection equation,[1] in our case the "diffusivity" is independent of $F_{\zeta\zeta}$. As a result, Eq. (2·26) is invariant to changes in the sign of $F$. This property is due to the symmetry of (1·1), (1·2) under the transformation

$$\Theta \to -\Theta, \qquad \Psi \to -\Psi\,, \qquad y \to -y\,. \qquad (2\cdot 27)$$

Once we know the function $F$, the temperature field of the liquid layer is determined by the following equation (cf., (2·16)):

$$T = T_- + (T_- - T_+)(-\eta + F(\zeta, \tau)). \qquad (2\cdot 28)$$

Thus, near the critical Rayleigh number the temperature disturbance turns out to be a slowly varying function with amplitude of the order of $(T_- - T_+)$. For the velocity components of the liquid we have (cf., (2·16))

$$u = \Psi_y = 60\sqrt{\varepsilon}\, F_\zeta (2\eta^3 - 3\eta^2 + \eta),$$
$$v = -\Psi_x = -30\varepsilon F_{\zeta\zeta}(\eta^4 - 2\eta^3 + \eta^2). \qquad (2\cdot 29)$$

## § 3. Temperature-dependence of viscosity

It is well known that the most interesting of the effects not included in the classical Oberbeck-Boussinesq model is the temperature-dependence of the viscosity of the liquid. We now wish to examine how inclusion of this effect influences the form of the equation for the function $F(\zeta, \tau)$. We replace the term $\nabla^4 \Psi$ in Eq. (1·2) by

$$(\nu \Psi_{xx})_{xx} + 2(\nu \Psi_{xy})_{xy} + (\nu \Psi_{yy})_{yy}\,,$$

where $\nu$ is the temperature-dependent nondimensional viscosity. We shall assume that the temperature-dependence is weak, in the sense that

$$\nu = 1 + \varepsilon\mu\left(\frac{1}{2} - \eta + \Theta\right), \qquad \varepsilon \ll 1\,, \qquad (3\cdot 1)$$

where $\mu$ is a parameter of the order of unity, and $-\eta+\Theta$ is the non-dimensional disturbed temperature of the liquid layer. Thus, the average viscosity of the undisturbed layer ($0\leq\eta\leq 1$) is equal to unity. The temperature-dependence of the viscosity (3·1) obviously has no effect on the form of the first-approximation equations. In the second approximation, however, an additional term appears on the right of the equation for $\psi^1$ (2·20):

$$-\mu[(1/2-\eta)\psi^0_{\eta\eta}]_{\eta\eta}-\mu F\psi^0_{\eta\eta\eta\eta}\ . \tag{3·2}$$

The first of these terms makes no contribution to the integral relation (2·25), since the corresponding element of the function $\psi^1$ is an odd function of $\eta-\frac{1}{2}$. The element of $\psi^1$ generated by the second term of (3·2) is obviously $-\mu F\psi^0$ (i.e., an even function of $\eta-\frac{1}{2}$). The contribution from this term to the integral relation (2·25) is $-\mu FF_\zeta$. Hence the equation for $F$ allowing for the temperature-dependence of viscosity is

$$\frac{\partial F}{\partial\tau}+\frac{17}{462}\frac{\partial^4 F}{\partial\zeta^4}+\frac{\partial}{\partial\zeta}\left[\left(1-\mu F-\frac{10}{7}\left(\frac{\partial F}{\partial\zeta}\right)^2\right)\frac{\partial F}{\partial\zeta}\right]+2\beta F=0\ . \tag{3·3}$$

Using the transformation

$$F=\sqrt{17/660}\,\Phi\ ,\qquad \zeta=\sqrt{17/231}\,\bar{\zeta}\ ,\qquad \tau=(34/231)\,\bar{\tau}\ ,$$
$$\beta=(231/68)\alpha\ ,\qquad \mu=\sqrt{165/17}\,\gamma\ , \tag{3·4}$$

one brings Eq. (3·3) to the following form, which is more convenient for analysis:

$$\frac{\partial\Phi}{\partial\bar{\tau}}+\frac{\partial^4\Phi}{\partial\bar{\zeta}^4}+\frac{\partial}{\partial\bar{\zeta}}\left[\left(2-\gamma\Phi-\left(\frac{\partial\Phi}{\partial\bar{\zeta}}\right)^2\right)\frac{\partial\Phi}{\partial\bar{\zeta}}\right]+\alpha\Phi=0\ . \tag{3·5}$$

## § 4. Normal and inverted bifurcation

The dispersion relation corresponding to linear analysis of the stability of the trivial solution ($\Phi\equiv 0$) is

$$\omega=2k^2-k^4-\alpha\qquad \Phi\sim\exp(\omega\bar{\tau}+ik\bar{\zeta}). \tag{4·1}$$

Thus, instability sets in at $\alpha<\alpha_c=1$. Near $\alpha=\alpha_c=1$, Eq. (3·5) is amenable to analytical treatment.

We set

$$\alpha=\alpha_c(1-\lambda^2),\qquad \tilde{\tau}=\lambda^2\bar{\tau} \tag{4·2}$$

and seek a solution in the form of an asymptotic expansion

$$\Phi=\lambda\Phi_1(\tilde{\tau},\bar{\zeta})+\lambda^2\Phi_2(\tilde{\tau},\bar{\zeta})+\lambda^3\Phi_3(\tilde{\tau},\bar{\zeta})+\cdots . \quad (4\cdot3)$$

The solution of the first-approximation problem (up to translation along the $\bar{\zeta}$-axis) is

$$\Phi_1(\tilde{\tau},\bar{\zeta})=A(\tilde{\tau})\cos\bar{\zeta} . \quad (4\cdot4)$$

The equation for the amplitude $A(\tilde{\tau})$ appears as the condition for solvability of the third-approximation problem:

$$\frac{dA}{d\tilde{\tau}}=A-\left(\frac{3}{4}-\frac{\gamma^2}{18}\right)A^3 . \quad (4\cdot5)$$

Thus, when $\alpha\lesssim\alpha_c$ and $\gamma^2<27/2$ (i.e., when the temperature-dependence of the viscosity is sufficiently weak), one has normal (supercritical) bifurcation. At $\gamma^2>27/2$, Eq. (4·5) does not have nontrivial time-independent solutions, an indication of the possibility of inverted (subcritical) bifurcation. In order to verify this, let us consider the case in which $\alpha$ is close to $\alpha_c=1$ and simultaneously $\gamma^2$ is close to ${\gamma_c}^2=27/2$. To this end, we set

$$\gamma=\gamma_c(1+\delta^2), \qquad \alpha=\alpha_c(1-\rho\delta^4), \qquad s=\delta^4\tilde{\tau} \quad (4\cdot6)$$

and construct a solution of Eq. (3·5) in the form of an asymptotic expansion

$$\Phi=\delta\varphi_1(\bar{\zeta},s)+\delta^2\varphi_2(\bar{\zeta},s)+\delta^3\varphi_3(\bar{\zeta},s)+\cdots . \quad (4\cdot7)$$

Solution of the zeroth-approximation problem (accurate up to translation along the $\bar{\zeta}$-axis) gives

$$\varphi_1(\bar{\zeta},s)=B(s)\cos\bar{\zeta} . \quad (4\cdot8)$$

The equations of the second, third and fourth approximations have solutions which are $2\pi$-periodic functions of $\bar{\zeta}$ for any $B(s)$. However, the fifth-approximation problem is solvable only if

$$\frac{dB}{ds}=\rho B+\frac{3}{2}B^3-\frac{249}{256}B^5 . \quad (4\cdot9)$$

Amplitude equation of the type of (4·9) is frequently proposed as a simple model illustrating subcritical bifurcation (see, e.g., Ref. 4)). In our case the equation appears as a rigorous asymptotic relation at $\delta\ll1$.

Figure 2 shows the results of numerical solution of the initial-value problem for Eq. (3·5) with different $\alpha$ and $\gamma$. The equation was solved in the interval $0\leq\bar{\zeta}\leq11\pi$ with periodic boundary conditions. The initial condition assumed was the antisymmetric perturbation $\Phi(0,\bar{\zeta})=(\bar{\zeta}-15)\exp[-(\bar{\zeta}-15)^2/10]$. With the passage of time, a steady cellular structure developed. At $\gamma\neq0$, adjacent cells are separated from one another by fairly sharp boundaries. At $\gamma>0$ (gas-type fluids) the flow is upward at the center of each cell and downward at the boundary

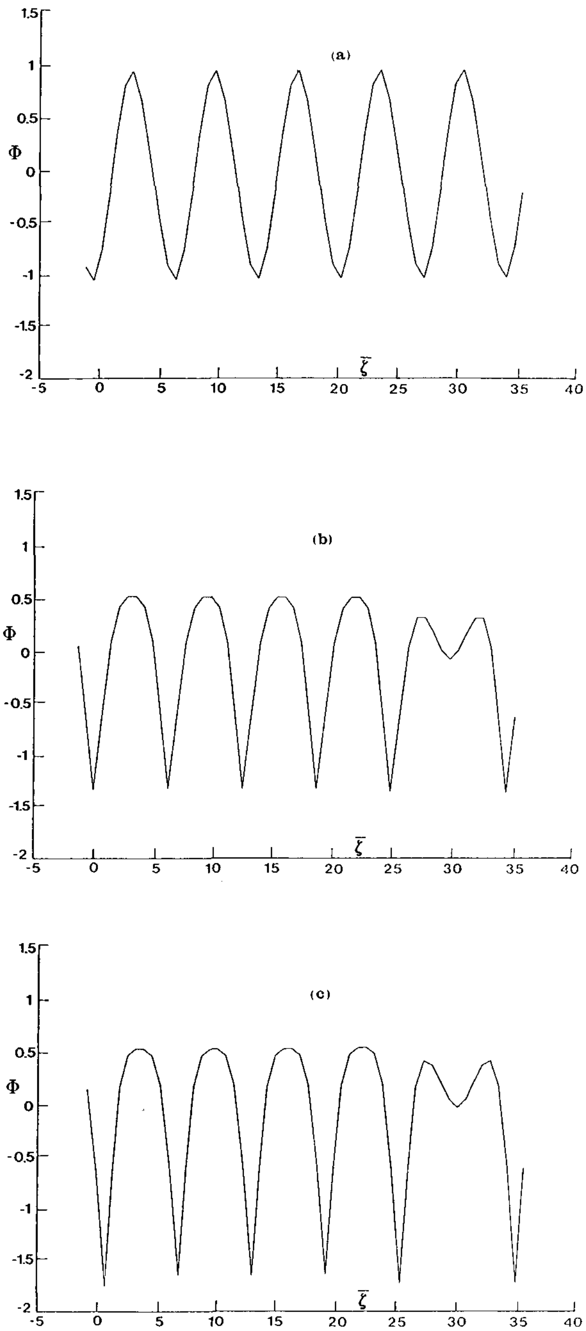


Fig. 2. Stationary cellular structure generated by Eq. (3·5).
(a) $\alpha=0.5$, $\gamma=0$. (b) $\alpha=0.9$, $\gamma=4$. (c) $\alpha=1.01$, $\gamma=5$.

(2·29). At $\gamma<0$ (liquid-type fluids) the pattern is, of course, the opposite.

We would like to direct attention to the fact that the above cellular structure, though stationary is not always regular. Since $\Phi_{\xi\xi}$ increases sharply on the cell boundaries with increasing $\gamma$ (represented by very sharp cusps on the graphs), it is not yet clear whether the observed irregularities are generated by the computation scheme or are indeed inherent in Eq. (3·5).

## § 5. Concluding remarks

Considerations of symmetry and invariance show that in the three-dimensional case the equation for $\Phi(\bar{\xi}, \bar{\zeta}, \bar{\tau})$ (in the case of constant viscosity) must have the form

$$\Phi_{\bar{\tau}}+\nabla^4\Phi+\nabla[(2-(\nabla\Phi)^2)\nabla\Phi]+\alpha\Phi=0\,. \tag{5·1}$$

Note that Eq. (5·1) may be expressed in variational notation as

$$\frac{\partial\Phi}{\partial\bar{\tau}}=-\frac{\delta F[\Phi]}{\delta\Phi}\,, \tag{5·2}$$

where the functional $F[\Phi]$ is defined as

$$F[\Phi]=-\int\Big[(\nabla\Phi)^2-\frac{1}{4}(\nabla\Phi)^4-\frac{1}{2}(\nabla^2\Phi)^2-\frac{\alpha}{2}\Phi^2\Big]d\bar{\xi}d\bar{\zeta}\,. \tag{5·3}$$

However, it is only in extremely exceptional cases that the variational formulation is admissible. For example, it is impossible in the case of Marangoni convection,[1] and also in buoyancy-driven convection, when one of the boundaries of the layer is free. In the latter case the equation for $F$ is identical (up to coefficients) with the equation obtained in I. Inclusion of the temperature dependence of viscosity (Eq. (3·5)) also makes the variational formulation impossible.

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## Appendix

We seek a solution of the linearized version of problem (1·1)~(1·4) in the form

$$\Theta = h(y)\exp(\Omega t + iKx),$$
$$\Psi_x = -v(y)\exp(\Omega t + iKx). \qquad (\mathrm{A}\cdot 1)$$

This gives the following boundary-value problem for $h$ and $v$:

$$\Omega h = h_{yy} + K^2 h - v = 0\,,$$
$$v_{yyyy} + 2K^2 v_{yy} + K^4 v - \mathrm{Ra}\ K^2 h = 0\,, \qquad (\mathrm{A}\cdot 2)$$
$$h_y = bh\,, \qquad v = v_y = 0\,, \qquad y = 0\,,$$
$$h_y = -bh\,, \qquad v = v_y = 0\,, \qquad y = 1\,. \qquad (\mathrm{A}\cdot 3)$$

To determine the dispersion relation $\Omega = \Omega(K)$, we apply Galerkin's method, taking as basis functions the simplest functions satisfying the boundary conditions (A·3):

$$v = Ay^2(1-y)^2\,, \qquad h = B(by^2 - by - 1). \qquad (\mathrm{A}\cdot 4)$$

We now substitute (A·4) into the left-hand side of Eqs. (A·2). Noting that the resulting expression must be orthogonal to the appropriate basis functions, we multiply the first and second equations of system (A·2) by $by^2 - by - 1$ and $y^2(1-y^2)$, respectively, and integrate the resulting expressions from 0 to 1. This gives

$$A\left(\frac{4}{5} + \frac{4}{105}K^2 + \frac{1}{640}K^4\right) + B\left[\mathrm{Ra}\ K^2\left(\frac{b}{140} + \frac{1}{130}\right)\right] = 0\,,$$
$$A\left(\frac{b}{140} + \frac{1}{30}\right) + B\left[2b + \frac{b^2}{3} + (\Omega + K^2)\left(\frac{b^2}{30} + \frac{b}{3} + 1\right)\right] = 0\,. \qquad (\mathrm{A}\cdot 5)$$

The equation of the neutral curve is obtained from (A·5) by putting $\Omega = 0$. Restricting ourselves to the first terms of the expansion of Ra in terms of $K^2$ and $b$, we obtain (see Fig. 1):

$$\mathrm{Ra} = 720 + \frac{240}{7}K^2 + 1440\frac{b}{K^2}\,. \qquad (\mathrm{A}\cdot 6)$$

Hence $\mathrm{Ra}_c = 720$.

The dispersion relation is obtained from the solvability condition for system (A·5). Again limiting ourselves to the first terms of the expansion, we get

$$\Omega = \left(\frac{\mathrm{Ra} - \mathrm{Ra}_c}{\mathrm{Ra}_c}\right)K^2 - \frac{1}{21}K^4 - 2b\,. \qquad (\mathrm{A}\cdot 7)$$

Although Eqs. (A·6) and (A·7) are approximate, they nevertheless provide a correct estimate of the order of magnitude of the characteristic values of $\Omega$, $K$ and $b$ when $(\mathrm{Ra} - \mathrm{Ra}_c)/\mathrm{Ra}_c \ll 1$.

**References**

1) G. I. Sivashinsky, Physica **D** (1981) (to appear).
2) F. H. Busse and N. Riahi, J. Fluid Mech. **96** (1980), 243.
3) S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press., Oxford, 1961).
4) C. Normand, Y. Pomeau and M. G. Velarde, Rev. Mod. Phys. **49** (1977), 581.