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# LARGE CHARACTER DEGREES OF GROUPS OF ODD ORDER

BY

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## 1. Introduction

In his paper [4], Gluck conjectures that the index of the Fitting subgroup of a finite solvable group $G$ is bounded by the square of the largest degree of an irreducible character (over the complex field) of $G$, i.e.,

$$|G:F(G)| \leq b(G)^2 \quad \text{where } b(G) = \max\{\psi(1) | \psi \in \mathrm{Irr}(G)\}.$$

(See p. 447 of [4].)

He shows in Theorem B of [4] that $|G:F(G)| \leq b(G)^{13/2}$ holds for a solvable group $G$. Here we prove his conjecture with the additional assumption that $|G|$ is odd. As a corollary, we show that if $G$ is a (solvable) group of odd order, then $G$ has an abelian subgroup $A$ such that $|G:A| \leq b(G)^6$. This improves (for groups of odd order) the bound $|G:A| \leq b(G)^{21/2}$ obtained in Theorem C of [4].

Our method consists of obtaining a regular orbit theorem for groups of odd order acting on vector spaces of odd characteristic (see Lemma 2.1 below). As Gluck points out, this method cannot be used for even order solvable groups (see p. 447 of [4]).

Our regular orbit theorem may be also used to prove a conjecture due to Huppert in some cases. Let $G$ be a finite solvable group and, for a natural number $n$, let $\pi(n)$ denote the set of prime divisors of $n$. For a group $H$, we write $\pi(H)$ for $\pi(|H|)$. Define

$$\sigma(G) = \max_{\chi \in \mathrm{Irr}(G)} |\pi(\chi(1))|,$$

$$\rho(G) = \bigcup_{\chi \in \mathrm{Irr}(G)} \pi(\chi(1))$$

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Huppert's conjecture asserts that $|\rho(G)| \leq 2\sigma(G)$. The best results up to now about this question are due to Gluck and Manz and appear as Theorem A and Theorem B of [5]. They prove that $|\rho(G)| \leq 3\sigma(G) + 32$ holds in general and $|\rho(G)| \leq 2\sigma(G) + 32$ holds when every normal Sylow subgroup of $G$ is abelian. We show that in the latter case and with the additional assumption that $|G|$ is odd, Huppert's conjecture holds (see Theorem 4.1 below). The reader interested in the difficult problem of dropping the restriction about the Sylow subgroups of $G$ is referred to [6].

A quite different case of Huppert's conjecture is obtained using Theorem A of [1]. We prove it for groups of odd order having a faithful quasiprimitive character (see Theorem 4.4 below). In this case, we obtain two characters $\chi$ and $\psi$ such that

$$\rho(G) = \pi(\chi(1)) \cup \pi(\psi(1)).$$

## 2. A regular orbit theorem

In this section we prove the main result of the paper. We obtain a regular orbit theorem for quasiprimitive modules of groups of odd order.

LEMMA 2.1. *Let $G$ be a group of odd order having a faithful and irreducible quasiprimitive module $V$ over a finite field $F$ of odd characteristic. Suppose that $F(G)$ is noncyclic. Then $V$ contains at least two regular $G$-orbits.*

*Proof.* Put $r = |F|$.

Observe that every normal abelian subgroup of $G$ is cyclic. As $|G|$ is odd, applying Corollary 2.4 of [8], we get:

(i) $F(G) = EU$, where $U$ is cyclic and the Sylow subgroups of $E$ are of prime order or extraspecial of prime exponent. Furthermore $U = Z(F(G))$.

Let $p_1, \dots, p_n$ be the distinct prime divisors of $|F(G)|$ and let $Z \leq U$ with $|Z| = p_1 \dots p_n$. Put $A = C_G(Z)$. Then:

(ii) $U = C_G(E)$ and $F(G) = C_A(E/Z)$.

(iii) Each Sylow subgroup of $E/Z$ is a completely reducible $A/F(G)$-module. Observe that our hypothesis ensures that $E \neq 1$.

Let $K$ be the algebraic closure of $F$. For each irreducible $KG$-submodule of $V \otimes K$, the number of homogeneous $F(G)$-components is divisible by $|G : A|$. Thus

$$\dim_F(V) = \dim_K(V \otimes K) = a|G : A|e \tag{1}$$

where $a$ is an integer and $e^2 = |E : Z|$.

Let $W$ be an irreducible $U$-submodule of $V$. Then, as $V$ is quasiprimitive, $W$ is faithful and $|U|\,|(|W|-1)/2$. Thus $|W|\geq 7$ since $|U|$ and $|W|$ are odd. It is well known that each irreducible $F(G)$-submodule of $V$ has order $|W|^e$. Hence

$$(2) \qquad |V| = |W|^{be} \quad \text{for some integer } b.$$

To prove that $G$ has at least two regular orbits, it suffices to show that

$$(*) \qquad \left| V - \bigcup_{1\neq g\in G} C_V(g) \right| > |G|.$$

By Lemma 1.3 of [1], we have $|C_V(g)| \leq |V|^{4/9}$ for $1 \neq g \in G$. To prove $(*)$, we will show that $|V| > |G| + |V|^{4/9}|G| = (|V|^{4/9}+1)|G|$. As $|W|\geq 7$ and $e\geq 3$, we have $|V|\geq 7^3$ by (2). Thus $|V|^{4/9}+1\leq |V|^{1/2}$. Thus it suffices to show that

$$(**) \qquad |G| < |V|^{1/2}.$$

Define $n = |G:A|$. By (iii), each Sylow subgroup of $E/Z$ is a completely reducible $A/F(G)$-module. By Theorem A of [1], $E/Z$ contains at least two regular $A/F(G)$-orbits. Thus $|A/F(G)| \leq e^2/2$. Thus

$$|G| \leq n\cdot e^2/2\cdot e^2|U| = |U|n\cdot e^4/2$$

As $|U|$ divides $(|W|-1)/2$, we have

$$|U|\,|(|W|^b-1)/2 = (r^{an}-1)/2 \leq r^{an}/2$$

by (1) and (2). Thus it is sufficient to prove

$$(***) \qquad ne^4 < 4r^{an(e/2-1)}.$$

The case $n=1$ appears as case 1 in Theorem 1 of [2]. (The hypothesis $p \nmid q-1$ for $p, q\in\pi(G)$ is used only to ensure that $G=A$). Thus we may suppose that $n\neq 1$ and hence $n\geq 3$.

Suppose that $e\geq 7$. Then it is easy to see that $(***)$ holds. Consider the case $e=5$. As $G\neq A$ and the cyclic groups of order 3 and 5 have not any odd order automorphism, we have $|U|\geq 35$. Thus $r^{an}-1\geq 70$. As $n$ is odd and $r\neq 5$, then either $r=3$ and $n\geq 5$, or $r\geq 7$. Hence $(***)$ holds.

Consider finally the case $e=3$. Here $E$ is extraspecial of order $3^3$. Now $A=EU$ since $E/Z(E)$ is a completely reducible $A/F(G)$-module. The group $G/A$ acts on $E/Z(E)$. Clearly the action cannot be irreducible. Now, as the exponent of $E$ is 3, $E$ contains a normal abelian subgroup of $G$

isomorphic to $C_3 \times C_3$. This is a contradiction since $V$ is faithful and quasiprimitive.

## 3. Gluck's conjecture

In this section we apply the result of Section 2 to prove (for groups of odd order) the conjecture due to Gluck and mentioned in the introduction. Our method is similar to his.

THEOREM 3.1. *Let $G$ be a (solvable) group of odd order and let $V$ be a faithful and completely reducible $G$-module over a finite field $F$ of odd characteristic. Then there exists $v \in V$ such that $|C_G(v)| \leq |V|^{1/2}$.*

*Proof.* We reason by induction on $|G| + \dim_F(V)$.

First we show that $V$ is $G$-irreducible. The argument appears in Gluck's paper [4]. We repeat it here for the convenience of the reader.

If $V$ is not $G$-irreducible, then we have $V = V_1 \oplus V_2$, where each $V_i$ is a nonzero $G$-module. Let $N = C_G(V_1)$. The induction hypothesis ensures that there exists $v_1 \in V_1$ such that $|C_{G/N}(v_1)| \leq |G/N|^{1/2}$. Now, as $N$ is normal in $G$, the $N$-module $V_2$ is completely reducible. By induction, we have $v_2 \in V_2$ such that $|C_N(v_2)| \leq |N|^{1/2}$. Take $v = (v_1, v_2) \in V$. Now $C_G(v)N/N \leq C_{G/N}(v_1)$. Thus $|C_G(v) : C_G(v) \cap N| \leq |G : N|^{1/2}$. Furthermore $C_G(v) \cap N = C_N(v_2)$. Hence $|C_G(v)| \leq |G : N|^{1/2}|N|^{1/2} = |G|^{1/2}$, as claimed.

Now suppose that $V \simeq W^H$, where $W$ is $H$-primitive, $H$ being a subgroup of $G$. It is possible that $H = G$. Define $N = \bigcap_{g \in G} H^g$. Take coset representatives $1 = x_1, \ldots, x_n$ of $\{xH | x \in G\}$. Define

$$\overline{H}_i = H^{x_i}/C_{x_i}(W^{x_i})$$

and let $\overline{N}_i$ denote the group

$$NC_{H^{x_i}}(W^{x_i})/C_{H^{x_i}}(W^{x_i}).$$

First, suppose that $F(\overline{H}_1)$ is cyclic. Then each $F(\overline{H}_i)$ is cyclic and $\overline{H}_i/F(\overline{H}_i)$ is abelian. As $\overline{N}_i \cap F(\overline{H}_i) = F(\overline{N}_i)$, $\overline{N}_i/F(\overline{N}_i)$ is abelian, too. Now $N$ is isomorphic to a subgroup of $\Pi_{i=1}^n \overline{N}_i$. Thus $N/F(N)$ is abelian. Now it is easy to see that $N/F(N)$ has a regular orbit on the completely reducible module $F(N)/\Phi(N)$. In particular, $|N/F(N)| \leq |F(N)|$. Observe that every nontrivial element of $W$ generates a regular $F(\overline{H}_1)$-orbit (in particular, a regular $F(\overline{N}_1)$-orbit). Observe that $W - \{0\}$ contains at least two $\overline{H}_1$-orbits since $|\overline{H}_1|$ and $|W|$ are odd. Take representatives $v$ and $w$ of two different $\overline{H}_1$-orbits on $W - \{0\}$. Identify the isomorphic groups $\overline{H}_1 W$ and $\overline{H}_i W^{x_i}$, $i = 2, \ldots, n$.

Consider the group $G/N$ acting as a permutation group on

$$\Omega = \{x_i H | i = 1, \dots, n\}.$$

By Corollary 1 of [3], $G/N$ has a regular orbit on the power set $2^{\Omega}$. Take $A \subseteq \Omega$ generating such an orbit. Consider $u = (u_1, \dots, u_n)$, where $u_i = v$ if $x_i H \in A$ and $u_i = w$ if $x_i H \notin A$.

Next we show that $C_G(u) \subseteq N - F(N)$. As there is not any element of $H_1$ sending $v$ to $w$, we have $C_G(u) \leq N$. If $x \in F(N)$, then the image of $x$ in $\overline{N}_i$ belongs to $F(\overline{N}_i)$. We may choose an index $j$ such that $x$ does not centralize the module $W^{x_j}$. Hence $x$ does not centralize $u_j$ and $x \notin C_G(u)$, as claimed.

Now it is clear that $|C_G(u)| \leq |N/F(N)| \leq |N|^{1/2} \leq |G|^{1/2}$. Hence the theorem is proven in the case in which $F(\overline{H}_1)$ is cyclic.

Suppose that $F(\overline{H}_1)$ is noncyclic. By Lemma 2.1, $\overline{H}_1$ has at least two regular orbits on $W$. Take representatives $v$ and $w$ of such regular orbits. Identify the isomorphic groups $\overline{H}_1 W$ and $\overline{H}_i W^{x_i}$, $i = 2, \dots, n$. Let $A \subseteq \Omega$ be as before. Consider $u = (u_1, \dots, u_n)$, where $u_i = v$ if $x_i H \in A$ and $u_i = w$ if $x_i H \notin A$. Now $|u|$ generates a regular $G$-orbit. In particular, $|C_G(u)| \leq |G|^{1/2}$. The proof of the theorem is finished.

Now let $b(G) = \max\{\psi(1) | \psi \in \mathrm{Irr}(G)\}$.
We prove the main theorem of this section.

THEOREM 3.2. *Let $G$ be a (solvable) group of odd order. Then*

$$|G : F(G)| \leq b(G)^2.$$

*Proof.* Consider $\overline{G} = G/F(G)$ acting faithfully on the completely reducible module $V = \mathrm{Irr}(F(G)/\Phi(G))$. Applying Theorem 3.1 we get $\lambda \in V$ such that

$$\bar{I} = I_{\overline{G}}(\lambda) = \{\bar{g} \in \overline{G} | \lambda^{\bar{g}} = \lambda\}$$

satisfies

$$|\bar{I}| \leq |\overline{G}|^{1/2}.$$

Consider $\lambda$ as a character of $F(G)$ with kernel containing $\Phi(G)$. Let $I$ be the preimage of $\bar{I}$ in $G$. Now $I = I_G(\lambda) = \{g \in G | \lambda^g = \lambda\}$. Take $\mu \in \mathrm{Irr}(I|\lambda)$. Now $\psi = \mu^G \in \mathrm{Irr}(G)$. We have $\psi(1) \geq |G : I|$. As

$$|G : F(G)| = |G : I||I : F(G)| \leq |G : I|^2 = \psi(1)^2 \leq b(G)^2,$$

our claim is verified.

COROLLARY 3.3. *Let $G$ be a (solvable) group of odd order. Then $G$ has an abelian subgroup $A$ such that $|G:A| \leq b(G)^6$.*

*Proof.* By Theorem 12.26 of [7], we have an abelian subgroup $A$ of $F(G)$ such that $|F(G): A| \leq b(F(G))^4 \leq b(G)^4$. Now the result is obvious applying Theorem 3.2.

## 4. Huppert's conjecture

We refer the reader to the introduction for the definitions of $\rho(G)$ and $\sigma(G)$. Remember that Huppert's conjecture asserts that $|\rho(G)| \leq 2\sigma(G)$. Our first result is a remark about the paper by Gluck and Manz [5].

THEOREM 4.1. *Let $G$ be a (solvable) group of odd order. Assume that every normal Sylow subgroup of $G$ is abelian. Then $|\rho(G)| \leq 2\sigma(G)$.*

*Proof.* The argument of Theorem A of [5] works replacing their Proposition 4 by our Theorem 2.1 and their Lemma 7 by Corollary 1 of [3].

Finally we prove Huppert's conjecture in a different situation. We need a lemma.

LEMMA 4.2. *Let $G$ be a group acting on an elementary abelian $p$-group $V$. Suppose that $V$ is endowed with a non-singular symplectic form $\langle -, - \rangle$ fixed by $G$. Assume that $G$ has a regular orbit on $V$. Then $G$ has a regular orbit on $\mathrm{Irr}(V)$.*

*Proof.* Let $v \in V$ be a generator of a regular $G$-orbit. Consider

$$\langle v \rangle^{\perp} = \{w \in V | \langle v, w \rangle = 0\}.$$

We may consider an irreducible character $\chi$ of $V$ with kernel $\langle v \rangle^{\perp}$. Suppose that $g \in G$ fixes $\chi$ and we will show that $g = 1$. We may assume that $g$ is of $q$-power order, $q$ a prime. As $(\langle v \rangle^{\perp})^{\perp} = \langle v \rangle$, we know that $g$ normalizes $\langle v \rangle$. If $q = p$, then $g$ centralizes $\langle v \rangle$ and $g$ is trivial. If $q \neq p$, then we may find a $\langle g \rangle$-invariant complement $W$ to $\langle v \rangle^{\perp}$ in $V$. As $g$ fixes $\chi$, $g$ centralizes $W$. Now $\langle v^i, w \rangle \neq 0$ when $(p, i) = 1$ and $0 \neq w \in W$. Thus, if $w$ is a nontrivial element of $W$, we have

$$\langle v, w \rangle = \langle v^g, w^g \rangle = \langle v^g, w \rangle.$$

Hence $\langle v^{-1} v^g, w \rangle = 0$. Now $g$ fixes $v$ and $g = 1$.

THEOREM 4.3. *Let $G$ be a (solvable) group of odd order having a faithful quasiprimitive complex character. Then* $|\rho(G)| \leq 2\sigma(G)$.

*Proof.* The hypotheses ensure that every normal abelian subgroup of $G$ is cyclic and central. Now Corollary 2.4 of [8] proves that $F(G) = UE$, where $U$ is cyclic and the Sylow subgroups of $E$ are of prime order or extraspecial of prime exponent. Furthermore $U \leq Z(G)$. Let $Z = U \cap E$. Now each Sylow subgroup of $E/Z$ is a completely reducible $G/F(G)$-module. Also $C_G(E/Z) = F(G)$. By Theorem A of [1], $G/F(G)$ has a regular orbit on $E/Z$. By Lemma 4.2, $G/F(G)$ has a regular orbit on $\mathrm{Irr}(E/Z)$. Let $\lambda \in \mathrm{Irr}(E/Z)$ be a generator of such an orbit. Consider $\lambda$ as a character of $F(G)$ with kernel containing $U$. Now $\lambda^G \in \mathrm{Irr}(G)$ and $\lambda^G(1) = |G : F(G)|$. Let $\psi = \lambda^G$. Let $\mu$ be a faithful irreducible character of $F(G)$ and $\chi \in \mathrm{Irr}(G|\mu)$. Now $\rho(G) = \pi(\psi(1)) \cup \pi(\chi(1))$ by Ito's theorem and our claim is verified.

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