Large cliques or stable sets in graphs with no four-edge path and no five-edge path in the complement

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Abstract

Erdős and Hajnal [4] conjectured that, for any graph H, every graph on n vertices that does not have H as an induced subgraph contains a clique or a stable set of size $n^{\varepsilon(H)}$ for some $\varepsilon(H) > 0$. The conjecture is known to be true for graphs H with $|V(H)| \leq 4$. One of the two remaining open cases on five vertices is the case where H is a four-edge path, the other case being a cycle of length five. In this paper we prove that every graph on n vertices that does not contain a four-edge-path or the complement of a five-edge-path as an induced subgraph contains either a clique or a stable set of size at least $n^{1/6}$.

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1 Introduction

All graphs in this paper are finite and simple. A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise nonadjacent vertices. Let G be a graph. For a set of vertices X, we denote by $G|X$ the subgraph of G induced by X. For a graph H, we say that G *contains H as an induced subgraph* if G has an induced subgraph that is isomorphic to H. Let $\mathrm{Forb}(X_1, X_2, \ldots, X_k)$ be the set of all graphs G such that, for all $i \in \{1, 2, \ldots, k\}$, G does not contain X_i as an induced subgraph. For a vertex $v \in V(G)$, we let $N(v)$ denote the set of vertices adjacent to v and $M(v)$ the subset of vertices of $V(G)\backslash\{v\}$ that are nonadjacent to v. We say that two sets $X \subseteq V(G)$ and $Y \subseteq V(G)$ are *complete* to each other if every $x \in X$ and $y \in Y$ are adjacent. We say that X and Y are *anticomplete* to each other if every $x \in X$ and $y \in Y$ are nonadjacent. We denote by G^c the graph with vertex set $V(G)$ and edge set $\{\{u, v\} \in V(G)^2 \mid u \neq v, uv \notin E(G)\}$. We call G^c the *complement* of G. For a set $X \subseteq V(G)$, we denote by $G \setminus X$ the graph $G|(V(G) \setminus X)$. For $k \geq 0$, we denote by P_k the k-edge path (thus, $|V(P_k)| = k + 1$).

We say that a graph H has the *Erdős-Hajnal property* if there exists $\varepsilon(H) > 0$ such that every graph on n vertices that does not have H as an induced subgraph contains either a clique or a stable set of size at least $n^{\varepsilon(H)}$. Clearly, if H has the property, then so does H^c. Erdős and Hajnal [4] conjectured that all graphs have the property. It is known to be true for every graph H with $|V(H)| \leq 4$. In [1], it was shown that if two graphs H_1 and H_2 have the Erdős-Hajnal property, then so does the graph constructed from H_1 by replacing a vertex $x \in V(H_1)$ by H_2 and making $V(H_2)$ complete to the neighbors of x in H_1 and anticomplete to the non-neighbors of x in H_1 (this operation is known as the substitution operation). Moreover, it was shown in [3] that the triangle with two disjoint pendant edges (this graph is known as the *bull*) has the property. This leaves the four-edge-path P_4 and the cycle C_5 of length five as the remaining open cases for graphs on at most 5 vertices. This paper deals with the case where H is a four-edge path, where, in addition, we exclude the complement of a five-edge path. To be precise, we will prove the following theorem:

Theorem 1.1. *Every graph* $G \in \mathrm{Forb}(P_4, P_5^c) \cup \mathrm{Forb}(P_4^c, P_5)$ *contains a clique or a stable set of size at least* $|V(G)|^{1/6}$.

For a graph G, let $\omega(G)$ denote the size of the largest clique in G and let $\chi(G)$ denote the chromatic number of G. G is called *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G. It was shown in [2] that a graph G is perfect if and only if it does not contain an odd cycle of length at least five or the complement of an odd cycle of length at least five as an induced subgraph.

We say that a function $g : V(G) \to \mathbb{R}^+$ is a *covering function for G* if $\sum_{p \in V(P)} g(p) \leq 1$ for every perfect induced subgraph P of G. For $\beta \geq 1$, we say that a graph G is β-narrow if $\sum_{v \in V(G)} g^\beta(v) \leq 1$ for every covering function g. Notice that since a graph is perfect if and only if its complement is perfect, it follows that a graph is β-narrow if and only if its complement is β-narrow. It was shown in [3] that bull-free graphs are 2-narrow. We will take a similar approach and prove that

Theorem 1.2. *All graphs in* $\mathrm{Forb}(P_4^c, P_5)$ *are* 3*-narrow.*

This result suffices for proving Theorem 1.1, because of the following result:

(1.3) *Let G be a β-narrow graph. Then G has a clique or stable set of size at least $|V(G)|^{1/2\beta}$.*

Proof. Let $\mathcal{P}$ be the set of all perfect induced subgraphs of G. Let $K = \max_{P \in \mathcal{P}} |V(P)|$. Consider the function $g : V(G) \to \mathbb{R}^+$ with $g(v) = 1/K$ for all $v \in V(G)$. Clearly, $\sum_{v \in V(P)} g(v) \leq 1$ for all $P \in \mathcal{P}$. Therefore, since G is β-narrow, it follows that g satisfies

$$1 \geq \sum_{v \in V(G)} g(v)^\beta = \frac{|V(G)|}{K^\beta}.$$

Equivalently, we have $K \geq |V(G)|^{\frac{1}{\beta}}$. Thus, G has a perfect induced subgraph H with $|V(H)| \geq |V(G)|^{\frac{1}{\beta}}$. Since H is a perfect graph, H satisfies $|V(H)| \leq \chi(H)\alpha(H) = \omega(H)\alpha(H)$ and hence $\max(\omega(H), \alpha(H)) \geq \sqrt{|V(H)|} \geq |V(G)|^{1/2\beta}$. Therefore, H has a clique or stable set of size at least $|V(G)|^{1/2\beta}$. Since H is an induced subgraph of G, G has a clique or stable set of size at least $|V(G)|^{1/2\beta}$. This proves (1.3). $\square$

Notice that the proof of (1.3) also shows that a graph G is 1-narrow if and only if G is perfect. Jacob Fox [5] proved that the 'converse' of (1.3) is also true:

Theorem. *Let H be a graph that has the Erdős-Hajnal property. Then, every graph in* Forb(H) *is $\beta(H)$-narrow for some $\beta(H) \geq 1$.*

This implies that the Erdős-Hajnal conjecture is equivalent to the following conjecture:

Conjecture. *For every graph H, there exists $\beta(H) \geq 1$ such that every $G \in$* Forb(H) *is $\beta(H)$-narrow.*

This paper is organized as follows. In Section 2, we describe tools that we will use in the rest of the paper. Section 3 deals with graphs in Forb(P_4^c, P_5) for which we additionally require that they have no induced copy of C_6, the cycle of length six. Finally, in Section 4 we abandon this additional requirement and finish the proof of Theorem 1.1.

2 Decompositions

We start with a number of graph decompositions and their relationship to the narrowness of graphs.

Lemma 2.1. *Let G be a graph and let $\beta \geq 1$. Suppose that for every $v \in V(G)$, either*

(i) *$G|N(v)$ is β-narrow and $G|M(v)$ is $(\beta+1)$-narrow, or*

(ii) *$G|M(v)$ is β-narrow and $G|N(v)$ is $(\beta+1)$-narrow.*

Then G is $(\beta+1)$-narrow.

Proof. Let g be a covering function for G. Choose $u \in V(G)$ with $g(u)$ maximal. If $g(u) = 1$, then, because every 2-vertex induced subgraph of G is perfect, $g(u') = 0$ for all $u' \in V(G) \setminus \{u\}$ and thus $\sum_{v \in V(G)} g(v)^{\beta+1} \leq 1$ trivially holds. So we may assume that $g(u) < 1$. Let $M = M(u)$, $N = N(u)$,

$G_M = G|M$ and $G_N = G|N$. Since β-narrowness is invariant under taking complements, we may, possibly by passing to the complement, assume that G_M is $(\beta+1)$-narrow and G_N is β-narrow.

Define $f_M : V(G_M) \to \mathbb{R}^+$ by $f_M(v) = g(v)/\big[1-g(u)\big]$ for each $v \in V(G_M)$. Let P be a perfect induced subgraph of G_M. Since $G|(V(P) \cup \{u\})$ is perfect, it follows that $\sum_{v \in V(P)} f_M(v) \leq 1$. Since G_M is $(\beta+1)$-narrow, f_M satisfies $\sum_{v \in M} \big[f_M(v)\big]^{\beta+1} \leq 1$ and therefore

$$\sum_{v \in M} \big[g(v)\big]^{\beta+1} \leq \big[1-g(u)\big]^{\beta+1}.$$

By repeating the same argument for G_N, since G_N is β-narrow, it follows that

$$\sum_{v \in N} \big[g(v)\big]^{\beta} \leq \big[1-g(u)\big]^{\beta}.$$

Moreover, we have, by the choice of u,

$$\sum_{v \in N} \big[g(v)\big]^{\beta+1} \leq g(u) \sum_{v \in N} \big[g(v)]\big]^{\beta} \leq g(u)\big[1-g(u)\big]^{\beta}.$$

Hence,

$$\begin{aligned} \sum_{v \in V(G)} \big[g(v)\big]^{\beta+1} &= \big[g(u)\big]^{\beta+1} + \sum_{v \in M} \big[g(v)\big]^{\beta+1} + \sum_{v \in N} \big[g(v)\big]^{\beta+1} \\ &\leq \big[g(u)\big]^{\beta+1} + (1-g(u))^{\beta+1} + g(u)(1-g(u))^{\beta} \\ &= \big[g(u)\big]^{\beta+1} + (1-g(u))^{\beta} \leq 1, \end{aligned}$$

where the last inequality follows from the fact that the function $h(x) = x^{\beta+1} + (1-x)^{\beta}$ is strictly convex and $h(0) = h(1) = 1$. This proves Lemma 2.1. $\square$

Let G be a graph. We say that a set $Z \subseteq V(G)$ is a *homogeneous set* in G if $1 < |Z| < |V(G)|$ and $V(G) \setminus Z = A \cup C$ where A is anticomplete to Z and C is complete to Z. In this case, we say that (Z, A, C) is a *homogeneous set decomposition* of G. The following is a theorem from [6].

Lemma 2.2. *Let G be a graph and let (Z, A, C) be a homogeneous set decomposition of G. Construct G' from $G|(A \cup C)$ by adding a vertex z that is complete to C and anticomplete to A. Let P_1 be a perfect induced subgraph of G' with $z \in V(P_1)$ and let P_2 be a perfect induced subgraph of $G|Z$. Then $G|((V(P_1) \cup V(P_2)) \setminus \{z\})$ is perfect.*

It was shown in [3] that homogeneous set decompositions preserve β-narrowness. For our purposes, we will need a more general decomposition. We say that a set $Z \subseteq V(G)$ is a *quasi-homogeneous set* in G if there exist disjoint sets $A, C \subseteq V(G) \setminus Z$ with union $V(G) \setminus Z$ that satisfy the following properties:

(i) $1 < |Z| < |V(G)|$.

(ii) Z is complete to C.

(iii) Let G' be obtained from $G|(A \cup C)$ by adding a vertex z that is anticomplete to A and complete to C. Suppose that P_1 is a perfect induced subgraph of G' with $z \in V(P_1)$ and suppose P_2 is a perfect induced subgraph of $G|Z$. Then the graph $P = G|((V(P_1) \cup V(P_2)) \setminus \{z\})$ is perfect.

(iv) G contains G' as an induced subgraph.

We say that the triple (Z, A, C) is a *quasi-homogeneous set decomposition*. In the light of Lemma 2.2, it is easy to see that a homogeneous set decomposition is a special case of a quasi-homogeneous set decomposition. Just like homogeneous set decompositions, quasi-homogeneous set decompositions preserve β-narrowness:

Lemma 2.3. *Let G be a graph and let (Z, A, C) be a quasi-homogeneous set decomposition of G. Let H_1 be the graph obtained from $G|(A \cup C)$ by adding a vertex z that is anticomplete to A and complete to C and let $H_2 = G|Z$. If H_1 and H_2 are β-narrow, then G is β-narrow.*

Proof. The proof is essentially the same as the proof of **1.3** in [3], but we include it here for completeness. Let g be a covering function for G. For $i = 1, 2$, let $\mathcal{P}_i$ be the set of perfect induced subgraphs of H_i. Let $K = \max_{P \in \mathcal{P}_2} \sum_{v \in V(P)} g(v)$. Define $g_1 : V(H_1) \to \mathbb{R}^+$ as follows. For $v \in A \cup C$, let $g_1(v) = g(v)$ and let $g_1(z) = K$. Define $g_2 : V(H_2) \to \mathbb{R}^+$ by $g_2(v) = g(v)/K$ for $v \in V(H_2)$. From the definition of a quasi-homogeneous set decomposition, it follows that for every $P_1 \in \mathcal{P}_1$ with $z \in V(P_1)$ and every $P_2 \in \mathcal{P}_2$, $G|((V(P_1) \cup V(P_2)) \setminus \{z\})$ is perfect. It follows that g_1 is a covering function for H_1. Since H_1 is β-narrow, it follows that

$$1 \geq \sum_{v \in V(H_1)} \big[g_1(v)\big]^\beta = \sum_{v \in A \cup C} [g(v)]^\beta + K^\beta.$$

Clearly, g_2 is a covering function for H_2. Thus, since H_2 is β-narrow, it follows that

$$1 \geq \sum_{v \in V(H_2)} \big[g_2(v)\big]^\beta = \sum_{v \in Z} \frac{\big[g(v)\big]^\beta}{K^\beta}.$$

Therefore,

$$\sum_{v \in Z} \big[g(v)\big]^\beta \leq K^\beta.$$

Finally, it follows that

$$\sum_{v \in V(G)} \big[g(v)\big]^\beta \leq \sum_{v \in A \cup C} \big[g(v)\big]^\beta + \sum_{v \in Z} \big[g(v)\big]^\beta \leq (1 - K^\beta) + K^\beta = 1.$$

This proves Lemma 2.3. □

(Observe that the proof of Lemma 2.3 still goes through even without assuming property (iv) of a quasi-homogeneous set decomposition. This additional property is solely used for the inductive arguments in Sections 3 and 4.)

Let G be a graph. We say that G *admits a Σ-join* if there exist disjoint sets X_1, X_2, N_1, N_2, C, A with union $V(G)$ such that

- for $i = 1, 2$, $|X_i| \geq 2$ and X_i is a stable set, and
- for $\{i, j\} = \{1, 2\}$, X_i is complete to $C \cup N_i$ and anticomplete to $A \cup N_j$, and
- X_1 is not anticomplete to X_2.

We call $(X_1, X_2, N_1, N_2, C, A)$ a Σ-*join*. The following lemma states that Σ-joins preserve narrowness:

Lemma 2.4. *Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and suppose that G admits a Σ-join $(X_1, X_2, N_1, N_2, C, A)$. Let G' be obtained from $G \setminus (X_1 \cup X_2)$ by adding two adjacent vertices x_1 and x_2 such that, for $\{i, j\} = \{1, 2\}$, x_i is complete to $C \cup N_i$ and anticomplete to $A \cup N_j$. Then, $G' \in \mathrm{Forb}(P_4^c, P_5)$ and if, for some $\beta \geq 1$, G' is β-narrow, then G is β-narrow.*

Proof. Notice first that since X_1 is not anticomplete to X_2, G contains G' as an induced subgraph and therefore $G' \in \mathrm{Forb}(P_4^c, P_5)$. Now suppose that G' is β-narrow for some $\beta \geq 1$. For an induced subgraph P of G' and $\{i, j\} = \{1, 2\}$, let

$$P(X_1, X_2) = \begin{cases} G|((V(P) \setminus \{x_i\}) \cup X_i) & \text{if } x_i \in V(P) \text{ and } x_j \notin V(P); \\ G|((V(P) \setminus \{x_1, x_2\}) \cup X_1 \cup X_2) & \text{if } x_1, x_2 \in V(P). \end{cases}$$

We first claim the following:

($\star$) *If P is a perfect induced subgraph of G', then $P(X_1, X_2)$ is a perfect induced subgraph of G.*

Write $P' = P(X_1, X_2)$. Suppose that P' is not perfect. Then, P' contains either a cycle of odd length $k \geq 5$ or the complement of a cycle of length $k \geq 5$ as an induced subgraph. Since P' is an induced subgraph of G, it follows that $P' \in \mathrm{Forb}(P_4^c, P_5)$ and, thus, P' contains no cycle of odd length at least seven and no complement of a cycle of odd length at least seven as an induced subgraph. Thus, P' has an induced cycle of length five, say $F = f_1\text{-}f_2\text{-}\cdots\text{-}f_5\text{-}f_1$. Suppose that $V(F) \cap X_1 = \emptyset$. Then F is an induced subgraph of $P(X_1, X_2) \setminus X_1$. However, it follows from Lemma 2.2 applied to $G' \setminus \{x_1\}$, $P \setminus \{x_1\}$ and X_2 that $P(X_1, X_2) \setminus X_1$ is perfect, a contradiction. Thus, we may assume that $V(F) \cap X_i \neq \emptyset$ for $i = 1, 2$. We may assume that $f_1 \in X_1$, and either $f_2 \in X_2$ or $f_3 \in X_2$. First suppose that $f_3 \in X_2$. Because f_2 is complete to $\{f_1, f_3\}$, it follows from the definition of the Σ-join that $f_2 \in C$. Because f_4 is anticomplete to $\{f_1, f_2\}$ and adjacent to f_3, it follows that $f_4 \in N_2$ and, symmetrically, $f_5 \in N_1$. But now, $x_1\text{-}f_4\text{-}f_2\text{-}f_5\text{-}x_2$ is an induced four-edge antipath in G', a contradiction. This proves that $f_3 \notin X_2$ and hence $f_2 \in X_2$. We may also assume that no two nonadjacent $f, f' \in V(F)$ satisfy $f \in X_1$ and $f' \in X_2$. Therefore, since f_4 is anticomplete to $\{f_1, f_2\}$, it follows that $f_4 \in A$. This implies that $f_3 \in N_2$ and $f_5 \in N_1$. But now, $x_1\text{-}x_2\text{-}f_3\text{-}f_4\text{-}f_5\text{-}x_1$ is an induced cycle of length five in P, contrary to the fact that P is perfect. This proves ($\star$). □

To prove that G is β-narrow, let $g : V(G) \to \mathbb{R}_+$ be a covering function for G. Define $g' : V(G') \to \mathbb{R}_+$ as follows: for $i = 1, 2$, $g'(x_i) = \sum_{v \in X_i} g(v)$, and $g'(v) = g(v)$ for all $v \in V(G') \setminus \{x_1, x_2\}$. We claim that g' is a covering function for G'. Let P be a perfect induced subgraph of G'. Since

$P(X_1, X_2)$ is a perfect induced subgraph of G by $(\star)$, it follows that

$$\sum_{v \in V(P)} g'(v) = \sum_{\substack{i \in \{1,2\}: \\ x_i \in V(P)}} g'(x_i) + \sum_{\substack{v \in V(P) \\ v \neq x_1, x_2}} g'(v)$$
$$= \sum_{\substack{i \in \{1,2\}: \\ x_i \in V(P)}} \sum_{v \in X_i} g(v) + \sum_{\substack{v \in V(P) \\ v \neq x_1, x_2}} g(v) = \sum_{v \in V(P(X_1, X_2))} g(v) \leq 1.$$

This proves that g' is a covering function for G'. Since G' is β-narrow, it follows that

$$\sum_{v \in V(G)} \left[g(v)\right]^\beta \leq \left[\sum_{v \in X_1} g(v)\right]^\beta + \left[\sum_{v \in X_2} g(v)\right]^\beta + \sum_{v \in V(G) \setminus (X_1 \cup X_2)} \left[g(v)\right]^\beta = \sum_{v \in V(G')} \left[g'(v)\right]^\beta \leq 1,$$

where we have used the fact that for $x, y \geq 0$ and $\beta \geq 1$, $x^\beta + y^\beta \leq (x+y)^\beta$. This proves that G is β-narrow, thereby proving Lemma 2.4. $\square$

3 Graphs in Forb(P_4^c, P_5, C_6)

We start by additionally excluding the cycle of length six, C_6. Throughout the paper, we call an induced subgraph of a graph G that is a cycle of length k a *k-gon* in G. We will often denote the vertices of a k-gon H by, for example, $h_1, h_2, \ldots, h_k$ in order. Any arithmetic involving the subscripts of these vertices is modulo k. For a k-gon H, we say that $v \in V(G) \setminus V(H)$ is a *center* for H, if v is complete to $V(H)$. Analogously, v is an *anticenter* for H if v is anticomplete to $V(H)$.

We say that a graph $G \in \text{Forb}(P_4^c, P_5, C_6)$ is *composite* if there exist a 5-gon B in G and $a, c \in V(G) \setminus V(B)$ such that a is an anticenter for B and c is a center for B. We say that a graph $G \in \text{Forb}(P_4^c, P_5, C_6)$ is *basic* if G is not composite.

This section is organized as follows. We will first prove some basic properties of graphs in $\text{Forb}(P_4^c, P_5, C_6)$. Next, we will show that composite graphs admit a quasi-homogeneous set decomposition. Finally, we will show that basic graphs satisfy the assumptions of Lemma 2.1 with $\beta = 1$. This will imply that all graphs in $\text{Forb}(P_4^c, P_5, C_6)$ are 2-narrow.

3.1 Elementary properties

We will repeatedly use the following lemmas:

(3.1) *Let $G \in \text{Forb}(P_4^c)$ and let f_1-f_2-f_3-f_4 be an induced path. Then no vertex is complete to $\{f_1, f_2, f_4\}$ and nonadjacent to f_3.*

Proof. Suppose for a contradiction that x is adjacent to f_1, f_2, and f_4 and not to f_3. Then x-f_3-f_1-f_4-f_2 is a four-edge antipath, a contradiction. This proves (3.1). $\square$

For a 5-gon H in a graph G, we call a vertex $x \in V(G) \setminus V(H)$ that has a neighbor in $V(H)$ an *attachment of H*. The following lemma deals with attachments of 5-gons.

(3.2) *Let $G \in \text{Forb}(P_4^c, P_5)$ and let H be a 5-gon with vertices $h_1, h_2, \dots, h_5$ in order. Let $x \in V(G) \setminus V(H)$ be an attachment of H. Then, for some for some $i \in \{1, 2, \dots, 5\}$, one of the following holds:*

(1) *x is complete to $V(H)$ ("center"), or*

(2) *x is adjacent to h_i and x has no other neighbor in $V(H)$ ("leaf of type i"), or*

(3) *x is adjacent to h_{i+2}, h_{i+3} and x has no other neighbor in $V(H)$ ("hat of type i"), or*

(4) *x is adjacent to h_{i+4}, h_{i+1}, nonadjacent to h_{i+2}, h_{i+3} and the adjacency between x and h_i is arbitrary ("clone of type i").*

Proof. If x is complete to $V(H)$, then outcome (1) holds. From this and from the symmetry, we may assume that x is adjacent to h_1 and not to h_2. First, suppose that x is adjacent to h_3. From (3.1) applied first to x and h_1-h_2-h_3-h_4 and then to x and h_5-h_1-h_2-h_3, it follows that x is nonadjacent to h_4 and h_5 and thus outcome (4) holds. So we may assume that x is nonadjacent to h_3. If x is adjacent to h_4, then outcome (4) holds. So we may assume that x is nonadjacent to h_4. If x is nonadjacent to h_5, then outcome (2) holds. If x is adjacent to h_5, then outcome (3) holds. This proves (3.2). □

We call an attachment x of H a *small attachment* if x is a leaf or a hat for H. Let $i \in \{1, 2, \dots, 5\}$. We call a pair of vertices (a, b) a *pyramid of type i* for H if a and b are adjacent, a is a leaf of type i for H, and b is a hat of type i for H. We say that $\{a, b\}$ is a pyramid if (a, b) or (b, a) is a pyramid. It turns out that whenever two small attachments are adjacent, they are of the same type. The following lemma deals with combinations of small attachments:

(3.3) *Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ and let H be a 5-gon. Suppose that u and v are small attachments of H. Then the following two statements hold:*

(a) *If u and v are adjacent, then, up to interchanging u and v, for some $i \in \{1, 2, \dots, 5\}$, either*

  (A1) *u and v are leaves for H of type i; or*

  (A2) *u and v are hats for H of type i; or*

  (A3) *u is a leaf for H of type i, v is a hat for H of type i, and thus (u, v) is a pyramid of type i for H.*

(b) *If u and v are nonadjacent, then, up to interchanging u and v, for some $i \in \{1, 2, \dots, 5\}$, either*

  (B1) *u is a leaf of type i and v is a leaf of type $j \in \{i-1, i, i+1\}$; or*

  (B2) *u is a hat of type i and v is a hat of type $j \in \{i-2, i, i+2\}$; or*

  (B3) *u is a leaf of type i and v is a hat of type $j \in \{i-2, i, i+2\}$.*

Proof. Let $h_1, h_2, \dots, h_5$ be the vertices of H in order. Since u and v are small attachments, each of u, v is either a leaf or a hat for H.

For part (a), suppose that u and v are adjacent. First assume that u is a leaf. From the symmetry, we may assume that u is a leaf of type 1 and v is either a leaf of type 1, 2 or 3, or a hat of type 1, 4 or 5. If v is a leaf of type 1, then outcome (A1) holds. If v is a hat of type 1, then outcome (A3) holds. If v is a leaf of type 2 or a hat of type 4, then u-v-h_2-h_3-h_4-h_5 is an induced five-edge path, a contradiction. If v is a leaf of type 3 or a hat of type 5, then u-v-h_3-h_4-h_5-h_1-u is an induced cycle of length six, a contradiction. This finishes the case when u is a leaf. So we may now assume that both u and v are hats. From the symmetry, we may assume that u is a hat of type 1 and v is a hat of type 1, 2 or 3. If v is a hat of type 1, then outcome (A2) holds. If v is a hat of type 2, then u-v-h_5-h_1-h_2-h_3-u is an induced cycle of length six, a contradiction. If v is a hat of type 3, then the adjacencies of v with respect to the path u-h_4-h_5-h_1 contradict (3.1). This proves part (a).

For part (b), suppose that u and v are nonadjacent. First assume that u is a leaf. From the symmetry, we may assume that u is of type 1 and v is either a leaf of type 1, 2 or 3, or a hat of type 1, 4, 5. If v is a leaf of type 1 or 2, then (B1) holds. If v is a leaf of type 3 or a hat of type 5, then u-h_1-h_5-h_4-h_3-v is an induced five-edge path, a contradiction. If v is a hat of type 1 or 4, then outcome (B3) holds. This finishes the case when u is a leaf. We may therefore assume that u and v are both hats for H. From the symmetry, we may assume that u is a hat of type 1 and v is a hat of type 1, 2 or 3. If v is a hat of type 1 or 3, then (B2) holds. If v is a hat of type 2, then u-h_3-h_2-h_1-h_5-v is an induced five-edge path, a contradiction. This proves part (b), thereby completing the proof of (3.3). □

(3.4) *Let $G \in \mathrm{Forb}(P_4^c, P_5)$. Let H be a 5-gon in G and suppose that x is a small attachment of H. Then, every neighbor $y \in V(G) \setminus V(H)$ of x is an attachment of H.*

Proof. Suppose that $y \in V(G) \setminus V(H)$ is adjacent to x but y has no neighbor in $V(H)$. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. We may assume that x is adjacent to h_1 and anticomplete to $\{h_2, h_3, h_4\}$. Now y-x-h_1-h_2-h_3-h_4 is an induced five-edge path, a contradiction. This proves (3.4). □

(3.5) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and let H be a 5-gon. Let (a, b) and (a', b') be two disjoint pyramids for H. Then (a, b) and (a', b') are pyramids of the same type.*

Proof. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. From the symmetry, we may assume that (a, b) is a pyramid of type 1 for H and (a', b') is a pyramid of type 1, 2 or 3 for H. If (a', b') is of type 1, then the claim holds. If (a', b') is a pyramid of type 2 for H, then b is a hat of type 1 for H and b' is a hat of type 2 for H, contrary to (A2) and (B2) of (3.3). If (a', b') is a pyramid of type 3 for H, then a is a leaf of type 1 and a' is a leaf of type 3 for H, contrary to (A1) and (B1) of (3.3). This proves (3.5). □

3.2 Composite graphs

Let $G \in \text{Forb}(P_4^c, P_5, C_6)$ be a graph. Our goal is to produce a quasi-homogeneous set. In order to do so, we need to understand how different 5-gons interact with each other. To this end, we consider the following auxiliary graph. Let B be a 5-gon in G and let $\mathcal{W}$ be a graph with the following properties:

(a) The vertices of $\mathcal{W}$ are 5-gons in G, and B is a vertex of $\mathcal{W}$.

(b) Two 5-gons H and H' are adjacent if and only if one of the following holds:

  (b1) $|V(H) \cap V(H')| = 4$ and $x \in V(H) \setminus V(H')$ is a clone for H'. In this case, we say that H and H' are *clone neighbors* and we call the edge HH' a *clone edge*.

  (b2) $B \in \{H, H'\}$, $|V(H) \cap V(H')| = 3$ and $\{x, y\} = V(H) \setminus V(H')$ is a pyramid for H'. In this case, we say that H and H' are *pyramid neighbors* and we call the edge HH' a *pyramid edge*.

(c) $\mathcal{W}$ is connected.

We call such a graph $\mathcal{W}$ a C_5*-structure around B in G*. Note that we do not require that *all* 5-gons in G are vertices of $\mathcal{W}$. Also note that the adjacency of two 5-gons is well-defined because property (b) is symmetric. Further, observe that pyramid edges occur only between B and other 5-gons, and not between two 5-gons different from B. We say that a C_5-structure $\mathcal{W}$ is *maximal* if $|V(\mathcal{W})|$ is maximal. Let $U(\mathcal{W}) = \bigcup_{H \in V(\mathcal{W})} V(H)$ denote the set of vertices of G that are 'covered' by $\mathcal{W}$. We say that a vertex $x \in V(G) \setminus U(\mathcal{W})$ is a *center* for $\mathcal{W}$ if x is complete to $U(\mathcal{W})$.

Let $H \in V(\mathcal{W})$ and let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. Let $i \in \{1, 2, \ldots, 5\}$ and let x be a clone of type i for H. We will write $H/x = G|((V(H) \setminus \{h_i\}) \cup \{x\})$ and we will say that H/x is *obtained from H by cloning h_i* and x is a clone *in the position of h_i*. For two 5-gons $F, H \in V(\mathcal{W})$, let $\text{dist}(F, H)$ be the number of edges in a shortest path from F to H in $\mathcal{W}$.

Let us first prove a number of claims about C_5-structures:

(3.6) *Let $G \in \text{Forb}(P_4^c, P_5)$ and let B be a 5-gon in G. Let $\mathcal{W}$ be a C_5-structure around B. Suppose that $H \in V(\mathcal{W})$ and $H' \in V(\mathcal{W})$ are clone neighbors. If c is a center for H, then c is also a center for H'.*

Proof. Let c be a center for H. From the definition of a clone edge, it follows that $|V(H) \cap V(H')| = 4$. Since c is complete to $V(H)$, it follows that c has at least four neighbors in $V(H')$. Therefore, it follows from (3.2) that c is complete to $V(H')$. This proves (3.6). □

(3.7) *Let $G \in \text{Forb}(P_4^c, P_5)$ and let B be a 5-gon in G. Let $\mathcal{W}$ be a maximal C_5-structure around B. Let c be a center for some 5-gon in $V(\mathcal{W})$. Then either c is a center for every $H \in V(\mathcal{W})$ or $c \in U(\mathcal{W})$.*

Proof. If c is complete to all $H \in V(\mathcal{W})$, then the claim holds. So we may assume that c is not complete to at least one 5-gon in $V(\mathcal{W})$. Let $H_1, H_2 \in V(\mathcal{W})$ be such that c is complete to H_1 but not to H_2 and, subject to that, such that $\mathrm{dist}(H_1, H_2)$ is minimum. Clearly, since c is complete to $V(H_1)$ and not to $V(H_2)$, it follows that $H_1 \neq H_2$. Since $\mathrm{dist}(H_1, H_2)$ is minimum, it follows that H_1 and H_2 are neighbors. It follows from (3.6) that H_1 and H_2 are pyramid neighbors. We may write $H_1 = h_1\text{-}h_2\text{-}h_3\text{-}h_4\text{-}h_5\text{-}h_1$ and $H_2 = h_1\text{-}a\text{-}b\text{-}h_4\text{-}h_5\text{-}h_1$. Since c is complete to $V(H_1)$, it follows that c has at least three neighbors in $V(H_2)$. Hence, since c is not complete to $V(H_2)$, it follows from (3.2) that c is a clone for H_2. Therefore, H_2/c is a 5-gon. From the maximality of $\mathcal{W}$, it follows that $H_2/c \in V(\mathcal{W})$ and, thus, that $c \in U(\mathcal{W})$. This proves (3.7). □

(3.8) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and let B be a 5-gon in G. Let $\mathcal{W}$ be a maximal C_5-structure around B. Suppose that $H \in V(\mathcal{W})$ and $H' \in V(\mathcal{W})$ are clone neighbors and let x be such that $H' = H/x$. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. Let $i \in \{1, 2, \ldots, 5\}$ and suppose that (p, q) is a pyramid of type i for H. Then either*

(1) *(p, q) is also a pyramid of type i for H', or*

(2) *x is a clone of type $j \in \{i-1, i+1\}$ for H and x is complete to $\{p, q, h_j\}$.*

Proof. From the symmetry, we may assume that (p, q) is a pyramid of type 1 for H and x is a clone of type 1, 2 or 3 for H. First assume that x is a clone of type 1 for H. It follows from (3.2) applied to q and H' that x is not adjacent to q. Therefore, q is a hat for H'. Since p is a neighbor of q, it follows from (3.4) that p has a neighbor in $V(H')$. It follows that p is adjacent to x. Thus, (p, q) is a pyramid for H' and outcome (1) holds. Next, assume that x is a clone of type 2 for H. Then it follows from (3.2) applied to x and $h_1\text{-}h_2\text{-}h_3\text{-}q\text{-}p\text{-}h_1$ that x is either complete or anticomplete to $\{p, q\}$. If x is anticomplete to $\{p, q\}$, then (p, q) is a pyramid for H' and thus outcome (1) holds. If x is complete to $\{p, q\}$, then it follows from (3.3) that x is adjacent to h_2. Hence, outcome (2) holds. So we may assume that x is a clone of type 3 for H. First suppose that p is adjacent to x. From (3.2) applied to x and the 5-gon $h_1\text{-}h_2\text{-}h_3\text{-}q\text{-}p\text{-}h_1$, it follows that x is anticomplete to $\{q, h_3\}$. But now the adjacencies of q with respect to $h_3\text{-}h_4\text{-}x\text{-}p$ contradict (3.1). This proves that p is nonadjacent to x. But now, since p is a leaf of type 1 for H', q is a small attachment of H', and p and q are adjacent, it follows from (3.3) that q is a hat of type 1 for H' and (p, q) is a pyramid for H'. Hence, outcome (1) holds. This proves (3.8). □

(3.9) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and let B be a 5-gon. Let $\mathcal{W}$ be a maximal C_5-structure around B. If (a, b) is a pyramid for some $H \in V(\mathcal{W})$, then $\{a, b\} \subset U(\mathcal{W})$.*

Proof. Let $H^* \in V(\mathcal{W})$ be a 5-gon in G for which (a, b) is a pyramid and, subject to that, such that $\mathrm{dist}(H^*, B)$ is minimum. Let $h_1, h_2, \ldots, h_5$ be the vertices of H^* in order. From the symmetry, we may assume that (a, b) is a pyramid of type 1 for H^*.

Let P be a shortest path from H^* to B in $\mathcal{W}$. It follows from the definition of a maximal C_5-structure that, if $H^* = B$, then $\{a, b\} \subset U(\mathcal{W})$. So we may assume that $H^* \neq B$ and hence that $|E(P)| \geq 1$. Let H^1 be the neighbor of H^* in P. Since H^* was chosen with $\mathrm{dist}(H^*, B)$ minimum, it follows that $\{a, b\}$ is not a pyramid for H^1.

First suppose that H^1 is a clone neighbor of H^*. Let x be such that $H^1 = H^*/x$. From (3.8) and the fact that $\{a,b\}$ is not a pyramid for H^1, it follows that H^1 is obtained from H^* by cloning h_2 or h_5 and x is complete to $\{a,b\}$. But now, from the maximality of $\mathcal{W}$, H^1-H^1/b-$H^1/b/a$ is a path in $\mathcal{W}$ and hence $\{a,b\} \subset U(\mathcal{W})$.

Therefore, we may assume that H^1 is a pyramid neighbor of H^*. From the definition of a C_5-structure and the fact that $H^* \neq B$, it follows that $H^1 = B$. Let $\{p,q\} = V(B) \setminus V(H^*)$. We claim that either (p,q) or (q,p) is a pyramid of type 1 for H^*. If $\{p,q\} \cap \{a,b\} = \emptyset$, then, since (a,b) is a pyramid of type 1 for H^*, it follows from (3.5) that (p,q) or (q,p) is a pyramid of type 1 for H^*. If $\{p,q\} \cap \{a,b\} \neq \emptyset$, then it follows from the definition of a pyramid that (p,q) or (q,p) is a pyramid of type 1 for H^*. Hence, we may assume that $V(H^1) = V(B) = \{h_1, p, q, h_4, h_5\}$. This proves that (p,q) or (q,p) is a pyramid of type 1 for H^*. Possibly by swapping p and q, we may assume that (p,q) is a pyramid of type 1 for H^*.

If $(a,b) = (p,q)$, then $\{a,b\} \subset V(B)$ and hence $\{a,b\} \subset U(\mathcal{W})$. If $a \neq p$ and $b = q$, then a is a clone for B and $b \in V(B)$ and, therefore, $\{a,b\} \subset U(\mathcal{W})$. If $a = p$ and $b \neq q$, then b is a clone for B and $a \in V(B)$ and, therefore, $\{a,b\} \subset U(\mathcal{W})$.

So we may assume that $\{a,b\} \cap \{p,q\} = \emptyset$. Now first suppose that a is adjacent to q. Then a is a clone for B and b is a clone for B/a. Hence, by the maximality of $\mathcal{W}$, it follows that $B/a, B/a/b \in V(\mathcal{W})$ and, therefore, that $\{a,b\} \subset U(\mathcal{W})$. Next, suppose that b is adjacent to p. Then b is a clone for B and a is a clone for B/b. Hence, by the maximality of $\mathcal{W}$, it follows that $B/b, B/b/a \in V(\mathcal{W})$ and, therefore, that $\{a,b\} \subset U(\mathcal{W})$.

It follows that we may assume that the only possible edges between $\{a,b\}$ and $\{p,q\}$ are ap and bq. If ap and bq are both edges, then a and b are hats of different types for B, contrary to (3.3). If neither of ap and bq is an edge, then a and b are leaves of different types for B, contrary to (3.3). Thus, exactly one of ap and bq is an edge and hence $\{a,b\}$ is a pyramid for B. If a is adjacent to p, then (b,a) is a pyramid of type 4 for B, contrary to (3.5). If b is adjacent to q, then (a,b) is a pyramid of type 1 for B. By the maximality of $\mathcal{W}$, it follows that $\{a,b\} \subset U(\mathcal{W})$. This proves (3.9). $\square$

The goal in this section is to prove the following:

(3.10) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ be a composite graph. Then G admits a quasi-homogeneous set decomposition.*

As a first step in this direction, we prove the following lemma which states that $U(\mathcal{W})$ does not contain both all centers and all anticenters of B. This is useful, because in order for $U(\mathcal{W})$ to be a quasi-homogeneous set, we should have $|U(\mathcal{W})| < |V(G)|$.

(3.11) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and let B be a 5-gon in G with both a center and an anticenter. Let $\mathcal{W}$ be a maximal C_5-structure around B. Then $V(G) \setminus U(\mathcal{W}) \neq \emptyset$.*

Proof. We may assume that all centers and all anticenters for B are contained in $U(\mathcal{W})$, because otherwise the lemma holds.

(i) *B and every pyramid neighbor of B in $\mathcal{W}$ has a pyramid.*

We first claim that B has a pyramid. For suppose not. Then, all edges in $\mathcal{W}$ are clone edges. Let x be a center for B. Then it follows from (3.6) that x is a center for all $H \in V(\mathcal{W})$. In particular, for every $H \in V(\mathcal{W})$, $x \notin V(H)$. Therefore, $x \notin U(\mathcal{W})$, contrary to our assumption. Now let B' be any pyramid neighbor of B. Clearly, $\{p, q\} = V(B) \setminus V(B')$ is a pyramid for B'. This proves **(i)**. □

Now let a be an anticenter for B. We first show that:

(ii) *a is anticomplete to every pyramid (p, q) for B and a is an anticenter for every pyramid neighbor of B in $\mathcal{W}$.*

Let (p, q) be a pyramid for B. Suppose that $z \in \{p, q\}$ is adjacent to a. Since z is a small attachment of B, it follows from (3.4) that a has a neighbor in $V(B)$, contrary to the assumption that a is an anticenter for B. Since every pyramid neighbor H of B satisfies $V(H) \subseteq (V(B) \cup \{p', q'\})$ for some pyramid $\{p', q'\}$ for B, it follows from the above that a is an anticenter for every pyramid neighbor of B. This proves **(ii)**. □

Since $a \in U(\mathcal{W})$ there exists a 5-gon $H^* \in V(\mathcal{W})$ such that $a \in V(H^*)$ and, subject to that, such that $\operatorname{dist}(B, H^*)$ is minimum. Let P be a shortest path from H^* to B in $\mathcal{W}$ and write $P = H^*\text{-}H^1\text{-}H^2\text{-}\cdots\text{-}H^k$, where $H^k = B$ and $k = \operatorname{dist}(B, H^*)$. From the definition of a C_5-structure, it follows that all edges in P are clone edges, except possibly $H^{k-1}\text{-}H^k$.

(iii) *$H^* = H^1/a$, $k \geq 2$, and H^1 is not a pyramid-neighbor of B in $\mathcal{W}$.*

First suppose that $H^1 = B$. If H^* and B are pyramid neighbors, then it follows from **(ii)** that a is anticomplete to H^*, a contradiction. If H^* and B are clone neighbors, then, since $|V(B) \cap V(H^*)| = 4$ and a has two neighbors in $V(H^*)$, it follows that a has at least one neighbor in B, contradicting the fact that a is an anticenter for B. This proves that $H^1 \neq B$ and, thus, that $k \geq 2$. It follows from the definition of $\mathcal{W}$ that $H^*\text{-}H^1$ is a clone edge. Since $a \in V(H^*)$ and $a \notin V(H^1)$ by the minimality of k, it follows that $H^* = H^1/a$. Since a has a neighbor in $V(H^1)$, it follows from **(ii)** that H^1 is not a pyramid neighbor of B. This proves **(iii)**. □

(iv) *a is not a clone for H^i for $i \geq 2$.*

Suppose that a is a clone for H^i. Then $H^i/a\text{-}H^i\text{-}H^{i+1}\text{-}\cdots\text{-}H^k$ is a path between B and a 5-gon containing a that is shorter than P, contrary to the choice of H^*. This proves **(iv)**. □

Let $h_1, h_2, \ldots, h_5$ be the vertices of H^1 in order. From the symmetry and from **(iii)**, we may assume that a is adjacent to h_2 and h_5, and possibly to h_1. It follows from **(iii)** that we may now consider H^2. We claim that H^1 and H^2 are clone neighbors in $\mathcal{W}$. For suppose to the contrary that H^1 and H^2 are pyramid neighbors. It follows from the definitions of $\mathcal{W}$ and P that $H^2 = B$. But now, H^1 is a pyramid neighbor of B, contrary to **(iii)**. Thus, H^1 and H^2 are clone neighbors.

(v) *Up to symmetry, H^2 is obtained from H^1 by cloning h_2. Let h_2' be such that $H^2 = H^1/h_2'$. Then h_2' is nonadjacent to a, and either (see Figure 1)*

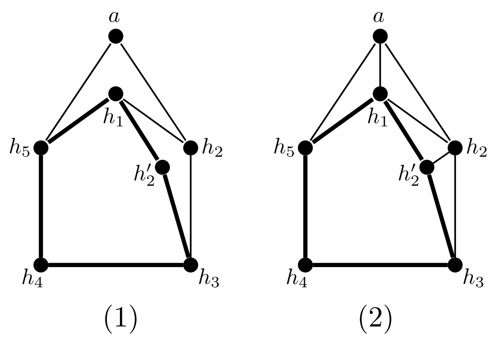


Figure 1: The outcomes of **(v)**.

(1) *ah_1 and h_2h_2' are both non-edges, or*

(2) *ah_1 and h_2h_2' are both edges.*

Moreover, $k \geq 3$ and H^2 is not a pyramid neighbor of B in $\mathcal{W}$.

We proved that H^1 and H^2 are clone neighbors. From the symmetry, we may assume that H^2 is obtained from H^1 by cloning h_1, h_2, or h_3. It follows from **(iv)** that H^2 is not obtained from H^1 by cloning h_1. Suppose next that H^2 is obtained from H^1 by cloning h_3. Let h_3' be such that $H^2 = H^1/h_3'$. It follows from (3.2) that a is a clone for H^2, contradicting **(iv)**. Therefore, we may assume that H^2 is obtained from H^1 by cloning h_2. Let h_2' be such that $H^2 = H^1/h_2'$. Because, from **(iv)**, a is not a clone for H^2, it follows that h_2' is nonadjacent to a. If h_2' is adjacent to h_2 and h_1 is nonadjacent to a, then h_2-h_5-h_2'-a-h_1 is an induced four-edge antipath, a contradiction. Likewise, if h_2' is nonadjacent to h_2 and h_1 is adjacent to a, then h_1-h_3-a-h_2'-h_2 is a four-edge antipath, a contradiction. This proves that ah_1 and h_2h_2' are either both edges or both non-edges.

Since a has a neighbor in H^2, it follows that $H^2 \neq B$ and hence that $k \geq 3$. Using **(ii)**, it follows that H^2 is not a pyramid neighbor of B. This proves **(v)**. □

Let H^2 and h_2' be as in **(v)**. It follows from **(v)** that we may now consider H^3. We claim that H^2 and H^3 are clone neighbors in $\mathcal{W}$. For suppose to the contrary that H^2 and H^3 are pyramid neighbors. It follows from the definitions of $\mathcal{W}$ and P that $H^3 = B$. But now, H^2 is a pyramid neighbor of B, contrary to **(v)**. Thus, H^2 and H^3 are clone neighbors.

(vi) *Up to symmetry, H^3 is either (see Figure 2)*

(1) *obtained from H^2 by cloning h_5, $h_5' \in V(H^3) \setminus V(H^2)$ is anticomplete to $\{a, h_2, h_5\}$, and $ah_1, h_2'h_2$ are non-edges; or*

(2) *obtained from H^2 by cloning h_5, $h_5' \in V(H^3) \setminus V(H^2)$ is adjacent to h_5 and anticomplete to $\{a, h_2\}$, and $ah_1, h_2'h_2$ are edges, or*

(3) *obtained from H^2 by cloning h_1, $h_1' \in V(H^3) \setminus V(H^2)$ is adjacent to h_1 and anticomplete to $\{a, h_2\}$, and $ah_1, h_2'h_2$ are edges.*

Moreover, $k \geq 4$ and H^3 is not a pyramid neighbor of B.

Since H^1 and H^2 are clone neighbors by **(v)**, we may assume that H^2 is obtained from H^1 by cloning h_2. It follows from **(v)** that h_2' is nonadjacent to a. H^3 is not obtained from H^2 by cloning h_2', because if it is, then H^3 is adjacent to H^1, contrary to the minimality of P.

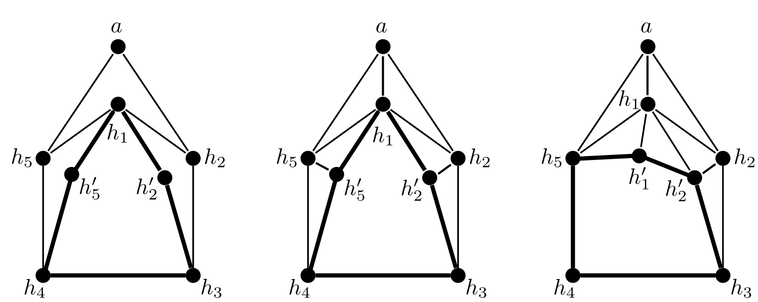
Also note that H^3 has no neighbor $H' \in V(\mathcal{W})$ such that a is a clone for H'. Because if so, then H'/a-H'-H^3-H^4-$\cdots$-H^k is a path between B and a 5-gon containing a that is shorter than P, a contradiction.

There are four cases to consider:

(a) H^3 is obtained from H^2 by cloning h_1. (see Figure 3.a.) Let h_1' be such that $H^3 = H^2/h_1'$. If h_1' is adjacent to h_2, then H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and a is a clone for H^3/h_2, a contradiction. Therefore, h_1' is nonadjacent to h_2. First suppose that H^2 satisfies outcome (1) of **(v)**. Since h_1-h_3-h_1'-h_2-h_2' is not an induced four-edge antipath, it follows that h_1' is nonadjacent to h_1. If h_1' is nonadjacent to a, then a and h_2 are adjacent leaves of different types for H^3, contrary to (3.3). Therefore, h_1' is adjacent to a. But now h_1'-h_1-a-h_2'-h_5 is an induced four-edge antipath, a contradiction. Next suppose that H^2 satisfies outcome (2) of **(v)**. From the fact that a-h_2'-h_5-h_2-h_1' is not an induced four-edge antipath, it follows that a is nonadjacent to h_1'. It follows, from the fact that h_2'-h_5-h_2-h_1'-h_1 is not a four-edge antipath, that h_1' is adjacent to h_1. Hence, outcome (3) holds.

(b) H^3 is obtained from H^2 by cloning h_3. (see Figure 3.b.) Let h_3' be such that $H^3 = H^2/h_3'$. Suppose that h_3' is adjacent to a. Then, it follows from (3.2) applied to a and H^2 that a is a clone for H^3, contrary to **(iv)**. Hence, h_3' is nonadjacent to a. It follows that a is either a leaf of type 5 or a hat of type 3 for H^3. If h_3' is adjacent to h_2, then H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and a is a clone for H^3/h_3, a contradiction. Therefore, h_3' is nonadjacent to h_2. But now a and h_2 are adjacent small attachments of H^3 and they have different types, contrary to (3.3).

(c) H^3 is obtained from H^2 by cloning h_4. (see Figure 3.c.) Let h_4' be such that $H^3 = H^2/h_4'$. From (3.2) applied to h_2 and H^3, it follows that h_4' is nonadjacent to h_2 and, in particular, that h_2 is a clone for H^3. But now H^3 is adjacent to H^3/h_2 in $\mathcal{W}$ and



(1) (2) (3)

Figure 2: The outcomes of **(vi)**.

a is a clone for H^3/h_2, a contradiction.

(d) H^3 is obtained from H^2 by cloning h_5. (see Figure 3.d.) Let h'_5 be such that $H^3 = H^2/h'_5$. From (3.2) applied to h_2 and H^3, it follows that h'_5 is nonadjacent to h_2 and, in particular, that h_2 is a clone for H^3. Since a is not a clone for H^3/h_2, it follows that a is nonadjacent to h'_5. If H^2 satisfies outcome (1) of **(v)**, then because h_1-a-h'_5-h_2-h_5 is not an induced four-edge antipath, it follows that h_5 is nonadjacent to h'_5 and hence outcome (1) holds. If H^2 satisfies outcome (2) of **(v)**, then since h_5-h'_5-a-h_4-h_1 is not an induced four-edge antipath, it follows that h_5 is adjacent to h'_5, and hence outcome (2) holds.

Now suppose that $H^3 = B$ or H^3 is a pyramid neighbor of B. Since a is an anticenter for B and for every pyramid neighbor of B, it follows that H^3 satisfies outcome (1). It follows from **(i)** and **(ii)** that H^3 has a pyramid (p, q) that is anticomplete to a. From the symmetry, we may assume that (p, q) is a pyramid of type 1, 2, or 3 for H^3. First suppose that (p, q) is a pyramid of type 1 for H^3. It follows from (3.8) that $\{p, q\}$ is anticomplete to $\{h_2, h_5\}$. But now h_2 is a leaf for the 5-gon $F = h_1$-p-q-h_4-h'_5-h_1, a is adjacent to h_2 and a has no neighbor in F, contrary to (3.4). Next suppose that (p, q) is a pyramid of type 2 for H^3. Then it follows from (3.8) that p is nonadjacent to h_5. Hence, a is a leaf of type 5 and p is a leaf of type 2 for H^3/h_5, contrary to (3.3). So we may assume that (p, q) is a pyramid of type 3 for H^3. It follows from (3.8) that p is nonadjacent to h_5. Hence, a is a leaf of type 5 and p is a leaf of type 3 for H^3/h_5, contrary to (3.3). This proves that H^3 is not B or a pyramid neighbor of B and therefore that $k \geq 4$. This proves **(vi)**. □

Let H^3 be as in **(vi)**. It follows from **(vi)** that we may now consider H^4, which is a clone neighbor of H^3. Now, again, since P is a shortest path from a 5-gon that contains a to B, it follows that there is no one- or two-edge path in $\mathcal{W}$ from H^4 to a 5-gon for which a is clone.

First, suppose that H^3 satisfies outcome (1) or (2) of **(vi)**. Let h'_5 be as in outcome (1) and (2) of **(vi)**. From the symmetry, we may assume that H^4 is obtained from H^3 by cloning h_1, h'_2, or h_3. We need to check a number of cases:

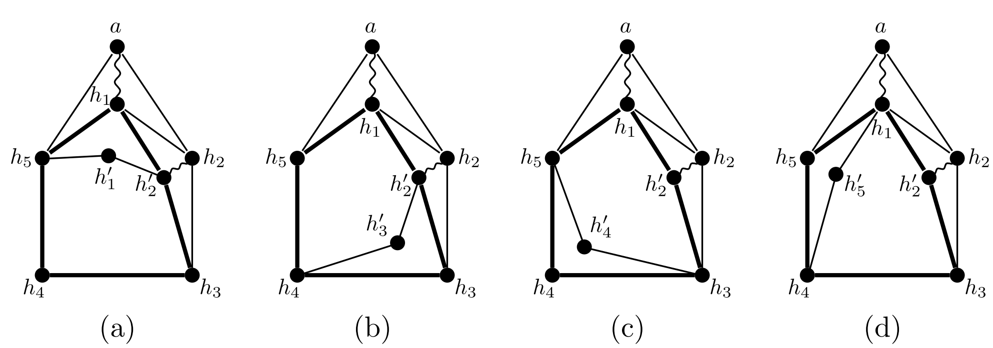


Figure 3: Potential neighbors of H^2 if H^2 satisfies (1) of **(vi)**.
The "wiggly" edges represent arbitrary adjacencies.

(a) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_1}$. Let h_1' be such that $H^4 = H^3/h_1'$. First suppose that h_1' is nonadjacent to h_2. It follows that h_2 is a leaf of type 3 or a hat of type 5 for H^4. Since a is adjacent to h_2, it follows from (3.4) that a is adjacent to h_1'. But now a is a leaf of type 1 for H^4 and a is adjacent to h_2, contrary to (3.3). Therefore, h_1' is adjacent to h_2 and, from the symmetry, h_1' is adjacent to h_5. But now the path H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(b) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_2'}$. Now H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(c) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_3}$. Let h_3' be such that $H^4 = H^3/h_3'$. From (3.2) applied to h_5 and H^4, it follows that h_3' is nonadjacent to h_5 and, in particular, that h_5 is a clone for H^4. Since, by the minimality of P, a is a not a clone for H^4/h_5, it follows from (3.2) that a is nonadjacent to h_3'. If h_3' is adjacent to h_2, then H^4-H^4/h_2-$H^4/h_2/h_5$ is a two-edge path from H^4 to a 5-gon for which a is a clone, a contradiction. Hence, h_3' is nonadjacent to h_2 and therefore h_2 is a small attachment of H^4. Since a is adjacent to h_2, it follows from (3.4) that a is adjacent to h_1 and hence that outcome (2) of **(vi)** holds. But now a is a leaf of type 1 for H^4, h_2 is a hat of type 4 for H^4, and a and h_2 are adjacent, contrary to (3.3).

This proves that H^3 does not satisfy outcome (1) or outcome (2) of **(vi)**. So next suppose that H^3 satisfies outcome (3) of **(vi)**. We need to check a number of cases:

(a) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_1'}$. H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(b) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_2'}$. Let h_2'' be such that $H^4 = H^3/h_2''$. Since a is not a clone for H^4, it follows that h_2'' is nonadjacent to a. If h_2'' is adjacent to h_1, then H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction. Therefore h_2'' is nonadjacent to h_1. But now h_1 is a hat of type 3 and a is a leaf of type 5 for H^4, and h_1 and a are adjacent, contrary to (3.3).

(c) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_3}$. Let h_3' be such that $H^4 = H^3/h_3'$. Since a is not a clone for H^4 by **(iv)**, it follows that a is nonadjacent to h_3'. From (3.2) applied to h_1 and H^4, it follows that h_3' is nonadjacent to h_1. If h_3' is nonadjacent to h_2, then h_2 and a are leaves of type 2 and 5, respectively, for H^4, and a and h_2 are adjacent, contrary to (3.3). Therefore, h_3' is adjacent to h_2. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(d) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_4}$. Let h_4' be such that $H^4 = H^3/h_4'$. By (3.2) applied to h_1 and H^4, it follows that h_4' is nonadjacent to h_1. By (3.2) applied to h_2 and H^4/h_1, it follows that h_4' is nonadjacent to h_2. By (3.2) applied to a and $H^4/h_1/h_2$, it follows that h_4' is nonadjacent to a. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

(e) $\underline{H^4 \text{ is obtained from } H^3 \text{ by cloning } h_5}$. Let h_5' be such that $H^4 = H^3/h_5'$. From (3.2) applied to h_2 and H^4, it follows that h_5' is nonadjacent to h_2. If h_5' is nonadjacent to h_1, then h_1 and h_2 are hats of type 4 and 5, respectively, and h_1 and h_2 are adjacent, contrary to (3.3).

Therefore, h'_5 is adjacent to h_1. Since h_2 is a hat for H^4 and a is adjacent to h_2, it follows from (3.4) that a is adjacent to h'_5. But now H^4-H^4/h_1-$H^4/h_1/h_2$ is a two-edge path from H^4 to a 5-gon for which a is clone, a contradiction.

This proves that H^3 does not satisfy any of the outcomes of **(vi)**, a contradiction. This completes the proof of (3.11). $\square$

Next, we are interested in how vertices in $V(G) \backslash U(\mathcal{W})$ can attach to $U(\mathcal{W})$ where $\mathcal{W}$ is a maximal C_5-structure.

(3.12) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and let B be a 5-gon. Let $\mathcal{W}$ be a maximal C_5-structure around B. Let $x \in V(G) \setminus U(\mathcal{W})$ and assume that x is not a center for $\mathcal{W}$. Let u and v be two nonadjacent neighbors of x and assume that $u \in U(\mathcal{W})$. Then, for every $H \in V(\mathcal{W})$ such that $u \in V(H)$, v is a clone for H in the same position as u and, in particular, $v \in U(\mathcal{W})$.*

Proof. Let $H \in V(\mathcal{W})$ such that $u \in V(H)$ and let h_1, h_2, h_3, h_4, h_5 be the vertices of H in order. From the symmetry, we may assume that $h_1 = u$. It follows from (3.7) and the assumption that x is not a center for $\mathcal{W}$ that x is not complete to $V(H)$. Moreover, since $\mathcal{W}$ is maximal and $x \notin U(\mathcal{W})$, it follows that x is not a clone for H. Therefore x is either a leaf or a hat for H. Because v is nonadjacent to u and adjacent to x, it follows that $v \notin V(H)$. From the symmetry, we may assume that x is anticomplete to $\{h_2, h_3, h_4\}$, but possibly adjacent to h_5. Because x is a small attachment of H and v is adjacent to x, it follows from (3.4) that v has at least one neighbor in $V(H)$. Since u and v are nonadjacent, v is not complete to H. Hence, it follows from (3.2) that v is either a small attachment or a clone for H.

First suppose that v is a small attachment of H. Then, from (3.3) and the fact that u and v are nonadjacent, it follows that $\{x, v\}$ is a pyramid for H. But now, by (3.9), $\{x, v\} \subset U(\mathcal{W})$, contradicting the fact that $x \notin U(\mathcal{W})$.

So we may assume that v is a clone for H. If v is adjacent to h_2 and h_5, then the claim holds. Therefore, we may assume that v is adjacent to at most one of h_2, h_5. Since u and v are nonadjacent, it follows that v is a clone of type 3 or 4 for H. If v is a clone of type 3 for H, then it follows from (3.2) that x is a clone for H/v and hence $x \in U(\mathcal{W})$, a contradiction. If v is a clone type 4, then again x is a clone for H/v and hence $x \in U(\mathcal{W})$, a contradiction. This proves (3.12). $\square$

We are now in a position to prove (3.10).

Proof of (3.10). Let B be a 5-gon with a center and an anticenter and let $\mathcal{W}$ be a maximal C_5-structure around B. Let $Z = U(\mathcal{W})$, let C be the set of centers for $\mathcal{W}$ and let A be $V(G) \setminus (Z \cup C)$. It follows from (3.11) that $A \cup C \neq \emptyset$.

(i) *There exists $z \in Z$ that is anticomplete to A.*

Let $b_1, b_2, \ldots, b_5$ be the vertices of B in order. Let $K_1, K_2, \ldots, K_q$ be the components of $G|A$. We may assume that $V(B) \cap \bigcup_{v \in A} N(v) = V(B)$, because otherwise the claim holds. It follows from (3.3) and the maximality of $U(\mathcal{W})$ that, for $j = 1, 2, \ldots, q$, every two vertices $u, v \in V(K_j)$ are either leaves of the same type or hats of the same type with respect to B.

In particular, for each $j = 1, 2, \ldots, q$, $V(B) \cap N(u) = V(B) \cap N(v)$ for all $u, v \in V(K_j)$. Since $V(B) \cap \bigcup_{v \in A} N(v) = V(B)$, it follows that there exist a stable set $S \subseteq A$ such that $V(B) \cap \bigcup_{v \in S} N(v) = V(B)$. First suppose that some $s_1 \in S$ is a leaf for B. From the symmetry, we may assume that $V(B) \cap N(s_1) = b_1$. For $i = 2, 5$, let $s_i \in S$ be a neighbor of b_i. It follows from (3.3) applied to s_1 and s_2 that s_2 is either a leaf of type 2 for B, or a hat of type 4. This implies that $V(B) \cap N(s_2) \subseteq \{b_1, b_2\}$ and, symmetrically, $V(B) \cap N(s_5) = \{b_1, b_5\}$. But now, s_2-b_2-b_3-b_4-b_5-s_5 is an induced five-edge path, a contradiction. Thus, we may assume that every vertex in S is a hat for B. Again, consider $s_1 \in S$. From the symmetry, we may assume that s_1 is a hat of type 1 for B. For $i = 2, 5$, let $s_i \in S$ be a neighbor of b_i. It follows from (3.3) applied to s_1 and s_2 that s_2 is a hat of type 4 for B. Symmetrically, s_5 is a hat of type 3 for B. But now, s_2 and s_5 contradict (3.3). This proves **(i)**. □

We claim that (Z, A, C) is a quasi-homogeneous set decomposition of G. Clearly, $1 < |Z| < |V(G)|$ and C is complete to Z. Construct G' from $G|(A \cup C)$ by adding a new vertex z that is complete to C and anticomplete to A. It follows from **(i)** that there exists a vertex in Z that is complete to C and anticomplete to A. Therefore, G contains G' as an induced subgraph. This settles properties (i), (ii), and (iv) of the quasi-homogeneous set decomposition. To prove property (iii), let P_1 be a perfect induced subgraph of G' with $z \in V(P_1)$, let P_2 be a perfect induced subgraph of $G|Z$, and let $P = G|(V(P_1) \cup V(P_2) \setminus \{z\})$. We need to show that P is perfect. So suppose that P is not perfect. Since P is an induced subgraph of G, it does not have an induced four-edge antipath or an induced five-edge path. It follows that P contains an induced cycle F of length five. Let $f_1, f_2, \ldots, f_5$ be the vertices of F in order.

(ii) *No edge of F has one endpoint in Z and one endpoint in C.*

From the symmetry, we may assume that $f_1 \in Z$ and $f_2 \in C$. Since C is complete to Z, and f_4 is nonadjacent to f_1 and f_2, it follows that $f_4 \in A$. Moreover, since f_5 is nonadjacent to f_2, it follows for the same reason that $f_5 \in A \cup C$. If $f_5 \in A$, then (3.12) with $x = f_5$, $u = f_1$ and $v = f_4$, implies that $f_4 \in Z$, a contradiction. Therefore, we may assume that $f_5 \in C$. Because f_3 is nonadjacent to f_1 and f_5, it follows that $f_3 \notin C \cup Z$, and hence that $f_3 \in A$. But now z-f_2-f_3-f_4-f_5-z is an induced cycle of length five in P_1, contradicting the fact that P_1 is perfect. This proves **(ii)**. □

Let P^* be obtained from P by deleting all edges between $A \cap V(P)$ and $Z \cap V(P)$. It follows from Lemma 2.2 that P^* is perfect. Therefore, F is not an induced subgraph of P^*. It follows that some edge of F has one endpoint in Z and one endpoint in A, say $f_1 \in Z$ and $f_2 \in A$.

Let $H \in V(\mathcal{W})$ be such that $f_1 \in V(H)$. Let $h_1, h_2, \ldots, h_5$ be the vertices of H in order. We may assume that $f_1 = h_1$.

(iii) *No vertex $w \in A$ is a clone or a center for H.*

If w is a clone for H, then it follows from the maximality of $\mathcal{W}$ that $w \in Z$, a contradiction. If w is a center for H, then it follows from (3.7) that $w \in Z \cup C$, a contradiction. This proves **(iii)**. □

(iv) *f_3 is a clone of type 1 for H and $\{f_3, f_4, f_5\} \subset Z$.*

Since f_1 is nonadjacent to f_3, it follows from (3.12) that $f_3 \in Z$ and f_3 is a clone in the same position as f_1 for H. It follows from **(ii)** that $f_5 \in A \cup Z$. Suppose that $f_5 \in A$. Since f_4 is nonadjacent to f_1, it follows from (3.12) that f_4 is also a clone of type 1 for H. If f_5 is adjacent to both h_5 and h_2, then it follows from (3.2) that f_5 is a clone or a center for H, contrary to **(iii)**. Therefore, from the symmetry, we may assume that f_5 is nonadjacent to h_2. But now h_2-f_5-f_3-f_1-f_4 is an induced four-edge antipath, a contradiction. This proves that $f_5 \in Z$ and, from the symmetry, that $f_4 \in Z$, and hence this proves **(iv)**. □

Since f_5 is adjacent to f_1, but not to f_3, it follows that $f_5 \notin V(H)$. Since f_4 is adjacent to f_3 but not to f_1, it follows that $f_4 \notin V(H)$. It follows from **(iii)** that f_2 is not a clone or a center for H and hence that f_2 is nonadjacent to h_3 and h_4.

We claim that $\{f_4, f_5\}$ is anticomplete to $\{h_2, h_5\}$. For suppose not. From the symmetry, we may assume that f_4 is adjacent to h_2. If f_4 is nonadjacent to h_5, then f_3-f_1-f_4-h_5-h_2 is an induced four-edge antipath, a contradiction. Therefore, f_4 is adjacent to h_5. If f_2 is adjacent to both h_2 and h_5, then it follows from (3.2) that f_2 is a clone or a center for H, contrary to **(iii)**. Hence, from the symmetry, we may assume that f_2 is nonadjacent to h_2. But now f_3-f_1-f_4-f_2-h_2 is an induced four-edge antipath, a contradiction. This proves that $\{f_4, f_5\}$ is anticomplete to $\{h_2, h_5\}$.

It follows from (3.4) applied to h_3, h_4 and h_2-f_3-f_4-f_5-f_1-h_2 that there is at least one edge between $\{h_3, h_4\}$ and $\{f_4, f_5\}$. From the symmetry, we may assume that f_5 is adjacent to h_4. It follows from (3.2) applied to h_4 and h_5-f_3-f_4-f_5-f_1-h_5 that h_4 is nonadjacent to f_4. It follows from (3.2) applied to f_5 and H that f_5 is nonadjacent to h_3. By applying (3.4) to h_4, h_3 and F, h_3 has a neighbor in $V(F)$. Therefore, h_3 is adjacent to f_4. But now h_3 and h_4 are adjacent leaves for F that have different types, contradicting (3.3). This proves (3.10). □

3.3 Basic graphs

In the previous section, we showed that composite graphs in $\mathrm{Forb}(P_4^c, P_5, C_6)$, i.e. graphs that have a 5-gon with both a center and an anticenter, admit a quasi-homogeneous set decomposition. In this section, we will analyze basic graphs. It turns out that if a graph does not contain a 5-gon with both a center and an anticenter, then a 'dual' statement is also true: there is no vertex that simultaneously serves as a center for some 5-gon in G and as an anticenter for some other 5-gon in G (we will prove this shortly). In particular, this implies that for every $v \in V(G)$, either $G|N(v)$ or $G|M(v)$ is perfect (and, equivalently, 1-narrow).

(3.13) *Let $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and suppose that no 5-gon has both a center and an anticenter. Then there do not exist v, A and B such that $v \in V(G)$, A and B are 5-gons in B, and v is a center for A and an anticenter for B.*

Proof. Suppose that v is a center for a 5-gon A and an anticenter for a 5-gon B. Since v is complete to $V(A)$ and anticomplete to $V(B)$, it follows that $V(A) \cap V(B) = \emptyset$. Let $a_1, a_2, \ldots, a_5$

and $b_1, b_2, \ldots, b_5$ be the vertices of A and B, respectively, in order.

(i) *Every $x \in V(B)$ is a small attachment of A and all $x \in V(B)$ are of the same type.*

It follows from (3.2) that x is either an anticenter, or a small attachment, or a clone, or a center for A. Since G is basic, A does not have an anticenter and hence x is not an anticenter for A. Now suppose that x is a clone for A. It follows from (3.2) applied to A/x that v is adjacent to x, contradicting the fact that v is anticomplete to $V(B)$. This proves that every vertex in $V(B)$ is either a small attachment or a center for A.

Suppose that some vertex in $V(B)$ is complete to $V(A)$. Since B has no center, not all vertices in $V(B)$ are centers for A. Therefore, there are adjacent $y, z \in V(B)$ such that y is complete to $V(A)$ and z is not. Therefore, z is a small attachment of A. Let $a \in V(A)$ be a neighbor of z and let $a' \in V(A)$ be a non-neighbor of a. Since z is a small attachment of A, it follows that a' is nonadjacent to z. But now a-a'-z-v-y is an induced four-edge antipath, a contradiction. This proves that every vertex in $V(B)$ is a small attachment of A. Now suppose that not all vertices of $V(B)$ are of the same type with respect to A. Then there exist adjacent $b, b' \in V(B)$ such that b and b' are small attachments for A, but of different types, contradicting (3.3). This proves **(i)**. □

(ii) *Let $x \in V(A)$. Then x is either a clone or an anticenter for B.*

Suppose that x is not a clone or an anticenter for B. Since G is basic, B does not have a center and hence x is not complete to $V(B)$. Then it follows from (3.2) that x is a small attachment of B. But now v is a neighbor of a small attachment of B and v has no neighbor in $V(B)$, contrary to (3.4). This proves **(ii)**. □

From **(i)** and the symmetry, we may assume that all $b \in V(B)$ are of type 1 for A. That is, for every $b \in V(B)$, b is either adjacent to a_1 and anticomplete to $\{a_2, a_3, a_4, a_5\}$, or b is adjacent to a_3 and a_4 and anticomplete to $\{a_1, a_2, a_5\}$. Since B does not have a center, at least one of the vertices of B is a leaf and at least one of them is a hat. From the symmetry, we may assume that b_1 is a leaf for A that is adjacent to a_1. Since from **(ii)** every vertex of A is either a clone or an anticenter for B, it follows that we may assume that a_1 is adjacent to b_4 and a_1 is anticomplete to $\{b_2, b_3\}$. Since a_1 is anticomplete to $\{b_2, b_3\}$, it follows from **(i)** that b_2 and b_3 are complete to $\{a_3, a_4\}$. Because b_1 and b_4 are leaves, it follows that $\{b_1, b_4\}$ is anticomplete to $\{a_3, a_4\}$. Therefore, it follows from (3.2) applied to a_3 and B that a_3 is a hat for B, contradicting **(ii)**. This proves (3.13). □

We can now prove that

Theorem 3.14. *Every graph $G \in \mathrm{Forb}(P_4^c, P_5, C_6)$ is 2-narrow.*

Proof. We prove this by induction on $|V(G)|$. If G is perfect, then G is 1-narrow and there is nothing to prove. So we may assume that G is not perfect. From the fact that G has no induced four-edge antipath and no induced five-edge path, it follows that G contains a 5-gon. First suppose that G contains a 5-gon with a center and an anticenter. Then, by (3.10), G admits a quasi-homogeneous set decomposition (Z, A, C). Let G' be the graph obtained from $G|(A \cup C)$ by adding a vertex z that is anticomplete to A and complete to C. Notice that G' is an induced subgraph of G by property (iv) of a quasi-homogeneous set decomposition, and hence $G' \in \mathrm{Forb}(P_4^c, P_5, C_6)$. By the induction hypothesis, G' and $G|Z$ are 2-narrow. It follows from Lemma 2.3 that G is 2-narrow. So we may assume that G has no 5-gon that has both a center and an anticenter. Let $v \in V(G)$. It follows from the induction hypothesis that $G|N(v)$ and $G|M(v)$ are both 2-narrow. Moreover, it follows from (3.13) that either $G|N(v)$ or $G|M(v)$ is perfect and hence 1-narrow. Since this is true for every $v \in V(G)$, it follows from Lemma 2.1 that G is 2-narrow. This proves Theorem 3.14. □

4 Graphs in $\mathrm{Forb}(P_4^c, P_5)$

In this section, we will prove that every graph in $\mathrm{Forb}(P_4^c, P_5)$ is 3-narrow. Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and suppose that G does not contain a 6-gon with a center. Then it follows that $G|N(v) \in \mathrm{Forb}(P_4^c, P_5, C_6)$ for every $v \in V(G)$. So, we can conclude by Theorem 3.14 that $G|N(v)$ is 2-narrow for every $v \in V(G)$. If G is a minimal counterexample to Theorem 1.2, then, by the minimality of G, it follows that $G|M(v)$ is 3-narrow for every $v \in V(G)$ and hence G is 3-narrow by Lemma 2.1 (for details, see the proof of Theorem 1.2 at the end of this section). Thus, it remains to consider the case when G does contain a 6-gon with a center. We deal with this case in (4.2). We will start with a lemma that deals with attachments of 6-gons.

(4.1) *Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and let H be a 6-gon in G with vertices $h_1, h_2, \ldots, h_6$ in order. Let $v \in V(G) \setminus V(H)$ and suppose that v has a neighbor and a non-neighbor in $V(H)$. Then, up to symmetry, either*

- (x) *v is complete to $\{h_1, h_3, h_5\}$ and v is anticomplete to $\{h_2, h_4, h_6\}$, or*
- (y) *v is complete to $\{h_3, h_6\}$, v is anticomplete to $\{h_1, h_2\}$ and v is either complete or anticomplete to $\{h_4, h_5\}$, or*
- (z) *v is complete to $\{h_1, h_3\}$, anticomplete to $\{h_4, h_5, h_6\}$, and the adjacency between v and h_2 is arbitrary.*

Proof. We may assume that v is adjacent to h_1 and nonadjacent to h_2. Suppose that v is adjacent to h_3. Since h_1-h_2-h_3-h_4 is an induced path, and v is complete to $\{h_1, h_3\}$ and nonadjacent to h_2, it follows from (3.1) that v is nonadjacent to h_4. From the symmetry, it follows that v is nonadjacent to h_6. If v is adjacent to h_5, then (x) holds. If v is nonadjacent to h_5, then (z) holds. So we may assume that v is nonadjacent to h_3. If v is nonadjacent to h_4, then, since v-h_1-h_2-h_3-h_4-h_5 is not an induced five-edge path, it follows that v is adjacent to h_5 and (z) holds. So we may assume that v is adjacent to h_4. Because h_4-h_5-h_6-h_1 is an induced path and v is adjacent to h_1 and h_4, it follows from (3.1) that v is either complete or anticomplete to $\{h_5, h_6\}$. Therefore, (y) holds. This proves (4.1). □

Let $G \in \mathrm{Forb}(P_4^c, P_5)$ and let H be a 6-gon in G. We call a vertex $v \in V(G) \setminus V(H)$ an (x)-*vertex,* (y)-*vertex, or* (z)-*vertex for H* if v satisfies (x), (y), or (z) of (4.1), respectively. Let $z \in V(G) \setminus V(H)$ be a (z)-vertex for H. Then, there exists a unique vertex $h \in V(H)$ such that

$H' = G|((V(H) \setminus \{h\}) \cup \{z\})$ is a 6-gon. We say that H' is *the 6-gon obtained from rerouting H through z.*

(4.2) *Let $G \in \text{Forb}(P_4^c, P_5)$ and suppose that G contains a 6-gon with a center. Then G admits either a quasi-homogeneous set decomposition, or a Σ-join.*

Proof. Let H be a 6-gon with a center and let $h_1, h_2, \ldots, h_6$ be the vertices of H in order. Let C be the set of vertices that are complete to $V(H)$. Notice that $C \neq \emptyset$. Let X, Y, and Z be the sets of (x)-vertices, (y)-vertices, and (z)-vertices for H, respectively.

(i) *C is complete to $X \cup Y \cup Z$ and X is anticomplete to Y.*

Let $c \in C$ and $z \in Z$. Let H' be the 6-gon obtained from rerouting H through z. Then c has at least five neighbors in $V(H')$ and hence (4.1) implies that c is adjacent to z. This proves that C is complete to Z. Now let $x \in X$. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$ and anticomplete to $\{h_2, h_4, h_6\}$. Since h_6-h_1-x-h_3 is an induced path and c is complete to $\{h_1, h_3, h_6\}$, it follows from (3.1) that c is adjacent to x. Hence, C is complete to X. Next, let $y \in Y$. We may assume that y is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Then h_1-h_6-y-h_3 is an induced path and c is complete to $\{h_1, h_3, h_6\}$. It follows from (3.1) that y is adjacent to c and hence that Y is complete to C. This proves that C is complete to $X \cup Y \cup Z$.

Next, suppose that $x \in X$ and $y \in Y$ are adjacent. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$ and anticomplete to $\{h_2, h_4, h_6\}$, and that y is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Now, h_1-h_2-h_3-y is an induced path, x is complete to $\{h_1, h_3, y\}$ and x is nonadjacent to h_2, contrary to (3.1). This proves **(i)**. □

Let Y' be the set of vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z)$ with a neighbor in Y. Let X' be the set of vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y')$ with a neighbor in X. Let X'' be the set of the vertices in $V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y' \cup X')$ with a neighbor in X'. Let $A = V(G) \setminus (V(H) \cup C \cup X \cup Y \cup Z \cup Y' \cup X' \cup X'')$. Since $(A \cup X' \cup X'' \cup Y') \cap (X \cup Y \cup Z \cup C) = \emptyset$, (4.1) implies that $A \cup Y' \cup X' \cup X''$ is anticomplete to $V(H)$. It follows from the definition of Y', X', X'', and A that $X' \cup X'' \cup A$ is anticomplete to Y, X is anticomplete to $X'' \cup A$, and X' is anticomplete to A.

(ii) *Z is anticomplete to $A \cup X' \cup X'' \cup Y'$, Y' is anticomplete to $A \cup X' \cup X''$, and A is anticomplete to X''.*

First, suppose that $z \in Z$ is adjacent to $a \in A \cup X' \cup X'' \cup Y'$. Let H' be obtained from rerouting H through z. Then it follows that a has exactly one neighbor in $V(H')$, contrary to (4.1). This proves that Z is anticomplete to $A \cup X' \cup X'' \cup Y'$.

Next, suppose that $y' \in Y'$ is adjacent to $a \in A \cup X' \cup X''$. Let $y \in Y$ be a neighbor of y'. We may assume that y is adjacent to h_3 and not to h_1 and h_2. Now h_1-h_2-h_3-y-y'-a is an induced five-edge path, a contradiction. This proves that Y' is anticomplete to $A \cup X' \cup X''$.

Finally, suppose that $x'' \in X''$ is adjacent to $a \in A$. Then let $x' \in X'$ be a neighbor of x'' and let $x \in X$ be a neighbor of x'. From the symmetry, we may assume that x is adjacent to

h_1 and not to h_2. Then h_2-h_1-x-x'-x''-a is an induced five-edge path, a contradiction. This proves that A is anticomplete to X'', thus proving **(ii)**. $\square$

The following two claims deal with the case when $Y \neq \emptyset$.

(iii) *Suppose that $Y \neq \emptyset$. Then there do not exist x, p, q such that $x \in X \cup Y$, $p, q \in X' \cup X'' \cup Y'$, and x-p-q is an induced path.*

Suppose that $Y \neq \emptyset$ and suppose that such x, p, q exist. First suppose that $x \in Y$. We may assume that x is complete to $\{h_3, h_6\}$ and anticomplete to $\{h_1, h_2\}$. Now h_1-h_2-h_3-x-p-q is an induced five-edge path, a contradiction. We may therefore assume that $x \in X$. Let $y \in Y$. It follows from **(i)** that y is nonadjacent to x. From the symmetry, we may assume that x is complete to $\{h_1, h_3, h_5\}$, y is complete to $\{h_3, h_6\}$ and y is anticomplete to $\{h_1, h_2\}$. Since q-p-x-h_1-h_6-y is not an induced five-edge path, it follows that y is adjacent to at least one of p and q. Because we already proved that no vertex in Y forms a two-edge induced path with p and q, it follows that y is complete to $\{p, q\}$. But now x-h_3-y-q is an induced path, p is complete to $\{x, y, q\}$, and p is nonadjacent to h_3, contrary to (3.1). This proves **(iii)**. $\square$

(iv) *If $Y \neq \emptyset$, then the lemma holds.*

Suppose that $Y \neq \emptyset$. We claim that $X'' = \emptyset$. For suppose that $x'' \in X''$. Then let $x' \in X'$ be a neighbor of x'', and let $x \in X$ be a neighbor of x'. Then x-x'-x'' is an induced path with $x \in X$ and $x', x'' \in X' \cup X''$, contrary to **(iii)**. This proves that $X'' = \emptyset$.

Let A' be the union of all the components K of $G|(X' \cup Y')$ such that C is not complete to K. Let $N = A \cup A'$ and $U = (V(H) \cup X \cup Y \cup Z \cup X' \cup Y') \setminus A'$. We claim that (U, N, C) is a quasi-homogeneous set decomposition of G.

Clearly, $1 < |U| < |V(G)|$. Construct G' from $G|(N \cup C)$ by adding a new vertex z that is complete to C and anticomplete to N. First notice that any vertex in $V(H)$ is complete to C and anticomplete to N, and therefore G contains G' as an induced subgraph. Next, it follows from **(i)** and the definition of A' that C is complete to U. This settles properties (i), (ii), and (iv) of the quasi-homogeneous set decomposition.

To prove property (iii), let P_1 be a perfect induced subgraph of G' with $z \in V(P_1)$, let P_2 be a perfect induced subgraph of $G|Z$, and let $P = G|((V(P_1) \cup V(P_2)) \setminus \{z\})$. We need to show that P is perfect. So suppose that P is not perfect. Since P is an induced subgraph of G, it does not have an induced four-edge antipath or an induced five-edge path. It follows that P contains an induced cycle F of length five. Let $f_1, f_2, \ldots, f_5$ be the vertices of F in order. Let P^* be obtained from P by deleting all edges between $U \cap V(P)$ and $N \cap V(P)$. It follows from Lemma 2.2 that P^* is perfect. Therefore, F is not an induced subgraph of P^*. It follows that some edge of F has one endpoint in U and one endpoint in N, say $f_1 \in U$ and $f_2 \in N$.

It follows from **(ii)** that A is anticomplete to U. Hence, because f_1 and f_2 are adjacent, it follows that $f_2 \notin A$ and therefore $f_2 \in A'$. It follows from the definition of A' that $f_1 \notin V(H) \cup X' \cup Y' \cup Z$ and hence $f_1 \in X \cup Y$. Now let us consider f_3. Since f_3 is adjacent to f_2, it follows that $f_3 \in X \cup Y \cup A' \cup C$. If $f_3 \in A'$, then f_1-f_2-f_3 is an induced path with $f_1 \in X \cup Y$ and $f_2, f_3 \in X' \cup Y'$, contrary to **(iii)**. Since $f_1 \in X \cup Y$, C is complete to $X \cup Y$, and f_3 is nonadjacent to f_1, it follows that $f_3 \notin C$, and therefore $f_3 \in X \cup Y$. Now let us consider f_4 and f_5. If both f_4 and f_5 are in $X' \cup Y'$, then f_3-f_4-f_5 is an induced path with $f_3 \in X \cup Y$ and $f_4, f_5 \in X' \cup Y'$, contrary to **(iii)**. Therefore, from the symmetry, we may assume that $f_4 \notin X' \cup Y'$. Since f_4 is adjacent to f_3, this implies that $f_4 \in V(H) \cup C \cup X \cup Y \cup Z$. Since f_4 is not adjacent to f_1 and C is complete to f_1, it follows that $f_4 \notin C$. Therefore, **(i)** implies that f_4 is complete to C. This proves that C is complete to $\{f_1, f_3, f_4\}$.

Let K' be the component of A' that contains f_2. We first claim that no vertex in $X \cup Y$ has both a neighbor and a non-neighbor in K'. For suppose otherwise. Then, there exist $x \in X \cup Y$ and adjacent $k_1, k_2 \in K'$ such that x is adjacent to k_1 and nonadjacent to k_2. But now f_1-k_1-k_2 is an induced path that contradicts **(iii)**.

Since f_1 and f_3 are adjacent to $f_2 \in K'$, it follows that f_1 and f_3 are complete to K'. Next, we claim that f_4 is anticomplete to K'. If $f_4 \in V(H) \cup Z$, then this follows from the fact that $V(H) \cup Z$ is anticomplete to $X' \cup Y'$. If $f_4 \in X \cup Y$, then this follows from the above and the fact that f_4 is nonadjacent to $f_2 \in K'$. Thus, f_4 is anticomplete to K'.

Since K' is not complete to C by the definition of A', we may choose $f_2' \in K'$ and $c \in C$ such that f_2' is nonadjacent to c (perhaps by choosing $f_2' = f_2$). It follows from the above that f_2' is adjacent to f_1 and f_3 and nonadjacent to f_4. Therefore, f_1-f_2'-f_3-f_4 is an induced path. It follows from the above that c is complete to $\{f_1, f_3, f_4\}$ and nonadjacent to f_2', contrary to (3.1). This proves **(iv)**. □

In view of **(iv)**, we may from now on assume that no 6-gon with a center in G has a (y)-vertex.

(v) *If $Z \neq \emptyset$, then the lemma holds.*

Suppose that $Z \neq \emptyset$. From the symmetry, we may assume that there exists $z \in Z$ such that z is adjacent to h_2 and h_6. Let Z_1' be the set of vertices in Z that are adjacent to h_2 and h_6 and let $Z_1 = Z_1' \cup \{h_1\}$. It follows from the definition of Z_1 that $|Z_1| \geq 2$. Let R be the set of vertices in $V(G) \setminus Z_1$ with a neighbor in Z_1 and let $S = V(G) \setminus (Z_1 \cup R)$. We claim that (Z_1, S, R) is a homogeneous set decomposition of G. For suppose not. Then there exist $v \in V(G) \setminus Z_1$ and $x, y \in Z_1$ such that v is adjacent to x and nonadjacent to y. It follows from the definition of Z_1 that $v \notin V(H)$. Let $H' = x$-h_2-h_3-$\ldots$-h_6-x. It follows from **(i)** that C is complete to Z_1. Thus, H' has a center and, therefore, since no 6-gon with a center has a (y)-vertex, H' has no (y)-vertex. It follows from (4.1) that v is either an (x)-vertex or a (z)-vertex for H'. It follows that v is nonadjacent to h_4 and, since $v \notin Z_1$, v is adjacent to at least one of h_3, h_5. From the symmetry, we may assume that v is adjacent to h_3. It follows from the fact that v is either an (x)-vertex or a (z)-vertex for H', that v is nonadjacent to h_6. Since y-h_6-x-v-h_3-h_4 is not an induced five-edge path, it follows that x is adjacent to y. If v is nonadjacent to h_2, then x-h_3-y-v-h_2 is an induced four-edge antipath, a contradiction. Thus, v is adjacent to h_2 and hence v is a (z)-vertex for H', and v is nonadjacent to h_5. Now, the adjacency of v with respect to the 6-gon y-h_2-h_3-$\ldots$-h_6-y contradicts (4.1). This proves that (Z_1, R, S) is a homogeneous set decomposition, and hence a quasi-homogeneous set decomposition, of G. This proves **(v)**. $\square$

In view of **(v)**, we may from now on assume that $Z = \emptyset$. Let X_1 and X_2 be the vertices in X that are complete to $\{h_1, h_3, h_5\}$ and $\{h_2, h_4, h_6\}$, respectively. Now, $(\{h_1, h_3, h_5\}, \{h_2, h_4, h_6\}, X_1, X_2, C, A \cup X' \cup X'')$ is a Σ-join. This proves (4.2). $\square$

We are now in a position to prove Theorem 1.2:

Proof of Theorem 1.2. We prove the theorem by induction on $|V(G)|$. Let $G \in \mathrm{Forb}(P_4^c, P_5)$. Suppose first that G contains a 6-gon with a center. Then it follows from (4.2) that G admits either a quasi-homogeneous set decomposition or a Σ-join. If G admits a quasi-homogeneous set decomposition, then it follows from Lemma 2.3 and the induction hypothesis that G is 3-narrow. Otherwise, G admits a Σ-join and it follows from Lemma 2.4 and the induction hypothesis that G is 3-narrow. So we may assume that G contains no 6-gon with a center. Now let $v \in V(G)$. Clearly, $G|N(v)$ does not have C_6 as an induced subgraph. Therefore, $G|N(v) \in \mathrm{Forb}(P_4^c, P_5, C_6)$ and hence, by Theorem 3.14, $G|N(v)$ is 2-narrow. By the induction hypothesis, it follows that $G|M(v)$ is 3-narrow. Since this is true for every $v \in V(G)$, it follows from Lemma 2.1 that G is 3-narrow. This proves Theorem 1.2. $\square$

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