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LARGE DEVIATION PRINCIPLE FOR SOLUTIONS TO SDE DRIVEN BY MARTINGALE MEASURE

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ABSTRACT. We consider a type of large deviation principle(LDP) using Freidlin-Wentzell exponential estimates for the solutions to perturbed stochastic differential equations(SDEs) driven by Martingale measure(Gaussian noise). We are using exponential tail estimates and exit probability of a diffusion process. Referring to Freidlin- Wentzell inequality, we want to show another approach to get LDP for the solutions to SDEs.

1. Introduction

The large deviation principle(LDP) characterizes the limiting behavior of a family of probability measure $\{\mu_\epsilon\}$ on $(\mathcal{X}, \mathcal{B})$ as $\epsilon \to 0$ in terms of a rate function.

Recall that if $\{\mu_\epsilon\}$ is a family of probability measure on a Polish space $\mathcal{P}$ with metric ρ, then μ_ϵ is said to have an LDP with rate functional $I : \mathcal{P} \longrightarrow [0, \infty]$, which is a lower semi-continuous function such that the sets

$$\mathcal{K}(r) = \{x \in \mathcal{P} : I(x) \leqq r\}, \quad r \geqq 0$$

are compact and satisfies the following:

$(L1)$ for each open set $O \subset \mathcal{P}$

$$\liminf_{\epsilon \longrightarrow 0} 2\epsilon^2 \log \mu_\epsilon(O) \geqq -\inf\{I(x) : x \in O\}$$

$(L2)$ for each closed set $C \subset \mathcal{P}$

$$\limsup_{\epsilon \longrightarrow 0} 2\epsilon^2 \log \mu_\epsilon(C) \leqq -\inf\{I(x) : x \in C\}.$$

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Instead of directly trying to show the above inequalities, (L1) and (L2) we will follow a reformulated equivalent form of the so-called Freidlin-Wentzell exponential estimates. (see [7])

$(L1')$ $\forall x \in \mathcal{P}$, $\forall \delta > 0$ and $\forall \gamma > 0$ $\exists \epsilon_0$ such that $\forall \epsilon \in (0, \epsilon_0]$

$$\mu_\epsilon\{y : \rho(y, x) \leqq \delta\} \geqq \exp\{-\frac{I(x) + \gamma}{2\epsilon^2}\}$$

$(L2')$ $\forall r$, $\forall \delta > 0$ and $\forall \gamma > 0$ $\exists \epsilon_0$ such that $\forall \epsilon \in (0, \epsilon_0]$

$$\mu_\epsilon\{y : \rho(y, \mathcal{K}(r)) \geqq \delta\} \leqq \exp\{-\frac{r - \gamma}{2\epsilon^2}\}.$$

In this paper we consider a large deviation principle(LDP) for a mild solution to stochastic partial differential equations(SPDEs) driven by martingale measure. This extends the results proved by Freidlin and Wentzell [4] for diffusion processes, Peszat [7], and Cho [3].

Let L be the usual elliptic operator defined by

$$Lf = f_{xx} - f_t$$

and let $M(dt, dx)$ be a continuous orthogonal martingale measure defined on a probability space $(\Omega, \mathcal{F}, P)$. For any $\epsilon > 0$, let $X^\epsilon(t, x)$ be the solution of the parabolic SPDE under the following conditions:

$$\begin{aligned} \partial_t X^\epsilon(t, x) &= LX^\epsilon(t, x) + b(t, x, X^\epsilon(t, x)) \\ &\qquad + \epsilon\sigma(t, x, X^\epsilon(t, x))M(dt, dx) \\ X^\epsilon(0, \cdot) &= u_0(\cdot) \end{aligned} \tag{1.1}$$

for all $(t, x) \in [0, T] \times [0, 1]$ and $u_0 \in C_0^\infty$ with Dirichlet's or Neuman's boundary conditions. We also suppose that there exist positive constants C_1 and C_2 such that

$$|b(t, x, y) - b(t, x, z)| \leqq C_1|y - z| \tag{1.2}$$

$$|\sigma(t, x, y) - \sigma(t, x, z)| \leqq C_2|y - z|, \tag{1.3}$$

for all $(t, x) \in [0, T] \times [0, 1]$ and $y, z \in R$, and that there exists a constant C_3 such that for every $z \in R$,

$$\sup_{(t,x)\in[0,1]\times[0,T]} |\sigma(t, x, z) + b(t, x, z)| \leqq C_3(1 + |z|). \tag{1.4}$$

Finally, we also assume that the function

$$(1.5) \qquad (t,x,y) \longmapsto \sigma(t,x,y) \qquad \text{is continuous.}$$

It is J. Walsh [9] who first gave a rigorous meaning to Eq. (1.1): that is $X^\epsilon(t,x)$ is the solution of the following evolution equation:

$$\begin{aligned} X^\epsilon(t,x) = G_t(x,u_0) &+ \int_0^t \int_0^1 G_{t-s}(x,y) b(s,y,X^\epsilon(s,y)) ds dy \\ &+ \epsilon \int_0^t \int_0^1 G_{t-y}(x,y)\sigma(s,y,X^\epsilon(s,y)) M(ds,dy), \end{aligned}$$

where $G_t(x,u_0) = \int_0^1 G_t(x,y)u_0(y)dy$ for fixed $u_0 \in C[0,1]$, and $G_t(\cdot,\cdot)$ is the Green kernel associated with the partial differential equation $\partial_t u = Lu$ and with the same boundary conditions as those of X^ϵ.

It is well known that for given $u_0 \in C[0,1]$ the trajectories of the process, X^ϵ are known to be almost surely continuous in t and x.

Let M be a continuous martingale measure and $\pi(ds,dx)$ be its covariance measure. We assume that there exists a predictable process $h(s,x)$ such that

$$(1.6) \qquad \pi(t,A) \leqq \int_{A\times[0,t]} h(s,x) ds dx,$$

for every borel set A, and for $q' = \frac{p}{p-2}, 2 < p \leqq 3$ and for some $p_0 \geqq 0$,

$$E\int_0^T \|\frac{1}{1+|x|^{p_0}} h(s,\cdot)\|_{q'} ds < \infty,$$

where $\|\cdot\|_{q'}$ is the usual $L_{q'}$-norm.

REMARK. Condition(1.6) includes the case in which

$$M^2(t,A) - \int_{A\times[0,t]} h(s,x) ds dx$$

is a martingale. If $h(s,x) \equiv 1$ the associated martingale measure is a white noise based on Lebesgue measure. If $h(s,x) \equiv h(x)$ is non-random, the associated martingale measure having continuous sample paths is a Brownian motion with the variance $\int_A h(x) m(dx)$.

Let H denote the Cameron-Martin space associated with M, i.e.,

$$H = \{h(t,x) = \int_0^t \int_0^x \dot{h}(u,z)\pi(du,dz) : \dot{h} \in L_2\},$$

and let

$$\mathcal{H}_T = \{h \in H : \int_0^T \int_0^1 |\dot{h}(u,z)|^2 \pi(du,dz) < \infty\}.$$

We also let $h \in \mathcal{H}_T$ and z^h be defined by the solution of the following integral equation:

$$\begin{aligned} z^h(t,x) = G_t(x,u_0) &+ \int_0^t \int_0^1 G_{t-s}(x,y) b(s,y,z^h(s,y)) dsdy \\ &+ \int_0^t \int_0^1 G_{t-u}(x,z)\sigma(s,y,z^h(s,y))\dot{h}(s,y)\pi(ds,dy), \end{aligned}$$

for all $(t,x) \in [0,T] \times [0,1]$.

Let $I : C[0,T] \times [0.1] \to [0,\infty]$ be given by

$$I(z) = \inf\{|h|^2_{\mathcal{H}_T} : z = z^h\},$$

where $|h|^2_{\mathcal{H}_T} = \int_0^T \int_0^1 |\dot{h}(u,z)|^2 \pi(du,dz)$. This I is our candidate for the rate function.

Now, if $X^\epsilon(t,x)$ converges to a deterministic limit, $z^h(t,x)$, we want to know what the rate of this convergence is. This problem is related to the Freidlin-Wentzell theory for the analysis of dynamical systems and to the study of PDE with small perturbations.

2. Preliminaries

F. Chenal and A. Millet [2] show a LDP for the solutions to a parabolic SPDE perturbed by a small non-linear white noise. D. Marquez-Carreras and M. Sarra [6] prove a LDP for a perturbed stochastic heat equation driven by a Gaussian noise. Both of their proofs are based on a classical result given by Azencott [1].

Azencott's method is the following:

THEOREM [1]. *Let* $(E_i, d_i), i = 1, 2$ *be two Polish spaces and* $X_i^\epsilon : \Omega \to E_i$, $\epsilon > 0$, $i = 1, 2$ *be two families of random variables. Suppose the following requirements:*

1. $\{X_1^\epsilon > 0\}$ *obeys a LDP with the rate function* $I_1 : E_1 \to [0, \infty]$.

2. *There exists a function* $K : \{I_1 < \infty\} \to E_2$ *such that for every* $a < \infty$, *the function* $K : \{I_1 < a\} \to E_2$ *is continuous.*

3. *For every* r, ρ, $a > 0$, *there exist* $\theta > 0$ *and* $\epsilon_0 > 0$ *such that, for* $h \in E_1$ *satisfying* $I_1(h) \leqq a$ *and* $\epsilon \leqq \epsilon_0$, *we have*

$$P\{d_2(X_2^\epsilon, K(h)) \geqq \rho, d_1(X_1^\epsilon, h) < \theta\} \leqq \exp(-\frac{r}{\epsilon^2}).$$

Then, the family $\{X_2^\epsilon > 0\}$ *obeys a LDP with the rate function*

$$I_2(\phi) = \inf\{I_1(h) : K(h) = \phi\}.$$

If we deduce (from Schilder's theorem) that the family $\{\epsilon M : \epsilon > 0\}$ satisfies a LDP with a good rate function I and show the continuity of the skeleton, z, then we may have the LDP due to the above theorem. However, we are going to prove our LDP problem by showing $(L1')$ and $(L2')$, which are the equivalent forms of (L1) and (L2).

We briefly review the Green kernel. Recall that the Green kernel G associated with satisfies the following inequalities under the case of Dirichlet's condition: $(X^\epsilon(t, 0) = X^\epsilon(t, 1) = 0)$ or Neumann's boundary conditions :$(\partial_x X^\epsilon(t, 0) = \partial_x X^\epsilon(t, 1) = 0)$. For all $x, y \in [0, 1]$ and $s, t \in [0, T]$,

$$|G_t(x, y)| \leqq \frac{C}{\sqrt{t}} \exp(-\frac{(y - x)^2}{4t}), \tag{2.1}$$

$$\sup_{t \in [0,T]} \int_0^t \int_0^1 |G_u(x, z) - G_u(y, z)|^p du dz \leqq C|x - y|^{3-p}, \ \frac{3}{2} < p < 3 \tag{2.2}$$

$$\sup_{x \in [0,1]} \int_0^s \int_0^1 |G_{t-u}(x, z) - G_{s-u}(y, z)|^p du dz \leqq C|t - s|^{\frac{3-p}{2}}, \ 1 < p < 3 \tag{2.3}$$

$$\sup_{x \in [0,1]} \int_s^t \int_0^1 |G_u(x, z)|^p du dz \leqq C|t - s|^{\frac{3-p}{2}} \leqq C_T, \ 1 < p < 3. \tag{2.4}$$

Also for convenience, we let $M(t, [0, x]) \equiv M(t, x)$ and $|\cdot|_\infty = \|\cdot\|$.

3. Lower bound

LEMMA 3.1. *Assume that β is a predictable process and there exists a constant $l < \infty$ such that*

$$\int_0^T \int_0^1 \beta^2(s,x)\pi(ds,dx) \leqq l,\ P - \text{almost surely.}$$

Then for all $r > 0$ one has

$$\mathbf{P}\{\sup_{0\leqq t\leqq T} |\int_0^t \int_0^1 \beta(s,x)M(ds,dx)| \geqq r\} \leqq 3\cdot\exp\{-\frac{r^2}{4l}\}. \tag{3.1}$$

PROOF. Let

$$Z(t) = \int_0^t \int_0^1 \beta(s,x)M(ds,dx) \tag{3.2}$$

For the following estimations let

$$\phi_\lambda(x) = (1+\lambda x^2)^{\frac{1}{2}}, \quad \lambda, x \in R_+. \tag{3.3}$$

We again apply the Ito formula to (3.2).

$$\begin{aligned}
&\phi_\lambda(z(t)) \\
&= 1 + \int_0^t \int_0^1 \phi'_\lambda(z(s))\beta(s,x))M(ds,dx) \\
&\quad + \int_0^t \int_0^1 \frac{1}{2}\phi''_\lambda(z(s))\beta(s)^2\pi(ds,dx) \\
&= 1 + \int_0^t \int_0^1 \phi'_\lambda(z(s))\beta(s,x)M(ds,dx) \\
&\quad - \frac{1}{2}\int_0^t \int_0^1 (\phi'_\lambda(z(s))\beta(s,x))^2\pi(ds,dx) \\
&\quad + \frac{1}{2}\int_0^t \int_0^1 (\phi'_\lambda(z(s))\beta(s,x))^2\pi(ds,dx) \\
&\quad + \frac{1}{2}\int_0^t \int_0^1 (\phi''_\lambda(z(s))\beta^2(s,x))\pi(ds,dx).
\end{aligned} \tag{3.4}$$

Note that $\phi'_\lambda(x) = \frac{\lambda x}{\phi_\lambda}$ and $\frac{1}{\phi_\lambda(x)} \leqq 1$. For every $\lambda \in R_+$, let

$$\eta_t^\lambda \equiv \int_0^t \int_0^1 \phi'_\lambda(z(s))\beta(s,x)M(ds,dx) \tag{3.5}$$
$$-\frac{1}{2}\int_0^t \int_0^1 (\phi'_\lambda(z(s))\beta(s,x))^2\pi(ds,dx).$$

Then

$$\phi_\lambda(z(t)) \leqq 1 + \eta_t^\lambda + \frac{1}{2}\lambda \cdot l + \frac{1}{2}\int_0^t \int_0^1 \frac{\lambda \cdot \lambda x^2}{1+\lambda x^2}\beta^2(s,x)\pi(ds,dx)$$
$$\leqq 1 + \eta_t^\lambda + \lambda \cdot l. \tag{3.6}$$

Let $Z_t^\lambda \equiv \exp\{\eta_t^\lambda\}$ for every real λ. Then Z_t^λ is a local martingale and it follows easily from the Ito formula that $EZ_t^\lambda = 1$. Hence using Doob's martingale inequality, we have

(3.7)
$$P\{\sup_{0\leqq s\leqq t} |z(s)| \geqq r\} \leqq P\{\sup_{0\leqq s\leqq t} \phi_\lambda(z(s)) \geqq (1+\lambda r^2)^{\frac{1}{2}}\}$$
$$\leqq P\{\sup_{0\leqq s\leqq t} Z_s^\lambda \geqq \exp\{(1+\lambda r)^{\frac{1}{2}} - 1 - \lambda l\}$$
$$\leqq \exp\{-[(1+\lambda r^2)^{\frac{1}{2}} - 1 - \lambda l]\}$$
$$\leqq \exp\{-(1+\lambda r^2)^{\frac{1}{2}} + 1 + \lambda l\}.$$

If $r^2 > 2l$ then taking $\lambda = \frac{r^4 - 4l^2}{4l^2 r^2}$ we obtain the desired bound (3.1). If $r^2 < 2l$, then $3\exp(\frac{-r^2}{4l}) \geqq 1$. Thus (3.1) holds for any $r > 0$. □

LEMMA 3.2. *If there exists a constant $\eta < \infty$ such that*

$$\sup_{x\in[0,1]} \int_0^T \int_0^1 |G_{t-s}(x,y)\beta(s,y)|^2\pi(ds,dy) < \eta, \quad P-\textit{almost surely}$$

then

$$P\{\sup_{0\leqq t\leqq T}\sup_{x\in[0,1]} |\int_0^t \int_0^1 G_{t-s}(x,y)\beta(s,y)M(ds,dy)| \geqq \delta\}$$
$$< 4T\exp\{-\frac{\delta^2}{9\eta}\}.$$

PROOF. Let $\psi_t(s,x,y) = G_{t-s}(x,y)\beta(s,y)$ for $0 \leqq s < t \leqq T$ and 0 otherwise. Then by Lemma 3.1

$$P\{\exp(\frac{1}{9\eta}\sup_{0\leqq t\leqq T}\|\int_0^t\int_0^1 \psi_t(s,x,y)M(ds,dx))\|^2 \geqq u\}$$
$$\leqq 3\exp(-\frac{9}{4}\ln u) \leqq 3u^{-\frac{9}{4}}.$$

Hence

$$E\exp\{\frac{1}{9\eta}\sup_{0\leqq t\leqq T}\|\int_0^t\int_0^1 \psi_t(s,x,y)M(ds,dx))\|^2\}$$
$$= \int_0^\infty P\{\exp(\frac{1}{9\eta}\sup_{0\leqq t\leqq T}\|\int_0^t\int_0^1 \psi_t(s,x,y)M(ds,dx))\|^2) \geqq u\}du$$
$$\leqq 1 + 3\int_1^\infty u^{-\frac{9}{4}}du \leqq 4$$
$$\times \int_0^T E\exp\{\frac{1}{9\eta}\sup_{0\leqq t\leqq T}\|\int_0^t\int_0^1 \psi_t(s,x,y)M(ds,dx))\|^2\}dt \leqq 4T.$$

$$P\{\sup_{0\leqq t\leqq T}\|\int_0^t\int_0^1 G_{t-s}(x,y)\beta(s,y)M(ds,dy)\|^2 \geqq \delta^2\}$$
$$\leqq P\{\sup_{0\leqq t\leqq T}\exp(\frac{1}{9\eta}\|\int_0^t\int_0^1 \psi_t(s,x,y)M(ds,dx))\|^2) \geqq \exp(\frac{\delta^2}{9\eta})\}$$
$$\leqq 4T\exp(-\frac{\delta^2}{9\eta}).$$

□

THEOREM 3.3 (LOWER BOUND). *We assume (1.2)-(1.5).* $\forall T > 0$, $\forall l > 0$, $\forall\delta > 0$, $\forall\gamma > 0$, *there exists* $\epsilon_0 > 0$ *such that* $\forall\epsilon \in [0,\epsilon_0]$ *and* $\forall h \in \mathcal{H}_T$ *satisfying* $|h|^2_{\mathcal{H}_T} \leqq l$, *we have*

$$\mathbf{P}\{\sup_{0\leqq t\leqq T}\|X^\epsilon(t,\cdot) - z^h(t,\cdot)\|_\infty \leqq \delta\} \geqq \exp\{-\frac{|h|^2_{\mathcal{H}_T}+\gamma}{2\epsilon^2}\}. \tag{3.8}$$

PROOF. Let $l > 0$, $r > 0$, $\delta > 0$ and $h \in \mathcal{H}_T$ satisfying $|h|^2_{\mathcal{H}_T} \leqq l$. Define

$$M^\epsilon(t,x) = M(t,x) - \frac{1}{\epsilon}\int_0^t \int_0^x \dot{h}(s,y)\pi(ds,dy).$$

According to the Girsanov theorem, for fixed x $M^\epsilon(\cdot,x)$ is a martingale process on the probability space $(\Omega, \mathcal{F}, P^\epsilon)$, where

$$\frac{dP}{dP^\epsilon} = \exp\{-\frac{1}{\epsilon}\int_0^T \int_0^1 h(s,y)M(ds,dy) - \frac{1}{2\epsilon^2}\int_0^T \int_0^1 \dot{h}^2(s,y)\pi(ds,dy)\}.$$

Note that $P \ll P^\epsilon$.

Now, consider the following:

$$\begin{aligned}
&|X^\epsilon(t,x) - z^h(t,x)| \\
&\leqq C\int_0^t \int_0^1 |G_{t-s}(x,y)(X^\epsilon(s,y)) - z^h(s,y))|dsdy \\
&+ \epsilon|\int_0^t \int_0^1 G_{t-s}(x,y)\sigma(X^\epsilon(s,y))M^\epsilon(ds,dy)| \\
&+ \int_0^t \int_0^1 G_{t-s}(x,y)|X^\epsilon(s,y) - z^h(s,y)||\dot{h}(s,y)|\pi(ds,dy).
\end{aligned}$$

By (1.2)-(1.5) and the Gronwall inequality, there exists a constant K_1 such that

$$\begin{aligned}
&\sup_{0\leqq t\leqq T} |X^\epsilon(t,x) - z^h(t,x)| \\
&\leqq K_1 \sup_{0\leqq t\leqq T} \|\epsilon\int_0^t \int_0^1 G_{t-s}(x,y)\sigma(X^\epsilon(s,y))M^\epsilon(ds,dy)\|.
\end{aligned}$$

Let $\tilde{l} \in (0,l)$, and

$$\begin{aligned}
\mathcal{G}(\epsilon,t,x) &= \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(X^\epsilon(s,y))M^\epsilon(ds,dy) \\
\mathcal{A}(\epsilon) &= \{\sup_{0\leqq t\leqq T} \|X^\epsilon(t,\cdot) - z^h(t,\cdot)\| \leqq \delta\} \\
\mathcal{B}(\epsilon) &= \{\|\epsilon\int_0^T \int_0^1 \dot{h}(s,y)M^\epsilon(ds,dy)\| \leqq \frac{\tilde{l}}{2}\}.
\end{aligned}$$

Then

$$\mathbf{P}\{\mathcal{A}(\epsilon)\} = E^\epsilon[\frac{dP}{dP^\epsilon} : \mathcal{A}(\epsilon)] \geqq \exp\{-\frac{|h|^2_{\mathcal{H}} + \tilde{l}}{2\epsilon^2}\}P^\epsilon\{\mathcal{A}(\epsilon) \cap \mathcal{B}(\epsilon)\}.$$

We also want to show that $P^\epsilon\{\mathcal{A}(\epsilon) \cap B(\epsilon)\} \to 1$ uniformly with respect to x and h on bounded sets. Now,

$$\begin{aligned} &P^\epsilon\{\mathcal{A}(\epsilon) \cap \mathcal{B}(\epsilon)\} \\ &\geqq 1 - P^\epsilon\{\sup_{0 \leqq t \leqq T} \|\mathcal{G}(\epsilon, t, \cdot)\| > \frac{\delta}{K_1 \epsilon}\} - P^\epsilon\{\mathcal{B}^c(\epsilon)\}. \end{aligned}$$

Note that by condition (1.4) and (2.4), there exists a constant l_0 such that

$$\sup_{x \in [0,1]} \int_0^T \int_0^1 |G_{t-s}(x, y)\sigma(X^\epsilon(s, y))|^2 \pi(ds, dy) < l_0.$$

Lemma 3.2 shows that

$$P^\epsilon\{\sup_{0 \leqq t \leqq T} \|\mathcal{G}(\epsilon, t, \cdot)\| > \frac{\delta}{K_1 \epsilon}\} \leqq 3\exp\{-\frac{\delta^2}{4K_1^2 \epsilon^2 l_0}\} \to 0 \tag{3.9}$$

and $P^\epsilon\{\mathcal{B}^c(\epsilon)\} \to 0$ as $\epsilon \to 0$ by Lemma 3.1. Hence, we get the lower bound. □

4. Upper bound

For convenience, let

$$I^\epsilon(t, x) \equiv \int_0^t \int_0^1 G_{t-s}(x, y)\sigma(s, y, X^\epsilon(s, y))M(ds, dx).$$

LEMMA 4.1. *$\forall r >, \exists M > 0$ such that $\forall \epsilon \in (0, 1]$ we have*

$$\mathbf{P}\{\sup_{0 \leqq t \leqq T} \|X^\epsilon(t, \cdot)\| \geqq M\} \leqq \exp\{-\frac{r}{\epsilon^2}\}. \tag{4.1}$$

PROOF. From (1.2)-(1.3), (2.4) and Gronwall's inequality, there exist $C_1, C_2 > 0$ such that $\forall\epsilon$ and $x \in B(x_0, R)$ (for fixed x_0),

$$\sup_{0\leqq t\leqq T} \|X^\epsilon(t,\cdot)\|^2 \leqq C_1 + C_2 \sup_{0\leqq t\leqq T} \|\epsilon I^\epsilon(t,\cdot)\|^2.$$

Let $M > 0$ be such that $M^2 > C_1$. By Lemma 3.2

$$\begin{aligned} &P\{\sup_{0\leqq t\leqq T} \|X^\epsilon(t,\cdot)\| \geqq M\} \\ \leqq\; &P\{\sup_{0\leqq t\leqq T} \|I^\epsilon(t,\cdot)\| \geqq (\frac{M^2 - C_1}{C_2})^{\frac{1}{2}}\} \\ \leqq\; &\exp\{-\frac{1}{\epsilon^2}(\frac{M^2 - C_1}{9C_2\eta} - \log 4T)\} \\ \leqq\; &\exp(-\frac{r}{\epsilon^2}), \end{aligned}$$

where r is chosen satisfying $r = \frac{M^2 - C_1}{9C_2\eta} - \log 4T$. □

The proofs of the following two propositions are simple modifications of Proposition 6.2 and Proposition 6.1 in [7] respectively in our setting. For the compactness of this paper we omit them.

PROPOSITION 4.2. *$\forall r > 0$, $\forall\delta > 0$ $\exists b > 0$ and $\exists\epsilon_0 > 0$ such that $\forall\epsilon \in (0, \epsilon_0]$ we have*

$$\mathbf{P}\{\sup_{0\leqq t\leqq T} \|\epsilon I^\epsilon(t,\cdot)\| \geqq \delta, \sup_{0\leqq t\leqq T} \|\epsilon M(t,\cdot)\| \leqq b\} \leqq \exp\{-\frac{r}{\epsilon^2}\}. \tag{4.2}$$

To prove (4.2), X^ϵ is being approximated by the simpler process X_n^ϵ which is defined to be stopped on small intervals.

For $n \in N$ and $k = 0, \cdots, 2^n - 1$ set $t_{n,k} = k \cdot 2^{-n}T$ and $d_{n,k} = [t_{n,k}, t_{n,k+1})$. Let $\pi_n(t) = t_{n,k}$ for $t \in d_{n,k}$. Finally, let

$$X_n^\epsilon(t,x) \equiv \int_0^1 G_{t-\pi_n(t)}(x,y) X^\epsilon(\pi_n(t), y)dy.$$

PROPOSITION 4.3. *$\forall r > 0$, $\forall\delta > 0$ $\exists n_0$, $\forall n \geqq n_0$, and $\exists\epsilon_n > 0$ such that $\forall\epsilon \in (0, \epsilon_n]$ we have*

$$\mathbf{P}\{\sup_{0\leqq t\leqq T} \|X^\epsilon(t,\cdot) - X_n^\epsilon(t,\cdot)\| \geqq \delta\} \leqq \exp\{-\frac{r}{\epsilon^2}\}.$$

PROPOSITION 4.4. *Under the assumption* (1.2)-(1.5), $\forall r > 0$, $\forall \delta > 0$, $\forall h \in \mathcal{H}_T$, *there exists* $\epsilon_0 > 0$ *and* $b > 0$ *such that* $\forall \epsilon \in (0, \epsilon_0]$ *we have*

$$\begin{aligned} &\mathbf{P}\{\sup_{0\leqq t\leqq T} \|X^\epsilon(t,\cdot) - z^h(t,\cdot)\| \geqq \delta, \\ &\qquad \sup_{0\leqq t\leqq T} \|\epsilon M(t,\cdot) - \int_0^t \int_0^\cdot \dot{h}(s,y)\pi(ds,dy)\| \leqq b\} \\ &\leqq \exp\{-\frac{r}{\epsilon^2}\}. \end{aligned} \tag{4.3}$$

PROOF. Let

$$\begin{aligned} \mathcal{N}(\epsilon, b) \equiv &\{\sup_{0\leqq t\leqq T} \|X^\epsilon(t,x) - z^h(t,x)\| \geqq \delta, \\ &\qquad \sup_{0\leqq t\leqq T} \|\epsilon M(t,\cdot) - \int_0^t \int_0^\cdot \dot{h}(s,y)\pi(ds,dy)\| \leqq b\} \end{aligned} \tag{4.4}$$

and $h \in \mathcal{H}_T$. Let W^ϵ and P^ϵ be defined as in the proof of Theorem 3.4. For $\lambda > 0$ set

$$\mathcal{M}(\epsilon, \lambda) = \{\int_0^T \int_0^1 \dot{h}(t,x) M(ds,dx) \geqq -\frac{\lambda}{\epsilon}\}.$$

Obviously, for an arbitrary $\lambda > 0$, we have

$$\mathbf{P}\{\mathcal{N}(\epsilon,b)\} \leqq \mathbf{P}\{\mathcal{N}(\epsilon,b) \cap \mathcal{M}(\epsilon,\lambda)\} + \mathbf{P}\{\mathcal{M}^c(\epsilon,\lambda)\}. \tag{4.5}$$

By Theorem 3.1, we have

$$\mathbf{P}\{\mathcal{M}^c(\epsilon,\lambda)\} \leqq 3\exp\{-\frac{\lambda^2}{4\epsilon^2 |h|^2_{\mathcal{H}_T}}\}. \tag{4.6}$$

Note that

$$\begin{aligned} \mathbf{P}\{\mathcal{N}(\epsilon,b) \cap \mathcal{M}(\epsilon,\lambda)\} &= E^\epsilon[\frac{dP}{dP^\epsilon} : \mathcal{N}(\epsilon,b) \cap \mathcal{M}(\epsilon,\lambda)] \\ &\leqq \exp\{\frac{\lambda}{\epsilon^2} + \frac{|h|^2_{\mathcal{H}_T}}{2\epsilon^2}\}\mathbf{P}^\epsilon\{\mathcal{N}(\epsilon,b)\}. \end{aligned} \tag{4.7}$$

We have

$$\mathbf{P}^\epsilon\{\mathcal{N}(\epsilon,b)\} = \mathbf{P}\{\sup_{0\leqq t\leqq T} \|\tilde{X}^\epsilon(t,\cdot) - z^h(t,\cdot)\| > \delta,\ \|\epsilon M(t,\cdot)\| \leqq b\},$$

where $\tilde{X}(t,x)$ is the solution of the SDE

$$\partial_t \tilde{X}^\epsilon(t,x) = L\tilde{X}^\epsilon(t,x) + b(t,x,\tilde{X}^\epsilon(t,x)) + \epsilon\sigma(t,x,\tilde{X}^\epsilon(t,x))M(dt,dx)$$
$$X^\epsilon(0,\cdot) = u_0(\cdot).$$

Using the same argument as in the proof of Theorem 3.2 we get

(4.8)
$$\begin{aligned}
&\mathbf{P}^\epsilon\{\mathcal{N}(\epsilon,b)\} \\
&\leqq \mathbf{P}\{\sup_{0\leqq t\leqq T} \|\epsilon \int_0^t \int_0^1 G_{t-s}(\cdot,y)\sigma(\tilde{X}^\epsilon(s,y))M^\epsilon(ds,dy)\| \geqq \frac{\delta}{C_1}, \\
&\qquad \sup_{0\leqq t\leqq T} \|\epsilon M(t,\cdot)\| \leqq b\} \\
&\leqq \mathbf{P}\{\sup_{0\leqq t\leqq T} \| \int_0^t \int_0^1 G_{t-s}(\cdot,y)\sigma(\tilde{X}^\epsilon(s,y))M^\epsilon(ds,dy)\| \geqq \frac{\delta}{\epsilon C_1}\} \\
&\leqq 3\exp\{-\frac{\delta^2}{4\epsilon^2 C_1^2 l_0}\},
\end{aligned}$$

where C_1 is a constant and l_0 is a constant defined as (3.9). Hence for each $\tilde{a} > a$ we can find b such that

$$\mathbf{P}^\epsilon\{\mathcal{N}(\epsilon,b)\} \leqq \exp\{-\frac{\tilde{a}}{\epsilon^2}\}. \tag{4.9}$$

Combining (4.5) to (4.6), we have

$$\mathbf{P}\{\mathcal{N}(\epsilon,b)\} \leqq 3\exp\{-\frac{\lambda^2}{4\epsilon^2|h|^2_{\mathcal{H}_T}}\} + \exp\{\frac{2\lambda + |h|^2_{\mathcal{H}_T} - 2\tilde{a}}{2\epsilon^2}\}.$$

If we first choose λ and then $\tilde{a}$ we get (4.4). □

Let

$$\mathcal{H}_T^r = \{h \in H : |h|^2_{\mathcal{H}_T} \leqq r\}. \tag{4.10}$$

PROPOSITION 4.5. *$\forall T > 0$, $\forall r > 0$, $\forall \delta > 0$, $\forall \gamma > 0$, there exists $\epsilon_0 > 0$ such that $\forall \epsilon \in (0, \epsilon_0]$ we have*

$$\mathbf{P}\{\mathrm{dist}_{C_{[0,T]\times[0,1]}}(\epsilon M, \mathcal{H}_T^r) \geqq \delta\} \leqq \exp\{-\frac{r-\gamma}{2\epsilon^2}\}, \tag{4.11}$$

where $\mathrm{dist}_{C_{[0,T]\times[0,1]}}(\epsilon M, \mathcal{H}_T^r)$ means the distance of ϵM to the space, $\mathcal{H}_T^r$ in the metric of $C_{[0,T]\times[0,1]}$.

PROOF. As before, let

$$\frac{dP}{dP^\epsilon} = \exp\{-\frac{1}{\epsilon}\int_0^T\int_0^1 1\, M(ds,dy) - \frac{1}{2\epsilon^2}\int_0^T\int_0^1 \dot{h}^2(s,y)\pi(ds,dy)\}.$$

Also let

$$\mathcal{C}_r(\epsilon) = \{\inf_{h\in\mathcal{H}_T^r} |\epsilon M(T,1) - \int_0^T\int_0^1 \dot{h}(s,y)\pi(ds,dy)| \geqq \delta, |h|^2_{\mathcal{H}_T} \leqq r\}.$$

Then

$$\begin{aligned} P\{\mathcal{C}_r(\epsilon)\} &= E^\epsilon\{\frac{dP}{dP^\epsilon} : \mathcal{C}_r(\epsilon)\} \\ &\leqq \exp\{-\frac{r-\gamma}{2\epsilon^2}\}. \end{aligned}$$

□

THEOREM 4.6 (UPPER BOUND). *Under the condition (1.2)-(1.5), $\forall T > 0$, $\forall y \in R$, $\forall \delta > 0$, $\forall R > 0$, $\forall r \geqq 0$, $\forall \gamma > 0$, there exists $\epsilon_0 > 0$ such that $\forall x \in R : |x-y| \leqq R$ and $\forall \epsilon \in (0,\epsilon_0]$ we have*

$$\mathbf{P}\{\mathrm{dist}_{C_{[0,T]\times[0,1]}}(X^\epsilon(\cdot,x), \mathcal{K}(r)) \geqq \delta\} \leqq \exp\{-\frac{r-\gamma}{2\epsilon^2}\}.$$

PROOF. By Proposition 4.4, $\forall h \in \mathcal{H}_T$, there exist $b_h > 0$ and $\epsilon_h > 0$ such that $\forall \epsilon \in (0, \epsilon_h]$ and $\forall x \in B(x_0, R)$,

$$\begin{aligned} &P\{\sup_{0\leqq t\leqq T} \|X^\epsilon(t,\cdot) - z^h(t,\cdot)\| \geqq \delta, \\ &\qquad \sup_{0\leqq t\leqq T} \|\epsilon M(t,\cdot) - \int_0^t\int_0^{\cdot} \dot{h}(s,y)\pi(ds,dy)\| \leqq b_h\} \qquad (4.12) \\ &\leqq \exp\{-\frac{a}{\epsilon^2}\}. \end{aligned}$$

If we sum over all $h \in \mathcal{H}_T$ satisfying $|h|^2_{\mathcal{H}_T} \leqq r$,

$$\mathcal{H}^r_T \subset \bigcup \{u \in C_{[0,T]\times[0,1]} : \sup_{0\leqq t\leqq T} \|u(t,\cdot) - \int_0^t \int_0^{\cdot} \dot{h}(s,y)\pi(ds,dy)\| \leqq b_h\}.$$

Since $\mathcal{H}^r_T$ is compact, we may find $\{h_1,\dots,h_l\}$, where each of h_i satisfies $|h_i|^2_{\mathcal{H}_T} \leqq r$ such that

$$\mathcal{H}^r_T \subset \bigcup_{i=1}^{l} \{u \in C_{[0,T]\times[0,1]} : \sup_{0\leqq t\leqq T} \|u(t,\cdot) - \int_0^t \int_0^{\cdot} \dot{h}(s,y)\pi(ds,dy)\| \leqq b_{h_i}\} \equiv \tilde{\mathcal{H}}.$$

(4.13)

$$\begin{aligned} &P\{\mathrm{dist}_{C_{[0,T]\times[0,1]}}(X^\epsilon(\cdot,\cdot).\mathcal{K}(r)) \geqq \delta\} \\ &\leqq P\{\mathrm{dist}_{C_{[0,T]\times[0,1]}}(X^\epsilon(\cdot,\cdot).\mathcal{K}(r)) \geqq \delta, \epsilon M \in \tilde{\mathcal{H}}\} + P\{\epsilon M \notin \tilde{\mathcal{H}}\}. \end{aligned}$$

Let $\epsilon_1 = \min\{\epsilon_{h_1},\dots,\epsilon_{h_l}\}$. From (4.12) for $\epsilon \in (0,\epsilon_1]$ we have

(4.14)

$$\begin{aligned} &P\{\mathrm{dist}_{C_{[0,T]\times[0,1]}}\big(X^\epsilon(\cdot,\cdot),\mathcal{K}(r)\big) \geqq \delta, \epsilon M \in \tilde{\mathcal{H}}\} \\ &\leqq \sum_{i=1}^{l} P\{\sup_{0\leqq t\leqq T} \|X^\epsilon(t,\cdot) - z^h(t,\cdot)\| \geqq \delta, \\ &\qquad \sup_{0\leqq t\leqq T} \|\epsilon M(t,\cdot) - \int_0^t \int_0^{\cdot} \dot{h}(s,y)\pi(ds,dy)\| \leqq b_{h_i}\} \\ &\leqq l \cdot \exp\{-\frac{a}{\epsilon^2}\}. \end{aligned}$$

Since $\mathcal{H}^r_T$ is compact, there exists $\tilde{\delta} > 0$ such that

$$\{u \in C_{[0,T]\times[0,1]} : \mathrm{dist}_{C_{[0,T]\times[0,1]}}(\epsilon M, \mathcal{H}^r_T) < \tilde{\delta}\} \subset \tilde{\mathcal{H}}.$$

By Proposition 4.5, there exists $\epsilon_2 > 0$ such that $\forall \epsilon \in [0,\epsilon_2]$,

$$\begin{aligned} P\{\epsilon M \notin \tilde{\mathcal{H}}\} &\leqq P\{\mathrm{dist}_{C_{[0,T]\times[0,1]}}(\epsilon M, \mathcal{H}^r_T) \geqq \tilde{\delta}\} \\ &\leqq \exp\{\frac{r-\tilde{\gamma}}{2\epsilon^2}\}. \end{aligned} \tag{4.15}$$

Let $\epsilon_3 = \epsilon_1 \wedge \epsilon_2$. (4.13)-(4.15) imply that for all $\epsilon \in (0, \epsilon_3]$ and $x \in \mathcal{B}(x_0, R)$, letting $2a > r - \tilde{\gamma}$,

$$\begin{aligned} & P\{\mathrm{dist}_{C_{[0,T]\times[0,1]}}\big(X^\epsilon(\cdot,\cdot), \mathcal{K}(r)\big) \geqq \delta\} \\ & \leqq l \exp\{-\frac{a}{\epsilon^2}\} + \exp\{-\frac{r-\tilde{\gamma}}{2\epsilon^2}\} \\ & \leqq \exp\{-\frac{r-\gamma}{2\epsilon^2}\}, \end{aligned}$$

for all $\epsilon \in (0, \epsilon_0]$ and $\forall x \in B(x_0, R)$. □

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