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Large Deviation Principles of Obstacle Problems for Quasilinear Stochastic PDEs

Anis Matoussi[1] · **Wissal Sabbagh**[2] · **Tusheng Zhang**[3]

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Abstract

In this paper, we first present a sufficient condition(a variant) for the large deviation criteria of Budhiraja, Dupuis and Maroulas for functionals of Brownian motions. The sufficient condition is particularly more suitable for stochastic differential/partial differential equations with reflection. We then apply the sufficient condition to establish a large deviation principle for obstacle problems of quasi-linear stochastic partial differential equations. It turns out that the backward stochastic differential equations will also play an important role.

Keywords Stochastic partial differential equation · Obstacle problems · Large deviations · Weak convergence · Backward stochastic differential equations

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1 Introduction

Consider the following obstacle problems for quasilinear stochastic partial differential equations (SPDEs) in $\mathbb{R}^d$:

✉ Tusheng Zhang
tusheng.zhang@manchester.ac.uk

Anis Matoussi
anis.matoussi@univ-lemans.fr

Wissal Sabbagh
wissal.sabbagh@univ-evry.fr

1 Risk and Insurance Institute of Le Mans, Laboratoire Manceau de Mathématiques, Le Mans University, Le Mans, France

2 Laboratoire de Mathématiques et Modélisation d'Evry, University of Evry, Èvry, France

3 School of Mathematics, University of Manchester, Oxford Road, Manchester, England M13 9PL, UK

$$
\begin{aligned}
& dU(t,x)+\frac{1}{2}\Delta U(t,x)+\sum_{i=1}^{d}\partial_i g_i(t,x,U(t,x),\nabla U(t,x))dt \\
& \quad + f(t,x,U(t,x),\nabla U(t,x))dt \\
& \quad + \sum_{j=1}^{\infty} h_j(t,x,U(t,x),\nabla U(t,x))dB_t^j = -R(dt,dx), \qquad (1.1) \\
& U(t,x)\geq L(t,x), \qquad (t,x)\in \mathbb{R}^+\times\mathbb{R}^d, \\
& U(T,x)=\Phi(x), \qquad x\in\mathbb{R}^d, \qquad (1.2)
\end{aligned}
$$

where B_t^j, $j = 1, 2, \ldots$ are independent real-valued standard Brownian motions, the stochastic integral against Brownian motions is interpreted as the backward Ito integral, Δ is the Laplacian operator, f, g_i, h_j are appropriate measurable functions specified later, $L(t,x)$ is the given barrier/obstacle function, $R(dt,dx)$ is a random measure which is a part of the solution pair (U, R). The random measure R plays a similar role as a local time which prevents the solution $U(t,x)$ from falling below the barrier L.

Such SPDEs appear in various applications like pathwise stochastic control problems, the Zakai equations in filtering and stochastic control with partial observations. Existence and uniqueness of the above stochastic obstacle problems were established in [13] based on an analytical approach. Existence and uniqueness of the obstacle problems for quasi-linear SPDEs on the whole space $\mathbb{R}^d$ and driven by finite dimensional Brownian motions were studied in [20] using the approach of backward stochastic differential equations (BSDEs). Obstacle problems for nonlinear stochastic heat equations driven by space-time white noise were studied by several authors, see [23,28] and references therein.

In this paper, we are concerned with the small noise large deviation principle(LDP) of the following obstacle problems for quasilinear SPDEs:

$$
\begin{aligned}
& dU^\varepsilon(t,x)+\frac{1}{2}\Delta U^\varepsilon(t,x)+\sum_{i=1}^{d}\partial_i g_i(t,x,U^\varepsilon(t,x),\nabla U^\varepsilon(t,x))dt \\
& \quad + f(t,x,U^\varepsilon(t,x),\nabla U^\varepsilon(t,x))dt \\
& \quad + \sqrt{\varepsilon}\sum_{j=1}^{\infty} h_j(t,x,U^\varepsilon(t,x),\nabla U^\varepsilon(t,x))dB_t^j = -R^\varepsilon(dt,dx), \qquad (1.3) \\
& U^\varepsilon(t,x)\geq L(t,x), \quad (t,x)\in \mathbb{R}^+\times\mathbb{R}^d, \\
& U^\varepsilon(T,x)=\Phi(x), \quad x\in\mathbb{R}^d. \qquad (1.4)
\end{aligned}
$$

Large deviations for stochastic evolution equations and stochastic partial differential equations driven by Brownian motions have been investigated in many papers, see e.g. [3,5,6,8,11,18,19,25,27] and references therein.

To obtain the large deviation principle, we will adopt the weak convergence approach introduced by Budhiraja, Dupuis and Maroulas in [2–4]. We refer the reader to [2,11,18,19], [3,25] for large deviation principles of various dynamical systems driven by Gaussian noises.

In order to apply the weak convergence method to the obstacle problems, because of the singularity introduced by the reflection/local time, it seems difficult to directly use the criteria in [2]. We therefore first need to provide a sufficient condition to verify the criteria of Budhiraja–Dupuis–Maroulas. This sufficient condition turns out to be particularly suitable for stochastic dynamics generated by stochastic differential equations and stochastic partial differential equations with reflection. The advantage of the new sufficient condition is to shift the difficulty of proving the tightness of the perturbations of stochastic differential(partial differential) equations to a study of the continuity (with respect to the driving signals) of deterministic skeleton equations associated with the stochastic equations. This new sufficient condition is recently successfully applied to obtain a large deviation principle for stochastic conservation laws (Ref. [9]), which otherwise could not (at least very difficult) be established using the original form of the criteria in [2].

The important part of the current work is to study the continuity of the deterministic obstacle problems driven by the elements in the Cameron–Martin space of the driving Brownian motions. We need to show that if the driving signals converge weakly in the Cameron–Martin space, then the corresponding solutions of the skeleton equations converge in the appropriate state space. This turns out to be hard because of the singularity caused by the obstacle. To overcome the difficulties, we have to appeal to the penalized approximation of the skeleton equation and to establish some uniform estimate for the solutions of the approximating equations with the help of the backward stochastic differential equation representation of the solutions. This is purely due to the technical reason because primarily the LDP problem has not much to do with backward stochastic differential equations.

The rest of the paper is organized as follows. In Sect. 2, we introduce the stochastic obstacle problem and the precise framework. In Sect. 3, we recall the weak convergence approach of large deviations and present a sufficient condition. Section 4 is devoted to the study of skeleton obstacle problems. We will show that the solution of the skeleton problem is continuous with respect to the driving signal. The proof of the large deviation principle is in Sect. 5.

2 The Framework

2.1 Obstacle Problems

Let $H := \mathbf{L}^2(\mathbb{R}^d)$ be the Hilbert space of square integrable functions with respect to the Lebesgue measure on $\mathbb{R}^d$. The associated scalar product and the norm are denoted by

$$(u, v) = \int_{\mathbb{R}^d} u(x)\, v(x)\, dx, \quad |u| = \left(\int_{\mathbb{R}^d} u^2(x) dx\right)^{1/2}.$$

Let $V := H(\mathbb{R}^d)$ denote the first order Sobolev space, endowed with the norm and the inner product:

$$\|u\| = \left(\int_{\mathbb{R}^d} |\nabla u|^2(x)dx + \int_{\mathbb{R}^d} |u|^2(x)dx \right)^{1/2},$$
$$\langle u, v \rangle = \int_{\mathbb{R}^d} (\nabla u) \cdot (\nabla v)(x)dx + \int_{\mathbb{R}^d} u(x)v(x)dx.$$

V^* will denote the dual space of V. When causing no confusion, we also use $\langle u, v\rangle$ to denote the dual pair between V and V^*.

Our evolution problem will be considered over a fixed time interval $[0, T]$. Now we introduce the following assumptions.

Assumption 2.1 (i) $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $h = (h_1, \ldots, h_i, \ldots) : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^\infty$ and $g = (g_1, \ldots, g_d) : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable in (t, x, y, z) and satisfy $f^0, h^0, g^0 \in \mathbf{L}^2\left([0, T] \times \mathbb{R}^d\right) \cap \mathbf{L}^\infty\left([0, T] \times \mathbb{R}^d\right)$ where $f^0(t, x) := f(t, x, 0, 0)$, $h^0(t, x) := (\sum_{j=1}^\infty h_j(t, , x, 0, 0)^2)^{\frac{1}{2}}$ and $g^0(t, x) := (\sum_{j=1}^d g_j(t, , x, 0, 0)^2)^{\frac{1}{2}}$.

(ii) There exist constants $c > 0, 0 < \alpha < 1$ and $0 < \beta < 1$ such that for any $(t, x) \in [0, T] \times \mathbb{R}^d$; $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$

$$|f(t, x, y_1, z_1) - f(t, x, y_2, z_2)| \le c\big(|y_1 - y_2| + |z_1 - z_2|\big)$$
$$\left(\sum_{i=1}^\infty |h_i(t, x, y_1, z_1) - h_i(t, x, y_2, z_2)|^2\right)^{1/2} \le c|y_1 - y_2| + \beta|z_1 - z_2|$$
$$\left(\sum_{i=1}^d |g_i(t, x, y_1, z_1) - g_i(t, x, y_2, z_2)|^2\right)^{1/2} \le c|y_1 - y_2| + \alpha|z_1 - z_2|.$$

(iii) There exists a function $\bar{h} \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that for $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

$$\left(\sum_{i=1}^\infty |h_i(t, x, y, z)|^2\right)^{1/2} \le \bar{h}(x).$$

(iv) The contract property: $\alpha + \dfrac{\beta^2}{2} < \dfrac{1}{2}$.

(v) The barrier function $L(t, x) : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\frac{\partial L(t, x)}{\partial t}, \quad \nabla L(t, x), \quad \Delta L(t, x) \in L^2([0, T] \times \mathbb{R}^d) \cap L^\infty([0, T] \times \mathbb{R}^d),$$

where the gradient ∇ and the Laplacian Δ act on the space variable x.

Let $H_T := C([0, T], H) \cap L^2([0, T], V)$ be the Banach space endowed with the norm

$$\|u\|_{H_T} = \sup_{0 \le t \le T} |u_s| + \left(\int_0^T \|u_s\|^2 ds\right)^{1/2}.$$

We denote by $\mathcal{H}_T$ the space of predictable, processes $(u_t, t \geq 0)$ such that $u \in H_T$ and that

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|u_s|_2^2 + \int_0^T \|u_s\|^2 ds\right] < \infty.$$

The space of test functions is $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^\infty(\mathbb{R}^d)$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ denotes the space of real-valued infinitely differentiable functions with compact supports in $\mathbb{R}^+$ and $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable functions with compact supports in $\mathbb{R}^d$.

Let B_t^j, $j = 1, 2, \ldots$ be a sequence of independent real-valued standard Brownian motions on a complete filtrated probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We now precise the definition of solutions for the reflected quasilinear SPDE (1.1):

Definition 2.1 We say that a pair (U, R) is a solution of the obstacle problem (1.1) if

(1) $U \in \mathcal{H}_T$, $U(t,x) \geq L(t,x)$, $dP \otimes dt \otimes dx$-a.e. and $U(T,x) = \Phi(x)$, $dx - a.e.$
(2) R is a random regular measure on $[0, T) \times \mathbb{R}^d$,
(3) for every $\varphi \in \mathcal{D}$

$$\begin{aligned}
&(U_t, \varphi_t) - (\Phi, \varphi_T) + \int_t^T (U_s, \partial_s\varphi_s)ds + \frac{1}{2}\int_t^T \langle \nabla U_s, \nabla\varphi_s\rangle ds \\
&= \int_t^T (f_s(U_s, \nabla U_s), \varphi_s)ds + \sum_{j=1}^\infty \int_t^T (h_s^j(U_s, \nabla U_s), \varphi_s)dB_s^j \\
&\quad - \sum_{i=1}^d \int_t^T (g_s^i(U_s, \nabla U_s), \partial_i\varphi_s)ds + \int_t^T \int_{\mathbb{R}^d} \varphi_s(x)R(dx, ds),
\end{aligned} \tag{2.1}$$

(4) U admits a quasi-continuous version $\tilde{U}$, and

$$\int_0^T \int_{\mathbb{R}^d} (\tilde{U}(s,x) - L(s,x))R(dx, ds) = 0 \qquad a.s.$$

Remark 2.1 We refer the reader to [13] for the precise definition of regular measures and quasi-continuity of functions on the space $[0, T] \times \mathbb{R}^d$.

Let us recall the following result from [13,20].

Theorem 2.1 *Let Assumption 2.1 hold and assume $\Phi(x) \geq L(T,x)$ dx-a.e.. Then there exists a unique solution (U, R) to the obstacle problem (1.1).*

2.2 The Measures $\mathbb{P}^m$

The operator $\partial_t + \frac{1}{2}\Delta$, which represents the main linear part in the equation (1.1), is associated with the Bownian motion in $\mathbb{R}^d$. The sample space of the Brownian motion is $\Omega' = \mathcal{C}([0,\infty);\mathbb{R}^d)$, the canonical process $(W_t)_{t\geq 0}$ is defined by $W_t(\omega) = \omega(t)$, for any $\omega \in \Omega'$, $t \geq 0$ and the shift operator, $\theta_t : \Omega' \longrightarrow \Omega'$, is defined by $\theta_t(\omega)(s) = \omega(t+s)$, for any $s \geq 0$ and $t \geq 0$. The canonical filtration $\mathcal{F}_t^W = \sigma(W_s; s \leq t)$ is completed by the standard procedure with respect to the probability measures produced by the transition function

$$P_t(x,dy) = q_t(x-y)dy, \quad t > 0, \quad x \in \mathbb{R}^d,$$

where $q_t(x) = (2\pi t)^{-\frac{d}{2}} \exp(-|x|^2/2t)$ is the Gaussian density. Thus we get a continuous Hunt process $(\Omega', W_t, \theta_t, \mathcal{F}, \mathcal{F}_t^W, \mathbb{P}^x)$. We shall also use the backward filtration of the future events $\mathcal{F}'_t = \sigma(W_s;\ s \geq t)$ for $t \geq 0$. $\mathbb{P}^0$ is the Wiener measure, which is supported by the set $\Omega'_0 = \{\omega \in \Omega',\ \omega(0) = 0\}$. We also set $\Pi_0(\omega)(t) = \omega(t) - \omega(0)$, $t \geq 0$, which defines a map $\Pi_0 : \Omega' \to \Omega'_0$. Then $\Pi = (W_0, \Pi_0) : \Omega' \to \mathbb{R}^d \times \Omega'_0$ is a bijection. For each probability measure μ on $\mathbb{R}^d$, the probability $\mathbb{P}^\mu$ of the Brownian motion started with the initial distribution μ is given by

$$\mathbb{P}^\mu = \Pi^{-1}\left(\mu \otimes \mathbb{P}^0\right).$$

In particular, for the Lebesgue measure in $\mathbb{R}^d$, which we denote by $m = dx$, we have

$$\mathbb{P}^m = \Pi^{-1}\left(dx \otimes \mathbb{P}^0\right).$$

Notice that $\{W_{t-r}, \mathcal{F}'_{t-r}, r \in [0,t]\}$ is a backward local martingale under $\mathbb{P}^m$. Let $J(\cdot,\cdot) : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ be a measurable function such that $J \in \mathbf{L}^2([0,T]\times\mathbb{R}^d \to \mathbb{R}^d)$ for every $T > 0$. We recall the forward and backward stochastic integral defined in [20,27] under the measure $\mathbb{P}^m$.

$$\int_s^t J(r,W_r) * dW_r = \int_s^t \langle J(r,W_r), dW_r\rangle + \int_s^t \langle J(r,W_r), d\overleftarrow{W}_r\rangle.$$

When J is smooth, one has

$$\int_s^t J(r,W_r) * dW_r = -2\int_s^t div(J(r,\cdot))(W_r)dr. \tag{2.2}$$

We refer the reader to [20,27] for more details.

3 A Sufficient Condition for LDP

In this section we recall the criteria obtained in [2] for proving a large deviation principle and we will provide a sufficient condition to verify the criteria.

Let $\mathcal{E}$ be a Polish space with the Borel σ-field $\mathcal{B}(\mathcal{E})$. Recall

Definition 3.1 *(Rate function)* A function $I : \mathcal{E} \to [0,\infty]$ is called a rate function on $\mathcal{E}$, if for each $M < \infty$, the level set $\{x \in \mathcal{E} : I(x) \leq M\}$ is a compact subset of $\mathcal{E}$.

Definition 3.2 *(Large deviation principle)* Let I be a rate function on $\mathcal{E}$. A family $\{X^\varepsilon\}$ of $\mathcal{E}$-valued random elements is said to satisfy a large deviation principle on $\mathcal{E}$ with rate function I if the following two claims hold.

(a) (Upper bound) For each closed subset F of $\mathcal{E}$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{x\in F} I(x).$$

(b) (Lower bound) For each open subset G of $\mathcal{E}$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq -\inf_{x\in G} I(x).$$

3.1 A Criteria of Budhiraja–Dupuis

The Cameron–Martin space associated with the Brownian motion $\{B_t = (B_t^1, \ldots, B_t^j, \ldots), t \in [0, T]\}$ is isomorphic to the Hilbert space $K := L^2([0, T]; l^2)$ with the inner product:

$$\langle h_1, h_2\rangle_K := \int_0^T \langle h_1(s), h_2(s)\rangle_{l^2} ds,$$

where

$$l^2 = \left\{ a = (a_1, \ldots, a_j, \ldots); \sum_{i=1}^{\infty} a_i^2 < \infty \right\}.$$

l^2 is a Hilbert space with inner product $\langle a, b\rangle_{l^2} = \sum_{i=1}^{\infty} a_i b_i$ for $a, b \in l^2$.

Let $\tilde{K}$ denote the class of l^2-valued $\{\mathcal{F}_t\}$-predictable processes ϕ that belong to the space K a.s.. Let $S_N = \{k \in K; \int_0^T \|k(s)\|_{l^2}^2 ds \leq N\}$. The set S_N endowed with the weak topology is a compact Polish space. Set $\tilde{S}_N = \{\phi \in \tilde{K}; \phi(\omega) \in S_N, \mathbb{P}\text{-a.s.}\}$.

The following result was proved in [2].

Theorem 3.1 *For $\varepsilon > 0$, let Γ^ε be a measurable mapping from $C([0, T]; \mathbb{R}^\infty)$ into $\mathcal{E}$. Set $X^\varepsilon := \Gamma^\varepsilon(B(\cdot))$. Suppose that there exists a measurable map $\Gamma^0 : C([0, T]; \mathbb{R}^\infty) \to \mathcal{E}$ such that*

(a) *for every $N < +\infty$ and any family $\{k^\varepsilon; \varepsilon > 0\} \subset \tilde{S}_N$ satisfying that k^ε converges in law as S_N-valued random elements to some element k as $\varepsilon \to 0$, $\Gamma^\varepsilon \left(B(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot k^\varepsilon(s) ds \right)$ converges in law to $\Gamma^0(\int_0^\cdot k(s) ds)$ as $\varepsilon \to 0$;*

(b) *for every $N < +\infty$, the set*

$$\left\{ \Gamma^0 \left(\int_0^\cdot k(s) ds \right); k \in S_N \right\}$$

is a compact subset of $\mathcal{E}$.

Then the family $\{X^\varepsilon\}_{\varepsilon>0}$ *satisfies a large deviation principle in* $\mathcal{E}$ *with the rate function* I *given by*

$$I(g) := \inf_{\{k\in K;\, g=\Gamma^0(\int_0^\cdot k(s)ds)\}} \left\{ \frac{1}{2}\int_0^T \|k(s)\|_{l^2}^2 ds \right\},\ g \in \mathcal{E}, \tag{3.1}$$

with the convention $\inf\{\emptyset\} = \infty$.

3.2 A Sufficient Condition

Here is a sufficient condition for verifying the assumptions in Theorem 3.1.

Theorem 3.2 *For* $\varepsilon > 0$*, let* Γ^ε *be a measurable mapping from* $C([0,T];\mathbb{R}^\infty)$ *into* $\mathcal{E}$*. Set* $X^\varepsilon := \Gamma^\varepsilon(B(\cdot))$*. Suppose that there exists a measurable map* $\Gamma^0 : C([0,T];\mathbb{R}^\infty) \to \mathcal{E}$ *such that*

(i) *for every* $N < +\infty$*, any family* $\{k^\varepsilon; \varepsilon > 0\} \subset \tilde{S}_N$ *and any* $\delta > 0$*,*

$$\lim_{\varepsilon\to 0} P(\rho\left(Y^\varepsilon, Z^\varepsilon\right) > \delta) = 0,$$

where $Y^\varepsilon = \Gamma^\varepsilon(B(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot k^\varepsilon(s)ds)$*,* $Z^\varepsilon = \Gamma^0\left(\int_0^\cdot k^\varepsilon(s)ds\right)$ *and* $\rho(\cdot,\cdot)$ *stands for the metric in the space* $\mathcal{E}$

(ii) *for every* $N < +\infty$ *and any family* $\{k^\varepsilon; \varepsilon > 0\} \subset S_N$ *satisfying that* k^ε *converges weakly to some element* k *as* $\varepsilon \to 0$*,* $\Gamma^0\left(\int_0^\cdot k^\varepsilon(s)ds\right)$ *converges to* $\Gamma^0(\int_0^\cdot k(s)ds)$ *in the space* $\mathcal{E}$*.*

Then the family $\{X^\varepsilon\}_{\varepsilon>0}$ *satisfies a large deviation principle in* $\mathcal{E}$ *with the rate function* I *given by*

$$I(g) := \inf_{\{k\in K;\, g=\Gamma^0(\int_0^\cdot k(s)ds)\}} \left\{ \frac{1}{2}\int_0^T \|k(s)\|_{l^2}^2 ds \right\},\ g \in \mathcal{E}, \tag{3.2}$$

with the convention $\inf\{\emptyset\} = \infty$.

Remark 3.1 When proving a small noise large deviation principle for stochastic differential equations/stochastic partial differential equations, condition (i) is usually not difficult to check because the small noise disappears when $\varepsilon \to 0$.

Proof We will show that the conditions in Theorem 3.1 are fulfilled. Condition (b) in Theorem 3.1 follows from condition (ii) because S_N is compact with respect to the weak topology. Condition (i) implies that for any bounded, uniformly continuous function $G(\cdot)$ on $\mathcal{E}$,

$$\lim_{\varepsilon\to 0} E[|G(Y^\varepsilon) - G(Z^\varepsilon)|] = 0.$$

Thus, condition (a) will be satisfied if Z^ε converges in law to $\Gamma^0(\int_0^\cdot k(s)ds)$ in the space $\mathcal{E}$. This is indeed true since the mapping Γ^0 is continuous by condition (ii) and k^ε converge in law as S_N-valued random elements to k. The proof is complete. □

4 Skeleton Equations

Recall $K := L^2([0,T], l^2)$. Let $k = (k^1, \dots, k^j, \dots) \in K$ and consider the deterministic obstacle problem:

$$\begin{aligned} &du^k(t,x) + \frac{1}{2}\Delta u^k(t,x) + \sum_{i=1}^{d} \partial_i g_i(t,x,u^k(t,x),\nabla u^k(t,x))dt \\ &\quad + f(t,x,u^k(t,x),\nabla u^k(t,x))dt \\ &\quad + \sum_{j=1}^{\infty} h_j(t,x,u^k(t,x),\nabla u^k(t,x))k_t^j dt = -\nu^k(dt,dx), \qquad (4.1) \\ &u^k(t,x) \geq L(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ &u^k(T,x) = \Phi(x), \qquad x \in \mathbb{R}^d. \qquad (4.2) \end{aligned}$$

The existence and uniqueness of the solution of the deterministic obstacle problem (4.1) can be obtained similarly as the random obstacle problem (1.1) (but simpler). We refer the reader to [13] for more details. Denote by u^{k^ε} the solution of equation (4.1) with k^ε replacing k. The main purpose of this section is to show that u^{k^ε} converges to u^k in the space H_T if $k^\varepsilon \to k$ weakly in the Hilbert space K. To this end, we first need to establish a number of preliminary results.

Consider the penalized equation:

$$\begin{aligned} &du^{k,n}(t,x) + \frac{1}{2}\Delta u^{k,n}(t,x) + \sum_{i=1}^{d} \partial_i g_i(t,x,u^{k,n}(t,x),\nabla u^{k,n}(t,x))dt \\ &+ f(t,x,u^{k,n}(t,x),\nabla u^{k,n}(t,x))dt \\ &+ \sum_{j=1}^{\infty} h_j(t,x,u^{k,n}(t,x),\nabla u^{k,n}(t,x))k_t^j dt = -n(u^{k,n}(t,x) - L(t,x))^- dt, \end{aligned} \qquad (4.3)$$

$$u^{k,n}(T,x) = \Phi(x), \quad x \in \mathbb{R}^d. \qquad (4.4)$$

It is known that $u^{k,n} \to u^k$ as $n \to \infty$ for a fixed $k \in K$ (please see [13]). For later use, we need to show that for any $M > 0$, $u^{k,n} \to u^k$ uniformly over the bounded subset $\{k; \|k\|_K \leq M\}$ as $n \to \infty$. For this purpose, it turns out that we have to appeal to the BSDE representation of the solutions. Let $Y_t^{k,n} := u^{k,n}(t, W_t)$, $Z_t^{k,n} = \nabla u^{k,n}(t, W_t)$. Then it was shown in [20] that $(Y^{k,n}, Z^{k,n})$ is the solution of the backward stochastic differential equation under $\mathbb{P}^m$:

$$\begin{aligned} Y_t^{k,n} &= \Phi(W_T) + \int_t^T f(r,W_r,Y_r^{k,n},Z_r^{k,n})dr + \sum_{j=1}^{\infty}\int_t^T h_j(r,W_r,Y_r^{k,n},Z_r^{k,n})k_r^j dr \\ &\quad + n\int_t^T (Y_r^{k,n} - S_r)^- dr + \frac{1}{2}\int_s^T g(r,W_r,Y_r^{k,n},Z_r^{k,n}) * dW_r - \int_t^T Z_r^{k,n} dW_r. \end{aligned} \qquad (4.5)$$

where $S_r = L(r, W_r)$ satisfies

$$dS_r = \frac{\partial L}{\partial r}(r, W_r)dr + \frac{1}{2}\Delta L(r, W_r)dr + \nabla L(r, W_r)dW_r. \tag{4.6}$$

The following result is a uniform estimate for $(Y^{k,n}, Z^{k,n})$.

Lemma 4.1 *For $M > 0$, we have the following estimate:*

$$\begin{aligned}
&\sup_{\{k\in K;\|k\|_K\leq M\}} \sup_n \left\{ \mathbb{E}^m\left[\sup_{0\leq t\leq T}|Y_t^{k,n}|^2\right] + \mathbb{E}^m\left[\int_0^T |Z_t^{k,n}|^2 dt\right]\right.\\
&\quad \left. + \mathbb{E}^m\left[\left(n\int_0^T (Y_t^{k,n} - S_t)^- dt\right)^2\right]\right\}\\
&\leq c_M\left[|\Phi|^2 + \mathbb{E}^m\left[\sup_{0\leq t\leq T}|S_t|^2\right] + \int_{\mathbb{R}^d}\int_0^T [|f^0(t,x)|^2\right.\\
&\quad \left. + |g^0(t,x)|^2 + |h^0(t,x)|^2]dtdx\right]
\end{aligned} \tag{4.7}$$

The proof of this lemma is a repeat of the proof of Lemma 6 in [20]. One just needs to notice that when applying the Grownwall's inequality, the constant c_M on on right of (4.7) only depends on the norm of k which is bounded by M.

We also need the following estimate.

Lemma 4.2

$$\sup_n \sup_{\{k\in K;\|k\|_K\leq M\}} \mathbb{E}^m\left[n\int_0^T [(Y_t^{k,n} - S_t)^-]^2 dt\right] \leq C_M. \tag{4.8}$$

Proof Let $F(z) = z^2$. Applying the Ito's formula (see [20]) we have

$$\begin{aligned}
F(Y_t^{k,n} - S_t) &= F(\Phi(W_T) - S_T) + \int_t^T F'(Y_r^{k,n} - S_r)f(r, W_r, Y_r^{k,n}, Z_r^{k,n})dr\\
&\quad + \sum_{j=1}^{\infty}\int_t^T F'(Y_r^{k,n} - S_r)h_j(r, W_r, Y_r^{k,n}, Z_r^{k,n})k_r^j dr\\
&\quad + n\int_t^T F'(Y_r^{k,n} - S_r)(Y_r^{k,n} - S_r)^- dr\\
&\quad + \frac{1}{2}\int_s^T F'(Y_r^{k,n} - S_r)g(r, W_r, Y_r^{k,n}, Z_r^{k,n}) * dW_r\\
&\quad + \int_t^T \left\langle \nabla(F'(u^{k,n}(r,\cdot) - L(r,\cdot))), g(r,\cdot,u^{k,n}(r,\cdot),\nabla u^{k,n}(r,\cdot))\right\rangle(W_r)dr\\
&\quad - \int_t^T F'(Y_r^{k,n} - S_r)Z_r^{k,n}dW_r + \int_t^T F'(Y_r^{k,n} - S_r)\frac{\partial L}{\partial r}(r, W_r)dr
\end{aligned}$$

$$+\int_t^T F'(Y_r^{k,n}-S_r)\frac{1}{2}\Delta L(r,W_r)dr+\int_t^T F'(Y_r^{k,n}-S_r)\nabla L(r,W_r)dW_r$$
$$-\frac{1}{2}\int_t^T F''(Y_r^{k,n}-S_r)|Z_r^{k,n}-\nabla L(r,W_r)|^2dr. \tag{4.9}$$

Rearranging the terms we get

$$\begin{aligned}
&(Y_t^{k,n}-S_t)^2+\int_t^T |Z_r^{k,n}-\nabla L(r,W_r)|^2dr+2n\int_t^T [(Y_r^{k,n}-S_r)^-]^2dr\\
&=(\Phi(W_T)-S_T)^2+2\int_t^T (Y_r^{k,n}-S_r)f(r,W_r,Y_r^{k,n},Z_r^{k,n})dr\\
&+2\sum_{j=1}^\infty\int_t^T (Y_r^{k,n}-S_r)h_j(r,W_r,Y_r^{k,n},Z_r^{k,n})k_r^j dr\\
&+\int_t^T (Y_r^{k,n}-S_r)g(r,W_r,Y_r^{k,n},Z_r^{k,n})*dW_r\\
&+2\int_t^T \left\langle Z_r^{k,n}-\nabla L(r,W_r),g(r,W_r,Y_r^{k,n},Z_r^{k,n})\right\rangle dr\\
&-2\int_t^T (Y_r^{k,n}-S_r)Z_r^{k,n}dW_r+2\int_t^T (Y_r^{k,n}-S_r)\frac{\partial L}{\partial r}(r,W_r)dr\\
&+\int_t^T (Y_r^{k,n}-S_r)\frac{1}{2}\Delta L(r,W_r)dr+2\int_t^T (Y_r^{k,n}-S_r)\nabla L(r,W_r)dW_r.
\end{aligned} \tag{4.10}$$

Using the conditions on h in the Assumption 2.1, for any given positive constant ε_1 we have

$$\begin{aligned}
&2\sum_{j=1}^\infty\int_t^T (Y_r^{k,n}-S_r)h_j(r,W_r,Y_r^{k,n},Z_r^{k,n})k_r^j dr\\
&=2\int_t^T (Y_r^{k,n}-S_r)\sum_{j=1}^\infty(h_j(r,W_r,Y_r^{k,n},Z_r^{k,n})-h_j(r,W_r,S_r,\nabla L(r,W_r)))k_r^j dr\\
&+2\int_t^T (Y_r^{k,n}-S_r)\sum_{j=1}^\infty(h_j(r,W_r,S_r,\nabla L(r,W_r))-h_j(r,W_r,0,0))k_r^j dr\\
&+2\int_t^T (Y_r^{k,n}-S_r)\sum_{j=1}^\infty h_j(r,W_r,0,0)k_r^j dr\\
&\le 2\int_t^T |Y_r^{k,n}-S_r|\left(\sum_{j=1}^\infty(h_j(r,W_r,Y_r^{k,n},Z_r^{k,n})-h_j(r,W_r,S_r,\nabla L(r,W_r)))^2\right)^{\frac{1}{2}}\|k_r\|_{l^2}dr\\
&+2\int_t^T |Y_r^{k,n}-S_r|\left(\sum_{j=1}^\infty(h_j(r,W_r,S_r,\nabla L(r,W_r))-h_j(r,W_r,0,0))^2\right)^{\frac{1}{2}}\|k_r\|_{l^2}dr
\end{aligned}$$

$$
\begin{aligned}
&+2\int_t^T |Y_r^{k,n}-S_r|\left(\sum_{j=1}^{\infty} h_j(r,W_r,0,0)^2\right)^{\frac{1}{2}} \|k_r\|_{l^2}dr\\
&\leq C\int_t^T |Y_r^{k,n}-S_r|^2\|k_r\|_{l^2}^2dr+\varepsilon_1\int_t^T |Z_r^{k,n}-\nabla L(r,W_r)|^2dr\\
&+C\int_t^T [L(r,W_r)^2+|\nabla L(r,W_r)|^2+h^0(r,W_r)^2]dr. \qquad (4.11)
\end{aligned}
$$

By the assumptions on g, for any given positive constant ε_2 we have

$$
\begin{aligned}
&2\int_t^T \left\langle Z_r^{k,n}-\nabla L(r,W_r), g(r,W_r,Y_r^{k,n},Z_r^{k,n})\right\rangle dr\\
&=2\int_t^T \left\langle Z_r^{k,n}-\nabla L(r,W_r), g(r,W_r,Y_r^{k,n},Z_r^{k,n})-g(r,W_r,S_r,\nabla L(r,W_r))\right\rangle dr\\
&\quad+2\int_t^T \left\langle Z_r^{k,n}-\nabla L(r,W_r), g(r,W_r,S_r,\nabla L(r,W_r))-g(r,W_r,0,0)\right\rangle dr\\
&\quad+2\int_t^T \left\langle Z_r^{k,n}-\nabla L(r,W_r), g(r,W_r,0,0)\right\rangle dr\\
&\leq 2C\int_t^T |Z_r^{k,n}-\nabla L(r,W_r)||Y_r^{k,n}-S_r|dr+2\alpha\int_t^T |Z_r^{k,n}-\nabla L(r,W_r)|^2dr\\
&\quad+C\int_t^T |Z_r^{k,n}-\nabla L(r,W_r)|[|L(r,W_r)|+|\nabla L(r,W_r)|+g^0(r,W_r)]dr\\
&\leq C\int_t^T |Y_r^{k,n}-S_r|^2dr+(2\alpha+\varepsilon_2)\int_t^T |Z_r^{k,n}-\nabla L(r,W_r)|^2dr\\
&\quad+C\int_t^T [|L(r,W_r)|^2+|\nabla L(r,W_r)|^2+g^0(r,W_r)^2]dr. \qquad (4.12)
\end{aligned}
$$

By a similar calculation, we have for any given $\varepsilon_3>0$,

$$
\begin{aligned}
&2\int_t^T (Y_r^{k,n}-S_r)f(r,W_r,Y_r^{k,n},Z_r^{k,n})dr\\
&\quad\leq C\int_t^T |Y_r^{k,n}-S_r|^2dr+\varepsilon_3\int_t^T |Z_r^{k,n}-\nabla L(r,W_r)|^2dr\\
&\qquad+C\int_t^T [|L(r,W_r)|^2+|\nabla L(r,W_r)|^2+f^0(r,W_r)^2]dr. \qquad (4.13)
\end{aligned}
$$

Substitute (4.11), (4.12) and (4.13) back into to (4.10), choose $\varepsilon_1,\varepsilon_2,\varepsilon_3$ sufficiently small to obtain

$$
\begin{aligned}
&\mathbb{E}^m[(Y_t^{k,n}-S_t)^2]+\mathbb{E}^m\left[\int_t^T |Z_r^{k,n}-\nabla L(r,W_r)|^2dr\right]+2n\mathbb{E}^m\left[\int_t^T [(Y_r^{k,n}-S_r)^-]^2dr\right]\\
&\leq C\mathbb{E}^m[(\Phi(W_T)-S_T)^2]+C\mathbb{E}^m\left[\int_t^T \{f^0(r,W_r)^2+h^0(r,W_r)^2+g^0(r,W_r)^2\}dr\right]
\end{aligned}
$$

$$
+C\int_t^T \mathbb{E}^m[(Y_r^{k,n}-S_r)^2]\|k_r\|_{l^2}^2dr + C\mathbb{E}^m\left[\int_t^T (Y_r^{k,n}-S_r)^2dr\right]
$$
$$
+C\mathbb{E}^m\left[\int_t^T\left[\left(\frac{\partial L}{\partial r}(r,W_r)+\Delta L(r,W_r)\right)^2+|L(r,W_r)|^2+|\nabla L(r,W_r)|^2\right]dr, \quad (4.14)
$$

where the condition on α in the Assumption 2.1 was used. Now the desired conclusion (4.8) follows from the Grownwall's inequality. □

Lemma 4.3 *For $M>0$, we have*

$$
\lim_{n\to\infty}\sup_{\{k\in K;\|k\|_K\le M\}}\mathbb{E}^m[\sup_{0\le t\le T}[(Y_t^{k,n}-S_t)^-]^4]=0. \quad (4.15)
$$

Proof Let $G(z)=(z^-)^4$. By the Ito's formula we have

$$
\begin{aligned}
G(Y_t^{k,n}-S_t) &= \int_t^T G'(Y_r^{k,n}-S_r)f(r,W_r,Y_r^{k,n},Z_r^{k,n})dr \\
&+\sum_{j=1}^{\infty}\int_t^T G'(Y_r^{k,n}-S_r)h_j(r,W_r,Y_r^{k,n},Z_r^{k,n})k_r^j dr \\
&+n\int_t^T G'(Y_r^{k,n}-S_r)(Y_r^{k,n}-S_r)^-dr \\
&+\frac{1}{2}\int_s^T G'(Y_r^{k,n}-S_r)g(r,W_r,Y_r^{k,n},Z_r^{k,n})*dW_r \\
&+\int_t^T\Big\langle\nabla(G'(u^{k,n}(r,\cdot)-L(r,\cdot))),g(r,\cdot,u^{k,n}(r,\cdot),\nabla u^{k,n}(r,\cdot))\Big\rangle(W_r)dr \\
&-\int_t^T G'(Y_r^{k,n}-S_r)Z_r^{k,n}dW_r+\int_t^T G'(Y_r^{k,n}-S_r)\frac{\partial L}{\partial r}(r,W_r)dr \\
&+\int_t^T G'(Y_r^{k,n}-S_r)\frac{1}{2}\Delta L(r,W_r)dr+\int_t^T G'(Y_r^{k,n}-S_r)\nabla L(r,W_r)dW_r \\
&-\frac{1}{2}\int_t^T G''(Y_r^{k,n}-S_r)|Z_r^{k,n}-\nabla L(r,W_r)|^2dr. \qquad (4.16)
\end{aligned}
$$

Rearrange the terms in the above equation to get

$$
\begin{aligned}
&[(Y_t^{k,n}-S_t)^-]^4+6\int_t^T[(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n} \\
&\quad-\nabla L(r,W_r)|^2dr+4n\int_t^T[(Y_r^{k,n}-S_r)^-]^4dr \\
&= -4\int_t^T[(Y_r^{k,n}-S_r)^-]^3f(r,W_r,Y_r^{k,n},Z_r^{k,n})dr \\
&\quad-4\sum_{j=1}^{\infty}\int_t^T[(Y_r^{k,n}-S_r)^-]^3h_j(r,W_r,Y_r^{k,n},Z_r^{k,n})k_r^j dr \\
&\quad-2\int_t^T[(Y_r^{k,n}-S_r)^-]^3g(r,W_r(x),Y_r^{k,n},Z_r^{k,n})*dW_r
\end{aligned}
$$

$$+12\int_t^T[(Y_r^{k,n}-S_r)^-]^2\Big\langle Z_r^{k,n}-\nabla L(r,W_r),g(r,W_r,Y_r^{k,n},Z_r^{k,n})\Big\rangle dr$$
$$+4\int_t^T[(Y_r^{k,n}-S_r)^-]^3Z_r^{k,n}dW_r-4\int_t^T[(Y_r^{k,n}-S_r)^-]^3\frac{\partial L}{\partial r}(r,W_r)dr$$
$$-2\int_t^T[(Y_r^{k,n}-S_r)^-]^3\Delta L(r,W_r)dr-4\int_t^T[(Y_r^{k,n}-S_r)^-]^3\nabla L(r,W_r)dW_r. \tag{4.17}$$

By Assumption 2.1, for any given positive constant ε_1 we have

$$\begin{aligned}
&12\int_t^T[(Y_r^{k,n}-S_r)^-]^2\Big\langle Z_r^{k,n}-\nabla L(r,W_r),g(r,W_r,Y_r^{k,n},Z_r^{k,n})\Big\rangle dr\\
&=12\int_t^T[(Y_r^{k,n}-S_r)^-]^2\langle Z_r^{k,n}-\nabla L(r,W_r),g(r,W_r,Y_r^{k,n},Z_r^{k,n})\\
&\quad-g(r,W_r,S_r,\nabla L(r,W_r))\rangle dr\\
&\quad+12\int_t^T[(Y_r^{k,n}-S_r)^-]^2\langle Z_r^{k,n}-\nabla L(r,W_r),g(r,W_r,S_r,\nabla L(r,W_r))\\
&\quad-g(r,W_r,0,0)\rangle dr\\
&\quad+12\int_t^T[(Y_r^{k,n}-S_r)^-]^2\Big\langle Z_r^{k,n}-\nabla L(r,W_r),g(r,W_r,0,0)\Big\rangle dr\\
&\le C\int_t^T[(Y_r^{k,n}-S_r)^-]^3|Z_r^{k,n}-\nabla L(r,W_r)|dr\\
&\quad+12\alpha\int_t^T[(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n}-\nabla L(r,W_r)|^2dr\\
&\quad+C\int_t^T[(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n}-\nabla L(r,W_r)|[|L(r,W_r)|+|\nabla L(r,W_r)|]dr\\
&\quad+C\int_t^T[(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n}-\nabla L(r,W_r)|g^0(r,W_r)dr\\
&\le \varepsilon_1\int_t^T[(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n}-\nabla L(r,W_r)|^2dr\\
&\quad+12\alpha\int_t^T[(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n}-\nabla L(r,W_r)|^2dr\\
&\quad+C\int_t^T[(Y_r^{k,n}-S_r)^-]^4[|L(r,W_r)|^2+|\nabla L(r,W_r)|^2\\
&\quad+g^0(r,W_r)^2]dr+C\int_t^T[(Y_r^{k,n}-S_r)^-]^4dr
\end{aligned}$$

$$\leq (\varepsilon_1 + 12\alpha) \int_t^T [(Y_r^{k,n} - S_r)^-]^2 |Z_r^{k,n} - \nabla L(r, W_r)|^2 dr$$
$$+ C \int_t^T [(Y_r^{k,n} - S_r)^-]^4 dr \tag{4.18}$$

Using again Assumption 2.1 and the similar computation as above we can show that for any constants $\varepsilon_2 > 0$, $\varepsilon_3 > 0$,

$$- 4 \sum_{j=1}^{\infty} \int_t^T [(Y_r^{k,n} - S_r)^-]^3 h_j(r, W_r, Y_r^{k,n}, Z_r^{k,n}) k_r^j dr$$
$$\leq \varepsilon_2 \int_t^T [(Y_r^{k,n} - S_r)^-]^2 |Z_r^{k,n} - \nabla L(r, W_r)|^2 dr + C \int_t^T [(Y_r^{k,n} - S_r)^-]^4 dr$$
$$+ C \int_t^T [(Y_r^{k,n} - S_r)^-]^4 \|k_r\|_{l^2}^2 dr + C \int_t^T [(Y_r^{k,n} - S_r)^-]^2 dr, \tag{4.19}$$

and

$$- 4 \int_t^T [(Y_r^{k,n} - S_r)^-]^3 f(r, W_r, Y_r^{k,n}, Z_r^{k,n}) dr$$
$$\leq \varepsilon_3 \int_t^T [(Y_r^{k,n} - S_r)^-]^2 |Z_r^{k,n} - \nabla L(r, W_r)|^2 dr + C \int_t^T [(Y_r^{k,n} - S_r)^-]^4 dr$$
$$+ C \int_t^T [(Y_r^{k,n} - S_r)^-]^2 dr. \tag{4.20}$$

Put (4.20), (4.19), (4.18) and (4.17) together, select the constants ε_1, ε_2 and ε_3 sufficiently small, and take expectation to get

$$\mathbb{E}^m[[(Y_t^{k,n} - S_t)^-]^4] + \mathbb{E}^m \left[\int_t^T [(Y_r^{k,n} - S_r)^-]^2 |Z_r^{k,n} - \nabla L(r, W_r)|^2 dr \right]$$
$$+ 4n\mathbb{E}^m \left[\int_t^T [(Y_r^{k,n} - S_r)^-]^4 dr \right]$$
$$\leq + C \int_t^T \mathbb{E}^m[[(Y_r^{k,n} - S_r)^-]^4] dr$$
$$+ C \int_t^T \mathbb{E}^m[[(Y_r^{k,n} - S_r)^-]^4] \|k_r\|_{l^2}^2 dr + C\mathbb{E}^m \left[\int_t^T [(Y_r^{k,n} - S_r)^-]^2 dr \right] \tag{4.21}$$

Applying the Grownwall's inequality and Lemma 4.2 we obtain

$$\lim_{n\to\infty} \sup_{\{k\in K;\, \|k\|_K \leq M\}} \sup_{0\leq t\leq T} \mathbb{E}^m[[(Y_t^{k,n} - S_t)^-]^4]$$
$$\leq C_M \lim_{n\to\infty} \sup_{\{k\in K;\, \|k\|_K \leq M\}} \mathbb{E}^m \left[\int_0^T [(Y_r^{k,n} - S_r)^-]^2 dr \right] = 0, \tag{4.22}$$

and

$$\lim_{n\to\infty}\sup_{\{k\in K;\|k\|_K\le M\}}\mathbb{E}^m\left[\int_0^T [(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n}-\nabla L(r,W_r)|^2dr\right]=0. \quad (4.23)$$

Observe that by the assumptions on the function g,

$$\begin{aligned}
&2\mathbb{E}^m\left[\sup_{0\le t\le T}|\int_t^T [(Y_r^{k,n}-S_r)^-]^3 g(r,W_r(x),Y_r^{k,n},Z_r^{k,n})*dW_r|\right]\\
&\le C\mathbb{E}^m\left[\left(\int_0^T [(Y_r^{k,n}-S_r)^-]^6|g|^2(r,W_r(x),Y_r^{k,n},Z_r^{k,n})dr\right)^{\frac{1}{2}}\right]\\
&\le \frac{1}{4}\mathbb{E}^m\left[\sup_{0\le r\le T}[(Y_r^{k,n}-S_r)^-]^4\right]\\
&\quad+C\mathbb{E}^m\left[\int_0^T [(Y_r^{k,n}-S_r)^-]^2|g|^2(r,W_r(x),Y_r^{k,n},Z_r^{k,n})dr\right]\\
&\le \frac{1}{4}\mathbb{E}^m\left[\sup_{0\le r\le T}[(Y_r^{k,n}-S_r)^-]^4\right]+C\mathbb{E}^m\left[\int_0^T [(Y_r^{k,n}-S_r)^-]^4dr\right]\\
&\quad+C\mathbb{E}^m\left[\int_0^T [(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n}-\nabla L(r,W_r)|^2dr\right]\\
&\quad+C\mathbb{E}^m\left[\int_0^T [(Y_r^{k,n}-S_r)^-]^2dr\right],
\end{aligned} \quad (4.24)$$

and

$$\begin{aligned}
&4\mathbb{E}^m\left[\sup_{0\le t\le T}|\int_t^T [(Y_r^{k,n}-S_r)^-]^3\langle Z_r^{k,n}-\nabla L(r,W_r),dW_r\rangle|\right]\\
&\le C\mathbb{E}^m\left[\left(\int_0^T [(Y_r^{k,n}-S_r)^-]^6|Z_r^{k,n}-\nabla L(r,W_r)|^2dr\right)^{\frac{1}{2}}\right]\\
&\le \frac{1}{4}\mathbb{E}^m\left[\sup_{0\le r\le T}[(Y_r^{k,n}-S_r)^-]^4\right]\\
&\quad+C\mathbb{E}^m\left[\int_0^T [(Y_r^{k,n}-S_r)^-]^2|Z_r^{k,n}-\nabla L(r,W_r)|^2dr\right].
\end{aligned} \quad (4.25)$$

Using (4.23)–(4.25) and taking supremum over the interval $[0,T]$ in (4.17) we further deduce that

$$\lim_{n\to\infty}\sup_{\{k\in K;\|k\|_K\le M\}}\mathbb{E}^m\left[\sup_{0\le t\le T}[(Y_t^{k,n}-S_t)^-]^4\right]=0.$$

completing the proof. □

Proposition 4.1 *For any $M > 0$, we have*

$$\lim_{n\to\infty} \sup_{\{k\in K;\|k\|_K\le M\}} |u^{k,n} - u^k|_{H_T} = 0. \tag{4.26}$$

Proof We note that for any $n, q \ge 1$,

$$\begin{aligned} &|u^{k,n} - u^{k,q}|^2_{H_T} \\ &\le \mathbb{E}^m\left[\sup_{0\le r\le T}(Y_r^{k,n} - Y_r^{k,q})^2\right] + C\mathbb{E}^m\left[\int_0^T |Z_r^{k,n} - Z_r^{k,q}|^2 dr\right]. \end{aligned} \tag{4.27}$$

(4.27) follows from the fact that the law of W_t under $\mathbb{P}^m$ is the Lebesgue measure m for any $t \ge 0$. Please also consult [20] (Theorem 3, Corollary 1) for details. Recall that for each $k \in K$, $u^{k,n} \to u^k$ as $n \to \infty$. Thus, to prove (4.26), it is sufficient to show

$$\lim_{n,q\to\infty} \sup_{\{k\in K;\|k\|_K\le M\}} \mathbb{E}^m\left[\sup_{0\le t\le T}(Y_t^{k,n} - Y_t^{k,q})^2\right] = 0, \tag{4.28}$$

and

$$\lim_{n,q\to\infty} \sup_{\{k\in K;\|k\|_K\le M\}} \mathbb{E}^m\left[\int_0^T |Z_r^{k,n} - Z_r^{k,q}|^2 dr\right] = 0. \tag{4.29}$$

We will achieve this with the help of backward stochastic differential equations satisfied by $Y_t^{k,n} = u^{k,n}(t, W_t)$. Applying Ito's formula we have

$$\begin{aligned} &(Y_t^{k,n} - Y_t^{k,q})^2 + \int_t^T |Z_r^{k,n} - Z_r^{k,q}|^2 dr \\ &= 2\int_t^T (Y_r^{k,n} - Y_r^{k,q})(f(r, W_r, Y_r^{k,n}, Z_r^{k,n}) - f(r, W_r, Y_r^{k,q}, Z_r^{k,q}))dr \\ &\quad + 2\sum_{j=1}^{\infty}\int_t^T (Y_r^{k,n} - Y_r^{k,q})(h_j(r, W_r, Y_r^{k,n}, Z_r^{k,n}) - h_j(r, W_r, Y_r^{k,q}, Z_r^{k,q}))k_r^j dr \\ &\quad + 2n\int_t^T (Y_r^{k,n} - Y_r^{k,q})(Y_r^{k,n} - S_r)^- dr - 2q\int_t^T (Y_r^{k,n} - Y_r^{k,q})(Y_r^{k,q} - S_r)^- dr \\ &\quad + \int_t^T (Y_r^{k,n} - Y_r^{k,q})(g(r, W_r, Y_r^{k,n}, Z_r^{k,n}) - g(r, W_r, Y_r^{k,m}, Z_r^{k,q})) * dW_r \\ &\quad + 2\int_t^T \left\langle Z_r^{k,n} - Z_r^{k,q}, g(r, W_r, Y_r^{k,n}, Z_r^{k,n}) - g(r, W_r, Y_r^{k,q}, Z_r^{k,q})\right\rangle dr \\ &\quad - 2\int_t^T (Y_r^{k,n} - Y_r^{k,q})\langle Z_r^{k,n} - Z_r^{k,q}, dW_r\rangle \\ &:= I_1^{k,n,q}(t) + I_2^{k,n,q}(t) + I_3^{k,n,q}(t) + I_4^{k,n,q}(t) + I_5^{k,n,q}(t) + I_6^{k,n,q}(t) + I_7^{k,n,q}(t). \end{aligned} \tag{4.30}$$

Note that

$$\begin{aligned}
&I_3^{k,n,q}(t) + I_4^{k,n,q}(t)\\
&= 2n\int_t^T (Y_r^{k,n} - Y_r^{k,q})(Y_r^{k,n} - S_r)^- dr - 2q\int_t^T (Y_r^{k,n} - Y_r^{k,q})(Y_r^{k,q} - S_r)^- dr\\
&\le 2n\int_t^T (Y_r^{k,q} - S_r)^-(Y_r^{k,n} - S_r)^- dr + 2q\int_t^T (Y_r^{k,n} - S_r)^-(Y_r^{k,q} - S_r)^- dr\\
&\le 2\sup_{0\le r\le T}(Y_r^{k,q} - S_r)^- n\int_0^T (Y_r^{k,n} - S_r)^- dr\\
&\quad + 2\sup_{0\le r\le T}(Y_r^{k,n} - S_r)^- q\int_0^T (Y_r^{k,q} - S_r)^- dr.
\end{aligned} \tag{4.31}$$

By Young's inequality, we have for any $\delta_1 > 0$,

$$I_1^{k,n,q}(t) \le \delta_1\int_t^T |Z_r^{k,n} - Z_r^{k,q}|^2 dr + C\int_t^T |Y_r^{k,n} - Y_r^{k,q}|^2 dr. \tag{4.32}$$

Moreover for any $\delta_2 > 0$, we have

$$I_2^{k,n,q}(t) \le \delta_2\int_t^T |Z_r^{k,n} - Z_r^{k,q}|^2 dr + C\int_t^T |Y_r^{k,n} - Y_r^{k,q}|^2(1 + \|k_r\|_{l^2}^2) dr. \tag{4.33}$$

Using Young's inequality again, we have for any $\delta_3 > 0$,

$$I_6^{k,n,q}(t) \le (\delta_3 + 2\alpha)\int_t^T |Z_r^{k,n} - Z_r^{k,q}|^2 dr + C\int_t^T |Y_r^{k,n} - Y_r^{k,q}|^2 dr. \tag{4.34}$$

Substitute (4.31)–(4.34) back to (4.30), choose constants $\delta_i, i = 1, 2, 3$ sufficiently small and take expectation to obtain

$$\begin{aligned}
&\mathbb{E}^m[(Y_t^{k,n} - Y_t^{k,q})^2] + \mathbb{E}^m\left[\int_t^T |Z_r^{k,n} - Z_r^{k,q}|^2 dr\right]\\
&\le C\mathbb{E}^m\left[\int_t^T |Y_r^{k,n} - Y_r^{k,q}|^2(1 + \|k_r\|_{l^2}^2) dr\right]\\
&\quad + C\left(\mathbb{E}^m\left[\sup_{0\le r\le T}[(Y_r^{k,q} - S_r)^-]^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^m\left[(n\int_0^T (Y_r^{k,n} - S_r)^- dr)^2\right]\right)^{\frac{1}{2}}\\
&\quad + C\left(\mathbb{E}^m\left[\sup_{0\le r\le T}[(Y_r^{k,n} - S_r)^-]^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^m\left[\left(q\int_0^T (Y_r^{k,q} - S_r)^- dr\right)^2\right]\right)^{\frac{1}{2}}
\end{aligned} \tag{4.35}$$

Using Lemmas 4.1, 4.3 and applying the Grownwall's inequality we deduce that

$$\lim_{n,q\to\infty}\sup_{\{k\in K;\|k\|_K\le M\}}\sup_{0\le t\le T}\mathbb{E}^m[(Y_t^{k,n}-Y_t^{k,q})^2]=0, \tag{4.36}$$

and

$$\lim_{n,q\to\infty}\sup_{\{k\in K;\|k\|_K\le M\}}\mathbb{E}^m\left[\int_0^T|Z_t^{k,n}-Z_t^{k,q}|^2dt\right]=0. \tag{4.37}$$

Next we will strengthen the convergence in (4.36) to

$$\lim_{n,q\to\infty}\sup_{\{k\in K;\|k\|_K\le M\}}\mathbb{E}^m\left[\sup_{0\le t\le T}(Y_t^{k,n}-Y^{k,q})^2\right]=0. \tag{4.38}$$

We notice that by the Burkhölder's inequality, for any $\delta_4>0$ we have

$$\begin{aligned}
&\mathbb{E}^m[\sup_{0\le t\le T}|I_5^{k,n,q}(t)|]\\
&\le C\mathbb{E}^m\left[\left(\int_0^T(Y_r^{k,n}-Y_r^{k,q})^2|g(r,W_r,Y_r^{k,n},Z_r^{k,n})-g(r,W_r,Y_r^{k,q},Z_r^{k,q})|^2dr\right)^{\frac{1}{2}}\right]\\
&\le \delta_4\mathbb{E}^m\left[\sup_{0\le r\le T}(Y_r^{k,n}-Y_r^{k,q})^2\right]\\
&\quad+C\mathbb{E}^m\left[\int_0^T|g(r,W_r,Y_r^{k,n},Z_r^{k,n})-g(r,W_r,Y_r^{k,q},Z_r^{k,q})|^2dr\right]\\
&\le \delta_4\mathbb{E}^m\left[\sup_{0\le r\le T}(Y_r^{k,n}-Y_r^{k,q})^2\right]+C\mathbb{E}^m\left[\int_0^T|Z_r^{k,n}-Z_r^{k,q}|^2dr\right]\\
&\quad+C\mathbb{E}^m\left[\int_0^T|Y_r^{k,n}-Y_r^{k,q}|^2dr\right].
\end{aligned} \tag{4.39}$$

Similarly, we have for $\delta_5>0$

$$\begin{aligned}
&\mathbb{E}^m\left[\sup_{0\le t\le T}|I_7^{k,n,q}(t)|\right]\\
&\le C\mathbb{E}^m\left[\left(\int_0^T(Y_r^{k,n}-Y_r^{k,q})^2|Z_r^{k,n}-Z_r^{k,q}|^2dr\right)^{\frac{1}{2}}\right]\\
&\le \delta_5\mathbb{E}^m\left[\sup_{0\le r\le T}(Y_r^{k,n}-Y_r^{k,q})^2\right]+C\mathbb{E}^m\left[\int_0^T|Z_r^{k,n}-Z_r^{k,q}|^2dr\right].
\end{aligned} \tag{4.40}$$

Now use the above two estimates (4.39) and (4.40) and the already proved (4.36) to obtain (4.38). This completes the proof. □

Theorem 4.1 *Let Assumptions 2.1 hold. Assume that $k^\varepsilon \to k$ weakly in the Hilbert space K as $\varepsilon \to 0$. Then u^{k^ε} converges to u^k in the space H_T, where u^{k^ε} denotes the solution of equation (4.1) with k^ε replacing k.*

Proof We will first prove a similar convergence result for the corresponding penalized PDEs and then combined with the uniform convergence proved in Proposition 4.1 we complete the proof of Theorem 4.1. Let $u^{k^\varepsilon,n}$ be the solution to the following penalized PDE:

$$\begin{aligned} du^{k^\varepsilon,n}(t,x) &+ \frac{1}{2}\Delta u^{k^\varepsilon,n}(t,x) + \sum_{i=1}^{d} \partial_i g_i(t,x,u^{k^\varepsilon,n}(t,x),\nabla u^{k^\varepsilon,n}(t,x))dt \\ &+ f(t,x,u^{k^\varepsilon,n}(t,x),\nabla u^{k^\varepsilon,n}(t,x))dt \\ &+ \sum_{j=1}^{\infty} h_j(t,x,u^{k^\varepsilon,n}(t,x),\nabla u^{k^\varepsilon,n}(t,x))k_t^{\varepsilon,j}dt \\ &= -n(u^{k^\varepsilon,n}(t,x) - L(t,x))^- dt, \end{aligned} \tag{4.41}$$

$$u^{k^\varepsilon,n}(T,x) = \Phi(x), \qquad x \in \mathbb{R}^d. \tag{4.42}$$

We first fix the integer n and show $\lim_{\varepsilon\to 0} \|u^{k^\varepsilon,n} - u^{k,n}\|_{H_T} = 0$, $u^{k,n}$ is the solution of equation (4.41) with k^ε replaced by k. To this end, we first prove that the family $\{u^{k^\varepsilon,n}, \varepsilon > 0\}$ is tight in the space $L^2([0,T], L^2_{loc}(\mathbb{R}^d))$. Using the chain rule and Gronwall's inequality, as in Lemma 4.1 , we can show that

$$\sup_\varepsilon \|u^{k^\varepsilon,n}\|^2_{H_T} = \sup_\varepsilon \left\{ \sup_{0\le t\le T} |u^{k^\varepsilon,n}(t)|^2 + \int_0^T \|u^{k^\varepsilon,n}(t)\|^2 dt \right\} < \infty. \tag{4.43}$$

For $\beta \in (0,1)$, recall that $W^{\beta,2}([0,T],V^*)$ is the space of mappings $v(\cdot) : [0,T] \to V^*$ that satisfy

$$\|v\|^2_{W^{\beta,2}([0,T],V^*)} = \int_0^T \|v(t)\|^2_{V^*} + \int_0^T \int_0^T \frac{\|v(t)-v(s)\|^2_{V^*}}{|t-s|^{1+2\beta}} < \infty. \tag{4.44}$$

It is well known (see e.g. [15]) that the imbedding

$$L^2([0,T],V) \cap W^{\beta,2}([0,T],V^*) \hookrightarrow L^2([0,T],L^2_{loc}(\mathbb{R}^d))$$

is compact. As an equation in V^*, we have

$$\begin{aligned} u^{k^\varepsilon,n}(t) \; = \; \Phi &+ \frac{1}{2}\int_t^T \Delta u^{k^\varepsilon,n}(s)ds + \int_t^T \sum_{i=1}^{d} \partial_i g_i(s,x,u^{k^\varepsilon,n}(s,x),\nabla u^{k^\varepsilon,n}(s,x))ds \\ &+ \int_t^T f(s,x,u^{k^\varepsilon,n}(s,x),\nabla u^{k^\varepsilon,n}(s,x))ds \end{aligned}$$

$$+\sum_{j=1}^{\infty}\int_t^T h_j(s,x,u^{k^\varepsilon,n}(s,x),\nabla u^{k^\varepsilon,n}(s,x))k_s^{\varepsilon,j}ds$$
$$+n\int_t^T (u^{k^\varepsilon,n}(s,x)-L(s,x))^- ds$$
$$:=\Phi+I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t). \tag{4.45}$$

In view of (4.43), we have

$$\begin{aligned}\|I_1(t)-I_1(s)\|_{V^*}^2 &\le C\int_s^t \|\Delta u^{k^\varepsilon,n}(r)\|_{V^*}^2 dr(|t-s|)\\ &\le C\int_0^T \|u^{k^\varepsilon,n}(r)\|^2 dr(|t-s|)\le C|t-s|. \end{aligned}\tag{4.46}$$

Using the condition (iii) in Assumption 2.1, we have

$$\|I_4(t)-I_4(s)\|_{V^*}^2\le C\left(\int_0^T \|k_r^\varepsilon\|_{l^2}^2 dr\right)|\bar{h}|^2|t-s|\le C|t-s|. \tag{4.47}$$

By (4.43) and the similar calculations as above we also have

$$\|I_i(t)-I_i(s)\|_{V^*}^2\le C|t-s|,\quad i=2,3,5. \tag{4.48}$$

Thus, for $\beta\in(0,\frac{1}{2})$, it follows from (4.45) –(4.48) that

$$\sup_\varepsilon\|u^{k^\varepsilon,n}\|_{W^{\beta,2}([0,T],V^*)}^2<\infty. \tag{4.49}$$

Combining (4.49) with (4.43), we conclude that $\{u^{k^\varepsilon,n},\varepsilon>0\}$ is tight in the space $L^2([0,T],L^2_{loc}(\mathbb{R}^d))$. Now, applying the chain rule, we obtain

$$\begin{aligned}&|u^{k^\varepsilon,n}(t)-u^{k,n}(t)|^2\\ &=-\int_t^T |\nabla(u^{k^\varepsilon,n}(s)-u^{k,n}(s))|^2 ds\\ &\quad-2\int_t^T \Big\langle g(s,\cdot,u^{k^\varepsilon,n}(s,\cdot),\nabla u^{k^\varepsilon,n}(s,\cdot))\\ &\quad-g(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot)),\nabla(u^{k^\varepsilon,n}(s)-u^{k,n}(s))\Big\rangle ds\\ &\quad+2\int_t^T \Big\langle f(s,\cdot,u^{k^\varepsilon,n}(s,\cdot),\nabla u^{k^\varepsilon,n}(s,\cdot))\\ &\quad-f(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot)),u^{k^\varepsilon,n}(s)-u^{k,n}(s)\Big\rangle ds\\ &\quad+2\int_t^T \Big\langle u^{k^\varepsilon,n}(s)-u^{k,n}(s),\sum_{j=1}^{\infty}(h_j(s,\cdot,u^{k^\varepsilon,n}(s,\cdot),\nabla u^{k^\varepsilon,n}(s,\cdot))\end{aligned}$$

$$
\begin{aligned}
&- h_j(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot)))k_s^{\varepsilon,j}\Big\rangle ds\\
&+2\int_t^T \Big\langle u^{k^\varepsilon,n}(s)-u^{k,n}(s),\sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon,j}-k_s^j)\Big\rangle ds\\
&+2n\int_t^T \Big\langle u^{k^\varepsilon,n}(s)-u^{k,n}(s),(u^{k^\varepsilon,n}(s)-L(s,\cdot))^- -(u^{k,n}(s,\cdot)-L(s,\cdot))^-\Big\rangle ds
\end{aligned}
\tag{4.50}
$$

By the assumptions on h_j and Young's inequality, we see that for any given $\delta_1>0$,

$$
\begin{aligned}
&2\int_t^T \Big\langle u^{k^\varepsilon,n}(s)-u^{k,n}(s),\sum_{j=1}^{\infty}(h_j(s,\cdot,u^{k^\varepsilon,n}(s,\cdot),\nabla u^{k^\varepsilon,n}(s,\cdot))\\
&\quad -h_j(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot)))k_s^{\varepsilon,j}\Big\rangle ds\\
&\le \delta_1\int_t^T |\nabla(u^{k^\varepsilon,n}(s)-u^{k,n}(s))|^2ds + C\int_t^T |u^{k^\varepsilon,n}(s)-u^{k,n}(s)|^2(1+\|k_s^\varepsilon\|_{l^2}^2)ds.
\end{aligned}
\tag{4.51}
$$

Using the assumptions on f, g and (4.51) it follows from (4.50) that there exist positive constants δ, C such that

$$
\begin{aligned}
&|u^{k^\varepsilon,n}(t)-u^{k,n}(t)|^2+\delta\int_t^T |\nabla(u^{k^\varepsilon,n}(s)-u^{k,n}(s))|^2ds\\
&\le C\int_t^T |u^{k^\varepsilon,n}(s)-u^{k,n}(s)|^2(1+\|k_s^\varepsilon\|_{l^2}^2)ds\\
&\quad +2\int_t^T \Big\langle u^{k^\varepsilon,n}(s)-u^{k,n}(s),\sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon,j}-k_s^j)\Big\rangle ds.
\end{aligned}
\tag{4.52}
$$

By Gronwall's inequality, (4.52) yields that

$$
\begin{aligned}
&\sup_{0\le t\le T}\Big|u^{k^\varepsilon,n}(t)-u^{k,n}(t)\Big|^2\\
&\le \exp(C\int_0^T (1+\|k_s^\varepsilon\|_{l^2}^2)ds)\sup_{0\le t\le T}\Bigg|\int_t^T \Big\langle u^{k^\varepsilon,n}(s)\\
&\quad - u^{k,n}(s),\sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon,j}-k_s^j)\Big\rangle ds\Bigg|\\
&\le C\sup_{0\le t\le T}\Bigg|\int_t^T \Big\langle u^{k^\varepsilon,n}(s)-u^{k,n}(s),
\end{aligned}
$$

$$\times \sum_{j=1}^{\infty} h_j(s, \cdot, u^{k,n}(s,\cdot), \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon,j} - k_s^j)\Big\rangle ds\Bigg|. \tag{4.53}$$

To show $\lim_{\varepsilon\to 0} \|u^{k^\varepsilon,n} - u^{k,n}\|_{H_T} = 0$, in view of (4.52) and (4.53), it suffices to prove

$$\lim_{\varepsilon\to 0} \sup_{0\le t\le T} \left|\int_t^T \Big\langle u^{k^\varepsilon,n}(s) - u^{k,n}(s), \sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot), \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon,j} - k_s^j)\Big\rangle ds\right| = 0. \tag{4.54}$$

This will be achieved if we show that for any sequence $\varepsilon_m \to 0$, one can find a subsequence $\varepsilon_{m_i} \to 0$ such that

$$\lim_{i\to\infty} \sup_{0\le t\le T} \left|\int_t^T \Big\langle u^{k^{\varepsilon_{m_i}},n}(s) - u^{k,n}(s), \sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot), \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j} - k_s^j)\Big\rangle ds\right| = 0 \tag{4.55}$$

Now fix a sequence $\varepsilon_m \to 0$. Since $\{u^{k^{\varepsilon_m},n}, m \ge 1\}$ is tight in $L^2([0,T], L^2_{loc}(\mathbb{R}^d))$, there exist a subsequence $m_i, i \ge 1$ and a mapping $\tilde{u}$ such that $u^{k^{\varepsilon_{m_i}},n} \to \tilde{u}$ in $L^2([0,T], L^2_{loc}(\mathbb{R}^d))$. Moreover, because of the uniform bound of $u^{k^{\varepsilon_{m_i}},n}$ in (4.43), $\tilde{u}$ belongs to $L^2([0,T], H)$. Now,

$$\begin{aligned}
&\sup_{0\le t\le T}\left|\int_t^T \Big\langle u^{k^{\varepsilon_{m_i}},n}(s) - u^{k,n}(s), \sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot), \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j} - k_s^j)\Big\rangle ds\right.\\
&\le \sup_{0\le t\le T}\left|\int_t^T \Big\langle u^{k^{\varepsilon_{m_i}},n}(s) - \tilde{u}(s), \sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot), \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j} - k_s^j)\Big\rangle ds\right.\\
&\quad + \sup_{0\le t\le T}\left|\int_t^T \Big\langle \tilde{u}(s) - u^{k,n}(s), \sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot), \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j} - k_s^j)\Big\rangle ds.\right.
\end{aligned} \tag{4.56}$$

Since $k^{\varepsilon_{m_i}} \to k$ weakly in $L^2([0,T], l^2)$, for every $t > 0$, it holds that

$$\lim_{i\to\infty} \int_t^T \Big\langle \tilde{u}(s) - u^{k,n}(s), \sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot), \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j} - k_s^j)\Big\rangle ds = 0. \tag{4.57}$$

On the other hand, using the assumption on h, for $0 < t_1 < t_2 \le T$, we have

$$\begin{aligned}
&\left|\int_{t_1}^{t_2} \left\langle \tilde{u}(s) - u^{k,n}(s), \sum_{j=1}^{\infty} h_j(s,\cdot,u^{k,n}(s,\cdot), \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j} - k_s^j)\right\rangle ds\right|\\
&\le C\left(\int_{t_1}^{t_2} |\tilde{u}(s) - u^{k,n}(s)|^2 ds\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|k_s^{\varepsilon_{m_i}} - k_s\|_{l^2}^2 ds\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C\left(\int_{t_1}^{t_2}|\tilde{u}(s)-u^{k,n}(s)|^2ds\right)^{\frac{1}{2}}. \tag{4.58}$$

Combing (4.57) and (4.58) we deduce that

$$\lim_{i\to\infty}\sup_{0\leq t\leq T}\left|\int_t^T\left\langle\tilde{u}(s)-u^{k,n}(s),\sum_{j=1}^{\infty}h_j(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j}-k_s^j)\right\rangle ds\right|=0. \tag{4.59}$$

By Hölder's inequality and the assumption on h, we have

$$\begin{aligned}
&\left|\int_t^T\left\langle u^{k^{\varepsilon_{m_i}},n}(s)-\tilde{u}(s),\sum_{j=1}^{\infty}h_j(s,\cdot,u^{k,n}(s,\cdot),\nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j}-k_s^j)\right\rangle ds\right|\\
&\leq\int_0^T\int_{\mathbb{R}^d}|u^{k^{\varepsilon_{m_i}},n}(s,x)-\tilde{u}(s,x)|\\
&\quad\times\left(\sum_{j=1}^{\infty}h_j^2(s,\cdot,u^{k,n}(s,x),\nabla u^{k,n}(s,x))^2\right)^{\frac{1}{2}}dx(\|k_s^{\varepsilon_{m_i}}\|_{l^2}+\|k_s\|_{l^2})ds)\\
&\leq\left(\int_0^T(\|k_s^{\varepsilon_{m_i}}\|_{l^2}^2+\|k_s\|_{l^2}^2)ds\right)^{\frac{1}{2}}\left(\int_0^T ds\left(\int_{\mathbb{R}^d}|u^{k^{\varepsilon_{m_i}},n}(s,x)-\tilde{u}(s,x)|\bar{h}(x)dx\right)^2\right)^{\frac{1}{2}}\\
&\leq C\left(\int_0^T ds\left(\int_{\mathbb{R}^d}|u^{k^{\varepsilon_{m_i}},n}(s,x)-\tilde{u}(s,x)|\bar{h}(x)dx\right)^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.60}$$

For any $M>0$, denote by B_M the ball in $\mathbb{R}^d$ centered at zero with radius M. We can bound the right side of (4.60) as follows:

$$\begin{aligned}
&\int_0^T ds\left(\int_{\mathbb{R}^d}|u^{k^{\varepsilon_{m_i}},n}(s,x)-\tilde{u}(s,x)|\bar{h}(x)dx\right)^2\\
&\quad\leq C\int_0^T ds\left(\int_{B_M}|u^{k^{\varepsilon_{m_i}},n}(s,x)-\tilde{u}(s,x)|^2dx\right)\left(\int_{\mathbb{R}^d}\bar{h}^2(x)dx\right)\\
&\qquad+C\int_0^T ds\left(\int_{\mathbb{R}^d}|u^{k^{\varepsilon_{m_i}},n}(s,x)-\tilde{u}(s,x)|^2dx\right)\left(\int_{B_M^c}\bar{h}^2(x)dx\right.\\
&\quad\leq C\int_0^T ds\left(\int_{B_M}|u^{k^{\varepsilon_{m_i}},n}(s,x)-\tilde{u}(s,x)|^2dx\right)+C\int_{B_M^c}\bar{h}^2(x)dx,
\end{aligned} \tag{4.61}$$

where the uniform $L^2([0,T]\times\mathbb{R}^d)$-bound of $u^{k^{\varepsilon_{m_i}},n}$ has been used. Now given any constant $\delta>0$, we can pick a constant M such that $C\int_{B_M^c}\bar{h}^2(x)dx\leq\delta$. For the chosen constant M, we have

$$\lim_{i\to\infty}\int_0^T ds\left(\int_{B_M}|u^{k^{\varepsilon_{m_i}},n}(s,x)-\tilde{u}(s,x)|^2dx\right)=0.$$

Thus, it follows from (4.60), (4.61) that

$$\lim_{i\to\infty}\sup_{0\le t\le T}\Big|\int_t^T\Big\langle u^{k^{\varepsilon_{m_i}},n}(s)-\tilde{u}(s),\sum_{j=1}^{\infty}h_j(s,\cdot,u^{k,n}(s,\cdot),\\ \nabla u^{k,n}(s,\cdot))(k_s^{\varepsilon_{m_i},j}-k_s^j)\Big\rangle ds\le\delta^{\frac{1}{2}}. \tag{4.62}$$

Since δ is arbitrary, (4.55) follows from (4.56), (4.59) and (4.62). Hence we have proved $\lim_{\varepsilon\to 0}\|u^{k^\varepsilon,n}-u^{k,n}\|_{H_T}=0$.

Now we are ready to complete the last step of the proof. For any $n\ge 1$, we have

$$\|u^{k^\varepsilon}-u^k\|_{H_T}\\ \le\|u^{k^\varepsilon}-u^{k^\varepsilon,n}\|_{H_T}+\|u^{k^\varepsilon,n}-u^{k,n}\|_{H_T}+\|u^{k,n}-u^k\|_{H_T}. \tag{4.63}$$

For any given $\delta>0$, by Proposition 4.1 there exists an integer n_0 such that $\sup_\varepsilon\|u^{k^\varepsilon}-u^{k^\varepsilon,n_0}\|_{H_T}\le\frac{\delta}{2}$ and $\|u^k-u^{k,n_0}\|_{H_T}\le\frac{\delta}{2}$. Replacing n in (4.63) by n_0 we get

$$\|u^{k^\varepsilon}-u^k\|_{H_T}\le\delta+\|u^{k^\varepsilon,n_0}-u^{k,n_0}\|_{H_T}.$$

As we just proved

$$\lim_{\varepsilon\to 0}\|u^{k^\varepsilon,n_0}-u^{k,n_0}\|_{H_T}=0,$$

we obtain that

$$\lim_{\varepsilon\to 0}\|u^{k^\varepsilon}-u^k\|_{H_T}\le\delta.$$

Since the constant δ is arbitrary, the proof is complete. □

5 Large Deviations

After the preparations in Sect. 4, we are ready to state and to prove the large deviation result. Recall that U^ε is the solution of the obstacle problem:

$$dU^\varepsilon(t,x)+\frac{1}{2}\Delta U^\varepsilon(t,x)+\sum_{i=1}^{d}\partial_i g_i(t,x,U^\varepsilon(t,x),\nabla U^\varepsilon(t,x))dt\\ +f(t,x,U^\varepsilon(t,x),\nabla U^\varepsilon(t,x))dt\\ +\sqrt{\varepsilon}\sum_{j=1}^{\infty}h_j(t,x,U^\varepsilon(t,x),\nabla U^\varepsilon(t,x))dB_t^j=-R^\varepsilon(dt,dx), \tag{5.1}$$

$$U^\varepsilon(t,x)\ge L(t,x),\qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ U^\varepsilon(T,x)=\Phi(x),\qquad x\in\mathbb{R}^d. \tag{5.2}$$

For $k \in K = L^2([0,T], l^2)$, denote by u^k the solution of the following deterministic obstacle problem:

$$
\begin{aligned}
&du^k(t,x) + \frac{1}{2}\Delta u^k(t,x) + \sum_{i=1}^{d} \partial_i g_i(t,x,u^k(t,x),\nabla u^k(t,x))dt \\
&\quad + f(t,x,u^k(t,x),\nabla u^k(t,x))dt \\
&\quad + \sum_{j=1}^{\infty} h_j(t,x,u^k(t,x),\nabla u^k(t,x))k_t^j dt = -\nu^k(dt,dx), \qquad (5.3)\\
&u^k(t,x) \geq L(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
&u^k(T,x) = \Phi(x), \qquad x \in \mathbb{R}^d. \qquad (5.4)
\end{aligned}
$$

Define a measurable mapping $\Gamma^0 : C([0,T];\mathbb{R}^\infty) \to H_T$ by

$$
\Gamma^0\left(\int_0^{\cdot} k_s ds\right) := u^k \qquad \text{for} \qquad k \in K,
$$

where u^k is the solution of (5.3). Here is the main result:

Theorem 5.1 *Let the Assumption 2.1 hold. Then the family $\{U^\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle on the space H_T with the rate function I given by*

$$
I(g) := \inf_{\{k\in K;\, g=\Gamma^0(\int_0^{\cdot} k_s ds)\}} \left\{ \frac{1}{2}\int_0^T \|k_s\|_{l^2}^2 ds \right\}, \ g \in H_T, \qquad (5.5)
$$

with the convention $\inf\{\emptyset\} = \infty$.

Proof The existence of a unique strong solution of the obstacle problem (5.1) implies that for every $\varepsilon > 0$, there exists a measurable mapping $\Gamma^\varepsilon(\cdot) : C([0,T];\mathbb{R}^\infty) \to H_T$ such that

$$
U^\varepsilon = \Gamma^\varepsilon(B(\cdot)).
$$

To prove the theorem, we are going to show that the conditions (i) and (ii) in Theorem 3.2 are satisfied. Condition (ii) is exactly the statement of Theorem 4.1. It remains to establish the condition (i) in Theorem 3.2. Recall the definitions of the spaces S_N and $\tilde{S}_N$ given in Sect. 3. Let $\{k^\varepsilon, \varepsilon > 0\} \subset \tilde{S}_N$ be a given family of stochastic processes. Applying Girsanov theorem it is easy to see that $U^{\varepsilon,k^\varepsilon} = \Gamma^\varepsilon\left(B(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^{\cdot} k^\varepsilon(s)ds\right)$ is the solution of the stochastic obstacle problem:

$$
\begin{aligned}
&dU^{\varepsilon,k^\varepsilon}(t,x) + \frac{1}{2}\Delta U^{\varepsilon,k^\varepsilon}(t,x) + \sum_{i=1}^{d} \partial_i g_i(t,x,U^{\varepsilon,k^\varepsilon}(t,x),\nabla U^{\varepsilon,k^\varepsilon}(t,x))dt \\
&\quad + f(t,x,U^{\varepsilon,k^\varepsilon}(t,x),\nabla U^{\varepsilon,k^\varepsilon}(t,x))dt
\end{aligned}
$$

$$+\sqrt{\varepsilon}\sum_{j=1}^{\infty}h_j(t,x,U^{\varepsilon,k^\varepsilon}(t,x),\nabla U^{\varepsilon,k^\varepsilon}(t,x))dB_t^j$$

$$+\sum_{j=1}^{\infty}h_j(t,x,U^{\varepsilon,k^\varepsilon}(t,x),\nabla U^{\varepsilon,k^\varepsilon}(t,x))k_t^{\varepsilon,j}dt=-\nu^\varepsilon(dt,dx), \quad (5.6)$$

$$U^{\varepsilon,k^\varepsilon}(t,x)\geq L(t,x), \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,$$
$$U^{\varepsilon,k^\varepsilon}(T,x)=\Phi(x), \qquad x\in\mathbb{R}^d. \quad (5.7)$$

Moreover, $V^{k^\varepsilon}=\Gamma^0\left(\int_0^\cdot k^\varepsilon(s)ds\right)$ is the solution of the random obstacle problem:

$$dV^{k^\varepsilon}(t,x)+\frac{1}{2}\Delta V^{k^\varepsilon}(t,x)+\sum_{i=1}^{d}\partial_i g_i(t,x,V^{k^\varepsilon}(t,x),\nabla V^{k^\varepsilon}(t,x))dt$$

$$+f(t,x,V^{k^\varepsilon}(t,x),\nabla V^{k^\varepsilon}(t,x))dt$$

$$+\sum_{j=1}^{\infty}h_j(t,x,V^{k^\varepsilon}(t,x),\nabla V^{k^\varepsilon}(t,x))k_t^{\varepsilon,j}dt=-\mu^\varepsilon(dt,dx), \quad (5.8)$$

$$V^{k^\varepsilon}(t,x)\geq L(t,x), \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,$$
$$V^{k^\varepsilon}(T,x)=\Phi(x), \qquad x\in\mathbb{R}^d. \quad (5.9)$$

The condition (ii) in Theorem 3.2 will be satisfied if we prove

$$\lim_{\varepsilon\to 0}\left\{E[\sup_{0\leq t\leq T}|U_t^{\varepsilon,k^\varepsilon}-V_t^{k^\varepsilon}|^2]+E[\int_0^T\|U_t^{\varepsilon,k^\varepsilon}-V_t^{k^\varepsilon}\|^2dt]\right\}=0, \quad (5.10)$$

here $U_t^{\varepsilon,k^\varepsilon}=U^{\varepsilon,k^\varepsilon}(t,\cdot)$ and $V_t^{k^\varepsilon}=V^{k^\varepsilon}(t,\cdot)$. The rest of the proof is to establish (5.10). By Ito formula, we have

$$|U_t^{\varepsilon,k^\varepsilon}-V_t^{k^\varepsilon}|^2+\int_t^T|\nabla(U_s^{\varepsilon,k^\varepsilon}-V_s^{k^\varepsilon})|^2ds$$
$$=-2\int_t^T\left\langle\nabla(U_s^{\varepsilon,k^\varepsilon}-V_s^{k^\varepsilon}),g(s,\cdot,U_s^{\varepsilon,k^\varepsilon},\nabla U_s^{\varepsilon,k^\varepsilon})-g(s,\cdot,V_s^{k^\varepsilon},\nabla V_s^{k^\varepsilon})\right\rangle ds$$
$$+2\int_t^T\left\langle U_s^{\varepsilon,k^\varepsilon}-V_s^{k^\varepsilon},f(s,\cdot,U_s^{\varepsilon,k^\varepsilon},\nabla U_s^{\varepsilon,k^\varepsilon})-f(s,\cdot,V_s^{k^\varepsilon},\nabla V_s^{k^\varepsilon})\right\rangle ds$$
$$+2\int_t^T\sum_{j=1}^{\infty}<U_s^{\varepsilon,k^\varepsilon}-V_s^{k^\varepsilon},h_j(s,\cdot,U_s^{\varepsilon,k^\varepsilon},\nabla U_s^{\varepsilon,k^\varepsilon})-h_j(s,\cdot,V_s^{k^\varepsilon},\nabla V_s^{k^\varepsilon})>k_s^{\varepsilon,j}ds$$
$$+2\sqrt{\varepsilon}\sum_{j=1}^{\infty}\int_t^T\left\langle U_s^{\varepsilon,k^\varepsilon}-V_s^{k^\varepsilon},h_j(s,\cdot,U_s^{\varepsilon,k^\varepsilon},\nabla U_s^{\varepsilon,k^\varepsilon})\right\rangle dB_s^j$$
$$+2\int_t^T\left\langle U_s^{\varepsilon,k^\varepsilon}-V_s^{k^\varepsilon},d\nu_s^\varepsilon-d\mu_s^\varepsilon\right\rangle+\varepsilon\sum_{j=1}^{\infty}\int_t^T|h_j(s,\cdot,U_s^{\varepsilon,k^\varepsilon},\nabla U_s^{\varepsilon,k^\varepsilon})|^2ds$$
$$:=I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t). \quad (5.11)$$

Here

$$\int_t^T \langle U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon}, d\nu_s^\varepsilon - d\mu_s^\varepsilon \rangle$$
$$= \int_t^T \int_{\mathbb{R}^d} (U^{\varepsilon,k^\varepsilon}(s,x) - V^{k^\varepsilon}(s,x))[\nu^\varepsilon(ds,dx) - \mu^\varepsilon(ds,dx)].$$

With the assumptions on g in mind, applying Young's inequality we have for any $\delta_1 > 0$

$$I_1(t) \leq (\delta_1 + 2\alpha) \int_t^T |\nabla(U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon})|^2 ds + C \int_0^t |U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon}|^2 ds. \quad (5.12)$$

By the assumption on f, for any $\delta_2 > 0$, we have

$$I_2(t) \leq \delta_2 \int_t^T |\nabla(U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon})|^2 ds + C \int_0^t |U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon}|^2 ds. \quad (5.13)$$

Using the assumption on h, given any $\delta_3 > 0$, we also have

$$I_3(t) \leq \delta_3 \int_t^T |\nabla(U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon})|^2 ds + C \int_0^t |U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon}|^2 (1 + \|k_s^\varepsilon\|_{l^2}^2) ds. \quad (5.14)$$

For the term I_5 in (5.11), because $U_s^{\varepsilon,k^\varepsilon} - L(s,\cdot) \geq 0$, $V_s^{\varepsilon,k^\varepsilon} - L(s,\cdot) \geq 0$ and because that the random measures ν_s^ε, μ_s^ε are positive, we have

$$I_5(t) = 2 \int_t^T \left\langle U_s^{\varepsilon,k^\varepsilon} - L(s,\cdot) + L(s,\cdot) - V_s^{k^\varepsilon}, d\nu_s^\varepsilon - d\mu_s^\varepsilon \right\rangle \leq 0. \quad (5.15)$$

Substituting (5.12)–(5.15) back into (5.11), choosing $\delta_1, \delta_2, \delta_3$ sufficiently small and rearranging terms we can find a positive constant $\delta > 0$ such that

$$|U_t^{\varepsilon,k^\varepsilon} - V_t^{k^\varepsilon}|^2 + \delta \int_t^T |\nabla(U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon})|^2 ds$$
$$\leq C \int_t^T |U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon}|^2 (1 + \|k_s^\varepsilon\|_{l^2}^2) ds$$
$$+ 2\sqrt{\varepsilon} \sum_{j=1}^\infty \int_t^T \left\langle U_s^{\varepsilon,k^\varepsilon} - V_s^{k^\varepsilon}, h_j(s,x,U_s^{\varepsilon,k^\varepsilon}, \nabla U_s^{\varepsilon,k^\varepsilon}) \right\rangle dB_s^j$$
$$+ \varepsilon \sum_{j=1}^\infty \int_t^T |h_j(s,x,U_s^{\varepsilon,k^\varepsilon}, \nabla U_s^{\varepsilon,k^\varepsilon})|^2 ds. \quad (5.16)$$

By the Gronwall's inequality it follows that

$$\sup_{0\le t\le T}|U_t^{\varepsilon,k^\varepsilon}-V_t^{k^\varepsilon}|^2+\int_0^T\|U_t^{\varepsilon,k^\varepsilon}-V_t^{k^\varepsilon}\|^2dt$$
$$\le (M_1^\varepsilon+M_2^\varepsilon)\exp\left(C\int_0^T(1+\|k_s^\varepsilon\|_{l^2}^2)ds\right)\le C_M(M_1^\varepsilon+M_2^\varepsilon), \quad (5.17)$$

where

$$M_1^\varepsilon=\sup_{0\le t\le T}\left|2\sqrt{\varepsilon}\sum_{j=1}^\infty\int_t^T\left\langle U_s^{\varepsilon,k^\varepsilon}-V_s^{k^\varepsilon},h_j(s,\cdot,U_s^{\varepsilon,k^\varepsilon},\nabla U_s^{\varepsilon,k^\varepsilon})\right\rangle dB_s^j\right|,$$
$$M_2^\varepsilon=\varepsilon\sum_{j=1}^\infty\int_0^T|h_j(s,\cdot,U_s^{\varepsilon,k^\varepsilon},\nabla U_s^{\varepsilon,k^\varepsilon})|^2ds.$$

Using Burkholder's inequality and the boundedness of h, we see that

$$E[M_1^\varepsilon]\le C\sqrt{\varepsilon}E\left[\left(\sum_{j=1}^\infty\int_0^T\left\langle U_s^{\varepsilon,k^\varepsilon}-V_s^{k^\varepsilon},h_j(s,\cdot,U_s^{\varepsilon,k^\varepsilon},\nabla U_s^{\varepsilon,k^\varepsilon})\right\rangle^2ds\right)^{\frac12}\right]$$
$$\le C\sqrt{\varepsilon}E\left[\left(\int_0^T|U_t^{\varepsilon,k^\varepsilon}-V_t^{k^\varepsilon}|^2dt\right)^{\frac12}\right]$$
$$\to 0,\quad \text{as}\quad \varepsilon\to 0, \quad (5.18)$$

where we have used the fact that $\sup_\varepsilon\{E[|U_t^{\varepsilon,k^\varepsilon}|^2]+E[|V_t^{k^\varepsilon}|^2]\}<\infty$. By the condition on h in the Assumption 2.1, it is also clear that

$$E[M_2^\varepsilon]\le C\varepsilon E\left[\int_0^T(1+|U_s^{\varepsilon,k^\varepsilon}|^2+\|U_s^{\varepsilon,k^\varepsilon}\|^2)ds\right]$$
$$\to 0,\quad \text{as}\quad \varepsilon\to 0. \quad (5.19)$$

Assertion (5.10) follows from (5.17) to (5.19). □

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