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Large deviations for 2-D stochastic Navier–Stokes equations driven by multiplicative *Lévy* noises

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In this paper, we establish a large deviation principle for two-dimensional stochastic Navier–Stokes equations driven by multiplicative *Lévy* noises. The weak convergence method introduced by Budhiraja, Dupuis and Maroulas [*Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011) 725–747] plays a key role.

Keywords: Brownian motions; large deviations; Poisson random measures; Skorohod representation; stochastic Navier–Stokes equations; tightness

1. Introduction

Let D be a bounded open domain in $\mathbb{R}^2$ with smooth boundary ∂D. Denote by u and p the velocity and the pressure fields. The Navier–Stokes equation, an important model in fluid dynamics, is given as follows:

$$\partial_t u - \nu \triangle u + u \cdot \nabla u + \nabla p = h \qquad \text{in } D \times [0, T], \tag{1.1}$$

with the conditions

$$\begin{cases} \nabla \cdot u = 0 & \text{in } D \times [0, T], \\ u = 0 & \text{in } \partial D \times [0, T], \\ u(0) = x \in L^2(D), \end{cases} \tag{1.2}$$

where $\nu > 0$ is the viscosity, h stands for the external force.

To formulate the Navier–Stokes equations, we introduce the following standard spaces. Let $\mathcal{V}$ be the space of infinitely differentiable two-dimensional vector fields $g(\cdot)$ on D with compact support, satisfying $\operatorname{div}(g(\cdot)) = 0$. Denote by V_α the closure of $\mathcal{V}$ in $[H^\alpha(D)]^2$, for $\alpha \geq 0$, where $[H^\alpha(D)]^2$ stands for the Sobolev space of order α. Set in particular

$$H = V_0, \qquad V = V_1.$$

We denote by $\|\cdot\|_H$ and $\langle \cdot, \cdot \rangle_H$ the norm and inner product in H. Identifying H with its dual space H', and denoted by V'_α the dual space of V_α, we consider equation (1.1) in the framework of Gelfrand triples: $V \subset H \cong H' \subset V'$.

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Define the Stokes operator A in H by

$$Au = -P_H \triangle u, \qquad \forall u \in D(A) = \left[H^2(D)\right]^2 \cap V,$$

where the linear operator P_H (Helmhotz–Hodge projection) is the projection operator from $[L^2(D)]^2$ into H. Since V coincides with $D(A^{1/2})$, we can endow V with the norm $\|u\|_V = \|A^{1/2}u\|_H$. Because the operator A is positive self adjoint with compact resolvent, there is a complete orthonormal system $\{e_1, e_2, \ldots\}$ in H made of eigenvectors of A, with corresponding eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ ($Ae_i = \lambda_i e_i$). We will use fractional powers of the operator A, denoted by A^α, as well as their domains $D(A^\alpha)$ for $\alpha \in \mathbb{R}$. Note that $D(A^\alpha) = \{u = \sum_{i=1}^\infty u_i \cdot e_i : \sum_{i=1}^\infty \lambda_i^{2\alpha} u_i^2 < \infty\}$. We may endow $D(A^\alpha)$ with the inner product $\langle u, v\rangle_{D(A^\alpha)} = \langle A^\alpha u, A^\alpha v\rangle_H$. So $D(A^\alpha)$ is a Hilbert space with the inner product $\langle \cdot, \cdot\rangle_{D(A^\alpha)}$ and $\{e_i/\lambda_i^\alpha\}_{i\in\mathbb{N}}$ is a complete orthonormal system of $D(A^\alpha)$. By Riesz representation theorem, $D(A^{-\alpha})$ is the dual space of $D(A^\alpha)$.

Let $B(u, v): V \times V \to (V \cap [L^2(D)]^2)$ be the bilinear operator defined as (cf. Lions [22])

$$\left\langle B(u, v), z\right\rangle = \int_D z(x) \cdot \big(u(x) \cdot \nabla\big) v(x)\, \mathrm{d}x$$

for all $z \in V \cap [L^2(D)]^2$. Set $B(u, v, z) = \langle B(u, v), z\rangle$, and for $u = v$, we write $B(u) = B(u, u)$. By the incompressibility condition,

$$\left\langle B(u, v), v\right\rangle = 0, \qquad \left\langle B(u, v), z\right\rangle = -\left\langle B(u, z), v\right\rangle. \tag{1.3}$$

By Višik and Fursikov [35], B can be expanded to a continuous operator

$$B : H \times H \to D\big(A^{-\varrho}\big) \tag{1.4}$$

for some $\varrho > 1$. And we have

$$\big|B(u, v, w)\big| \leq 2\|u\|_V^{1/2} \cdot \|u\|_H^{1/2} \cdot \|v\|_V^{1/2} \cdot \|v\|_H^{1/2} \cdot \|w\|_V. \tag{1.5}$$

We also need the following inequalities(cf. e.g., Brzeźniak, Liu and Zhu [5], Temam [33])

$$\begin{aligned} &\left|\left\langle B(u) - B(v), u - v\right\rangle_{V', V}\right| \\ &\quad = \left|\left\langle B(u - v), v\right\rangle_{V', V}\right| \\ &\quad \leq \frac{\nu}{2}\|u - v\|_V^2 + \frac{32}{\nu^3}\|v\|_{L^4(D;\mathbb{R}^2)}^4 \|u - v\|_H^2, \qquad u, v \in V, \end{aligned} \tag{1.6}$$

and

$$\|v\|_{L^4(D;\mathbb{R}^2)}^4 \leq 2\|v\|_H^2 \|v\|_V^2, \qquad v \in V. \tag{1.7}$$

By applying the operator P_H to each term of the above Navier–Stokes equation (NSE) (1.1), we can rewrite the NSE in the following abstract form:

$$\mathrm{d}u(t) + \nu A u(t)\, \mathrm{d}t + B\big(u(t)\big)\, \mathrm{d}t = f \qquad \text{in } L^2\big([0, T]; V'\big), \tag{1.8}$$

with the initial condition

$$u(0) = x \qquad \text{in } H. \tag{1.9}$$

In this paper, we consider stochastic Navier–Stokes equations (SNSE) driven by the multiplicative *Lévy* noise, that is, the following random perturbations of Navier–Stokes equation:

$$\begin{cases} \mathrm{d}u^\varepsilon(t) = -\nu A\big(u^\varepsilon(t)\big)\,\mathrm{d}t - B\big(u^\varepsilon(t)\big)\,\mathrm{d}t + \sqrt{\varepsilon}\sigma\big(t, u^\varepsilon(t)\big)\,\mathrm{d}\beta(t) \\ \qquad\qquad + \varepsilon \displaystyle\int_{\mathbb{X}} G\big(t, u^\varepsilon(t-), v\big)\widetilde{N}^{\varepsilon^{-1}}(\mathrm{d}t\,\mathrm{d}v); \\ u^\varepsilon(0) = x \in H. \end{cases} \tag{1.10}$$

Here $\mathbb{X}$ is a locally compact Polish space. $\{\beta(t), t \geq 0\}$ is a H-cylindrical Brownian motion admitting the following representation:

$$\beta(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k,$$

where $\beta_k(t), k \geq 1$ are independent standard Brownian motions. $N^{\varepsilon^{-1}}$ is a Poisson random measure on $[0, T] \times \mathbb{X}$ with a σ-finite intensity measure $\varepsilon^{-1}\lambda_T \otimes \vartheta$, λ_T is the Lebesgue measure on $[0, T]$ and ϑ is a σ-finite measure on $\mathbb{X}$. $\widetilde{N}^{\varepsilon^{-1}}([0, t] \times O) = N^{\varepsilon^{-1}}([0, t] \times O) - \varepsilon^{-1} t \vartheta(O)$, $\forall O \in \mathcal{B}(\mathbb{X})$ with $\vartheta(O) < \infty$, is the compensated Poisson random measure. σ, G are measurable mappings specified later.

Stochastic Navier–Stokes equations have been intensively studied since the work of Bensoussan and Temam [2]. A good reference for stochastic Navier–Stokes equations driven by additive Gaussian noise is the book Da Prato and Zabczyk [15] and the references therein. The existence and uniqueness of solutions for the two dimensional stochastic Navier–Stokes equations with multiplicative Gaussian noise were obtained in Flandoli and Gatarek [18], Mikulevicius and Rozovskii [25], Sritharan and Sundar [31]. The ergodic properties and invariant measures of the 2-D stochastic Navier–Stokes equations were studied in Flandoli [17] and Hairer and Mattingly [19].

The purpose of this paper is to establish a large deviation principle (LDP) for SNSEs driven by multiplicative *Lévy* noises, that is, a LDP for the solution of (1.10) as $\varepsilon \to 0$ on $D([0, T]; H)$, the space of H-valued right continuous functions with left limits on $[0, T]$.

Large deviations for stochastic evolution equations and stochastic partial differential equations driven by Gaussian processes have been investigated in many papers, see, for example, Chow [13], Sowers [30], Chenal and Millet [12], Cardon-Weber [10], Zhang [39], Cerrai and Röckner [11], Bessaih and Millet [3], Budhiraja, Dupuis and Maroulas [8], Duan and Millet [16], Manna, Sritharan and Sundar [24], Wang and Duan [36], Liu [23], Röckner, Zhang and Zhang [29], Zhang [40], and references therein. The situations for stochastic evolution equations and stochastic partial differential equations driven by *Lévy* noise are drastically different because of the appearance of the jumps. There is not much study on this topic so far. The first paper on large deviations of SPDEs of jump type is Röckner and Zhang [28] where the additive noise is considered. The case of multiplicative *Lévy* noise was studied in Święch and Zabczyk [32] and Budhiraja, Chen and Dupuis [6] where the large deviation was obtained on a larger space (hence, with a weaker topology) than the actual state space of the solution. Recently, Yang, Zhai and Zhang [38] obtained the large deviation principles on the actual state space of stochastic evolution equations with regular coefficients driven by multiplicative *Lévy* noise.

Large deviation principles for the two-dimensional stochastic Navier–Stokes equations driven by Gaussian noise have been established in Sritharan and Sundar [31], Chueshov and Millet [14]. Regarding LDP for the 2-D stochastic Navier–Stokes equation driven by *Lévy* noise, to the best of our knowledge, Xu and Zhang [37] is the first paper on this topic, where additive *Lévy* noise is considered. Our main concern in this paper is SNSEs driven by multiplicative *Lévy* noises.

To obtain the large deviation principle, we will use the weak convergence approach/criterions introduced by Budhiraja, Dupuis and Maroulas [9] for the case of Poisson random measures (see Section 2 for details) and by Budhiraja, Dupuis and Maroulas [8] for the case of Gaussian noises (and also in the earlier work Budhiraja and Dupuis [7]). This approach is now a powerful tool which has been applied by many people to prove large deviation principles for various dynamical systems driven by Gaussian noises, see, for example, Budhiraja and Dupuis [7], Ren and Zhang [27], Ren and Zhang [26], Sritharan and Sundar [31], Zhang [40], Budhiraja, Dupuis and Maroulas [8], Bessaih and Millet [3], Duan and Millet [16], Manna, Sritharan and Sundar [24], Wang and Duan [36], Röckner, Zhang and Zhang [29], Liu [23], Chueshov and Millet [14]. The weak convergence method was first used in Budhiraja, Chen and Dupuis [6] to obtain large deviation principles for SPDEs on co-nuclear spaces driven by *Lévy* noises and in Yang, Zhai and Zhang [38] for SPDEs on Hilbert spaces with regular coefficients.

For the two-dimensional stochastic Navier–Stokes equations driven by multiplicative *Lévy* noise, in addition to the difficulties caused by the jumps, much of the problem is to deal with the nonlinear term $B(u,u)$. To obtain the LDP, similarly as in Budhiraja, Chen and Dupuis [6], among other things we need to study the weak convergence of the solutions of random perturbations (in certain directions) of the equation (1.10). This is highly nontrivial. We first establish the tightness of the solutions of the perturbed SNS equations in a larger space $D([0,T];D(A^{-\varrho}))$ and then via the Skorohod representation theorem we are able to show that the weak convergence actually takes place in the space $D([0,T];H)$.

The organization of this paper is as follows. In Section 2, we will recall the general criteria for a large deviation principle obtained in Budhiraja, Dupuis and Maroulas [9]. In Section 3, we prove a preliminary result on two-dimensional deterministic Navier–Stokes equation, which will play an important role in the rest of the paper. The entire Section 4 is devoted to establishing the large deviation principle for the stochastic Navier–Stokes equation.

We end this section with some notations. Denote by $\mathbb{N},\mathbb{R},\mathbb{R}^+,\mathbb{R}^d$ the set of positive integers, real numbers, positive real numbers and d-dimensional real vectors respectively. For a topological space $\mathcal{E}$, denote the corresponding Borel σ-field by $\mathcal{B}(\mathcal{E})$. For a metric space $\mathbb{Y}$, denote by $M_b(\mathbb{Y})$, $C_b(\mathbb{Y})$ the space of real valued bounded $\mathcal{B}(\mathbb{Y})/\mathcal{B}(\mathbb{R})$-measurable maps and real valued bounded continuous functions respectively. For $p>0$, a measure μ on $\mathbb{Y}$, and a Hilbert space $\mathbb{H}$, denote by $L^p(\mathbb{Y},\mu;\mathbb{H})$ the space of measurable functions f from $\mathbb{Y}$ to $\mathbb{H}$ such that $\int_{\mathbb{Y}}\|f(v)\|_{\mathbb{H}}^p\mu(\mathrm{d}v)<\infty$, where $\|\cdot\|_{\mathbb{H}}$ is the norm of $\mathbb{H}$. We say a collection $\{X^\varepsilon\}$ of $\mathbb{Y}$-valued random variables is tight if the probability distributions of X^ε are tight in $\mathcal{P}(\mathbb{Y})$ (the space of probability measures on $\mathbb{Y}$). We will use the symbol "$\Longrightarrow$" to denote convergence in distribution. For a Polish space $\mathbb{X}$, denote by $C([0,T],\mathbb{X})$, $D([0,T],\mathbb{X})$ the space of continuous functions and right continuous functions with left limits from [0,T] into $\mathbb{X}$, respectively.

2. Preliminaries

In this section, we will recall the general criteria for a large deviation principle given in Budhiraja, Dupuis and Maroulas [9]. To this end, we closely follow the framework and the notations in Budhiraja, Chen and Dupuis [6] and Budhiraja, Dupuis and Maroulas [9].

2.1. Large deviation principle

Let $\{X^\varepsilon, \varepsilon > 0\}$ be a family of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a Polish space $\mathcal{E}$. Denote expectation with respect to $\mathbb{P}$ by $\mathbb{E}$. The large deviation principle is concerned with exponential decay of $\mathbb{P}(X^\varepsilon \in O)$ as $\varepsilon \to 0$. Now we recall the definition.

Definition 2.1 (Rate function). *A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function on $\mathcal{E}$, if for each $M < \infty$ the level set $\{x \in \mathcal{E} : I(x) \le M\}$ is a compact subset of $\mathcal{E}$. For $O \in \mathcal{B}(\mathcal{E})$, we define $I(O) \doteq \inf_{x \in O} I(x)$.*

Definition 2.2 (Large deviation principle). *Let I be a rate function on $\mathcal{E}$. The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviation principle on $\mathcal{E}$ with rate function I if the following two conditions hold.*

(a) *Large deviation upper bound. For each closed subset F of $\mathcal{E}$,*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\big(X^\varepsilon \in F\big) \le -I(F).$$

(b) *Large deviation lower bound. For each open subset G of $\mathcal{E}$,*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\big(X^\varepsilon \in G\big) \ge -I(G).$$

2.2. Controlled Poisson random measure

The following notations will be used. Let $\mathbb{X}$ be a locally compact Polish space. Denote by $\mathcal{M}_{FC}(\mathbb{X})$ the space of all measures ϑ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ such that $\vartheta(K) < \infty$ for every compact K in $\mathbb{X}$, and set $C_c(\mathbb{X})$ be the space of continuous functions with compact supports. Endow $\mathcal{M}_{FC}(\mathbb{X})$ with the weakest topology such that for every $f \in C_c(\mathbb{X})$, the function $\vartheta \to \langle f, \vartheta \rangle = \int_{\mathbb{X}} f(u)\, \mathrm{d}\vartheta(u)$, $\vartheta \in \mathcal{M}_{FC}(\mathbb{X})$ is continuous. This topology can be metrized such that $\mathcal{M}_{FC}(\mathbb{X})$ is a Polish space (see, e.g., Budhiraja, Dupuis and Maroulas [9]). Fix $T \in (0, \infty)$ and let $\mathbb{X}_T = [0, T] \times \mathbb{X}$. Fix a measure $\vartheta \in \mathcal{M}_{FC}(\mathbb{X})$, and let $\vartheta_T = \lambda_T \otimes \vartheta$, where λ_T is Lebesgue measure on $[0, T]$.

We recall that a Poisson random measure $\mathbf{n}$ on $\mathbb{X}_T$ with intensity measure ϑ_T is a $\mathcal{M}_{FC}(\mathbb{X}_T)$ valued random variable such that for each $B \in \mathcal{B}(\mathbb{X}_T)$ with $\vartheta_T(B) < \infty$, $\mathbf{n}(B)$ is Poisson distributed with mean $\vartheta_T(B)$ and for disjoint $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{X}_T)$, $\mathbf{n}(B_1), \ldots, \mathbf{n}(B_k)$ are mutually independent random variables (cf. Ikeda and Watanabe [20]). Denote by $\mathbb{P}$ the measure induced by $\mathbf{n}$ on $(\mathcal{M}_{FC}(\mathbb{X}_T), \mathcal{B}(\mathcal{M}_{FC}(\mathbb{X}_T)))$. Then letting $\mathbb{M} = \mathcal{M}_{FC}(\mathbb{X}_T)$, $\mathbb{P}$ is the unique probability measure on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which the canonical map, $N : \mathbb{M} \to \mathbb{M}$, $N(m) \doteq m$, is a Poisson random measure with intensity measure ϑ_T. With applications to large deviations in mind, we also consider, for $\theta > 0$, probability measures $\mathbb{P}_\theta$ on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which N is a Poissson random measure with intensity $\theta\vartheta_T$. The corresponding expectation operators will be denoted by $\mathbb{E}$ and $\mathbb{E}_\theta$, respectively.

Set $\mathbb{Y} = \mathbb{X} \times [0, \infty)$ and $\mathbb{Y}_T = [0, T] \times \mathbb{Y}$. Similarly, let $\bar{\mathbb{M}} = \mathcal{M}_{FC}(\mathbb{Y}_T)$ and let $\bar{\mathbb{P}}$ be the unique probability measure on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$ under which the canonical map, $\bar{N} : \bar{\mathbb{M}} \to \bar{\mathbb{M}}$, $\bar{N}(m) \doteq m$, is a Poisson random measure with intensity measure $\bar{\vartheta}_T = \lambda_T \otimes \vartheta \otimes \lambda_\infty$, with λ_∞ being Lebesgue measure on $[0, \infty)$. The corresponding expectation operator will be denoted by $\bar{\mathbb{E}}$. Let $\mathcal{F}_t \doteq \sigma\{\bar{N}((0, s] \times O) : 0 \le s \le t, O \in \mathcal{B}(\mathbb{Y})\}$, and denote by $\bar{\mathcal{F}}_t$ the completion under $\bar{\mathbb{P}}$. Set $\bar{\mathcal{P}}$ be the predictable σ-field on $[0, T] \times \bar{\mathbb{M}}$ with the filtration $\{\bar{\mathcal{F}}_t : 0 \le t \le T\}$ on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$. Let $\bar{\mathcal{A}}$ be the class of all $(\bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{X}))/\mathcal{B}[0, \infty)$-measurable maps $\varphi : \mathbb{X}_T \times \bar{\mathbb{M}} \to [0, \infty)$. For $\varphi \in \bar{\mathcal{A}}$, define a counting process N^φ on $\mathbb{X}_T$ by

$$N^\varphi\big((0, t] \times U\big) = \int_{(0,t]\times U} \int_{(0,\infty)} 1_{[0,\varphi(s,x)]}(r)\bar{N}(\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}r), \qquad t \in [0, T], U \in \mathcal{B}(\mathbb{X}). \quad (2.1)$$

N^φ is the controlled random measure, with φ selecting the intensity for the points at location x and time s, in a possibly random but nonanticipating way. When $\varphi(s, x, \bar{m}) \equiv \theta \in (0, \infty)$, we write $N^\varphi = N^\theta$. Note that N^θ has the same distribution with respect to $\bar{\mathbb{P}}$ as N has with respect to $\mathbb{P}_\theta$.

2.3. PRM and BM

Set $\mathbb{W} = C([0, T], \mathbb{R}^\infty)$, $\mathbb{V} = \mathbb{W} \times \mathbb{M}$ and $\bar{\mathbb{V}} = \mathbb{W} \times \bar{\mathbb{M}}$. Then let the mapping $N^{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ be defined by $N^{\mathbb{V}}(w, m) = m$ for $(w, m) \in \mathbb{V}$, and let $\beta^{\mathbb{V}} = (\beta_i^{\mathbb{V}})_{i=1}^\infty$ by $\beta_i^{\mathbb{V}}(w, m) = w_i$ for $(w, m) \in \mathbb{V}$. The maps $\bar{N}^{\bar{\mathbb{V}}} : \bar{\mathbb{V}} \to \bar{\mathbb{M}}$ and $\beta^{\bar{\mathbb{V}}} = (\beta_i^{\bar{\mathbb{V}}})_{i=1}^\infty$ are defined analogously. Define the σ-filtration $\mathcal{G}_t^{\mathbb{V}} := \sigma\{N^{\mathbb{V}}((0, s] \times O), \beta_i^{\mathbb{V}}(s) : 0 \le s \le t, O \in \mathcal{B}(\mathbb{X}), i \ge 1\}$. For every $\theta > 0$, $\mathbb{P}_\theta^{\mathbb{V}}$ denotes the unique probability measure on $(\mathbb{V}, \mathcal{B}(\mathbb{V}))$ such that:

1. $(\beta_i^{\mathbb{V}})_{i=1}^\infty$ is an i.i.d. family of standard Brownian motions,
2. $N^{\mathbb{V}}$ is a PRM with intensity measure $\theta\vartheta_T$.

Analogously, we define $(\bar{\mathbb{P}}_\theta^{\bar{\mathbb{V}}}, \bar{\mathcal{G}}_t^{\bar{\mathbb{V}}})$ and denote $\bar{\mathbb{P}}_{\theta=1}^{\bar{\mathbb{V}}}$ by $\bar{\mathbb{P}}^{\bar{\mathbb{V}}}$. We denote by $\{\bar{\mathcal{F}}_t^{\bar{\mathbb{V}}}\}$ the $\bar{\mathbb{P}}^{\bar{\mathbb{V}}}$-completion of $\{\bar{\mathcal{G}}_t^{\bar{\mathbb{V}}}\}$ and $\bar{\mathcal{P}}^{\bar{\mathbb{V}}}$ the predictable σ-field on $[0, T] \times \bar{\mathbb{V}}$ with the filtration $\{\bar{\mathcal{F}}_t^{\bar{\mathbb{V}}}\}$ on $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}))$. Let $\bar{\mathbb{A}}$ be the class of all $(\bar{\mathcal{P}}^{\bar{\mathbb{V}}} \otimes \mathcal{B}(\mathbb{X}))/\mathcal{B}[0, \infty)$-measurable maps $\varphi : \mathbb{X}_T \times \bar{\mathbb{V}} \to [0, \infty)$. Define $l : [0, \infty) \to [0, \infty)$ by

$$l(r) = r \log r - r + 1, \qquad r \in [0, \infty).$$

For any $\varphi \in \bar{\mathbb{A}}$ the quantity

$$L_T(\varphi) = \int_{\mathbb{X}_T} l\big(\varphi(t, x, \omega)\big)\vartheta_T(\mathrm{d}t\,\mathrm{d}x) \quad (2.2)$$

is well defined as a $[0,\infty]$-valued random variable.

Denote by $\mathcal{I}_2$ the Hilbert space of real sequences $x=(x_i)$ satisfying $\|x\|^2=\sum_{i=1}^{\infty}x_i^2<\infty$, with the usual inner product. Define

$$\mathcal{L}_2:=\left\{\psi:\psi \text{ is } \bar{\mathcal{P}}^{\bar{\mathbb{V}}}\setminus\mathcal{B}\big(\mathbb{R}^\infty\big) \text{ measurable and } \int_0^T \big\|\psi(s)\big\|^2\,\mathrm{d}s<\infty,\ \text{a.s.-}\bar{\mathbb{P}}^{\bar{\mathbb{V}}}\right\}. \tag{2.3}$$

Set $\mathcal{U}=\mathcal{L}_2\times\bar{\mathbb{A}}$. Define $\tilde{L}_T(\psi):=\frac{1}{2}\int_0^T\|\psi(s)\|^2\,\mathrm{d}s$ for $\psi\in\mathcal{L}_2$, and $\bar{L}_T(u):=\tilde{L}_T(\psi)+L_T(\varphi)$ for $u=(\psi,\varphi)\in\mathcal{U}$.

2.4. A general criteria

In this subsection, we recall a general criteria for a large deviation principle established in Budhiraja, Dupuis and Maroulas [9]. Let $\{\mathcal{G}^\varepsilon\}_{\varepsilon>0}$ be a family of measurable maps from $\bar{\mathbb{V}}$ to $\mathbb{U}$, where $\bar{\mathbb{V}}$ is introduced in Section 2.3 and $\mathbb{U}$ is some Polish space. We present below a sufficient condition for large deviation principle (LDP in abbreviation) to hold for the family $Z^\varepsilon=\mathcal{G}^\varepsilon(\sqrt{\varepsilon}\beta,\varepsilon N^{\varepsilon^{-1}})$, as $\varepsilon\to 0$.

Define

$$S^\Upsilon=\big\{g:\mathbb{X}_T\to[0,\infty):L_T(g)\le\Upsilon\big\}$$

and

$$\tilde{S}^\Upsilon=\big\{f:L^2\big([0,T],\mathcal{I}_2\big):\tilde{L}_T(f)\le\Upsilon\big\}.$$

A function $g\in S^\Upsilon$ can be identified with a measure $\vartheta_T^g\in\mathbb{M}$, defined by

$$\vartheta_T^g(O)=\int_O g(s,x)\vartheta_T(\mathrm{d}s\,\mathrm{d}x),\qquad O\in\mathcal{B}(\mathbb{X}_T).$$

This identification induces a topology on S^Υ under which S^Υ is a compact space, see the Appendix of Budhiraja, Chen and Dupuis [6]. Throughout we use this topology on S^Υ. Set $\bar{S}^\Upsilon=\tilde{S}^\Upsilon\times S^\Upsilon$. Define $\mathbb{S}=\bigcup_{\Upsilon\ge 1}\bar{S}^\Upsilon$, and let

$$\mathcal{U}^\Upsilon=\big\{u=(\psi,\varphi)\in\mathcal{U}:u(\omega)\in\bar{S}^\Upsilon,\bar{\mathbb{P}}^{\bar{\mathbb{V}}}\text{ a.e. }\omega\big\},$$

where $\mathcal{U}$ is introduced in Section 2.3.

The following condition will be sufficient for establishing a LDP for a family $\{Z^\varepsilon\}_{\varepsilon>0}$ defined by $Z^\varepsilon=\mathcal{G}^\varepsilon(\sqrt{\varepsilon}\beta,\varepsilon N^{\varepsilon^{-1}})$.

Condition 2.1. *There exists a measurable map $\mathcal{G}^0:\bar{\mathbb{V}}\to\mathbb{U}$ such that the following hold.*

(a) *For $\forall\Upsilon\in\mathbb{N}$, let $(f_n,g_n),(f,g)\in\bar{S}^\Upsilon$ be such that $(f_n,g_n)\to(f,g)$ as $n\to\infty$. Then*

$$\mathcal{G}^0\bigg(\int_0^\cdot f_n(s)\,\mathrm{d}s,\vartheta_T^{g_n}\bigg)\to\mathcal{G}^0\bigg(\int_0^\cdot f(s)\,\mathrm{d}s,\vartheta_T^{g}\bigg)\qquad in\ \mathbb{U}.$$

(b) *For* $\forall \Upsilon \in \mathbb{N}$, *let* $u_\varepsilon = (\psi_\varepsilon, \varphi_\varepsilon), u = (\psi, \varphi) \in \mathcal{U}^\Upsilon$ *be such that* u_ε *converges in distribution to* u *as* $\varepsilon \to 0$. *Then*

$$\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}\beta + \int_0^\cdot \psi_\varepsilon(s)\,\mathrm{d}s, \varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}\right) \Rightarrow \mathcal{G}^0\left(\int_0^\cdot \psi(s)\,\mathrm{d}s, \vartheta_T^\varphi\right).$$

For $\phi \in \mathbb{U}$, define $\mathbb{S}_\phi = \{(f, g) \in \mathbb{S} : \phi = \mathcal{G}^0(\int_0^\cdot f(s)\,\mathrm{d}s, \vartheta_T^g)\}$. Let $I : \mathbb{U} \to [0, \infty]$ be defined by

$$I(\phi) = \inf_{q=(f,g)\in\mathbb{S}_\phi} \{\bar{L}_T(q)\}, \qquad \phi \in \mathbb{U}. \tag{2.4}$$

By convention, $I(\phi) = \infty$ if $\mathbb{S}_\phi = \varnothing$.

The following criteria was established in Budhiraja, Dupuis and Maroulas [9].

Theorem 2.1. *For* $\varepsilon > 0$, *let* Z^ε *be defined by* $Z^\varepsilon = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}\beta, \varepsilon N^{\varepsilon^{-1}})$, *and suppose that Condition* 2.1 *holds. Then* I *defined as in* (2.4) *is a rate function on* $\mathbb{U}$ *and the family* $\{Z^\varepsilon\}_{\varepsilon>0}$ *satisfies a large deviation principle with rate function* I.

For applications, the following strengthened form of Theorem 2.1 is useful. Let $\{K_n \subset \mathbb{X}, n = 1, 2, \dots\}$ be an increasing sequence of compact sets such that $\bigcup_{n=1}^\infty K_n = \mathbb{X}$. For each n let

$$\begin{aligned}\bar{\mathbb{A}}_{b,n} \doteq \big\{\varphi \in \bar{\mathbb{A}} : \text{ for all } (t, \omega) \in [0, T] \times \bar{\mathbb{M}},\\ n \ge \varphi(t, x, \omega) \ge 1/n \text{ if } x \in K_n \text{ and } \varphi(t, x, \omega) = 1 \text{ if } x \in K_n^c\big\},\end{aligned}$$

and let $\bar{\mathbb{A}}_b = \bigcup_{n=1}^\infty \bar{\mathbb{A}}_{b,n}$. Define $\tilde{\mathcal{U}}^\Upsilon = \mathcal{U}^\Upsilon \cap \{(\psi, \phi) : \phi \in \bar{\mathbb{A}}_b\}$.

Theorem 2.2. *Suppose Condition* 2.1 *holds with* $\mathcal{U}^\Upsilon$ *replaced by* $\tilde{\mathcal{U}}^\Upsilon$. *Then the conclusions of Theorem* 2.1 *continue to hold.*

3. Hypotheses

In this section, we will state the precise assumptions on the coefficients and collect some preliminary results from Budhiraja, Chen and Dupuis [6] and Yang, Zhai and Zhang [38], which will be used in the following sections.

Denote by $L_2(H)$ the space of all Hilbert–Schmidt operators from H to H. Let $\sigma : [0, T] \times H \to L_2(H)$, $G : [0, T] \times H \times \mathbb{X} \to H$ be given measurable maps. Introduce the following conditions:

Condition 3.1. *There exists* $K(\cdot) \in L^1([0, T], \mathbb{R}^+)$ *such that*

(1) (*Growth*) *For all* $t \in [0, T]$, *and* $u \in H$,

$$\|\sigma(t, u)\|_{L_2(H)}^2 + \int_{\mathbb{X}} \|G(t, u, v)\|_H^2 \vartheta(\mathrm{d}v) \le K(t)\big(1 + \|u\|_H^2\big);$$

(2) (*Lipschitz*) *For all* $t \in [0, T]$, *and* $u_1, u_2 \in H$,

$$\|\sigma(t, u_1) - \sigma(t, u_2)\|^2_{L_2(H)} + \int_{\mathbb{X}} \|G(t, u_1, v) - G(t, u_2, v)\|^2_H \vartheta(\mathrm{d}v) \le K(t)\|u_1 - u_2\|^2_H.$$

Let

$$\|G(t, v)\|_{0,H} = \sup_{u \in H} \frac{\|G(t, u, v)\|_H}{1 + \|u\|_H}, \qquad (t, v) \in [0, T] \times \mathbb{X}.$$

$$\|G(t, v)\|_{1,H} = \sup_{u_1, u_2 \in H, u_1 \neq u_2} \frac{\|G(t, u_1, v) - G(t, u_2, v)\|_H}{\|u_1 - u_2\|_H}, \qquad (t, v) \in [0, T] \times \mathbb{X}.$$

Condition 3.2 (Exponential integrability). *For* $i = 0, 1$, *there exists* $\delta_1^i > 0$ *such that for all* $E \in \mathcal{B}([0, T] \times \mathbb{X})$ *satisfying* $\vartheta_T(E) < \infty$, *the following holds*

$$\int_E \mathrm{e}^{\delta_1^i \|G(s,v)\|^2_{i,H}} \vartheta(\mathrm{d}v)\,\mathrm{d}s < \infty.$$

Remark 3.1. Condition 3.2 implies that, for every $\delta_2^i > 0$ and for all $E \in \mathcal{B}([0, T] \times \mathbb{X})$ satisfying $\vartheta_T(E) < \infty$,

$$\int_E \mathrm{e}^{\delta_2^i \|G(s,v)\|_{i,H}} \vartheta(\mathrm{d}v)\,\mathrm{d}s < \infty.$$

The first part of the following lemma was proved in Budhiraja, Chen and Dupuis [6]. For the second part of this lemma, the case $i = 0$ can be found in Remark 2 of Yang, Zhai and Zhang [38], and the case $i = 1$ can be proved similarly. We omit its proof.

Lemma 3.1. *Under Conditions* 3.1 *and* 3.2,

(i) *For* $i = 0, 1$ *and every* $\Upsilon \in \mathbb{N}$,

$$C^{\Upsilon}_{i,2} := \sup_{g \in S^{\Upsilon}} \int_{\mathbb{X}_T} \|G(s, v)\|^2_{i,H} \big(g(s, v) + 1\big) \vartheta(\mathrm{d}v)\,\mathrm{d}s < \infty, \tag{3.1}$$

$$C^{\Upsilon}_{i,1} := \sup_{g \in S^{\Upsilon}} \int_{\mathbb{X}_T} \|G(s, v)\|_{i,H} \big|g(s, v) - 1\big| \vartheta(\mathrm{d}v)\,\mathrm{d}s < \infty. \tag{3.2}$$

(ii) *For ever* $\eta > 0$, *there exists* $\delta > 0$ *such that for any* $A \subset [0, T]$ *satisfying* $\lambda_T(A) < \delta$

$$\sup_{g \in S^{\Upsilon}} \int_A \int_{\mathbb{X}} \|G(s, v)\|_{i,H} \big|g(s, v) - 1\big| \vartheta(\mathrm{d}v)\,\mathrm{d}s \le \eta. \tag{3.3}$$

The proof of the following lemma is almost identical to the proof of Lemma 3.3 in Yang, Zhai and Zhang [38], and so is omitted.

Lemma 3.2. (a) *If* $\sup_{t\in[0,T]}\|Y(t)\|_H<\infty$, *for any* $q=(f,g)\in\mathbb{S}$, *then*

$$\sigma\big(\cdot,Y(\cdot)\big)f(\cdot)\in L^1\big([0,T],H\big),\int_{\mathbb{X}}G\big(\cdot,Y(\cdot),v\big)\big(g(\cdot,v)-1\big)\vartheta(\mathrm{d}v)\in L^1\big([0,T],H\big);$$

(b) *If the family of mappings* $\{Y_n:[0,T]\to H, n\geq 1\}$ *satisfies* $C=\sup_n\sup_{s\in[0,T]}\|Y_n(s)\|_H<\infty$, *then*

$$\begin{aligned}\widetilde{C}_\Upsilon := \sup_{q=(f,q)\in\bar{S}^\Upsilon}\sup_n\bigg[&\int_0^T\bigg\|\int_{\mathbb{X}}G\big(s,Y_n(s),v\big)\big(g(s,v)-1\big)\vartheta(\mathrm{d}v)\bigg\|_H\mathrm{d}s\\&+\int_0^T\big\|\sigma\big(s,Y_n(s)\big)f(s)\big\|_H\mathrm{d}s\bigg]\\<\infty.&\end{aligned}$$

We also need the following lemma, the proof of which can be found in Budhiraja, Chen and Dupuis [6].

Lemma 3.3. *Let* $h:[0,T]\times\mathbb{X}\to\mathbb{R}$ *be a measurable function such that*

$$\int_{\mathbb{X}_T}\big|h(s,v)\big|^2\vartheta(\mathrm{d}v)\,\mathrm{d}s<\infty,$$

and for all $\delta\in(0,\infty)$ *and* $E\in\mathcal{B}([0,T]\times\mathbb{X})$ *satisfying* $\vartheta_T(E)<\infty$,

$$\int_E\exp\big(\delta\big|h(s,v)\big|\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s<\infty.$$

(a) *Fix* $\Upsilon\in\mathbb{N}$, *and let* $g_n,g\in S^\Upsilon$ *be such that* $g_n\to g$ *as* $n\to\infty$. *Then*

$$\lim_{n\to\infty}\int_{\mathbb{X}_T}h(s,v)\big(g_n(s,v)-1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s=\int_{\mathbb{X}_T}h(s,v)\big(g(s,v)-1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s.$$

(b) *Fix* $\Upsilon\in\mathbb{N}$. *Given* $\varepsilon>0$, *there exists a compact set* $K_\varepsilon\subset\mathbb{X}$, *such that*

$$\sup_{g\in S^\Upsilon}\int_{[0,T]}\int_{K_\varepsilon^c}\big|h(s,v)\big|\big|g(s,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}s\leq\varepsilon.$$

(c) *For every compact* $K\subset\mathbb{X}$,

$$\lim_{M\to\infty}\sup_{g\in S^\Upsilon}\int_{[0,T]}\int_K\big|h(s,v)\big|1_{\{h\geq M\}}g(s,v)\vartheta(\mathrm{d}v)\,\mathrm{d}s=0.$$

Let $\mathbb{H}$ be a separable Hilbert space. Given $p > 1$, $\alpha \in (0,1)$, let $W^{\alpha,p}([0,T];\mathbb{H})$ be the Sobolev space of all $u \in L^p([0,T];\mathbb{H})$ such that

$$\int_0^T \int_0^T \frac{\|u(t)-u(s)\|_{\mathbb{H}}^p}{|t-s|^{1+\alpha p}} \,\mathrm{d}t\,\mathrm{d}s < \infty$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}([0,T];\mathbb{H})}^p = \int_0^T \|u(t)\|_{\mathbb{H}}^p \,\mathrm{d}t + \int_0^T \int_0^T \frac{\|u(t)-u(s)\|_{\mathbb{H}}^p}{|t-s|^{1+\alpha p}} \,\mathrm{d}t\,\mathrm{d}s.$$

The following result represents a variant of the criteria for compactness proved in Lions [22], Section 5, Chapter I, and Temam [34], Section 13.3.

Lemma 3.4. *Let $\mathbb{H}_0 \subset \mathbb{H} \subset \mathbb{H}_1$ be Banach spaces, $\mathbb{H}_0$ and $\mathbb{H}_1$ reflexive, with compact embedding of $\mathbb{H}_0$ in $\mathbb{H}$. Let $p \in (1,\infty)$ and $\alpha \in (0,1)$ be given. Let Λ be the space*

$$\Lambda = L^p\big([0,T];\mathbb{H}_0\big) \cap W^{\alpha,p}\big([0,T];\mathbb{H}_1\big)$$

endowed with the natural norm. Then the embedding of Λ in $L^p([0,T];\mathbb{H})$ is compact.

4. Large deviation principle

First, we introduce the following definition.

Definition 4.1. *Let $(\bar{\mathbb{V}}, \mathcal{B}(\bar{\mathbb{V}}), \bar{\mathbb{P}}^{\bar{\mathbb{V}}}, \{\bar{\mathcal{F}}_t^{\bar{\mathbb{V}}}\})$ be the filtered probability space described in Section 2.2. Suppose that X_0 is a $\bar{\mathcal{F}}_0$-measurable H-valued random variable such that $\bar{\mathbb{E}}\|X_0\|_H^2 < \infty$. A stochastic process $\{X^\varepsilon(t)\}_{t\in[0,T]}$ defined on $\bar{\mathbb{V}}$ is said to be a H-valued solution to (1.10) with initial value X_0, if*

(a) *$X^\varepsilon(t)$ is a H-valued $\bar{\mathcal{F}}_t^{\bar{\mathbb{V}}}$-measurable random variable, for all $t \in [0,T]$;*
(b) *$X^\varepsilon \in D([0,T],H) \cap L^2([0,T],V)$ a.s.;*
(c) *For all $t \in [0,T]$,*

$$\begin{aligned} X^\varepsilon(t) = X_0 &- \nu \int_0^t A X^\varepsilon(s)\,\mathrm{d}s - \int_0^t B\big(X^\varepsilon(s)\big)\,\mathrm{d}s \\ &+ \sqrt{\varepsilon} \int_0^t \sigma\big(s, X^\varepsilon(s)\big)\,\mathrm{d}\beta(s) \qquad (4.1) \\ &+ \varepsilon \int_0^t \int_{\mathbb{X}} G\big(s, X^\varepsilon(s-), v\big) \widetilde{N}^{\varepsilon^{-1}}(\mathrm{d}s,\mathrm{d}v) \text{ in } L^2\big([0,T];V'\big) \qquad \text{a.s.} \end{aligned}$$

Definition 4.2 (Pathwise uniqueness). *We say that the stochastic evolution equation (1.10) admits the pathwise uniqueness if any two H-valued solutions X and X' defined on the same filtered probability space with respect to the same Poisson random measure and Brownian motion starting from the same initial condition X_0 coincide almost surely.*

Assume X_0 is deterministic. Let X^ε be the H-valued solution to (1.10) with initial value X_0. In this section, we will establish an LDP for $\{X^\varepsilon\}$ as $\varepsilon \to 0$.

4.1. Skeleton equations

We begin by introducing the map $\mathcal{G}^0$ that will be used to define the rate function and also used for verification of Condition 2.1. Recall that $\mathbb{S} = \bigcup_{\Upsilon \geq 1} \bar{S}^\Upsilon$, where $\bar{S}^\Upsilon$ is defined in last section. As a first step we show that under the conditions below, for every $q = (f, g) \in \mathbb{S}$, the deterministic integral equation

$$\begin{aligned}\widetilde{X}^q(t) = X_0 - \nu \int_0^t A\widetilde{X}^q(s)\,\mathrm{d}s - \int_0^t B\big(\widetilde{X}^q(s)\big)\,\mathrm{d}s \\ + \int_0^t \sigma\big(s, \widetilde{X}^q(s)\big) f(s)\,\mathrm{d}s \\ + \int_0^t \int_{\mathbb{X}} G\big(s, \widetilde{X}^q(s), v\big)\big(g(s, v) - 1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s\end{aligned} \tag{4.2}$$

has a unique continuous solution.

Theorem 4.1. *Fix $X_0 = x \in H$ and $q = (f, g) \in \mathbb{S}$. Suppose Conditions 3.1 and 3.2 hold. Then there exists a unique $\widetilde{X}^q \in C([0, T], H) \cap L^2([0, T], V)$ such that,*

$$\begin{aligned}\widetilde{X}^q(t) = x - \int_0^t \nu A\widetilde{X}^q(s)\,\mathrm{d}s - \int_0^t B\big(\widetilde{X}^q(s)\big)\,\mathrm{d}s \\ + \int_0^t \sigma\big(s, \widetilde{X}^q(s)\big) f(s)\,\mathrm{d}s \\ + \int_0^t \int_{\mathbb{X}} G\big(s, \widetilde{X}^q(s), v\big)\big(g(s, v) - 1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s \text{ in } L^2\big([0, T]; V'\big).\end{aligned} \tag{4.3}$$

Moreover, for fixed $\Upsilon \in \mathbb{N}$, there exists $C_\Upsilon > 0$ such that

$$\sup_{q \in S^\Upsilon} \left(\sup_{s \in [0,T]} \big\|\widetilde{X}^q(s)\big\|_H^2 + \int_0^T \big\|\widetilde{X}^q(s)\big\|_V^2\,\mathrm{d}s \right) \leq C_\Upsilon. \tag{4.4}$$

Proof. *Existence*. First, let $\Phi_n : \mathbb{R} \to [0, 1]$ be a smooth function such that $\Phi_n(t) = 1$ if $|t| \leq n$, $\Phi_n(t) = 0$ if $|t| > n + 1$. Set $\mathcal{X}_n(u) = \Phi_n(\|u\|_H)$, $u \in H$. Let P_n be the projection operator from H to H defined as

$$P_n u = \sum_{i=1}^n \langle u, e_i\rangle_H e_i, \qquad u \in H,$$

and

$$B_n(u, u) = \mathcal{X}_n(u) B(u, u), \qquad u \in P_n H.$$

Consider the following Faedo–Galerkin approximations: $X_n(t) \in P_n H$ denotes the solution of

$$\begin{aligned} \mathrm{d}X_n(t) = {} & -\nu A X_n(t)\,\mathrm{d}t - P_n B_n\big(X_n(t), X_n(t)\big)\,\mathrm{d}t + P_n\sigma\big(t, X_n(t)\big) f(t)\,\mathrm{d}t \\ & + P_n \int_{\mathbb{X}} G\big(t, X_n(t), v\big)\big(g(t,v)-1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}t, \end{aligned} \tag{4.5}$$

with initial condition $X_n(0) = P_n x$.

Since B_n is a Lipschitz operator from $P_n H$ into $P_n H$, the solution of equation (4.5) can be obtained through an iteration argument as follows.

Let $Y_0(t) = P_n x, t \in [0,T]$. Suppose that Y_{m-1} has been defined. Define $Y_m \in C([0,T]; P_n H) \cap L^2([0,T]; P_n V)$ as the unique solution of the equation

$$\begin{aligned} \mathrm{d}Y_m(t) = {} & -\nu A Y_m(t)\,\mathrm{d}t - P_n B_n\big(Y_m(t), Y_m(t)\big)\,\mathrm{d}t + P_n\sigma\big(t, Y_{m-1}(t)\big) f(t)\,\mathrm{d}t \\ & + P_n \int_{\mathbb{X}} G\big(t, Y_{m-1}(t), v\big)\big(g(t,v)-1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}t \end{aligned} \tag{4.6}$$

and $Y_m(0) = P_n x$. Using similar arguments as in the proof of Theorem 3.1 of Yang, Zhai and Zhang [38], we can show that the limit X_n of Y_m, as $m \to \infty$, is the unique strong solution $X_n \in C([0,T]; P_n H) \cap L^2([0,T]; P_n V)$ of (4.5).

The next thing is to show that

$$\sup_{n\ge 1}\left(\sup_{t\in[0,T]} \|X_n(t)\|_H^2 + \int_0^T \|X_n(s)\|_V^2\,\mathrm{d}s\right) \le C, \tag{4.7}$$

and for $\alpha \in (0, 1/2)$, there exists $C_\alpha > 0$ such that

$$\sup_{n\ge 1} \|X_n\|^2_{W^{\alpha,2}([0,T],V')} \le C_\alpha. \tag{4.8}$$

Notice that

$$\big\langle P_n B_n(u,u), u\big\rangle_{V',V} = \big\langle \mathcal{X}_n(u) B(u,u), u\big\rangle_{V',V} = 0, \qquad \forall u \in P_n H.$$

We have

$$\begin{aligned} & \|X_n(t)\|_H^2 + 2\nu \int_0^t \|X_n(s)\|_V^2\,\mathrm{d}s \\ & \quad = \|X_n(0)\|_H^2 + 2\int_0^t \big\langle P_n\sigma\big(s, X_n(s)\big) f(s), X_n(s)\big\rangle_H\,\mathrm{d}s \\ & \qquad + 2\int_0^t \left\langle P_n \int_{\mathbb{X}} G\big(s, X_n(s), v\big)\big(g(s,v)-1\big)\vartheta(\mathrm{d}v), X_n(s)\right\rangle_H \mathrm{d}s \\ & \quad \le \|x\|_H^2 + 2\int_0^t \big\|\sigma\big(s, X_n(s)\big)\big\|_{L^2(H)} \|f(s)\|_H \|X_n(s)\|_H\,\mathrm{d}s \\ & \qquad + 2\int_0^t \int_{\mathbb{X}} \big\|G\big(s, X_n(s), v\big)\big\|_H \big|\big(g(s,v)-1\big)\big| \|X_n(s)\|_H \vartheta(\mathrm{d}v)\,\mathrm{d}s. \end{aligned} \tag{4.9}$$

According to Condition 3.1, it follows that

$$
\begin{aligned}
&2\int_0^t \big\|\sigma\big(s, X_n(s)\big)\big\|_{L^2(H)}\big\|f(s)\big\|_H\big\|X_n(s)\big\|_H\,\mathrm{d}s\\
&\quad\le 2\int_0^t \sqrt{K(s)}\sqrt{1+\big\|X_n(s)\big\|_H^2}\,\big\|f(s)\big\|_H\big\|X_n(s)\big\|_H\,\mathrm{d}s\\
&\quad\le 2\int_0^t \sqrt{K(s)}\big(1+2\big\|X_n(s)\big\|_H^2\big)\big\|f(s)\big\|_H\,\mathrm{d}s\\
&\quad\le \int_0^t K(s)\,\mathrm{d}s+\int_0^t \big\|f(s)\big\|_H^2\,\mathrm{d}s+2\int_0^t \big\|X_n(s)\big\|_H^2\big[K(s)+\big\|f(s)\big\|_H^2\big]\,\mathrm{d}s
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
&2\int_0^t\int_{\mathbb{X}}\big\|G\big(s, X_n(s), v\big)\big\|_H\big|\big(g(s,v)-1\big)\big|\big\|X_n(s)\big\|_H\vartheta(\mathrm{d}v)\,\mathrm{d}s\\
&\quad\le 2\int_0^t\int_{\mathbb{X}}\frac{\|G(s, X_n(s), v)\|_H}{1+\|X_n(s)\|_H}\big(1+\big\|X_n(s)\big\|_H\big)\big|\big(g(s,v)-1\big)\big|\big\|X_n(s)\big\|_H\vartheta(\mathrm{d}v)\,\mathrm{d}s\\
&\quad\le 2\int_0^t\int_{\mathbb{X}}\big\|G(s,v)\big\|_{0,H}\big|\big(g(s,v)-1\big)\big|\big(1+2\big\|X_n(s)\big\|_H^2\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s\\
&\quad\le 2\int_0^t\int_{\mathbb{X}}\big\|G(s,v)\big\|_{0,H}\big|\big(g(s,v)-1\big)\big|\vartheta(\mathrm{d}v)\,\mathrm{d}s\\
&\qquad+4\int_0^t\big\|X_n(s)\big\|_H^2\int_{\mathbb{X}}\big\|G(s,v)\big\|_{0,H}\big|\big(g(s,v)-1\big)\big|\vartheta(\mathrm{d}v)\,\mathrm{d}s.
\end{aligned}
\tag{4.11}
$$

Hence by (4.9), (4.10) and (4.11), we have

$$
\begin{aligned}
&\big\|X_n(t)\big\|_H^2+2\nu\int_0^t\big\|X_n(s)\big\|_V^2\,\mathrm{d}s\\
&\quad\le \|x\|_H^2+\int_0^t K(s)\,\mathrm{d}s+\int_0^t\big\|f(s)\big\|_H^2\,\mathrm{d}s\\
&\qquad+2\int_0^t\int_{\mathbb{X}}\big\|G(s,v)\big\|_{0,H}\big|\big(g(s,v)-1\big)\big|\vartheta(\mathrm{d}v)\,\mathrm{d}s\\
&\qquad+\int_0^t\big\|X_n(s)\big\|_H^2\bigg[2K(s)+2\big\|f(s)\big\|_H^2\\
&\qquad\qquad+4\int_{\mathbb{X}}\big\|G(s,v)\big\|_{0,H}\big|\big(g(s,v)-1\big)\big|\vartheta(\mathrm{d}v)\bigg]\,\mathrm{d}s.
\end{aligned}
\tag{4.12}
$$

By Gronwall' inequality,

$$\begin{aligned}
&\sup_{t\in[0,T]}\|X_n(t)\|_H^2 \\
&\quad\le \biggl[\|x\|_H^2+\int_0^T K(s)\,\mathrm{d}s+\int_0^T \|f(s)\|_H^2\,\mathrm{d}s \\
&\qquad + 2\int_0^T\int_{\mathbb{X}}\|G(s,v)\|_{0,H}\bigl|\bigl(g(s,v)-1\bigr)\bigr|\vartheta(\mathrm{d}v)\,\mathrm{d}s\biggr] \qquad (4.13)\\
&\qquad \times\exp\biggl[\int_0^T 2K(s)+2\|f(s)\|_H^2 \\
&\qquad\qquad + 4\int_{\mathbb{X}}\|G(s,v)\|_{0,H}\bigl|\bigl(g(s,v)-1\bigr)\bigr|\vartheta(\mathrm{d}v)\,\mathrm{d}s\biggr].
\end{aligned}$$

Combining (4.12) and (4.13), we have (4.7).

Now we prove (4.8). Write X_n as

$$\begin{aligned}
X_n(t) &= P_n x-\nu\int_0^t AX_n(s)\,\mathrm{d}s-\int_0^t P_nB_n\bigl(X_n(s),X_n(s)\bigr)\,\mathrm{d}s \\
&\quad +\int_0^t P_n\sigma\bigl(s,X_n(s)\bigr)f(s)\,\mathrm{d}s \\
&\quad +\int_0^t P_n\int_{\mathbb{X}} G\bigl(s,X_n(s),v\bigr)\bigl(g(s,v)-1\bigr)\vartheta(\mathrm{d}v)\,\mathrm{d}s \qquad (4.14)\\
&= J_n^1+J_n^2(t)+J_n^3(t)+J_n^4(t)+J_n^5(t).
\end{aligned}$$

Using the same arguments as in the proof of Theorem 3.1 in Flandoli and Gatarek [18], we have

$$\begin{aligned}
\|J_n^1\|_H^2 &\le L_1, \\
\|J_n^2\|_{W^{1,2}([0,T];V')}^2 &\le L_2, \qquad (4.15)\\
\|J_n^3\|_{W^{1,2}([0,T];V')}^2 &\le L_3.
\end{aligned}$$

Since for $t>s$,

$$\begin{aligned}
\|J_n^4(t)-J_n^4(s)\|_H^2 &= \biggl\|\int_s^t P_n\sigma\bigl(l,X_n(l)\bigr)f(l)\,\mathrm{d}l\biggr\|_H^2 \\
&\le \biggl(\int_s^t \bigl\|P_n\sigma\bigl(l,X_n(l)\bigr)f(l)\bigr\|_H\,\mathrm{d}l\biggr)^2 \\
&\le \biggl(\int_s^t \bigl\|P_n\sigma\bigl(l,X_n(l)\bigr)\bigr\|_{L^2(H)}\|f(l)\|_H\,\mathrm{d}l\biggr)^2
\end{aligned}$$

$$\leq \left(\int_s^t \sqrt{K(l)} \sqrt{1 + \| X_n(l) \|_H^2} \| f(l) \|_H \, \mathrm{d}l \right)^2$$
$$\leq \left(1 + \sup_{l \in [0,T]} \| X_n(l) \|_H^2 \right) \int_s^t K(l) \, \mathrm{d}l \int_s^t \| f(l) \|_H^2 \, \mathrm{d}l,$$

we have

$$\int_0^T \| J_n^4(t) \|_H^2 \, \mathrm{d}t \leq T \left(1 + \sup_{l \in [0,T]} \| X_n(l) \|_H^2 \right) \int_0^T K(l) \, \mathrm{d}l \int_0^T \| f(l) \|_H^2 \, \mathrm{d}l \tag{4.16}$$

and

$$\begin{aligned} &\int_0^T \int_0^T \frac{\| J_n^4(t) - J_n^4(s) \|_H^2}{|t-s|^{1+2\alpha}} \, \mathrm{d}t \, \mathrm{d}s \\ &\leq \left(1 + \sup_{l \in [0,T]} \| X_n(l) \|_H^2 \right) \int_0^T K(l) \, \mathrm{d}l \int_0^T \int_0^T \int_s^t \frac{\| f(l) \|_H^2}{|t-s|^{1+2\alpha}} \, \mathrm{d}l \, \mathrm{d}t \, \mathrm{d}s. \end{aligned} \tag{4.17}$$

Note the fact that by elementary application of Fubini theorem, there exists $L_4^1 > 0$ such that

$$\int_0^T \int_0^T \int_s^t \frac{\| f(l) \|_H^2}{|t-s|^{1+2\alpha}} \, \mathrm{d}l \, \mathrm{d}t \, \mathrm{d}s \leq L_4^1 \int_0^T \| f(l) \|_H^2 \, \mathrm{d}l. \tag{4.18}$$

Combining (4.16), (4.17) and (4.18), we have

$$\| J_n^4 \|_{W^{\alpha,2}([0,T];H)}^2 \leq L_4. \tag{4.19}$$

Now for J_n^5, we have

$$\begin{aligned} \| J_n^5(t) - J_n^5(s) \|_H^2 &= \left\| \int_s^t P_n \int_{\mathbb{X}} G\big(l, X_n(l), v\big) \big(g(l,v) - 1\big) \vartheta(\mathrm{d}v) \, \mathrm{d}l \right\|_H^2 \\ &\leq \left(\int_s^t \int_{\mathbb{X}} \big\| G\big(l, X_n(l), v\big) \big\|_H \big| g(l,v) - 1 \big| \vartheta(\mathrm{d}v) \, \mathrm{d}l \right)^2 \\ &\leq \left(\int_s^t \int_{\mathbb{X}} \| G(l,v) \|_{0,H} \big| g(l,v) - 1 \big| \big(1 + \| X_n(l) \|_H \big) \vartheta(\mathrm{d}v) \, \mathrm{d}l \right)^2 \\ &\leq \left(1 + \sup_{i \in [0,T]} \| X_n(l) \|_H \right)^2 \left(\int_s^t \int_{\mathbb{X}} \| G(l,v) \|_{0,H} \big| g(l,v) - 1 \big| \vartheta(\mathrm{d}v) \, \mathrm{d}l \right)^2 \\ &\leq \left(1 + \sup_{i \in [0,T]} \| X_n(l) \|_H \right)^2 \int_0^T \int_{\mathbb{X}} \| G(l,v) \|_{0,H} \big| g(l,v) - 1 \big| \vartheta(\mathrm{d}v) \, \mathrm{d}l \\ &\quad \times \int_s^t \int_{\mathbb{X}} \| G(l,v) \|_{0,H} \big| g(l,v) - 1 \big| \vartheta(\mathrm{d}v) \, \mathrm{d}l. \end{aligned}$$

Using similar arguments as J_n^4 and by Lemma 3.1, we have

$$\|J_n^5\|^2_{W^{\alpha,2}([0,T];H)} \le L_5. \tag{4.20}$$

By (4.79), (4.77) and (4.82), we obtain (4.8).

The estimates (4.7) and (4.8) enable us to assert the existence of an element $X \in L^2([0,T]; V) \cap L^\infty([0,T];H)$ and a sub-sequence $X_{m'}$ such that, as $m' \to \infty$

1. $X_{m'} \to X$ weakly in $L^2([0,T];V)$,
2. $X_{m'} \to X$ in the weak-star topology of $L^\infty([0,T];H)$,
3. $X_{m'} \to X$ strongly in $L^2([0,T];H)$.

Finally, we show that X is the unique solution of (4.3).

We will use the similar arguments as in the proof of Theorem 3.1 in Temam [33], Section 3, Chapter III.

Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T)=0$. We multiply (4.5) by $\psi(t)e_j$, and then integrate by parts. This leads to the equation

$$\begin{aligned}
&-\int_0^T \langle X_n(t), \psi'(t)e_j\rangle_{H,H}\,\mathrm{d}t + \nu\int_0^T \langle X_n(t), \psi(t)e_j\rangle_{V,V}\,\mathrm{d}t \\
&\quad= \langle X_n(0), \psi(0)e_j\rangle_{H,H} \\
&\qquad - \int_0^T \langle P_n B_n\big(X_n(t), X_n(t)\big), \psi(t)e_j\rangle_{V',V}\,\mathrm{d}t \\
&\qquad + \int_0^T \langle P_n\sigma\big(t, X_n(t)\big)f(t), \psi(t)e_j\rangle_{H,H}\,\mathrm{d}t \\
&\qquad + \int_0^T \left\langle P_n\int_{\mathbb{X}} G\big(t, X_n(t), v\big)\big(g(t,v)-1\big)\vartheta(\mathrm{d}v), \psi(t)e_j\right\rangle_{H,H}\mathrm{d}t.
\end{aligned}$$

Recall the definition of B_n and (4.7), for every $n > \sup_{m\in\mathbb{N}}\sup_{t\in[0,T]}\|X_m(t)\|_H^2 \vee j$, we have

$$\begin{aligned}
&-\int_0^T \langle X_n(t), \psi'(t)e_j\rangle_{H,H}\,\mathrm{d}t + \nu\int_0^T \langle X_n(t), \psi(t)e_j\rangle_{V,V}\,\mathrm{d}t \\
&\quad= \langle X_n(0), \psi(0)e_j\rangle_{H,H} - \int_0^T \langle B\big(X_n(t), X_n(t)\big), \psi(t)e_j\rangle_{V',V}\,\mathrm{d}t \\
&\qquad + \int_0^T \langle \sigma\big(t, X_n(t)\big)f(t), \psi(t)e_j\rangle_{H,H}\,\mathrm{d}t \\
&\qquad + \int_0^T \left\langle \int_{\mathbb{X}} G\big(t, X_n(t), v\big)\big(g(t,v)-1\big)\vartheta(\mathrm{d}v), \psi(t)e_j\right\rangle_{H,H}\mathrm{d}t.
\end{aligned} \tag{4.21}$$

Next, let $n \to \infty$ to show that

$$\begin{aligned}
&-\int_0^T \langle X(t), \psi'(t)e_j\rangle_{H,H}\,\mathrm{d}t + \nu\int_0^T \langle X(t), \psi(t)e_j\rangle_{V,V}\,\mathrm{d}t \\
&= \langle x, \psi(0)e_j\rangle_{H,H} - \int_0^T \langle B(X(t), X(t)), \psi(t)e_j\rangle_{V',V}\,\mathrm{d}t \\
&\quad + \int_0^T \langle \sigma(t, X(t)) f(t), \psi(t)e_j\rangle_{H,H}\,\mathrm{d}t \\
&\quad + \int_0^T \left\langle \int_{\mathbb{X}} G(t, X(t), v)(g(t,v)-1)\vartheta(\mathrm{d}v), \psi(t)e_j\right\rangle_{H,H}\mathrm{d}t.
\end{aligned} \tag{4.22}$$

First, using the similar argument as in the proof of Theorem 3.1 in Temam [33], Chapter III, and passing to the limit with the sequence m', we see that

$$\begin{aligned}
&\lim_{m'\to\infty}\Big[-\int_0^T \langle X_{m'}(t), \psi'(t)e_j\rangle_{H,H}\,\mathrm{d}t + \nu\int_0^T \langle X_{m'}(t), \psi(t)e_j\rangle_{V,V}\,\mathrm{d}t \\
&\qquad - \langle X_{m'}(0), \psi(0)e_j\rangle_{H,H} + \int_0^T \langle B(X_{m'}(t), X_{m'}(t)), \psi(t)e_j\rangle_{V',V}\,\mathrm{d}t\Big] \\
&= -\int_0^T \langle X(t), \psi'(t)e_j\rangle_{H,H}\,\mathrm{d}t \\
&\quad + \nu\int_0^T \langle X(t), \psi(t)e_j\rangle_{V,V}\,\mathrm{d}t - \langle x, \psi(0)e_j\rangle_{H,H} \\
&\quad + \int_0^T \langle B(X(t), X(t)), \psi(t)e_j\rangle_{V',V}\,\mathrm{d}t.
\end{aligned} \tag{4.23}$$

Next, we prove that

$$\lim_{m'\to\infty}\int_0^T \big\|\sigma(t, X_{m'}(t)) f(t) - \sigma(t, X(t)) f(t)\big\|_H\,\mathrm{d}t = 0 \tag{4.24}$$

and

$$\begin{aligned}
\lim_{m'\to\infty}\int_0^T \int_{\mathbb{X}} \big\| G(t, X_{m'}(t), v)(g(t,v)-1) \\
- G(t, X(t), v)(g(t,v)-1)\big\|_H \vartheta(\mathrm{d}v)\,\mathrm{d}t = 0.
\end{aligned} \tag{4.25}$$

For every $\varepsilon > 0$, let $A_{m',\varepsilon} = \{t \in [0,T] : \|X_{m'}(t) - X(t)\|_H > \varepsilon\}$, then we have

$$\lim_{m'\to\infty} \lambda_T(A_{m',\varepsilon}) \le \lim_{m'\to\infty} \frac{\int_0^T \|X_{m'}(t) - X(t)\|_H^2\,\mathrm{d}t}{\varepsilon^2} = 0. \tag{4.26}$$

Set $M=\sup_{i\in\mathbb{N}}\sup_{t\in[0,T]}\|X_i(t)\|_H\vee\sup_{t\in[0,T]}\|X(t)\|_H<\infty$. We have

$$\begin{aligned}
&\int_0^T \big\|\sigma\big(t,X_{m'}(t)\big)f(t)-\sigma\big(t,X(t)\big)f(t)\big\|_H\,\mathrm{d}t\\
&\quad\le\int_0^T \big\|\sigma\big(t,X_{m'}(t)\big)-\sigma\big(t,X(t)\big)\big\|_{L^2(H)}\big\|f(t)\big\|_H\,\mathrm{d}t\\
&\quad\le\int_0^T \sqrt{K(t)}\big\|X_{m'}(t)-X(t)\big\|_H\big\|f(t)\big\|_H\,\mathrm{d}t\\
&\quad\le 2M\int_{A_{m',\varepsilon}}\sqrt{K(t)}\big\|f(t)\big\|_H\,\mathrm{d}t+\varepsilon\int_{A^c_{m',\varepsilon}}\sqrt{K(t)}\big\|f(t)\big\|_H\,\mathrm{d}t\\
&\quad\le 2M\int_{A_{m',\varepsilon}}\sqrt{K(t)}\big\|f(t)\big\|_H\,\mathrm{d}t+\varepsilon\int_{A^c_{m',\varepsilon}}\sqrt{K(t)}\big\|f(t)\big\|_H\,\mathrm{d}t\\
&\quad\le 2M\sqrt{\int_{A_{m',\varepsilon}}K(t)\,\mathrm{d}t}\sqrt{\int_0^T\big\|f(t)\big\|_H^2\,\mathrm{d}t}+\varepsilon\bigg[\int_0^T K(t)\,\mathrm{d}t+\int_0^T\big\|f(t)\big\|_H^2\,\mathrm{d}t\bigg].
\end{aligned}\tag{4.27}$$

By (4.26) and $K(\cdot)\in L^1([0,T];\mathbb{R}^+)$, we have

$$\lim_{m'\to\infty}\int_{A_{m',\varepsilon}}K(t)\,\mathrm{d}t=0.\tag{4.28}$$

Combining (4.27) and (4.28), we arrive at (4.24).

Since

$$\begin{aligned}
&\int_0^T\int_{\mathbb{X}}\big\|G\big(t,X_{m'}(t),v\big)\big(g(t,v)-1\big)-G\big(t,X(t),v\big)\big(g(t,v)-1\big)\big\|_H\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{X}}\frac{\|G(t,X_{m'}(t),v)-G(t,X(t),v)\|_H}{\|X_{m'}(t)-X(t)\|_H}\big\|X_{m'}(t)-X(t)\big\|_H\big|g(t,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&\le\int_0^T\int_{\mathbb{X}}\big\|G(t,v)\big\|_{1,H}\big\|X_{m'}(t)-X(t)\big\|_H\big|g(t,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&\le 2M\int_{A_{m',\varepsilon}}\int_{\mathbb{X}}\big\|G(t,v)\big\|_{1,H}\big|g(t,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&\quad+\varepsilon\int_{A^c_{m',\varepsilon}}\int_{\mathbb{X}}\big\|G(t,v)\big\|_{1,H}\big|g(t,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}t,
\end{aligned}\tag{4.29}$$

together with Lemma 3.1, we obtain (4.25).

From here, using similar arguments as in the proof of Theorem 3.1 in Temam [33], Section 3, Chpter III, we can conclude that X is the solution desired.

Equation (4.4) can be proved similarly as (4.7). By Lemma 3.2 and using the same arguments as in the proof of Theorem 3.2 in Temam [33], Section 3, Chapter III, we also have

$$\frac{\mathrm{d}X}{\mathrm{d}t} \in L^2\big([0,T],V'\big) + L^1\big([0,T];H\big).$$

This implies $X \in C([0,T];H)$ by Lemma 1.2, (1.84) and (1.85) in Temam [33], Chapter III.

Uniqueness: Let us assume that X and Y are two solutions of (4.3), and let $Z = X - Y$. We have

$$\begin{aligned}
&\frac{\mathrm{d}\|Z(s)\|_H^2}{\mathrm{d}s} + 2\nu\|Z(s)\|_V^2 \\
&\quad = -2\big\langle B\big(X(s)\big) - B\big(Y(s)\big), X(s) - Y(s)\big\rangle_{V',V} \\
&\qquad + 2\big\langle \sigma\big(s, X(s)\big) f(s) - \sigma\big(s, Y(s)\big) f(s), X(s) - Y(s)\big\rangle_{H,H} \\
&\qquad + 2\int_{\mathbb{X}} \big\langle G\big(s, X(s), v\big) - G\big(s, Y(s), v\big), X(s) - Y(s)\big\rangle_{H,H}\big(g(s,v) - 1\big)\vartheta(\mathrm{d}v) \\
&\quad = I_1(s) + I_2(s) + I_3(s).
\end{aligned} \tag{4.30}$$

By (1.6), we have

$$I_1(s) \le \nu\|Z(s)\|_V^2 + \frac{64}{\nu^3}\|Y(s)\|_{L^4(D;\mathbb{R}^2)}^4\|Z(s)\|_H^2. \tag{4.31}$$

For I_2, we have

$$\begin{aligned}
I_2(s) &\le 2\big\|\sigma\big(s, X(s)\big) - \sigma\big(s, Y(s)\big)\big\|_{L^2(H)}\|f(s)\|_H\|X(s) - Y(s)\|_H \\
&\le 2\sqrt{K(s)}\|f(s)\|_H\|Z(s)\|_H^2 \\
&\le \big(K(s) + \|f(s)\|_H^2\big)\|Z(s)\|_H^2.
\end{aligned} \tag{4.32}$$

I_3 is bounded by

$$\begin{aligned}
I_3(s) &\le 2\int_{\mathbb{X}} \big\|G\big(s, X(s), v\big) - G\big(s, Y(s), v\big)\big\|_H\|X(s) - Y(s)\|_H\big|g(s,v) - 1\big|\vartheta(\mathrm{d}v) \\
&\le 2\|Z(s)\|_H^2\int_{\mathbb{X}}\|G(s,v)\|_{1,H}\big|g(s,v) - 1\big|\vartheta(\mathrm{d}v).
\end{aligned} \tag{4.33}$$

Setting

$$\psi(s) = \frac{64}{\nu^3}\|Y(s)\|_{L^4(D;\mathbb{R}^2)}^4 + K(s) + \|f(s)\|_H^2 + \int_{\mathbb{X}}\|G(s,v)\|_{1,H}\big|g(s,v) - 1\big|\vartheta(\mathrm{d}v),$$

we have

$$\frac{\mathrm{d}\|Z(s)\|_H^2}{\mathrm{d}s} \le -\nu\|Z(s)\|_V^2 + \psi(s)\|Z(s)\|_H^2,$$

which yields

$$
\begin{aligned}
&\exp\Big(-\int_0^t \psi(s)\,\mathrm{d}s\Big)\|Z(t)\|_H^2 \\
&\quad = -\int_0^t \exp\Big(-\int_0^s \psi(l)\,\mathrm{d}l\Big)\psi(s)\|Z(s)\|_H^2\,\mathrm{d}s \\
&\qquad + \int_0^t \exp\Big(-\int_0^s \psi(l)\,\mathrm{d}l\Big) d\|Z(s)\|_H^2 \qquad (4.34)\\
&\quad \le \int_0^t \exp\Big(-\int_0^s \psi(l)\,\mathrm{d}l\Big)\big(-\psi(s)\|Z(s)\|_H^2 - \nu\|Z(s)\|_V^2 + \psi(s)\|Z(s)\|_H^2\big)\,\mathrm{d}s \\
&\quad \le 0.
\end{aligned}
$$

Hence, $X = Y$.

The proof is complete. □

4.2. Large deviations

We are now ready to state the main result. Recall that for $q = (f, g) \in \mathbb{S}$, $\vartheta_T^g(\mathrm{d}s\,\mathrm{d}v) = g(s, v)\vartheta(\mathrm{d}v)\,\mathrm{d}s$. Define

$$\mathcal{G}^0\Big(\int_0^\cdot f(s)\,\mathrm{d}s, \vartheta_T^g\Big) = \widetilde{X}^q \qquad \text{for } q = (f, g) \in \mathbb{S} \text{ as given in Theorem 4.1.} \tag{4.35}$$

Let $I : D([0, T], H) \to [0, \infty]$ be defined as in (2.4).

Theorem 4.2. *Suppose that Conditions 3.1 and 3.2 hold. And assume*

$$K(\cdot) = C \quad \text{and} \quad \int_{\mathbb{X}} \|G(s, x, v)\|_H^4 \vartheta(\mathrm{d}v) \le C\big(1 + \|x\|_H^4\big).$$

Then the family $\{X^\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle on $D([0, T], H)$ with the good rate function I with respect to the topology of uniform convergence.

Proof. According to Theorem 2.2, we need to prove that Condition 2.1 is fulfilled. The verification of Condition 2.1(a) will be given by Proposition 4.1. Condition 2.1(b) will be established in Theorem 4.4. □

We now proceed to verify the first part of Condition 2.1. Recall the map $\mathcal{G}^0$ is defined by (4.35).

Proposition 4.1. *Fix $\Upsilon \in \mathbb{N}$, and let $q_n = (f_n, g_n), q = (f, g) \in \bar{S}^{\Upsilon}$ be such that $q_n \to q$ as $n \to \infty$. Then*

$$\mathcal{G}^0\left(\int_0^{\cdot} f_n(s)\,\mathrm{d}s, \vartheta_T^{g_n}\right) \to \mathcal{G}^0\left(\int_0^{\cdot} f(s)\,\mathrm{d}s, \vartheta_T^{g}\right) \qquad in\ C\big([0,T], H\big).$$

Proof. Recall $\mathcal{G}^0(\int_0^{\cdot} f_n(s)\,\mathrm{d}s, \vartheta_T^{g_n}) = \widetilde{X}^{q_n}$. For simplicity we denote $X_n = \widetilde{X}^{q_n}$.

Using similar arguments as for (4.7) and (4.8) and by Lemma 3.1, we can prove that there exist C_{Υ} and $C_{\alpha,\Upsilon}$ such that

1. $\sup_{t\in[0,T]} \|X_n(t)\|_H^2 + \nu \int_0^T \|X_n(s)\|_V^2\,\mathrm{d}s \le C_{\Upsilon}$,
2. $\|X_n\|^2_{W^{\alpha,2}([0,T],V')} \le C_{\alpha,\Upsilon}$, $\alpha \in (0, 1/2)$.

Hence, By Lemma 3.4, we can assert the existence of an element $X \in L^2([0,T]; V) \cap L^{\infty}([0,T]; H)$ and a sub-sequence $X_{m'}$ such that, as $m' \to \infty$

(a) $X_{m'} \to X$ in $L^2([0,T]; V)$ weakly,
(b) $X_{m'} \to X$ in $L^{\infty}([0,T]; H)$ weak-star,
(c) $X_{m'} \to X$ in $L^2([0,T]; H)$ strongly.

We will prove that $X = \widetilde{X}^q$.

Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T) = 0$. We multiply $X_n(t)$ by $\psi(t)e_j$, and then integrate by parts to obtain

$$\begin{aligned}
&-\int_0^T \big\langle X_n(t), \psi'(t)e_j\big\rangle_{H,H}\,\mathrm{d}t + \nu \int_0^T \big\langle X_n(t), \psi(t)e_j\big\rangle_{V,V}\,\mathrm{d}t \\
&= \big\langle x, \psi(0)e_j\big\rangle_{H,H} - \int_0^T \big\langle B\big(X_n(t), X_n(t)\big), \psi(t)e_j\big\rangle_{V',V}\,\mathrm{d}t \\
&\quad + \int_0^T \big\langle \sigma\big(t, X_n(t)\big) f_n(t), \psi(t)e_j\big\rangle_{H,H}\,\mathrm{d}t \\
&\quad + \int_0^T \left\langle \int_{\mathbb{X}} G\big(t, X_n(t), v\big)\big(g_n(t,v) - 1\big)\vartheta(\mathrm{d}v), \psi(t)e_j\right\rangle_{H,H}\mathrm{d}t.
\end{aligned} \tag{4.36}$$

Set

$$\begin{aligned}
I^1_{m'}(T) &= \int_0^T \int_{\mathbb{X}} \big\langle G\big(t, X_{m'}(t), v\big)\big(g_{m'}(t,v) - 1\big), \psi(t)e_j\big\rangle_{H,H}\vartheta(\mathrm{d}v)\,\mathrm{d}t, \\
I^2_{m'}(T) &= \int_0^T \int_{\mathbb{X}} \big\langle G\big(t, X_{m'}(t), v\big)\big(g(t,v) - 1\big), \psi(t)e_j\big\rangle_{H,H}\vartheta(\mathrm{d}v)\,\mathrm{d}t, \\
I(T) &= \int_0^T \int_{\mathbb{X}} \big\langle G\big(t, X(t), v\big)\big(g(t,v) - 1\big), \psi(t)e_j\big\rangle_{H,H}\vartheta(\mathrm{d}v)\,\mathrm{d}t.
\end{aligned}$$

We have

$$I^1_{m'}(T)-I(T)=I^1_{m'}(T)-I^2_{m'}(T)+I^2_{m'}(T)-I(T). \tag{4.37}$$

By (4.25), it is easy to see that

$$\lim_{m'\to\infty}\big(I^2_{m'}(T)-I(T)\big)=0. \tag{4.38}$$

Next, we claim that there exists subsequence $\{l'\}$ of $\{m'\}$ such that

$$\lim_{l'\to\infty}\big(I^1_{l'}(T)-I^2_{l'}(T)\big)=0. \tag{4.39}$$

By Lemmas 3.1 and 3.3, for any $\varepsilon>0$, there exists a compact subset $K_\varepsilon\subset\mathbb{X}$ such that

$$\begin{aligned}
&\int_0^T\int_{K_\varepsilon^c}\big|\big\langle G\big(t,X_{m'}(t),v\big)\big(g_{m'}(t,v)-1\big),\psi(t)e_j\big\rangle_{H,H}\big|\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&\quad\le\int_0^T\int_{K_\varepsilon^c}\big\|G\big(t,X_{m'}(t),v\big)\big\|_H\big|g_{m'}(t,v)-1\big|\big|\psi(t)\big|\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&\quad\le\sup_{t\in[0,T]}\big|\psi(t)\big|\Big(1+\sup_{t\in[0,T]}\big\|X_{m'}(t)\big\|_H\Big)\\
&\qquad\times\int_0^T\int_{K_\varepsilon^c}\big\|G(t,v)\big\|_{0,H}\big|g_{m'}(t,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&\quad\le\sup_{t\in[0,T]}\big|\psi(t)\big|(1+C_\Upsilon)\varepsilon
\end{aligned}\tag{4.40}$$

and

$$\begin{aligned}
&\int_0^T\int_{K_\varepsilon^c}\big|\big\langle G\big(t,X_{m'}(t),v\big)\big(g(t,v)-1\big),\psi(t)e_j\big\rangle_{H,H}\big|\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&\quad\le\sup_{t\in[0,T]}\big|\psi(t)\big|(1+C_\Upsilon)\varepsilon.
\end{aligned}\tag{4.41}$$

To prove the claim (4.39), applying the diagonal principle it suffices to show that, for every compact $K\in\mathbb{X}$, and $\eta=2\sup_{t\in[0,T]}|\psi(t)|(1+C_\Upsilon)\varepsilon>0$ there exists a subsequence $\{m'_K\}$ of $\{m'\}$ such that

$$\begin{aligned}
&\lim_{m'_K}\bigg|\int_0^T\int_K\big\langle G\big(t,X_{m'_K}(t),v\big)\big(g_{m'_K}(t,v)-1\big),\psi(t)e_j\big\rangle_{H,H}\vartheta(\mathrm{d}v)\,\mathrm{d}t\\
&\qquad-\int_0^T\int_K\big\langle G\big(t,X_{m'_K}(t),v\big)\big(g(t,v)-1\big),\psi(t)e_j\big\rangle_{H,H}\vartheta(\mathrm{d}v)\,\mathrm{d}t\bigg|\\
&\quad=\lim_{m'_K}\bigg|\int_0^T\int_K\big\langle G\big(t,X_{m'_K}(t),v\big)g_{m'_K}(t,v),\psi(t)e_j\big\rangle_{H,H}\vartheta(\mathrm{d}v)\,\mathrm{d}t
\end{aligned}\tag{4.42}$$

$$- \int_0^T \int_K \langle G(t, X_{m'_K}(t), v) g(t, v), \psi(t) e_j \rangle_{H,H} \vartheta(\mathrm{d}v)\,\mathrm{d}t \Big|$$
$$\le \eta.$$

Denote $A_M = \{(t, v) \in [0, T] \times K : \|G(t, v)\|_{0,H} \ge M\}$. By Lemma 3.3, for any $\varepsilon > 0$, there exists $M > 0$, such that

$$\begin{aligned} &\int_0^T \int_K \big|\langle G(t, X_{m'}(t), v) g_{m'}(t, v), \psi(t) e_j \rangle_{H,H}\big| 1_{A_M} \vartheta(\mathrm{d}v)\,\mathrm{d}t \\ &\quad \le \int_0^T \int_K \big|G(t, X_{m'}(t), v)\big|_H g_{m'}(t, v) \big|\psi(t)\big| 1_{A_M} \vartheta(\mathrm{d}v)\,\mathrm{d}t \\ &\quad \le \sup_{t\in[0,T]} \big|\psi(t)\big| \Big(1 + \sup_{t\in[0,T]} \big\|X_{m'}(t)\big\|_H\Big) \\ &\qquad \times \int_0^T \int_K \big|G(t, v)\big|_{0,H} g_{m'}(t, v) 1_{A_M} \vartheta(\mathrm{d}v)\,\mathrm{d}t \\ &\quad \le \sup_{t\in[0,T]} \big|\psi(t)\big| (1 + C_\Upsilon)\varepsilon \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} &\int_0^T \int_K \big|\langle G(t, X_{m'}(t), v) g(t, v), \psi(t) e_j \rangle_{H,H}\big| 1_{A_M} \vartheta(\mathrm{d}v)\,\mathrm{d}t \\ &\quad \le \sup_{t\in[0,T]} \big|\psi(t)\big| (1 + C_\Upsilon)\varepsilon. \end{aligned} \tag{4.44}$$

Denote

$$f_n(t, v) = \langle G(t, X_n(t), v), \psi(t) e_j \rangle_{H,H},$$
$$f(t, v) = \langle G(t, X(t), v), \psi(t) e_j \rangle_{H,H}.$$

We have

$$\begin{aligned} &\big|f_{m'}(t, v) 1_{A_M^c}\big| \\ &\quad \le \big|G(t, X_{m'}(t), v)\big|_H \big|\psi(t)\big| 1_{A_M^c} \\ &\quad \le \sup_{t\in[0,T]} \big|\psi(t)\big| \Big(1 + \sup_{t\in[0,T]} \big\|X_{m'}(t)\big\|_H\Big) \big|G(t, v)\big|_{0,H} 1_{A_M^c} \\ &\quad \le \sup_{t\in[0,T]} \big|\psi(t)\big| (1 + C_\Upsilon) M, \end{aligned} \tag{4.45}$$

and similarly

$$\big|f(t, v) 1_{A_M^c}\big| \le \sup_{t\in[0,T]} \big|\psi(t)\big| (1 + C_\Upsilon) M. \tag{4.46}$$

Let $\theta(\cdot) = \frac{\vartheta_T(\cdot\cap[0,T]\times K)}{\vartheta_T([0,T]\times K)}$, then θ is a probability measure on $[0, T] \times K$, and we have

$$
\begin{aligned}
&\int_0^T \int_K \left| f_{m'}(t,v) - f(t,v) \right| \theta(\mathrm{d}v, \mathrm{d}t) \\
&\quad \le \frac{1}{\vartheta_T([0,T]\times K)} \int_0^T \int_K \left| G\big(t, X_{m'}(t), v\big) - G\big(t, X(t), v\big) \right|_H \left| \psi(t) \right| \vartheta(\mathrm{d}v)\,\mathrm{d}t \\
&\quad \le \frac{\sup_{t\in[0,T]} |\psi(t)|}{\vartheta_T([0,T]\times K)} \int_0^T \int_K \left| G(t,v) - G(t,v) \right|_{1,H} \left\| X_{m'}(t) - X(t) \right\|_H \vartheta(\mathrm{d}v)\,\mathrm{d}t \\
&\quad \le \frac{\sup_{t\in[0,T]} |\psi(t)|}{\vartheta_T([0,T]\times K)} \sqrt{\int_0^T \int_{\mathbb{X}} \left| G(t,v) \right|^2_{1,H} \vartheta(\mathrm{d}v)\,\mathrm{d}t} \\
&\qquad \times \sqrt{\int_0^T \left\| X_{m'}(t) - X(t) \right\|_H^2 \mathrm{d}t\, \vartheta(K)} \\
&\quad \to 0, \qquad \text{as } m' \to \infty.
\end{aligned}
\tag{4.47}
$$

Hence, there exists a subsequence $\{m'_K\}$ of $\{m'\}$ such that

$$\lim_{m'_K\to\infty} f_{m'_K} = f, \qquad \theta\text{-a.s.} \tag{4.48}$$

By Lemma 2.8 of Boué and Dupuis [4] and noticing the proof of (3.25) in Budhiraja, Chen and Dupuis [6], we have

$$
\begin{aligned}
&\lim_{m'_K\to\infty} \int_0^T \int_K f_{m'_K}(t,v) 1_{A_M^c} g_{m'_K}(t,v) \vartheta(\mathrm{d}v)\,\mathrm{d}t \\
&\quad = \int_0^T \int_K f(t,v) 1_{A_M^c} g(t,v) \vartheta(\mathrm{d}v)\,\mathrm{d}t
\end{aligned}
\tag{4.49}
$$

and

$$\lim_{m'_K\to\infty} \int_0^T \int_K f_{m'_K}(t,v) 1_{A_M^c} g(t,v) \vartheta(\mathrm{d}v)\,\mathrm{d}t = \int_0^T \int_K f(t,v) 1_{A_M^c} g(t,v) \vartheta(\mathrm{d}v)\,\mathrm{d}t. \tag{4.50}$$

Hence combining (4.43), (4.44), (4.49) and (4.50), we get (4.42).

So by (4.40), (4.41) and (4.42), we have

$$\limsup_{m'_{K_\varepsilon}\to\infty} \left| I^1_{m'_{K_\varepsilon}}(T) - I^2_{m'_{K_\varepsilon}}(T) \right| \le 4 \sup_{t\in[0,T]} \left| \psi(t) \right| (1 + C_\Upsilon)\varepsilon. \tag{4.51}$$

Hence (4.39) follows. And then there exists $\{l'\}$ of $\{m'\}$ such that

$$\lim_{l'\to\infty} I^1_{l'}(T) = I(T). \tag{4.52}$$

By (4.36) and (4.52), using similar arguments as (4.22), we can prove that X satisfies

$$
\begin{aligned}
&-\int_0^T \langle X(t), \psi'(t)e_j\rangle_{H,H}\,\mathrm{d}t + \nu\int_0^T \langle X(t), \psi(t)e_j\rangle_{V,V}\,\mathrm{d}t \\
&= \langle x, \psi(0)e_j\rangle_{H,H} - \int_0^T \big\langle B\big(X(t),X(t)\big), \psi(t)e_j\big\rangle_{H,H}\,\mathrm{d}t \\
&\quad + \int_0^T \big\langle \sigma\big(t,X(t)\big)f(t), \psi(t)e_j\big\rangle_{H,H}\,\mathrm{d}t \\
&\quad + \int_0^T \Big\langle \int_{\mathbb{X}} G\big(t,X(t),v\big)\big(g(t,v)-1\big)\vartheta(\mathrm{d}v), \psi(t)e_j\Big\rangle_{H,H}\,\mathrm{d}t,
\end{aligned} \tag{4.53}
$$

and then using the same argument as in the proof of Theorem 3.1 of Temam [33], Section 3, Chapter III, we can conclude $X = \widetilde{X}^q$.

Next, we prove $X_{m'} \to X$ in $C([0,T],H)$.

Let $Z_{m'} = X_{m'} - X$. Then

$$
\begin{aligned}
&\frac{\mathrm{d}\|Z_{m'}(s)\|_H^2}{\mathrm{d}s} + 2\nu\big\|Z_{m'}(s)\big\|_V^2 \\
&= -2\big\langle B\big(X_{m'}(s)\big) - B\big(X(s)\big), X_{m'}(s) - X(s)\big\rangle_{V',V} \\
&\quad + 2\big\langle \sigma\big(s,X_{m'}(s)\big)f_{m'}(s) - \sigma\big(s,X(s)\big)f(s), X_{m'}(s) - X(s)\big\rangle_{H,H} \\
&\quad + 2\int_{\mathbb{X}} \big\langle G\big(s,X_{m'}(s),v\big)\big(g_{m'}(s,v)-1\big) \\
&\qquad\qquad - G\big(s,X(s),v\big)\big(g(s,v)-1\big), X_{m'}(s) - X(s)\big\rangle_{H,H}\vartheta(\mathrm{d}v) \\
&= -2\big\langle B\big(X_{m'}(s)\big) - B\big(X(s)\big), X_{m'}(s) - X(s)\big\rangle_{V',V} \\
&\quad + 2\big\langle \sigma\big(s,X_{m'}(s)\big)f(s) - \sigma\big(s,X(s)\big)f(s), X_{m'}(s) - X(s)\big\rangle_{H,H} \\
&\quad + 2\big\langle \sigma\big(s,X_{m'}(s)\big)f_{m'}(s) - \sigma\big(s,X_{m'}(s)\big)f(s), X_{m'}(s) - X(s)\big\rangle_{H,H} \\
&\quad + 2\int_{\mathbb{X}} \big\langle G\big(s,X_{m'}(s),v\big)\big(g(s,v)-1\big) \\
&\qquad\qquad - G\big(s,X(s),v\big)\big(g(s,v)-1\big), X_{m'}(s) - X(s)\big\rangle_{H,H}\vartheta(\mathrm{d}v) \\
&\quad + 2\int_{\mathbb{X}} \big\langle G\big(s,X_{m'}(s),v\big)\big(g_{m'}(s,v)-1\big) \\
&\qquad\qquad - G\big(s,X_{m'}(s),v\big)\big(g(s,v)-1\big), X_{m'}(s) - X(s)\big\rangle_{H,H}\vartheta(\mathrm{d}v) \\
&= I_1^{m'}(s) + I_2^{m'}(s) + I_3^{m'}(s) + I_4^{m'}(s) + I_5^{m'}(s).
\end{aligned} \tag{4.54}
$$

By (1.6), we have

$$I_1^{m'}(s) \le \nu \| Z_{m'}(s) \|_V^2 + \frac{64}{\nu^3} \| X(s) \|_{L^4(D;\mathbb{R}^2)}^4 \| Z_{m'}(s) \|_H^2. \tag{4.55}$$

For $I_2^{m'}$, we have

$$\begin{aligned} I_2^{m'}(s) &\le 2 \| \sigma(s, X_{m'}(s)) - \sigma(s, X(s)) \|_{L^2(H)} \| f(s) \|_H \| X_{m'}(s) - X(s) \|_H \\ &\le 2\sqrt{K(s)} \| f(s) \|_H \| Z_{m'}(s) \|_H^2 \\ &\le \big(K(s) + \| f(s) \|_H^2\big) \| Z_{m'}(s) \|_H^2. \end{aligned} \tag{4.56}$$

For $I_4^{m'}$, we have

$$\begin{aligned} I_4^{m'}(s) &\le 2 \int_{\mathbb{X}} \| G(s, X_{m'}(s), v) \\ &\qquad - G(s, X(s), v) \|_H \| X_{m'}(s) - X(s) \|_H |g(s, v) - 1| \vartheta(\mathrm{d}v) \\ &\le 2 \| Z_{m'}(s) \|_H^2 \int_{\mathbb{X}} \| G(s, v) \|_{1,H} |g(s, v) - 1| \vartheta(\mathrm{d}v). \end{aligned} \tag{4.57}$$

Setting

$$\psi(s) = \frac{64}{\nu^3} \| X(s) \|_{L^4(D;\mathbb{R}^2)}^4 + K(s) + \| f(s) \|_H^2 + \int_{\mathbb{X}} \| G(s, v) \|_{1,H} |g(s, v) - 1| \vartheta(\mathrm{d}v),$$

we have

$$\frac{\mathrm{d}\| Z_{m'}(s) \|_H^2}{\mathrm{d}s} \le -\nu \| Z_{m'}(s) \|_V^2 + \psi(s) \| Z_{m'}(s) \|_H^2 + I_3^{m'}(s) + I_5^{m'}(s).$$

Hence, we have

$$\begin{aligned} &\exp\Big(-\int_0^t \psi(s)\,\mathrm{d}s\Big) \| Z_{m'}(t) \|_H^2 \\ &\quad = -\int_0^t \exp\Big(-\int_0^s \psi(l)\,\mathrm{d}l\Big) \psi(s) \| Z_{m'}(s) \|_H^2\,\mathrm{d}s + \int_0^t \exp\Big(-\int_0^s \psi(l)\,\mathrm{d}l\Big)\,\mathrm{d}\| Z_{m'}(s) \|_H^2 \\ &\quad \le \int_0^t \exp\Big(-\int_0^s \psi(l)\,\mathrm{d}l\Big) \big(-\psi(s) \| Z(s) \|_H^2 - \nu \| Z(s) \|_V^2 + \psi(s) \| Z(s) \|_H^2\big)\,\mathrm{d}s \\ &\qquad + \int_0^t \exp\Big(-\int_0^s \psi(l)\,\mathrm{d}l\Big) \big(I_3^{m'}(s) + I_5^{m'}(s)\big)\,\mathrm{d}s \\ &\quad \le \int_0^T \big(|I_3^{m'}(s)| + |I_5^{m'}(s)|\big)\,\mathrm{d}s. \end{aligned} \tag{4.58}$$

Since

$$\int_0^T \psi(s)\,\mathrm{d}s < \infty,$$

we have

$$\sup_{t\in[0,T]} \left\| Z_{m'}(t) \right\|_H^2 \le \exp\left(\int_0^T \psi(s)\,\mathrm{d}s \right) \int_0^T \left(\left| I_3^{m'}(s) \right| + \left| I_5^{m'}(s) \right| \right) \mathrm{d}s. \tag{4.59}$$

Since $K(\cdot) = C$, $\sup_n \sup_{t\in[0,T]} \|X_n(t)\|_H^2 \le C_r$, and $X_{m'} \to X$ in $L^2([0,T], H)$, we have

$$\begin{aligned} &\int_0^T \left| I_3^{m'}(s) \right| \mathrm{d}s \\ &\le 2 \int_0^T \left\| \sigma(s, X_{m'}) \right\|_{L^2(H)} \left\| f_{m'}(s) - f(s) \right\|_H \left\| X_{m'}(s) - X(s) \right\|_H \mathrm{d}s \\ &\le 2 \int_0^T \sqrt{K(s)} \sqrt{1 + \left\| X_{m'}(s) \right\|_H^2} \left\| f_{m'}(s) - f(s) \right\|_H \left\| X_{m'}(s) - X(s) \right\|_H \mathrm{d}s \\ &\to 0, \qquad \text{as } m' \to \infty. \end{aligned} \tag{4.60}$$

Splitting the interval $[0, T]$ into two parts as in (4.29), we have

$$\begin{aligned} &\int_0^T \left| I_5^{m'}(s) \right| \mathrm{d}s \\ &\le 2 \int_0^T \int_{\mathbb{X}} \left| \left\langle G\left(s, X_{m'}(s), v\right)\left(g_{m'}(s,v) - 1\right), X_{m'}(s) - X(s) \right\rangle_{H,H} \right| \vartheta(\mathrm{d}v)\,\mathrm{d}s \\ &\quad + 2 \int_0^T \int_{\mathbb{X}} \left| \left\langle G\left(s, X_{m'}(s), v\right)\left(g(s,v) - 1\right), X_{m'}(s) - X(s) \right\rangle_{H,H} \right| \vartheta(\mathrm{d}v)\,\mathrm{d}s \\ &\le 2 \int_0^T \int_{\mathbb{X}} \left| G\left(s, X_{m'}(s), v\right) \right|_H \left| g_{m'}(s,v) - 1 \right| \left| X_{m'}(s) - X(s) \right|_H \vartheta(\mathrm{d}v)\,\mathrm{d}s \\ &\quad + 2 \int_0^T \int_{\mathbb{X}} \left| G\left(s, X_{m'}(s), v\right) \right|_H \left| g(s,v) - 1 \right| \left| X_{m'}(s) - X(s) \right|_H \vartheta(\mathrm{d}v)\,\mathrm{d}s \\ &\le 2 \int_0^T \int_{\mathbb{X}} \left| G(s,v) \right|_{0,H} \left(1 + \left\| X_{m'}(s) \right\|_H\right) \left| g_{m'}(s,v) - 1 \right| \left| X_{m'}(s) - X(s) \right|_H \vartheta(\mathrm{d}v)\,\mathrm{d}s \\ &\quad + 2 \int_0^T \int_{\mathbb{X}} \left| G(s,v) \right|_{0,H} \left(1 + \left\| X_{m'}(s) \right\|_H\right) \left| g(s,v) - 1 \right| \left| X_{m'}(s) - X(s) \right|_H \vartheta(\mathrm{d}v)\,\mathrm{d}s \\ &\le 2(1 + C_\Upsilon) \int_0^T \int_{\mathbb{X}} \left| G(s,v) \right|_{0,H} \left| g_{m'}(s,v) - 1 \right| \left| X_{m'}(s) - X(s) \right|_H \vartheta(\mathrm{d}v)\,\mathrm{d}s \end{aligned} \tag{4.61}$$

$$+2(1+C_\Upsilon)\int_0^T\int_{\mathbb{X}}\big|G(s,v)\big|_{0,H}\big|g(s,v)-1\big|\big|X_{m'}(s)-X(s)\big|_H\vartheta(\mathrm{d}v)\,\mathrm{d}s$$
$$\to 0 \qquad \text{as } m'\to\infty.$$

Therefore, by (4.59),

$$\lim_{m'\to\infty}\sup_{t\in[0,T]}\big\|Z_{m'}(t)\big\|_H^2=0. \tag{4.62}$$

□

The next theorem is contained in Theorem 1.2 and Example 4.7 in Brzeźniak, Liu and Zhu [5].

Theorem 4.3. *Assume Condition 3.1 with $K(\cdot)=C$ and that*

$$\int_{\mathbb{X}}\big\|G(s,x,v)\big\|_H^4\vartheta(\mathrm{d}v)\le C\big(1+\|x\|_H^4\big).$$

If $X_0\in H$, there exists a unique H-valued progressively measurable process $(X^\varepsilon(t))$ such that, $X^\varepsilon\in L^2(0,T;V)\cap D([0,T];H)$ for any $T>0$ and

$$\begin{aligned}X^\varepsilon(t)=X_0-\nu\int_0^t AX^\varepsilon(s)\,\mathrm{d}s-\int_0^t B\big(X^\varepsilon(s),X^\varepsilon(s)\big)\,\mathrm{d}s\\ +\sqrt{\varepsilon}\int_0^t\sigma\big(s,X^\varepsilon(s)\big)\,\mathrm{d}\beta(s)\\ +\varepsilon\int_0^t\int_{\mathbb{X}}G\big(s,X^\varepsilon(s-),v\big)\tilde{N}^{\varepsilon^{-1}}(\mathrm{d}s\,\mathrm{d}v),\qquad a.s.\end{aligned} \tag{4.63}$$

This theorem shows that the above equation admits a strong solution (in the probabilistic sense). In particular, for every $\varepsilon>0$, there exists a measurable map $\mathcal{G}^\varepsilon:\bar{\mathbb{V}}\to D([0,T];H)$ such that, for any Poisson random measure $n^{\varepsilon^{-1}}$ on $[0,T]\times\mathbb{X}$ with intensity measure $\varepsilon^{-1}\lambda_T\otimes\vartheta$ given on some probability space, $\mathcal{G}^\varepsilon(\sqrt{\varepsilon}\beta,\varepsilon n^{\varepsilon^{-1}})$ is the unique solution of (4.63) with $\tilde{N}^{\varepsilon^{-1}}$ replaced by $\tilde{n}^{\varepsilon^{-1}}$.

Let $\phi_\varepsilon=(\psi_\varepsilon,\varphi_\varepsilon)\in\tilde{\mathcal{U}}^\Upsilon$ and $\vartheta_\varepsilon=\frac{1}{\varphi_\varepsilon}$. The following lemma was proved in Budhiraja, Dupuis and Maroulas [9] (see Lemma 2.3 there).

Lemma 4.1.

$$\begin{aligned}\mathcal{E}_t^\varepsilon(\vartheta_\varepsilon):=\exp\bigg\{\int_{[0,t]\times\mathbb{X}\times[0,\varepsilon^{-1}]}\log\big(\vartheta_\varepsilon(s,x)\big)\bar{N}(\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}r)\\ +\int_{[0,t]\times\mathbb{X}\times[0,\varepsilon^{-1}]}\big(-\vartheta_\varepsilon(s,x)+1\big)\bar{\vartheta}_T(\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}r)\bigg\}\end{aligned}$$

and

$$\tilde{\mathcal{E}}_t^{\varepsilon}(\psi_\varepsilon) := \exp\left\{ \frac{1}{\sqrt{\varepsilon}} \int_0^t \psi_\varepsilon(s)\, \mathrm{d}\beta(s) - \frac{1}{2\varepsilon} \int_0^t \left\| \psi_\varepsilon(s) \right\|^2 \mathrm{d}s \right\}$$

are $\{\bar{\mathcal{F}}_t^{\bar{\mathbb{V}}}\}$*-martingales. Set*

$$\bar{\mathcal{E}}_t^{\varepsilon}(\psi_\varepsilon, \vartheta_\varepsilon) := \tilde{\mathcal{E}}_t^{\varepsilon}(\psi_\varepsilon)\mathcal{E}_t^{\varepsilon}(\vartheta_\varepsilon).$$

Then

$$\mathbb{Q}_t^{\varepsilon}(G) = \int_G \bar{\mathcal{E}}_t^{\varepsilon}(\psi_\varepsilon, \vartheta_\varepsilon)\, \mathrm{d}\bar{\mathbb{P}}^{\bar{\mathbb{V}}} \qquad \text{for } G \in \mathcal{B}(\bar{\mathbb{V}})$$

defines a probability measure on $\bar{\mathbb{V}}$.

Since $(\sqrt{\varepsilon}\beta + \int_0^\cdot \psi_\varepsilon(s)\,\mathrm{d}s, \varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ under $\mathbb{Q}_T^{\varepsilon}$ has the same law as that of $(\sqrt{\varepsilon}\beta, \varepsilon N^{\varepsilon^{-1}})$ under $\bar{\mathbb{P}}^{\bar{\mathbb{V}}}$, it follows that there exists a unique solution to the following controlled stochastic evolution equation, denoted by $\tilde{X}^{\varepsilon}$:

$$\begin{aligned}
\tilde{X}^{\varepsilon}(t) &= X_0 - \nu \int_0^t A\tilde{X}^{\varepsilon}(s)\,\mathrm{d}s - \int_0^t B\big(\tilde{X}^{\varepsilon}(s), \tilde{X}^{\varepsilon}(s)\big)\,\mathrm{d}s \\
&\quad + \int_0^t \sigma\big(s, \tilde{X}^{\varepsilon}(s)\big)\psi_\varepsilon(s)\,\mathrm{d}s + \sqrt{\varepsilon}\int_0^t \sigma\big(s, \tilde{X}^{\varepsilon}(s)\big)\,\mathrm{d}\beta(s) \\
&\quad + \int_0^t \int_{\mathbb{X}} G\big(s, \tilde{X}^{\varepsilon}(s-), v\big)\big(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}(\mathrm{d}s\,\mathrm{d}v) - \vartheta(\mathrm{d}v)\,\mathrm{d}s\big) \\
&= X_0 - \nu \int_0^t A\tilde{X}^{\varepsilon}(s)\,\mathrm{d}s - \int_0^t B\big(\tilde{X}^{\varepsilon}(s), \tilde{X}^{\varepsilon}(s)\big)\,\mathrm{d}s \\
&\quad + \int_0^t \sigma\big(s, \tilde{X}^{\varepsilon}(s)\big)\psi_\varepsilon(s)\,\mathrm{d}s + \sqrt{\varepsilon}\int_0^t \sigma\big(s, \tilde{X}^{\varepsilon}(s)\big)\,\mathrm{d}\beta(s) \\
&\quad + \int_0^t \int_{\mathbb{X}} G\big(s, \tilde{X}^{\varepsilon}(s), v\big)\big(\varphi_\varepsilon(s,v) - 1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{X}} \varepsilon G\big(s, \tilde{X}^{\varepsilon}(s-), v\big)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(\mathrm{d}s\,\mathrm{d}v) - \varepsilon^{-1}\varphi_\varepsilon(s,v)\vartheta(\mathrm{d}v)\,\mathrm{d}s\big),
\end{aligned} \tag{4.64}$$

and, we have

$$\mathcal{G}^{\varepsilon}\left(\sqrt{\varepsilon}\beta + \int_0^\cdot \psi_\varepsilon(s)\,\mathrm{d}s, \varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}\right) = \tilde{X}^{\varepsilon}. \tag{4.65}$$

The following estimates will be used later.

Lemma 4.2. *There exists $\varepsilon_0 > 0$ such that*

$$\sup_{0<\varepsilon<\varepsilon_0}\left[\mathbb{E}\sup_{0\le t\le T}\|\tilde{X}^\varepsilon(t)\|_H^2+\mathbb{E}\int_0^T\|\tilde{X}^\varepsilon(t)\|_V^2\,\mathrm{d}t\right]<\infty, \tag{4.66}$$

and, for $\alpha\in(0,1/2)$, there exits constant C_α such that

$$\sup_{0<\varepsilon<\varepsilon_0}\mathbb{E}\big[\|\tilde{X}^\varepsilon\|_{W^{\alpha,2}([0,T],V')}\big]\le C_\alpha<\infty. \tag{4.67}$$

Thus the family $\{\tilde{X}^\varepsilon, 0<\varepsilon<\varepsilon_0\}$ is tight in $L^2([0,T],H)$, by Lemma 3.4.

Proof. Equation (4.66) was proved in Brzeźniak, Liu and Zhu [5] (see Theorem 1.2 there). Now we prove (4.67).

Keep $\phi_\varepsilon=(\psi_\varepsilon,\varphi_\varepsilon)\in\tilde{\mathcal{U}}^\Upsilon$ in mind. Note (4.64),

$$\begin{aligned}\tilde{X}^\varepsilon(t)&=X_0-\nu\int_0^t A\tilde{X}^\varepsilon(s)\,\mathrm{d}s-\int_0^t B\big(\tilde{X}^\varepsilon(s),\tilde{X}^\varepsilon(s)\big)\,\mathrm{d}s\\&\quad+\int_0^t\sigma\big(s,\tilde{X}^\varepsilon(s)\big)\psi_\varepsilon(s)\,\mathrm{d}s+\sqrt{\varepsilon}\int_0^t\sigma\big(s,\tilde{X}^\varepsilon(s)\big)\,\mathrm{d}\beta(s)\\&\quad+\int_0^t\int_{\mathbb{X}}G\big(s,\tilde{X}^\varepsilon(s),v\big)\big(\varphi_\varepsilon(s,v)-1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s\\&\quad+\int_0^t\int_{\mathbb{X}}\varepsilon G\big(s,\tilde{X}^\varepsilon(s-),v\big)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(\mathrm{d}s\,\mathrm{d}v)-\varepsilon^{-1}\varphi_\varepsilon(s,v)\vartheta(\mathrm{d}v)\,\mathrm{d}s\big)\\&=J_\varepsilon^1+J_\varepsilon^2(t)+J_\varepsilon^3(t)+J_\varepsilon^4(t)+J_\varepsilon^5(t)+J_\varepsilon^6(t)+J_\varepsilon^7(t).\end{aligned} \tag{4.68}$$

By the same arguments as in the proof of Theorem 3.1 in Flandoli and Gatarek [18], we have

$$\begin{aligned}\sup_{0<\varepsilon<\varepsilon_0}\mathbb{E}\|J_\varepsilon^1\|_H^2&\le L_1,\\ \sup_{0<\varepsilon<\varepsilon_0}\mathbb{E}\|J_\varepsilon^2\|_{W^{1,2}([0,T];V')}^2&\le L_2,\\ \sup_{0<\varepsilon<\varepsilon_0}\mathbb{E}\|J_\varepsilon^3\|_{W^{1,2}([0,T];V')}&\le L_3.\end{aligned}$$

Since for $t>s$,

$$\begin{aligned}\mathbb{E}\|J_\varepsilon^4(t)-J_\varepsilon^4(s)\|_H^2&=\mathbb{E}\left\|\int_s^t\sigma\big(l,\tilde{X}^\varepsilon(l)\big)\psi_\varepsilon(l)\,\mathrm{d}l\right\|_H^2\\&\le\mathbb{E}\left(\int_s^t\|\sigma\big(l,\tilde{X}^\varepsilon(l)\big)\psi_\varepsilon(l)\|_H\,\mathrm{d}l\right)^2\end{aligned}$$

$$\le \mathbb{E}\Big(\int_s^t \big\|\sigma\big(l,\tilde{X}^\varepsilon(l)\big)\big\|_{L_2(H)}\big\|\psi_\varepsilon(l)\big\|_H\,\mathrm{d}l\Big)^2$$

$$\le \mathbb{E}\Big(\int_s^t \sqrt{K(l)}\sqrt{1+\big\|\tilde{X}^\varepsilon(l)\big\|_H^2}\big\|\psi_\varepsilon(l)\big\|_H\,\mathrm{d}l\Big)^2$$

$$\le \mathbb{E}\Big[\Big(1+\sup_{l\in[0,T]}\big\|\tilde{X}^\varepsilon(l)\big\|_H^2\Big)\int_s^t K(l)\,\mathrm{d}l\int_s^t\big\|\psi_\varepsilon(l)\big\|_H^2\,\mathrm{d}l\Big]$$

$$\le \Upsilon\mathbb{E}\Big(1+\sup_{l\in[0,T]}\big\|\tilde{X}^\varepsilon(l)\big\|_H^2\Big)\int_s^t K(l)\,\mathrm{d}l,$$

we have

$$\mathbb{E}\int_0^T\big\|J_\varepsilon^4(t)\big\|_H^2\,\mathrm{d}t \le \Upsilon T\mathbb{E}\Big(1+\sup_{l\in[0,T]}\big\|\tilde{X}^\varepsilon(l)\big\|_H^2\Big)\int_0^T K(l)\,\mathrm{d}l \tag{4.69}$$

and

$$\begin{aligned}&\mathbb{E}\int_0^T\int_0^T\frac{\|J_\varepsilon^4(t)-J_\varepsilon^4(s)\|_H^2}{|t-s|^{1+2\alpha}}\,\mathrm{d}t\,\mathrm{d}s\\ &\quad\le \Upsilon\mathbb{E}\Big(1+\sup_{l\in[0,T]}\big\|\tilde{X}^\varepsilon(l)\big\|_H^2\Big)\int_0^T\int_0^T\int_s^t\frac{K(l)}{|t-s|^{1+2\alpha}}\,\mathrm{d}l\,\mathrm{d}t\,\mathrm{d}s.\end{aligned} \tag{4.70}$$

By an elementary application of Fubini theorem, there exists $L_4^1>0$ such that

$$\int_0^T\int_0^T\int_s^t\frac{K(l)}{|t-s|^{1+2\alpha}}\,\mathrm{d}l\,\mathrm{d}t\,\mathrm{d}s \le L_4^1\int_0^T K(l)\,\mathrm{d}l. \tag{4.71}$$

Combining (4.69), (4.70), (4.71), we obtain

$$\sup_{0<\varepsilon<\varepsilon_0}\mathbb{E}\big\|J_\varepsilon^4\big\|_{W^{\alpha,2}([0,T];H)}^2 \le L_4. \tag{4.72}$$

Now for J_ε^5, since for $t>s$,

$$\begin{aligned}\mathbb{E}\big\|J_\varepsilon^5(t)-J_\varepsilon^5(s)\big\|_H^2 &= \mathbb{E}\Big\|\int_s^t\sqrt{\varepsilon}\sigma\big(l,\tilde{X}^\varepsilon(l)\big)\,\mathrm{d}\beta(l)\Big\|_H^2\\ &\le \varepsilon\mathbb{E}\Big(\int_s^t\big\|\sigma\big(l,\tilde{X}^\varepsilon(l)\big)\big\|_{L_2(H)}^2\,\mathrm{d}l\Big)\\ &\le \varepsilon\mathbb{E}\Big(\int_s^t K(l)\big(1+\big\|\tilde{X}^\varepsilon(l)\big\|_H^2\big)\,\mathrm{d}l\Big)\\ &\le \varepsilon\mathbb{E}\Big(1+\sup_{l\in[0,T]}\big\|\tilde{X}^\varepsilon(l)\big\|_H^2\Big)\int_s^t K(l)\,\mathrm{d}l,\end{aligned}$$

similar to (4.72), we have

$$\sup_{0<\varepsilon<\varepsilon_0} \mathbb{E}\|J_\varepsilon^5\|^2_{W^{\alpha,2}([0,T];H)} \le L_5. \tag{4.73}$$

For J_ε^6, we have

$$\begin{aligned}
&\mathbb{E}\|J_\varepsilon^6(t)-J_\varepsilon^6(s)\|_H^2\\
&\quad=\mathbb{E}\left\|\int_s^t\int_{\mathbb{X}} G\big(l,\tilde{X}^\varepsilon(l),v\big)\big(\varphi_\varepsilon(l,v)-1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}l\right\|_H^2\\
&\quad\le\mathbb{E}\left(\int_s^t\int_{\mathbb{X}}\big\|G\big(l,\tilde{X}^\varepsilon(l),v\big)\big\|_H\big|\varphi_\varepsilon(l,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}l\right)^2\\
&\quad\le\mathbb{E}\left(\int_s^t\int_{\mathbb{X}}\|G(l,v)\|_{0,H}\big|\varphi_\varepsilon(l,v)-1\big|\big(1+\|\tilde{X}^\varepsilon(l)\|_H\big)\vartheta(\mathrm{d}v)\,\mathrm{d}l\right)^2\\
&\quad\le\mathbb{E}\left[\Big(1+\sup_{l\in[0,T]}\|\tilde{X}^\varepsilon(l)\|_H\Big)^2\left(\int_s^t\int_{\mathbb{X}}\|G(l,v)\|_{0,H}\big|\varphi_\varepsilon(l,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}l\right)^2\right]\\
&\quad\le C_{0,1}^{\Upsilon}\mathbb{E}\left[\Big(1+\sup_{l\in[0,T]}\|\tilde{X}^\varepsilon(l)\|_H\Big)^2\int_s^t\int_{\mathbb{X}}\|G(l,v)\|_{0,H}\big|\varphi_\varepsilon(l,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}l\right],
\end{aligned}$$

where $C_{0,1}^{\Upsilon}$ appeared in (3.2), using the similar arguments as to bound J_ε^4 and by Lemma 3.1, we have

$$\sup_{0<\varepsilon<\varepsilon_0} \mathbb{E}\|J_\varepsilon^6\|^2_{W^{\alpha,2}([0,T];H)} \le L_6. \tag{4.74}$$

For J_ε^7, we have

$$\begin{aligned}
&\mathbb{E}\|J_\varepsilon^7(t)-J_\varepsilon^7(s)\|_H^2\\
&\quad=\mathbb{E}\left\|\int_s^t\int_{\mathbb{X}}\varepsilon G\big(l,\tilde{X}^\varepsilon(l-),v\big)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(\mathrm{d}l\,\mathrm{d}v)-\varepsilon^{-1}\varphi_\varepsilon(l,v)\vartheta(\mathrm{d}v)\,\mathrm{d}l\big)\right\|_H^2\\
&\quad\le\varepsilon\mathbb{E}\int_s^t\int_{\mathbb{X}}\big\|G\big(l,\tilde{X}^\varepsilon(l),v\big)\big\|_H^2\varphi_\varepsilon(l,v)\vartheta(\mathrm{d}v)\,\mathrm{d}l\\
&\quad\le\varepsilon\mathbb{E}\int_s^t\int_{\mathbb{X}}\|G(l,v)\|_{0,H}^2\varphi_\varepsilon(l,v)\big(1+\|\tilde{X}^\varepsilon(l)\|_H\big)^2\vartheta(\mathrm{d}v)\,\mathrm{d}l\\
&\quad\le\varepsilon\mathbb{E}\left[\sup_{l\in[0,T]}\big(1+\|\tilde{X}^\varepsilon(l)\|_H\big)^2\int_s^t\int_{\mathbb{X}}\|G(l,v)\|_{0,H}^2\varphi_\varepsilon(l,v)\vartheta(\mathrm{d}v)\,\mathrm{d}l\right]
\end{aligned}$$

using the similar arguments as to bound J_ε^4 and by Lemma 3.1, we obtain

$$\sup_{0<\varepsilon<\varepsilon_0} \mathbb{E}\| J_\varepsilon^7 \|^2_{W^{\alpha,2}([0,T];H)} \leq L_7. \tag{4.75}$$

Combining (4.69), (4.73), (4.74), and (4.75), we get (4.67). The proof is complete. □

To get our main results, we need to prove that $\{\tilde{X}^\varepsilon, 0 < \varepsilon < \varepsilon_0\}$ is tight in vector valued Skorokhod space $D([0,T], D(A^{-\varrho}))$, here ϱ is the positive number in (1.4). First, we recall the following two lemmas (see Jakubowski [21] and Aldous [1]).

Lemma 4.3. *Let E be a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. For an orthonormal basis $\{\chi_k\}_{k\in\mathbb{N}}$ in E, define the function $r_L^2 : E \to R^+$ by*

$$r_L^2(x) = \sum_{k\geq L+1} \langle x, \chi_k\rangle^2, \qquad L \in \mathbb{N}.$$

Let $\mathcal{D}$ be a total and closed under addition subset of E. Then a sequence $\{X_\varepsilon\}_{\varepsilon\in(0,1)}$ of stochastic processes with trajectories in $D([0,T], E)$ is tight iff the following two conditions hold:

1. *$\{X_\varepsilon\}_{\varepsilon\in(0,1)}$ is $\mathcal{D}$-weakly tight, that is, for every $h \in \mathcal{D}$, $\{\langle X_\varepsilon, h\rangle\}_{\varepsilon\in(0,1)}$ is tight in $D([0,T],\mathbb{R})$*;
2. *For every $\eta > 0$*

$$\lim_{L\to\infty} \lim_{\varepsilon\to 0} \sup P\big(r_L^2\big(X_\varepsilon(s)\big) > \eta \text{ for some } s \in [0,T]\big) = 0. \tag{4.76}$$

Let $\{Y_\varepsilon\}_{\varepsilon\in(0,1)}$ be a sequence of random elements of $D([0,T],\mathbb{R})$, and $\{\tau_\varepsilon, \delta_\varepsilon\}$ be such that:

(a) For each ε, τ_ε is a stopping time with respect to the natural σ-fields, and takes only finitely many values.
(b) The constant $\delta_\varepsilon \in [0,T]$ satisfies $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.

We introduce the following condition on $\{Y_\varepsilon\}$:

(A) For each sequence $\{\tau_\varepsilon, \delta_\varepsilon\}$ satisfying (a) and (b), $Y_\varepsilon(\tau_\varepsilon + \delta_\varepsilon) - Y_\varepsilon(\tau_\varepsilon) \to 0$, in probability, as $\varepsilon \to 0$.

For $f \in D([0,T],\mathbb{R})$, let $J(f)$ denote the maximum of the jump $|f(t) - f(t-)|$.

Lemma 4.4. *Suppose that $\{Y_\varepsilon\}_{\varepsilon\in(0,1)}$ satisfies* (A), *and either $\{Y_\varepsilon(0)\}$ and $\{J(Y_\varepsilon)\}$ are tight on the line*; *or $\{Y_\varepsilon(t)\}$ is tight on the line for each $t \in [0,T]$, then $\{Y_\varepsilon\}$ is tight in $D([0,T],\mathbb{R})$.*

Let $\tilde{X}^\varepsilon$ be defined as in (4.67). We have the following lemma.

Lemma 4.5. $\{\tilde{X}^\varepsilon, 0 < \varepsilon < \varepsilon_0\}$ *is tight in $D([0,T], D(A^{-\varrho}))$.*

Proof. Note that $\{\lambda_i^{\varrho} e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system of $D(A^{-\varrho})$. Since

$$\begin{aligned}
\lim_{L\to\infty}\limsup_{\varepsilon\to 0}\mathbb{E}\sup_{s\in[0,T]} r_L^2\big(\tilde{X}^\varepsilon(s)\big) &= \lim_{L\to\infty}\limsup_{\varepsilon\to 0}\mathbb{E}\sup_{s\in[0,T]}\sum_{i=L+1}^{\infty}\big\langle \tilde{X}^\varepsilon(s), \lambda_i^{\varrho} e_i\big\rangle_{D(A^{-\varrho})}^2 \\
&= \lim_{L\to\infty}\limsup_{\varepsilon\to 0}\mathbb{E}\sup_{s\in[0,T]}\sum_{i=L+1}^{\infty}\big\langle A^{-\varrho}\tilde{X}^\varepsilon(s), e_i\big\rangle_H^2 \\
&= \lim_{L\to\infty}\limsup_{\varepsilon\to 0}\mathbb{E}\sup_{s\in[0,T]}\sum_{i=L+1}^{\infty}\frac{\langle \tilde{X}^\varepsilon(s), e_i\rangle_H^2}{\lambda_i^{2\varrho}} \\
&\le \lim_{L\to\infty}\frac{\lim_{\varepsilon\to 0}\sup\mathbb{E}\sup_{s\in[0,T]}\|\tilde{X}^\varepsilon(s)\|_H^2}{\lambda_{L+1}^{2\varrho}} \\
&= 0,
\end{aligned}$$

(4.76) holds with $E = D(A^{-\varrho})$.

Choose $\mathcal{D} = D(A^{\varrho})$. We now prove $\{\tilde{X}^\varepsilon, 0<\varepsilon<\varepsilon_0\}$ is $\mathcal{D}$-weakly tight. Let $h\in D(A^{\varrho})$, and $\{\tau_\varepsilon, \delta_\varepsilon\}$ satisfies (a) and (b). It is easy to see $\{\langle \tilde{X}^\varepsilon(t), h\rangle_E, 0<\varepsilon<\varepsilon_0\}$ is tight on the real line for each $t\in[0,T]$.

We now prove that $\{\langle \tilde{X}^\varepsilon, h\rangle_E, 0<\varepsilon<\varepsilon_0\}$ satisfies (A). By (4.64), we have

$$\begin{aligned}
&\tilde{X}^\varepsilon(\tau_\varepsilon+\delta_\varepsilon) - \tilde{X}^\varepsilon(\tau_\varepsilon) \\
&\quad= -\nu\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} A\tilde{X}^\varepsilon(s)\,\mathrm{d}s - \int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} B\big(\tilde{X}^\varepsilon(s), \tilde{X}^\varepsilon(s)\big)\,\mathrm{d}s \\
&\qquad+ \int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\sigma\big(s, \tilde{X}^\varepsilon(s)\big)\psi_\varepsilon(s)\,\mathrm{d}s + \sqrt{\varepsilon}\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\sigma\big(s, \tilde{X}^\varepsilon(s)\big)\,\mathrm{d}\beta(s) \\
&\qquad+ \int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\int_{\mathbb{X}} G\big(s, \tilde{X}^\varepsilon(s), v\big)\big(\varphi_\varepsilon(s,v)-1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s \\
&\qquad+ \int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\int_{\mathbb{X}} \varepsilon G\big(s, \tilde{X}^\varepsilon(s-), v\big)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(\mathrm{d}s\,\mathrm{d}v) - \varepsilon^{-1}\varphi_\varepsilon(s,v)\vartheta(\mathrm{d}v)\,\mathrm{d}s\big) \\
&\quad= J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon + J_5^\varepsilon + J_6^\varepsilon.
\end{aligned}$$

It is easy to show

$$\lim_{\varepsilon\to 0}\mathbb{E}\big|\big\langle J_4^\varepsilon, h\big\rangle_E\big|^2 = 0 \tag{4.77}$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}\big|\big\langle J_6^\varepsilon, h\big\rangle_E\big|^2 = 0. \tag{4.78}$$

For J_1^ε, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\big|\langle J_1^\varepsilon, h\rangle_E\big| \le \lim_{\varepsilon\to 0}\nu\|h\|_{D(A)}\delta_\varepsilon\mathbb{E}\Big[\sup_{s\in[0,T]}\big\|\tilde{X}^\varepsilon(s)\big\|_H\Big]=0. \tag{4.79}$$

For J_2^ε,

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}\big|\langle J_2^\varepsilon, h\rangle_E\big| &\le \lim_{\varepsilon\to 0}\|h\|_V\mathbb{E}\Big[\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\big\|B\big(\tilde{X}^\varepsilon(s)\big)\big\|_{V'}\,\mathrm{d}s\Big]\\ &\le 2\|h\|_V\lim_{\varepsilon\to 0}\mathbb{E}\Big[\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\big\|\tilde{X}^\varepsilon(s)\big\|_H\big\|\tilde{X}^\varepsilon(s)\big\|_V\,\mathrm{d}s\Big]\\ &\le \|h\|_V\lim_{\varepsilon\to 0}\delta_\varepsilon^{1/2}\Big[\mathbb{E}\sup_{s\in[0,T]}\big\|\tilde{X}^\varepsilon(s)\big\|_H^2+\mathbb{E}\int_0^T\big\|\tilde{X}^\varepsilon(s)\big\|_V^2\,\mathrm{d}s\Big]\\ &=0.\end{aligned} \tag{4.80}$$

For J_3^ε, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\mathbb{E}\big|\langle J_3^\varepsilon, h\rangle_E\big|\\ &\quad\le \lim_{\varepsilon\to 0}\|h\|_H\mathbb{E}\Big[\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\big\|\sigma\big(s,\tilde{X}^\varepsilon(s)\big)\big\|_{L_2(H)}\big\|\psi_\varepsilon(s)\big\|_H\,\mathrm{d}s\Big]\\ &\quad\le \lim_{\varepsilon\to 0}\|h\|_H\mathbb{E}\Big[\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\big\|\sigma\big(s,\tilde{X}^\varepsilon(s)\big)\big\|_{L_2(H)}\big\|\psi_\varepsilon(s)\big\|_H\,\mathrm{d}s\Big]\\ &\quad\le \lim_{\varepsilon\to 0}\|h\|_H\mathbb{E}\Big[\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\sqrt{K(s)}\sqrt{1+\big\|\tilde{X}^\varepsilon(s)\big\|_H^2}\big\|\psi_\varepsilon(s)\big\|_H\,\mathrm{d}s\Big]\\ &\quad\le \lim_{\varepsilon\to 0}\|h\|_H\mathbb{E}\Big[\sup_{s\in[0,T]}\sqrt{1+\big\|\tilde{X}^\varepsilon(s)\big\|_H^2}\Big(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}K(s)\,\mathrm{d}s\Big)^{1/2}\Big(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\big\|\psi_\varepsilon(s)\big\|_H^2\,\mathrm{d}s\Big)^{1/2}\Big]\\ &\quad\le \lim_{\varepsilon\to 0}\|h\|_H\Upsilon^{1/2}\Big[\mathbb{E}\sup_{s\in[0,T]}\big(1+\big\|\tilde{X}^\varepsilon(s)\big\|_H^2\big)\Big]^{1/2}\Big[\mathbb{E}\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}K(s)\,\mathrm{d}s\Big]^{1/2}.\end{aligned}$$

By dominated convergence theorem, it follows that

$$\lim_{\varepsilon\to 0}\mathbb{E}\big|\langle J_3^\varepsilon, h\rangle_E\big|=0. \tag{4.81}$$

For J_5^ε, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\mathbb{E}\big|\langle J_5^\varepsilon, h\rangle_E\big|\\ &\quad\le \|h\|_H\lim_{\varepsilon\to 0}\mathbb{E}\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\int_{\mathbb{X}}\big\|G\big(s,\tilde{X}^\varepsilon(s),v\big)\big\|_H\big|\varphi_\varepsilon(s,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}s\end{aligned}$$

$$\le \|h\|_H \lim_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{s\in[0,T]}\big(1+\|\tilde{X}^\varepsilon(s)\|_H\big)\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\int_{\mathbb{X}}\|G(s,v)\|_{0,H}\big|\varphi_\varepsilon(s,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}s\Big]$$

$$\le \|h\|_H \lim_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{s\in[0,T]}\big(1+\|\tilde{X}^\varepsilon(s)\|_H\big)\sup_{g\in S^\Upsilon}\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\int_{\mathbb{X}}\|G(s,v)\|_{0,H}\big|g(s,v)-1\big|\vartheta(\mathrm{d}v)\,\mathrm{d}s\Big].$$

By Lemma 3.1, we obtain

$$\lim_{\varepsilon\to 0}\mathbb{E}\big|\langle J_5^\varepsilon, h\rangle_E\big| = 0. \tag{4.82}$$

Combining (4.77)–(4.82), we conclude that $\{\langle\tilde{X}^\varepsilon, h\rangle_E, 0<\varepsilon<\varepsilon_0\}$ satisfies (A) of Lemma 4.4.

The proof is complete. □

By Proposition 3.1 of Röckner and Zhang [28], there exists a unique solution $\widetilde{Y}^\varepsilon(t)$, $t\ge 0$ to the following equation:

$$\begin{aligned} d\tilde{Y}^\varepsilon(t) &= -\nu A\tilde{Y}^\varepsilon(t)\,\mathrm{d}t + \sqrt{\varepsilon}\sigma\big(t,\tilde{X}^\varepsilon(t)\big)\,\mathrm{d}\beta(t) \\ &\quad + \int_{\mathbb{X}}\varepsilon G\big(t,\tilde{X}^\varepsilon(t-),v\big)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(\mathrm{d}t\,\mathrm{d}v) - \varepsilon^{-1}\varphi_\varepsilon(t,v)\vartheta(\mathrm{d}v)\,\mathrm{d}t\big), \end{aligned} \tag{4.83}$$

and $\widetilde{Y}^\varepsilon \in D([0,T];H)\cap L^2([0,T],V)$, $\mathbb{P}$-a.s. We have the following estimate.

Lemma 4.6. *There exists constant C and $\widetilde{\varepsilon}_0$ such that for any* $0<\varepsilon\le\widetilde{\varepsilon}_0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|\tilde{Y}^\varepsilon(t)\|_H^2\Big] + \mathbb{E}\int_0^T\|\tilde{Y}^\varepsilon(t)\|_V^2\,\mathrm{d}t \le \sqrt{\varepsilon}C.$$

Proof. By Ito's formula,

$$\begin{aligned} &\|\tilde{Y}^\varepsilon(t)\|_H^2 + 2\nu\int_0^t\|\tilde{Y}^\varepsilon(s)\|_V^2\,\mathrm{d}s \\ &\quad = 2\int_0^t\big\langle\tilde{Y}^\varepsilon(s),\sqrt{\varepsilon}\sigma\big(s,\tilde{X}^\varepsilon(s)\big)\,\mathrm{d}\beta(s)\big\rangle \\ &\qquad + 2\int_0^t\int_{\mathbb{X}}\big\langle\tilde{Y}^\varepsilon(s-),\varepsilon G\big(s,\tilde{X}^\varepsilon(s-),v\big)\big(N^{\varepsilon^{-1}\varphi_\varepsilon}(\mathrm{d}s\,\mathrm{d}v)-\varepsilon^{-1}\varphi_\varepsilon(s,v)\vartheta(\mathrm{d}v)\,\mathrm{d}s\big)\big\rangle \\ &\qquad + \varepsilon^2\int_0^t\int_{\mathbb{X}}\big\|G\big(s,\tilde{X}^\varepsilon(s-),v\big)\big\|_H^2 N^{\varepsilon^{-1}\varphi_\varepsilon}(\mathrm{d}s\,\mathrm{d}v) + \varepsilon\int_0^t\big\|\sigma\big(s,\tilde{X}^\varepsilon(s)\big)\big\|_{L_2(H)}^2\,\mathrm{d}s \\ &\quad = I_1(t)+I_2(t)+I_3(t)+I_4(t). \end{aligned} \tag{4.84}$$

By Condition 3.1, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]} I_3(t)\Big] \le \varepsilon\int_0^T\big(1+K(s)\big)\,\mathrm{d}s\,\mathbb{E}\Big[\sup_{t\in[0,T]}\|\tilde{X}^\varepsilon(t)\|_H^2\Big] \tag{4.85}$$

and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} I_4(t)\Big] \le 2\varepsilon C^{\Upsilon}_{0,2}\Big(1+\mathbb{E}\Big[\sup_{t\in[0,T]}\|\tilde{X}^{\varepsilon}(t)\|^2_H\Big]\Big). \tag{4.86}$$

Similar to the proof of (3.45) and (3.50) in Yang, Zhai and Zhang [38], we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\big|I_1(t)\big|\Big] \le{}& 4\sqrt{\varepsilon}\mathbb{E}\Big[\sup_{t\in[0,T]}\|\tilde{Y}^{\varepsilon}(t)\|^2_H\Big] \\ &+4\sqrt{\varepsilon}\Big(\int_0^T K(s)\,\mathrm{d}s + T\mathbb{E}\Big[\sup_{t\in[0,T]}\|\tilde{X}^{\varepsilon}(t)\|^2_H\Big]\Big)\end{aligned} \tag{4.87}$$

and

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|I_2(t)\big|\Big] \le 4\sqrt{\varepsilon}\mathbb{E}\Big[\sup_{t\in[0,T]}\|\tilde{Y}^{\varepsilon}(t)\|^2_H\Big] + 8\sqrt{\varepsilon}C^{\Upsilon}_{0,2}\Big(1+\mathbb{E}\Big[\sup_{t\in[0,T]}\|\tilde{X}^{\varepsilon}(t)\|^2_H\Big]\Big). \tag{4.88}$$

By Lemma 4.2 and (4.84)–(4.88), there exist constants C and $\widetilde{\varepsilon}_0$ such that for any $0<\varepsilon\le\widetilde{\varepsilon}_0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|\tilde{Y}^{\varepsilon}(t)\|^2_H\Big] + 2\nu\mathbb{E}\int_0^T \|\tilde{Y}^{\varepsilon}(s)\|^2_V\,\mathrm{d}s \le \sqrt{\varepsilon}C.$$

□

The following theorem verifies the second part of Condition 2.1. Recall that $\mathcal{G}^{\varepsilon}$ is defined by (4.65).

Theorem 4.4. *Fix $\Upsilon\in\mathbb{N}$, and let $\phi_\varepsilon=(\psi_\varepsilon,\varphi_\varepsilon)$, $\phi=(\psi,\varphi)\in\widetilde{\mathcal{U}}^{\Upsilon}$ be such that ϕ_ε converges in distribution to ϕ as $\varepsilon\to 0$. Then*

$$\mathcal{G}^{\varepsilon}\Big(\sqrt{\varepsilon}\beta+\int_0^{\cdot}\psi_\varepsilon(s)\,\mathrm{d}s, \varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}\Big) \Rightarrow \mathcal{G}^0\Big(\int_0^{\cdot}\psi(s)\,\mathrm{d}s, \vartheta^{\varphi}\Big).$$

Proof. Note that $\tilde{X}^{\varepsilon}=\mathcal{G}^{\varepsilon}(\sqrt{\varepsilon}\beta+\int_0^{\cdot}\psi_\varepsilon(s)\,\mathrm{d}s, \varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$, $\varepsilon\in(0,\varepsilon_0)$. By Lemmas 4.2, 4.5 and 4.6, we know that

1. $\{\tilde{X}^{\varepsilon}, \varepsilon\in(0,\varepsilon_0)\}$ is tight in $L^2([0,T],H)\cap D([0,T],D(A^{-\varrho}))$,
2. $\lim_{\varepsilon\to0}\mathbb{E}[\sup_{t\in[0,T]}\|\tilde{Y}^{\varepsilon}(t)\|^2_H]+\mathbb{E}\int_0^T\|\tilde{Y}^{\varepsilon}(t)\|^2_V\,\mathrm{d}t=0$,

where $\tilde{Y}^{\varepsilon}$ is defined as in Lemma 4.6. Set

$$\Pi=\big(L^2\big([0,T],H\big)\cap D\big([0,T],D\big(A^{-\varrho}\big)\big);\widetilde{\mathcal{U}}^{\Upsilon};L^2\big([0,T],V\big)\cap D\big([0,T],H\big)\big).$$

Let $(\tilde{X},(\psi,\varphi),0)$ be any limit point of the tight family $\{(\tilde{X}^{\varepsilon},(\psi_\varepsilon,\varphi_\varepsilon),\tilde{Y}^{\varepsilon}), \varepsilon\in(0,\varepsilon_0)\}$. We must show that $\tilde{X}$ has the same law as $\mathcal{G}^0(\int_0^{\cdot}\psi(s)\,\mathrm{d}s,\vartheta^{\varphi})$, and actually $\tilde{X}^{\varepsilon}\Longrightarrow\tilde{X}$ in the smaller space $D([0,T];H)$.

By the Skorokhod representation theorem, there exit a stochastic basis $(\Omega^1,\mathcal{F}^1,\{\mathcal{F}^1_t\}_{t\in[0,T]},\mathbb{P}^1)$ and, on this basis, Π-valued random variables $(\tilde{X}_1,(\psi^1,\varphi^1),0)$, $(\tilde{X}^{\varepsilon}_1,(\psi^1_\varepsilon,\varphi^1_\varepsilon),\tilde{Y}^{\varepsilon}_1)$, $\varepsilon\in$

$(0, \varepsilon_0)$, such that $(\tilde{X}_1^\varepsilon, (\psi_\varepsilon^1, \varphi_\varepsilon^1), \tilde{Y}_1^\varepsilon)$ (respectively $(\tilde{X}_1, (\psi^1, \varphi^1), 0)$) has the same law as $(\tilde{X}^\varepsilon, (\psi_\varepsilon, \varphi_\varepsilon), \tilde{Y}^\varepsilon)$ (respectively, $(\tilde{X}, (\psi, \varphi), 0)$), and $(\tilde{X}_1^\varepsilon, (\psi_\varepsilon^1, \varphi_\varepsilon^1), \tilde{Y}_1^\varepsilon) \to (\tilde{X}_1, (\psi^1, \varphi^1), 0)$ in Π, $\mathbb{P}^1$-a.s.

From the equation satisfied by $(\tilde{X}^\varepsilon, (\psi_\varepsilon, \varphi_\varepsilon), \tilde{Y}^\varepsilon)$, we see that $(\tilde{X}_1^\varepsilon, (\psi_\varepsilon^1, \varphi_\varepsilon^1), \tilde{Y}_1^\varepsilon)$ satisfies the following integral equation

$$\begin{aligned}\tilde{X}_1^\varepsilon(t) - \tilde{Y}_1^\varepsilon(t) = X_0 - \nu \int_0^t A\big(\tilde{X}_1^\varepsilon(s) - \tilde{Y}_1^\varepsilon(s)\big)\,\mathrm{d}s - \int_0^t B\big(\tilde{X}_1^\varepsilon(s)\big)\,\mathrm{d}s \\ + \int_0^t \sigma\big(s, \tilde{X}_1^\varepsilon(s)\big)\psi_\varepsilon^1(s)\,\mathrm{d}s \\ + \int_0^t \int_{\mathbb{X}} G\big(s, \tilde{X}_1^\varepsilon(s), v\big)\big(\varphi_\varepsilon^1(s, v) - 1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}^1\big(\tilde{X}_1^\varepsilon - \tilde{Y}_1^\varepsilon \in C\big([0, T], H\big) \cap L^2\big([0, T], V\big)\big) \\ &= \bar{\mathbb{P}}^{\bar{\mathbb{V}}}\big(\tilde{X}^\varepsilon - \tilde{Y}^\varepsilon \in C\big([0, T], H\big) \cap L^2\big([0, T], V\big)\big) \\ &= 1.\end{aligned}$$

Let Ω_0^1 be the subset of Ω^1 such that $(\tilde{X}_1^\varepsilon, (\psi_\varepsilon^1, \varphi_\varepsilon^1), \tilde{Y}_1^\varepsilon) \to (\tilde{X}_1, (\psi^1, \varphi^1), 0)$ in Π, then $\mathbb{P}^1(\Omega_0^1) = 1$. Now we must prove that, for any fixed $\omega^1 \in \Omega_0^1$,

$$\sup_{t\in[0,T]} \big\|\tilde{X}_1^\varepsilon\big(\omega^1, t\big) - \tilde{X}_1\big(\omega^1, t\big)\big\|_H^2 \to 0. \tag{4.89}$$

Set $Z^\varepsilon = \tilde{X}_1^\varepsilon - \tilde{Y}_1^\varepsilon$, then $Z^\varepsilon(\omega^1) \in C([0, T], H) \cap L^2([0, T], V)$, and $Z^\varepsilon(\omega^1)$ satisfies

$$\begin{aligned}Z^\varepsilon(t) = X_0 - \nu \int_0^t AZ^\varepsilon(s)\,\mathrm{d}s - \int_0^t B\big(Z^\varepsilon(s) + \widetilde{Y}_1^\varepsilon(s)\big)\,\mathrm{d}s + \int_0^t \sigma\big(s, Z^\varepsilon(s) + \widetilde{Y}_1^\varepsilon(s)\big)\psi_\varepsilon^1(s)\,\mathrm{d}s \\ + \int_0^t \int_{\mathbb{X}} G\big(s, Z^\varepsilon(s) + \widetilde{Y}_1^\varepsilon(s), v\big)\big(\varphi_\varepsilon^1(s, v) - 1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s.\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0}\Big[\sup_{t\in[0,T]} \big\|\tilde{Y}_1^\varepsilon\big(\omega^1, t\big)\big\|_H^2 + \int_0^T \big\|\tilde{Y}_1^\varepsilon\big(\omega^1, t\big)\big\|_V^2\,\mathrm{d}t\Big] = 0,$$

by similar arguments as in the proof of Proposition 4.1, we can show that

$$\lim_{\varepsilon\to 0}\Big[\sup_{t\in[0,T]} \big\|\tilde{X}_1^\varepsilon\big(\omega^1, t\big) - \widehat{X}\big(\omega^1, t\big)\big\|_H^2\Big] = 0, \tag{4.90}$$

where

$$\widehat{X}(t) = X_0 - \nu \int_0^t A\widehat{X}(s)\,\mathrm{d}s - \int_0^t B\big(\widehat{X}(s)\big)\,\mathrm{d}s + \int_0^t \sigma\big(s, \widehat{X}(s)\big)\psi^1(s)\,\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{X}} G\big(s, \widehat{X}(s), v\big)\big(\varphi^1(s,v) - 1\big)\vartheta(\mathrm{d}v)\,\mathrm{d}s.$$

Hence $\tilde{X}_1 = \widehat{X} = \mathcal{G}^0(\int_0^\cdot \psi^1(s)\,\mathrm{d}s, \vartheta^{\varphi^1})$, and $\tilde{X}$ has the same law as $\mathcal{G}^0(\int_0^\cdot \psi(s)\,\mathrm{d}s, \vartheta^{\varphi})$. Because $\tilde{X}^\varepsilon = \tilde{X}_1^\varepsilon$ in law, (4.90) further implies that

$$\tilde{X}^\varepsilon \Longrightarrow \mathcal{G}^0\left(\int_0^\cdot \psi(s)\,\mathrm{d}s, \vartheta^\varphi\right)$$

completing the proof of the theorem. □

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References

[1] Aldous, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340. MR0474446
[2] Bensoussan, A. and Temam, R. (1973). Équations stochastiques du type Navier–Stokes. *J. Funct. Anal.* **13** 195–222. MR0348841
[3] Bessaih, H. and Millet, A. (2009). Large deviation principle and inviscid shell models. *Electron. J. Probab.* **14** 2551–2579. MR2570011
[4] Boué, M. and Dupuis, P. (1998). A variational representation for certain functionals of Brownian motion. *Ann. Probab.* **26** 1641–1659. MR1675051
[5] Brzeźniak, Z., Liu, W. and Zhu, J. (2014). Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise. *Nonlinear Anal. Real World Appl.* **17** 283–310. MR3158475
[6] Budhiraja, A., Chen, J. and Dupuis, P. (2013). Large deviations for stochastic partial differential equations driven by a Poisson random measure. *Stochastic Process. Appl.* **123** 523–560. MR3003362
[7] Budhiraja, A. and Dupuis, P. (2000). A variational representation for positive functionals of infinite dimensional Brownian motion. *Probab. Math. Statist.* **20** 39–61. MR1785237
[8] Budhiraja, A., Dupuis, P. and Maroulas, V. (2008). Large deviations for infinite dimensional stochastic dynamical systems. *Ann. Probab.* **36** 1390–1420. MR2435853
[9] Budhiraja, A., Dupuis, P. and Maroulas, V. (2011). Variational representations for continuous time processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 725–747. MR2841073
[10] Cardon-Weber, C. (1999). Large deviations for a Burgers'-type SPDE. *Stochastic Process. Appl.* **84** 53–70. MR1720097
[11] Cerrai, S. and Röckner, M. (2004). Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. *Ann. Probab.* **32** 1100–1139. MR2044675
[12] Chenal, F. and Millet, A. (1997). Uniform large deviations for parabolic SPDEs and applications. *Stochastic Process. Appl.* **72** 161–186. MR1486551

[13] Chow, P.L. (1992). Large deviation problem for some parabolic Itô equations. *Comm. Pure Appl. Math.* **45** 97–120. MR1135925

[14] Chueshov, I. and Millet, A. (2010). Stochastic 2D hydrodynamical type systems: Well posedness and large deviations. *Appl. Math. Optim.* **61** 379–420. MR2609596

[15] Da Prato, G. and Zabczyk, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge: Cambridge Univ. Press. MR1207136

[16] Duan, J. and Millet, A. (2009). Large deviations for the Boussinesq equations under random influences. *Stochastic Process. Appl.* **119** 2052–2081. MR2519356

[17] Flandoli, F. (1994). Dissipativity and invariant measures for stochastic Navier–Stokes equations. *NoDEA Nonlinear Differential Equations Appl.* **1** 403–423. MR1300150

[18] Flandoli, F. and Gatarek, D. (1995). Martingale and stationary solutions for stochastic Navier–Stokes equations. *Probab. Theory Related Fields* **102** 367–391. MR1339739

[19] Hairer, M. and Mattingly, J.C. (2006). Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. *Ann. of Math.* (2) **164** 993–1032. MR2259251

[20] Ikeda, N. and Watanabe, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. *North-Holland Mathematical Library* **24**. Amsterdam: North-Holland. MR0637061

[21] Jakubowski, A. (1986). On the Skorokhod topology. *Ann. Inst. Henri Poincaré Probab. Stat.* **22** 263–285. MR0871083

[22] Lions, J.-L. (1969). *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*. Paris: Dunod. MR0259693

[23] Liu, W. (2010). Large deviations for stochastic evolution equations with small multiplicative noise. *Appl. Math. Optim.* **61** 27–56. MR2575313

[24] Manna, U., Sritharan, S.S. and Sundar, P. (2009). Large deviations for the stochastic shell model of turbulence. *NoDEA Nonlinear Differential Equations Appl.* **16** 493–521. MR2525514

[25] Mikulevicius, R. and Rozovskii, B.L. (2005). Global L_2-solutions of stochastic Navier–Stokes equations. *Ann. Probab.* **33** 137–176. MR2118862

[26] Ren, J. and Zhang, X. (2005). Freidlin–Wentzell's large deviations for homeomorphism flows of non-Lipschitz SDEs. *Bull. Sci. Math.* **129** 643–655. MR2166732

[27] Ren, J. and Zhang, X. (2005). Schilder theorem for the Brownian motion on the diffeomorphism group of the circle. *J. Funct. Anal.* **224** 107–133. MR2139106

[28] Röckner, M. and Zhang, T. (2007). Stochastic evolution equations of jump type: Existence, uniqueness and large deviation principles. *Potential Anal.* **26** 255–279. MR2286037

[29] Röckner, M., Zhang, T. and Zhang, X. (2010). Large deviations for stochastic tamed 3D Navier–Stokes equations. *Appl. Math. Optim.* **61** 267–285. MR2585144

[30] Sowers, R.B. (1992). Large deviations for a reaction–diffusion equation with non-Gaussian perturbations. *Ann. Probab.* **20** 504–537. MR1143433

[31] Sritharan, S.S. and Sundar, P. (2006). Large deviations for the two-dimensional Navier–Stokes equations with multiplicative noise. *Stochastic Process. Appl.* **116** 1636–1659. MR2269220

[32] Święch, A. and Zabczyk, J. (2011). Large deviations for stochastic PDE with Lévy noise. *J. Funct. Anal.* **260** 674–723. MR2737394

[33] Temam, R. (1979). *Navier–Stokes Equations: Theory and Numerical Analysis*. Revised ed. *Studies in Mathematics and Its Applications* **2**. Amsterdam: North-Holland. With an appendix by F. Thomasset. MR0603444

[34] Temam, R. (1983). *Navier–Stokes Equations and Nonlinear Functional Analysis*. *CBMS-NSF Regional Conference Series in Applied Mathematics* **41**. Philadelphia, PA: SIAM. MR0764933

[35] Višik, M.I. and Fursikov, A.V. (1988). *Mathematical Problems of Statistical Hydromechanics*. *Mathematics and Its Applications* **9**. Dordrecht: Kluver. Translated from 1980 Russian original *Matematicheskie Zadachi Statisticheskoi Gidromekhaniki*. Moscow: Nauka. MR0591678

[36] Wang, W. and Duan, J. (2009). Reductions and deviations for stochastic partial differential equations under fast dynamical boundary conditions. *Stoch*. *Anal*. *Appl*. **27** 431–459. MR2523176
[37] Xu, T. and Zhang, T. (2009). Large deviation principles for 2-D stochastic Navier–Stokes equations driven by Lévy processes. *J*. *Funct*. *Anal*. **257** 1519–1545. MR2541279
[38] Yang, X., Zhai, J. and Zhang, T. (2014). Large deviations for SPDEs of jump type. Available at arXiv:1211.0466.
[39] Zhang, T.S. (2000). On the small time asymptotics of diffusion processes on Hilbert spaces. *Ann*. *Probab*. **28** 537–557. MR1782266
[40] Zhang, X. (2010). Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J*. *Funct*. *Anal*. **258** 1361–1425. MR2565842

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