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LARGE DEVIATIONS FOR A RANDOM WALK IN RANDOM ENVIRONMENT

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Let $\omega = (p_x)_{x \in \mathbb{Z}}$ be an i.i.d. collection of $(0,1)$-valued random variables. Given ω, let $(X_n)_{n \geq 0}$ be the Markov chain on $\mathbb{Z}$ defined by $X_0 = 0$ and $X_{n+1} = X_n + 1$ (resp. $X_n - 1$) with probability p_{X_n} (resp. $1 - p_{X_n}$). It is shown that X_n/n satisfies a large deviation principle with a continuous rate function, that is,

$$\lim_{n \to \infty} \frac{1}{n} \log P_\omega\big(X_n = \lfloor \theta_n n \rfloor\big) = -I(\theta) \quad \omega\text{-a.s. for } \theta_n \to \theta \in [-1,1].$$

First, we derive a representation of the rate function I in terms of a variational problem. Second, we solve the latter explicitly in terms of random continued fractions. This leads to a classification and qualitative description of the shape of I. In the recurrent case I is nonanalytic at $\theta = 0$. In the transient case I is nonanalytic at $\theta = -\theta_c, 0, \theta_c$ for some $\theta_c \geq 0$, with linear pieces in between.

0. Introduction, results and pictures.

0.1. *Motivation.* In this paper we obtain a large deviation principle for the position of a nearest-neighbor random walk on $\mathbb{Z}$ with random site-dependent transition probabilities. We calculate the rate function explicitly and find interesting dependence on underlying parameters. The main tools are some combinatorial and variational techniques from Greven and den Hollander (1992) and Baillon, Clément, Greven and den Hollander (1994).

Let $\omega = (p_x)_{x \in \mathbb{Z}}$ be an i.i.d. collection of $(0,1)$-valued random variables with marginal distribution α. For fixed ω, let $X = (X_n)_{n \geq 0}$ be the Markov chain on $\mathbb{Z}$ starting at $X_0 = 0$ and with transition probabilities

$$P_\omega\big(X_{n+1} = y \mid X_n = x\big) = \begin{cases} p_x, & \text{if } y = x+1, \\ 1 - p_x, & \text{if } y = x-1, \\ 0, & \text{otherwise.} \end{cases} \tag{0.1}$$

The symbol P_ω denotes the measure on path space given ω. The process (X, ω) is an example of a *random walk in random environment*, and X has law $P = \int \alpha^{\mathbb{Z}}(d\omega) P_\omega$.

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This model has been studied quite a bit and has been found to exhibit a number of phenomena not shared by classical random walk. We recall some of the literature.

Abbreviate

$$\rho = \frac{1-p}{p}, \qquad \langle f \rangle = \int f(p)\alpha(dp). \tag{0.2}$$

It was established by Solomon (1975) that X is ω-a.s.

$$\begin{array}{ll} \text{recurrent} & \text{if } \langle \log \rho \rangle = 0, \\ \text{transient} & \text{if } \langle \log \rho \rangle \neq 0. \end{array} \tag{0.3}$$

In the *transient* case

$$\lim_{n\to\infty} X_n = \begin{cases} +\infty, & P\text{-a.s. if } \langle \log \rho \rangle < 0, \\ -\infty, & P\text{-a.s. if } \langle \log \rho \rangle > 0. \end{cases} \tag{0.4}$$

Moreover, there are two speed regimes, namely,

$$\begin{array}{llll} \text{(i)} & \displaystyle\lim_{n\to\infty} \frac{X_n}{n} = \frac{1-\langle \rho \rangle}{1+\langle \rho \rangle} & & P\text{-a.s. if } \langle \rho \rangle < 1, \\ & \displaystyle\lim_{n\to\infty} \frac{X_n}{n} = \frac{1-\langle \rho^{-1} \rangle}{1+\langle \rho^{-1} \rangle} & & P\text{-a.s. if } \langle \rho^{-1} \rangle < 1, \\ \text{(ii)} & \displaystyle\lim_{n\to\infty} \frac{X_n}{n} = 0 & & P\text{-a.s. if } \langle \rho \rangle^{-1} \leq 1 \leq \langle \rho^{-1} \rangle. \end{array} \tag{0.5}$$

(By Jensen's inequality $\log\langle \rho^{-1} \rangle^{-1} \leq \langle \log \rho \rangle \leq \log\langle \rho \rangle$. Which case occurs depends on the location of 0 w.r.t. these three points.) Kesten, Kozlov and Spitzer (1975) supplemented the law of large numbers (0.5) by central limit type theorems. For instance, in regime (i) the classical central limit theorem holds if $\langle \rho^2 \rangle < 1$ (provided $\langle \rho^2 \log \rho \rangle < \infty$). In regime (ii), on the other hand, if $\langle \log \rho \rangle < 0 < \log\langle \rho \rangle$, then

$$\frac{1}{n^\kappa} X_n \Rightarrow Z \text{ in law w.r.t. } P \text{ with } 0 < \kappa < 1 \text{ the unique solution of } \langle \rho^\kappa \rangle = 1 \ \big(\text{provided } \langle \rho^\kappa \log \rho \rangle < \infty\big), \tag{0.6}$$

where Z is a random variable with a law related to a stable law of index κ. The result analogous to (0.6) in the negative direction holds after replacing ρ by ρ^{-1}.

In the *recurrent* case the motion is extremely slow. Sinai (1982) proved that

$$\frac{\sigma^2}{(\log n)^2} X_n \Rightarrow Z' \text{ in law w.r.t. } P \text{ with } 0 < \sigma^2 = \big\langle (\log \rho)^2 \big\rangle < \infty, \tag{0.7}$$

where Z' is a functional of the Wiener process. The explicit law of Z' was identified by Kesten (1986), namely, Z' is symmetric and $|Z'|$ has the law of the exit time of the standard Wiener process from the interval $[-1, 1]$.

Several other interesting results have been obtained, for which we refer the reader to the literature [see, e.g., Révész (1990)]. The purpose of the present paper is to do a *large deviation analysis*, that is, to compute

$$\lim_{n\to\infty}\frac{1}{n}\log P_\omega\big(X_n=\lfloor\theta n\rfloor\big)\quad\text{for }\theta\in[-1,1], \tag{0.8}$$

and to derive some properties of the limit as a function of θ.

0.2. *Large deviations.* We shall assume throughout the paper that $\langle\log\rho\rangle\le 0$. The case $\langle\log\rho\rangle>0$ follows after replacing ρ by ρ^{-1} and θ by $-\theta$. We shall also assume that $\mathrm{supp}(\alpha)$ is finite. This assumption will be removed at the end of subsection 0.3.

In order to formulate our large deviation result, we need to introduce the following notation:

$$i,j\in\mathbb{N},\qquad p,q\in\mathrm{supp}(\alpha), \tag{0.9}$$

$$\mathcal{P}(E)=\text{ set of probability measures on }E, \tag{0.10}$$

$$a\big((i,p),(j,q)\big)=i+j-1, \tag{0.11}$$

and the following three objects:

1.

$$M_{\theta,\alpha}=\Big\{Q\in\mathcal{P}\Big(\big[\mathbb{N}\times\mathrm{supp}(\alpha)\big]^{\mathbb{Z}}\Big):\ Q\text{ stationary},\ E_{\pi^2Q}(a)=\theta^{-1},\ \tilde{\pi}^{\mathbb{Z}}Q=\alpha^{\mathbb{Z}}\Big\}, \tag{0.12}$$

where $\theta\in(0,1]$, π^2Q is the projection of Q on $[\mathbb{N}\times\mathrm{supp}(\alpha)]^2$ (i.e., the two-dimensional marginal), E_{π^2Q} is the expectation w.r.t. π^2Q and $\tilde{\pi}^{\mathbb{Z}}Q$ is the projection of Q on $[\mathrm{supp}(\alpha)]^{\mathbb{Z}}$.

2.

$$I(Q\mid R_\alpha)=\text{ specific relative entropy of }Q\text{ w.r.t. }R_\alpha. \tag{0.13}$$

For the definition of specific relative entropy, see Georgii (1988), Sections 15.1 and 15.2.

3. R_α is the stationary Markov process with transition kernel $P_{R_\alpha}((i,p)\to(j,q))=A_q(i,j)\alpha(q)$ given by

$$A_q(i,j)=\binom{i+j-2}{i-1}q^i(1-q)^{j-1} \tag{0.14}$$

and with one-dimensional marginal given by

$$(0.15)\qquad \begin{aligned} &\langle \log \rho\rangle < 0: && \text{the invariant probability measure of } P_{R_\alpha},\\ &\langle \log \rho\rangle = 0: && \lambda \otimes \alpha, \ \text{ with } \lambda \text{ the counting measure on } \mathbb{N}. \end{aligned}$$

In subsection 1.3 we shall see that the kernel P_{R_α} describes the successive numbers of right jumps by the random walk X along the bonds of $\mathbb{Z}$. In subsection 3.2 we shall see that P_{R_α} is positive recurrent when X is transient ($\langle \log \rho\rangle < 0$) and null recurrent when X is recurrent ($\langle \log \rho\rangle = 0$). (In the latter case R_α should actually be called a Markov shift.)

We can now formulate our large deviation result:

THEOREM 1. *For every* $\theta \in [-1, 1]$ *and* $\theta_n \to \theta$,

$$(0.16)\qquad \lim_{n\to\infty} \frac{1}{n} \log P_\omega\big(X_n = \lfloor \theta_n n\rfloor\big) = -I(\theta) \quad \omega\text{-a.s.},$$

with

$$(0.17)\qquad I(\theta) = \theta \inf_{Q \in M_{\theta,\alpha}} I(Q \mid R_\alpha) \quad \text{if } \theta \in (0, 1]$$

and

$$(0.18)\qquad \begin{aligned} &I(\theta) = I(-\theta) + \theta\langle \log \rho\rangle \quad \text{if } \theta \in [-1, 0),\\ &I(0) = \lim_{\theta \to 0} I(\theta). \end{aligned}$$

We give an informal interpretation of the preceding results.

Imagine the random walk starting at $x = -\infty$. Consider the total number of steps to the right, $m(x)$, taken by the random walk at site x. We shall see in subsection 1.3 that, given ω, the sequence $\{m(x)\}_{x\in\mathbb{Z}}$ is Markov with a transition kernel at site x depending on ω through p_x. If together with $m(x)$ we record p_x and form the sequence $\{m(x), p_x\}_{x\in\mathbb{Z}}$, then we obtain a "double layer" process. In subsection 2.6 we shall see that the reverse of this process is Markov with transition kernel $A_{p_x}(i, j)\alpha(p_x)$. This is the process R_α defined in (0.14) and (0.15).

In Section 2 we prove that, on the event $X_n = \lfloor \theta n\rfloor$, the probability of the class of paths for which the empirical distribution of $\{m(x), p_x\}_{x=0}^{\lfloor\theta n\rfloor}$ converges to Q as $n \to \infty$ equals $\exp[-n\theta I(Q \mid R_\alpha) + o(n)]$. This explains the variational formula in (0.17). The two restrictions $E_{\pi^2 Q}(a) = \theta^{-1}$ and $\tilde{\pi}^{\mathbb{Z}} Q = \alpha^{\mathbb{Z}}$ in the set $M_{\theta,\alpha}$ in (0.12) mean the following: (i) the average time that the path spends on a typical site converges to θ^{-1} ("top layer"); (ii) the empirical distribution of the environment that the path sees converges to $\alpha^{\mathbb{Z}}$ ("bottom layer"). The minimizer of (0.17) therefore describes the *typical path and environment* realizing $X_n = \lfloor \theta n\rfloor$.

The identity $I(\theta) = I(-\theta) + \theta\langle \log \rho\rangle$ in (0.18) can be understood by reversing the path. Indeed, the ratio of the probability of a path from 0 to $\lfloor \theta n\rfloor$ and the probability of its reverse from $\lfloor \theta n \rfloor$ to 0 equals $\exp[-n\theta\langle \log \rho \rangle + o(n)]$ as $n \to \infty$. This follows from the observation that

$$A_{1-q}(j,i) = \frac{1-q}{q} A_q(i,j)$$

and from a Radon-Nikodym argument combined with the ergodic theorem applied to the medium (see the end of subsection 2.7).

0.3. *Solution of the variational problem.* In order to state our solution of (0.17), we need to introduce the following three objects:

1. Let $\rho_{\max}$ and $\rho_{\min}$ be the maximal resp. minimal value of $\rho = (1-p)/p$ over $\mathrm{supp}(\alpha)$. Note that $\rho_{\min} \le 1$ by our assumption that $\langle \log \rho \rangle \le 0$. Define

$$r_c = \begin{cases} 0, & \text{if } \rho_{\max} \ge 1 \text{ (case A)}, \\ \dfrac{1}{2} \log \dfrac{4\rho_{\max}}{(1+\rho_{\max})^2} < 0, & \text{if } \rho_{\max} < 1 \text{ (case B)}. \end{cases} \tag{0.19}$$

In case A the random environment has local drifts in both directions, whereas in case B the local drifts always point to the right. The distinction will be important because in the first case there are large regions where the random walk gets "trapped" (see Corollary 1), while in the second case there are not.

2. Let $f(r,\omega)$ be the *random continued fraction*

$$f(r,\omega) = \frac{1|}{|e^r(1+\rho_0)} - \frac{\rho_0|}{|e^r(1+\rho_1)} - \frac{\rho_1|}{|\cdots}, \qquad r \ge r_c, \tag{0.20}$$

with $\omega = (p_x)_{x \in \mathbb{Z}}$ and $\rho_x = (1-p_x)/p_x$. Define $\lambda(r)$ as

$$\log \lambda(r) = \int \alpha^{\mathbb{Z}}(d\omega) \log f(r,\omega), \qquad r \ge r_c. \tag{0.21}$$

In subsection 3.1 we shall see that $\lambda(r)$ is the Lyapunov exponent of a product of infinite random matrices drawn from an r-dependent family. The parameter r plays the role of the Lagrange multiplier needed to match the θ-restriction in (0.12) and (0.17). In subsection 4.1 we show that $f(r,\omega)$ a.s. exists and is positive iff $r \ge r_c$. In subsection 4.2 we show that $r \to \lambda(r)$ is continuous, strictly decreasing and strictly log convex on $[r_c, \infty)$, analytic on (r_c, ∞), and $\lambda(0) = 1$.

3. Let θ_c be given by

$$\theta_c^{-1} = \lim_{r \downarrow r_c} \left(-\frac{\lambda'(r)}{\lambda(r)} \right). \tag{0.22}$$

Define $r(\theta)$ as

$$\begin{aligned} &0 \le \theta \le \theta_c: \quad r(\theta) = r_c, \\ &\theta_c < \theta < 1: \quad r(\theta) \text{ is the unique solution of } \theta^{-1} = -\frac{\lambda'(r)}{\lambda(r)}. \end{aligned} \tag{0.23}$$

We shall see in subsection 3.4 that the parameter θ_c appears as the point where the infimum in (0.17) becomes a minimum.

We can now state our solution of the variational problem:

THEOREM 2. *For* $\theta \in (0,1)$,

$$I(\theta) = -r(\theta) - \theta \log \lambda\big(r(\theta)\big). \tag{0.24}$$

At the boundaries, $I(0) = -r_c$ *and* $I(1) = -\langle \log p\rangle = \langle \log(1+\rho)\rangle$.

The solution (0.24) is based on an explicit construction of a minimizing process $\overline{Q}_\theta$ and on the evaluation of the minimum $I(\overline{Q}_\theta \mid R_\alpha)$. The process $\overline{Q}_\theta$ has the following form. Let $\mathcal{G}_{r,\omega} = \{Q^{\chi}_{r,\omega}\}_{\chi}$ denote the collection of all Gibbs measures on $\mathbb{N}^{\mathbb{Z}}$, relative to the counting measure, with Hamiltonian

$$H_{r,\omega}\big((i_x)_{x\in\mathbb{Z}}\big) = \sum_x V_{\omega,x}(i_{x-1}, i_x) + \sum_x W_r i_x, \tag{0.25}$$

with random pair potential and external field

$$\begin{aligned} V_{\omega,x}(i,j) &= -\log A_{p_{x-1}}(i,j),\\ W_r &= 2r. \end{aligned} \tag{0.26}$$

(We use the standard convention that the Hamiltonian appears with a minus sign in the Boltzmann weight factor.) Let $\mathcal{G}_{r,\alpha}$ denote the set of all stationary measures $Q \in \mathcal{P}([\mathbb{N}\times \mathrm{supp}(\alpha)]^{\mathbb{Z}})$ satisfying $\tilde{\pi}^{\mathbb{Z}}Q = \alpha^{\mathbb{Z}}$ and admitting regular conditional measures $Q_\omega \in \mathcal{G}_{r,\omega}$, where Q_ω is defined by $Q(\cdot, d\omega) = Q_\omega(\cdot)\alpha^{\mathbb{Z}}(d\omega)$. We shall see in subsections 3.2 to 3.4 that $\{\bar{Q}_\theta\} = \mathcal{G}_{r(\theta),\alpha} \cap M_{\theta,\alpha} \neq \emptyset$. (This statement means that the minimizers precisely make up the set of Gibbs measures in the r.h.s.)

The reference process R_α defined in (0.14) and (0.15) is one of the Gibbs measures in $\mathcal{G}_{0,\alpha}$.

REMARK. The statements in subsections 0.2 and 0.3 continue to hold when $\mathrm{supp}(\alpha)$ is infinite, provided $\alpha([\delta, 1-\delta]) = 1$ for some $\delta > 0$. This can be seen as follows. First, given α and $\varepsilon > 0$, pick α' with finite support such that $\|\alpha - \alpha'\|_{\mathrm{var}} \le \varepsilon$. Then clearly $|P_\omega(X_n = \lfloor \theta n\rfloor) - P_{\omega'}(X_n = \lfloor \theta n \rfloor)| \le \exp(\varepsilon n)$, with ω and ω' drawn according to α (resp.α') and coupled in the obvious way. Hence the difference between the rate functions vanishes as $\varepsilon \to 0$. Second, the random continued fraction $f(r,\omega)$ in (0.20) is uniformly convergent in ω and analytic in r (see subsection 4.1). Consequently, the solution (0.20)–(0.24) remains valid for general α, and therefore also all the properties to be discussed in the next section.

0.4. *Properties of the rate function.* Define

$$\theta^* = \begin{cases} \dfrac{1-\langle,\rho\rangle}{1+\langle\rho\rangle}, & \text{if } \langle\rho\rangle < 1, \\ 0, & \text{if } \langle\rho\rangle \geq 1. \end{cases} \tag{0.27}$$

Recall (0.5): θ^* is the *typical speed* of the random walk.

The qualitative properties of the rate function are [see Figures 1 to 3 and recall (0.18)] stated in the following corollary.

COROLLARY 1. *Suppose that α is not a point mass.*

(a) $\theta \to I(\theta)$ *is continuous and convex on* $[0, 1]$, *with* $I(0) = -r_c$ *and* $I(1) = \langle\log(1+\rho)\rangle$.

(b) $\theta \to I(\theta)$ *is linear on* $[0, \theta_c]$, *strictly convex and analytic on* $(\theta_c, 1)$.

(c) $I(\theta^*) = 0$.

(d) *case A:* $0 \leq \theta_c = \theta^*$, $I(\theta) = 0$ *for* $\theta \in [0, \theta^*)$,
case B: $0 < \theta_c < \theta^*$, $I(\theta) = -r_c - \theta \log \lambda(r_c) > 0$ *for* $\theta \in [0, \theta^*)$.

(e) $\lim_{\theta \downarrow \theta_c} I'(\theta) = -\log \lambda(r_c)$.

(f) $\theta \to I(\theta)$ *is convex on* $[-1, 1]$.

We give an informal explanation of the linear pieces appearing in Figures 2 and 3.

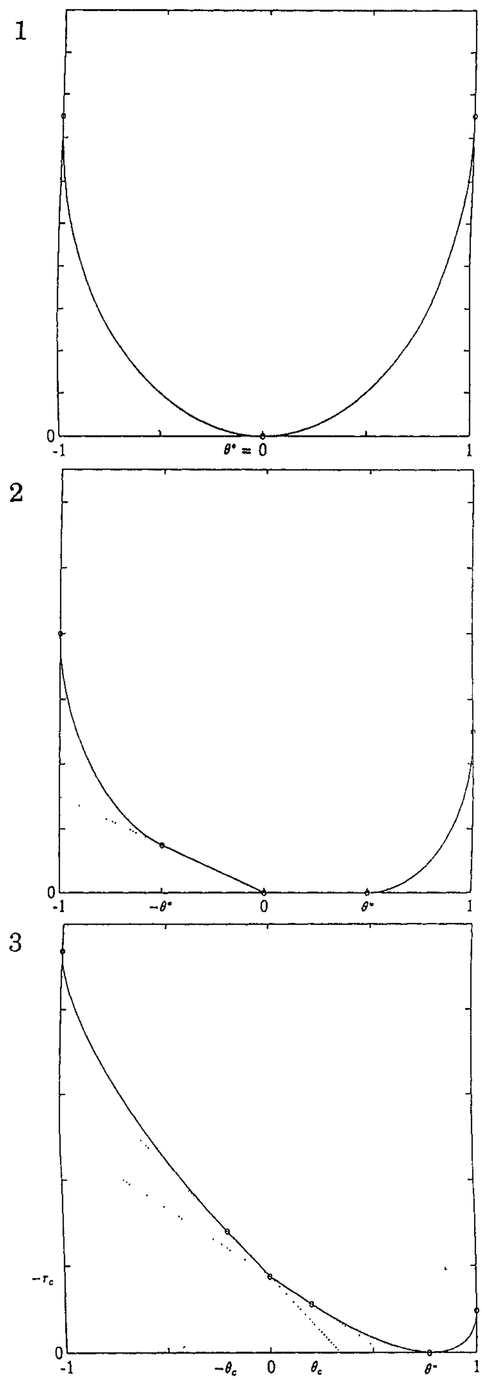
In case A the random environment contains arbitrarily long stretches where the local drifts point to the center. Such stretches tend to "trap" the walk, and the longer the stretch the easier it is for the walk to lose time inside. In the interval $[0, \lfloor\theta n\rfloor]$ the longest stretch has a length of order $\log n$. Now, in order to move at speed $0 < \theta < \theta^*(= \theta_c)$, the walk can decide to: (i) move at speed θ^* *outside* the longest stretches over a time $\theta n/\theta^*$ (which is the typical behavior with a subexponential rate); (ii) move at speed 0 *inside* the longest stretches over a time $(\theta^* - \theta)n/\theta^*$ (which also has subexponential rate). The result is $I(\theta) = I(\theta^*) = 0$.

In case B, on the other hand, stretches of the preceding type do not occur and therefore $I(\theta) > 0$ when $\theta < \theta^*$. The linear piece for $\theta < \theta_c$ in Figure 3 can be related to the flat piece in Figure 2 after a transformation of the environment (see subsection 3.4). This transformation changes the law on path space by a Radon-Nikodym factor $\exp[-n(-r_c - \theta \log \lambda(r_c)) + o(n)]$ for every path running from 0 to $\lfloor\theta n\rfloor$. The transformed environment has $\rho_{\max} = 1$, which is again case A. Moreover, θ_c is the typical drift of the random walk in the transformed environment.

The effect of the randomness of ω can be further illuminated by comparing with *effective media*. Let

$$\widehat{I}_\eta(\theta) = \tfrac{1}{2}(1+\theta)\log\left[\tfrac{1}{2}(1+\theta)(1+\eta)\right] + \tfrac{1}{2}(1-\theta)\log\left[\tfrac{1}{2}(1-\theta)(1+\eta^{-1})\right] \tag{0.28}$$

be the rate function corresponding to the homogeneous medium with $\rho_x \equiv \eta$ [i.e., $p_x = (1+\eta)^{-1}$ for all x].



FIGS. 1–3. *Qualitative pictures of the rate function* $I(\theta)$ *for* $\theta \in [-1, 1]$. *Three cases are plotted*: FIG. 1. *recurrent* ($\langle \log \rho \rangle = 0$). FIG. 2. *transient with positive drift* (*case A*: $\langle \rho \rangle < 1, \rho_{\max} \geq 1$). FIG. 3. *transient with positive drift* (*case B*: $\langle \rho \rangle < 1, \rho_{\max} < 1$). *The transient case with zero drift* (*case A*: $\langle \log \rho \rangle < 0, \langle \rho \rangle \geq 1$) *is the same as Fig.* 2 *but with the two linear pieces shrunk to zero.*

COROLLARY 2. *Suppose that α is not a point mass.*

(a) *If* $\langle \log \rho \rangle = 0$, *then* $I(\theta) \geq \widehat{I}_1(\theta)$ *for* $\theta \in [0,1]$ *with equality iff* $\theta = \theta^*(=0)$.

(b) *If* $\langle \rho \rangle < 1$, *then* $I(\theta) \leq \widehat{I}_{\langle \rho \rangle}(\theta)$ *for* $\theta \in [0, \theta^*]$ *with equality iff* $\theta = \theta^*(>0)$.

Figures 4 to 6 are simulations of the rate functions of Figures 1 to 3 for some choices of α with two atoms.

0.5. *Open problems.* Here are some interesting questions that should be solvable from (0.19)–(0.24).

1. Is $\overline{Q}_\theta$ defined through (0.25) and (0.26) the unique minimizer of (0.17)? In other words, is $\mathcal{G}_{r,\omega}$ a singleton ω-a.s. for all $r \geq r_c$? In subsection 3.3 we show that the set of minimizers is given by

$$\left\{ Q \in M_{\theta,\alpha} \colon I(Q \mid \overline{Q}_\theta) = 0 \right\}. \tag{0.29}$$

   It can be shown that (0.29) coincides with $\mathcal{G}_{r(\theta),\alpha} \cap M_{\theta,\alpha}$, i.e., the Gibbs variational principle holds for the random Hamiltonian in (0.25) [see Greven and den Hollander (1994), Section 2.6]. To prove for *random* Hamiltonians that the phase is unique is a somewhat delicate matter [see Zegarlinski (1991)]. A proof would depend on the specific form of the interaction.
2. Is it true that $\lim_{\theta \to 0} I''(\theta) = \infty$ in the recurrent case ($\langle \log \rho \rangle = 0$)? We know from (0.7) that X_n is of order $(\log n)^2$. This is slower than central limit behavior, which typically corresponds to $I(\theta)$ having finite curvature at $\theta = 0$. From (0.23) and (0.24) it follows that $\lim_{\theta \downarrow 0} I''(\theta) = \infty$ is equivalent to

$$\lim_{r \downarrow 0} \frac{\left(\log \lambda(r)\right)''}{\left[\left(\log \lambda(r)\right)'\right]^3} = 0. \tag{0.30}$$

3. Is it true that $I(\theta) < \widehat{I}_{\langle \rho \rangle}(\theta)$ for $\theta > \theta^*$ in the transient case with positive drift ($\langle \rho \rangle < 1$)? For this it would suffice to show that

$$-\frac{\lambda'(r)}{\lambda(r)} > \frac{1 + \langle \rho \rangle \lambda^2(r)}{1 - \langle \rho \rangle \lambda^2(r)} \quad \text{for } r > 0, \tag{0.31}$$

   as is explained in subsection 4.4. There we prove that the opposite inequality holds for $r \in [r_c, 0)$ in case B and equality holds for $r = 0$, which is what produces Corollary 2(b).
4. Is it possible to compute $\lambda(r_c)$ and $\theta_c = -\lambda(r_c)/\lambda'(r_c)$ explicitly in case B? [In case A we have $r_c = 0, \lambda(0) = 1, \theta_c = 0^*$.]

It is a natural question to investigate the large deviations of X_n/n also under the measure $P = \int \alpha^{\mathbb{Z}}(d\omega) P_\omega$ instead of P_ω. This is particularly interesting in the recurrent case and in the case of zero speed, because (0.6) and (0.7) do not have an analogue for fixed ω. It is not hard to guess how Theorems 1 and

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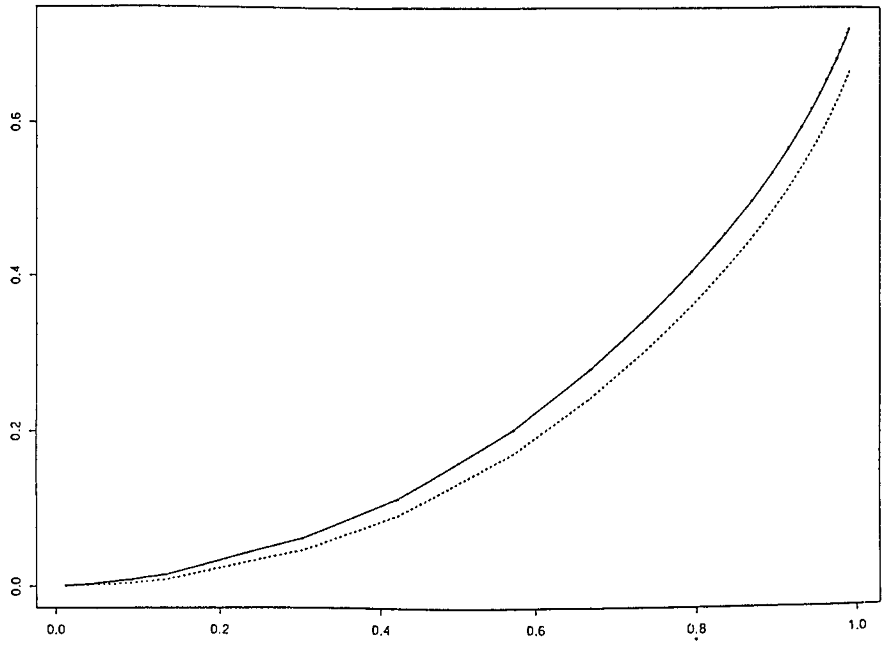

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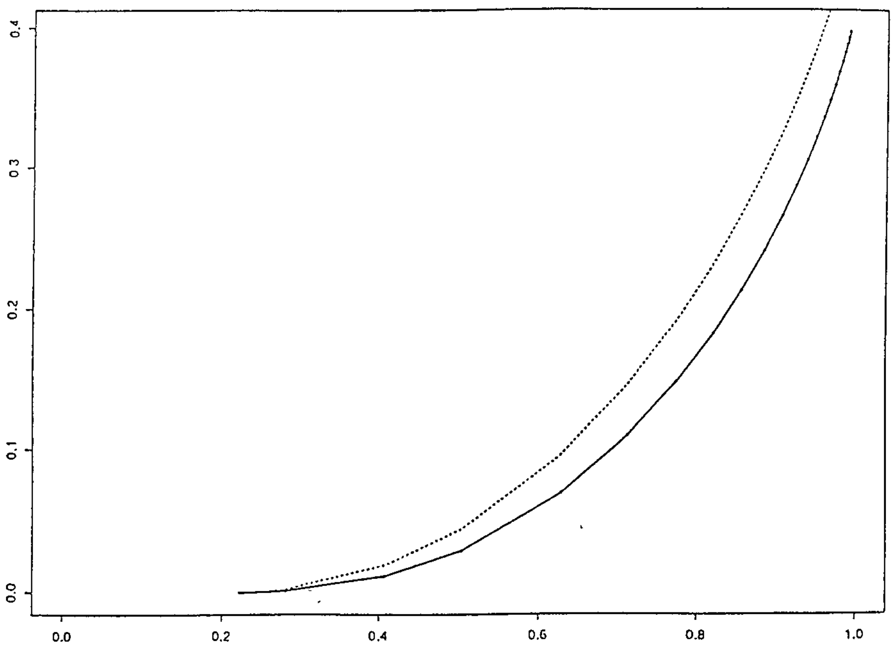

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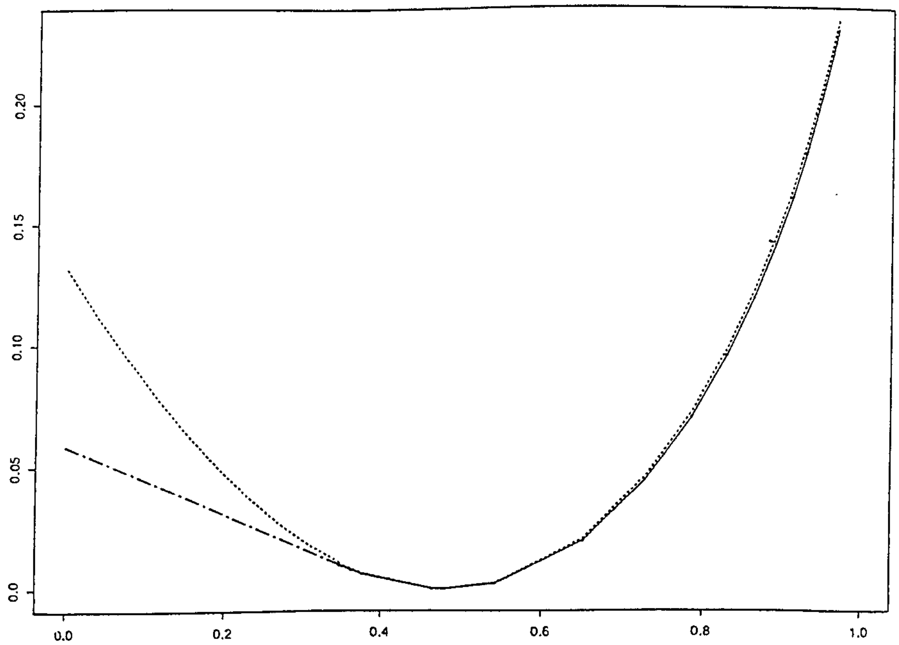

FIGS. 4–6. *Simulations of the rate function* $I(\theta)$ *for* $\theta_c \in [\theta_c, 1]$. *Three cases are plotted that correspond to Figures* 1–3, *respectively*: FIG. 4. $\alpha = \frac{1}{2}(\delta_{1/3} + \delta_{2/3})$: $r_c = 0$, $\theta_c = \theta^* = 0$. *The dotted line is the lower bound* $\hat{I}_1(\theta)$. FIG. 5. $\alpha = \frac{1}{2}(\delta_{11/24} + \delta_{11/12})$: $r_c = 0, \theta_c = \theta^* = \frac{2}{9}$. *The gap is where* $\theta < \theta_c$ *and* $I(\theta) = 0$. *The dotted line is the conjectured upper bound* $\hat{I}_{7/11}(\theta)$ *(see subsection* 0.5*)*. FIG. 6. $\alpha = \frac{1}{2}(\delta_{2/3} + \delta_{5/6})$: $r_c = \frac{1}{2}\log\frac{8}{9}, 0 < \theta_c < \theta^* = \frac{13}{27}$. *The dashed line is where* $\theta < \theta_c$ *and* $I(\theta) = r_c - \theta \log \lambda(r_c)$. *Numerically,* $\theta_c = 0.3528$ *and* $\lambda(r_c) = 1.1506$. *The dotted line is the conjectured upper bound* $\hat{I}_{7/20}(\theta)$, *proven only for* $\theta \leq \theta^*$. *Each curve was obtained by picking* 25 *values of* r, *computing* $\lambda(r)$ *and* $-\lambda'(r)/\lambda(r)$, *and plotting* $\theta = [-\lambda(r)/\lambda'(r)]^{-1}$ *vs.* $I(\theta) = -r - \theta \log \lambda(r)$. *The computation of* $\lambda(r)$ *was based on* (0.20) *and* (0.21). *To compute* $-\lambda'(r)/\lambda(r)$, *we need not estimate differences, but instead we can use* (3.12), (3.13) *and the first expression in* (3.21) *to compute this quantity directly. For each value of* r *we simulated* 10^4 *random continued fractions, each of length* 40.

The simulations only give $I(\theta)$ *for* $\theta \in [\theta_c, 1]$. *The degenerate part* $\theta < \theta_c$ *corresponds to* $r < r_c$, *where the random continued fractions have a positive probability to become negative so that* $\lambda(r)$ *fails to exist (see the remark following Lemma* 9 *in subsection* 4.1*). The simulations do not always give a sharp indication of this failure. In Fig.* 6, *for instance, due to the truncation after* 40 *terms the computations break down only for* r *below* -0.078, *whereas* $r_c = -0.058$.

2 transform, but the derivation is open. The rate functions will be *different* because P allows additional large deviations in the random environment.

The outline of the rest of the paper is as follows. In Sections 1 and 2 we prove Theorem 1. The derivation uses large deviation theory and ideas from Greven and den Hollander (1992). In Section 3 we prove Theorem 2. Here we encounter random maps and Gibbs measures, and we use ideas from Baillon, Clément, Greven and den Hollander (1994). In Section 4 we analyze the random continued fraction appearing in (0.20) and (0.21) and prove Corollaries 1 and 2. Sections 1 and 2 are the most technical parts of the paper. Sections 3 and 4 are more easily accessible.

1. Reformulation as a large deviation problem for an associated Markov chain. The aim of this section is to transform the original problem into a large deviation problem for the total bond crossing numbers of the random walk, which turn out to have a nice Markov representation. In Sections 1 and 2 we fix $\theta > 0$. At the end of Section 2 we shall show how to treat $\theta < 0$ and $\theta = 0$.

1.1. *Bond crossing numbers.* In this subsection we rewrite $P_\omega(X_n = \lfloor \theta n \rfloor)$ as the expectation of an exponential functional of the *bond crossing numbers* under the law of the n-step path of the random walk with drift θ.

Let

$$\Omega^n = \{S^n = (S_i)_{i=0}^n \colon S_0 = 0,\ |S_{i+1} - S_i| = 1 \text{ for } 0 \le i < n\}, \tag{1.1}$$

$$m_n^+(x, S^n) = \sum_{i=0}^{n-1} 1\{S_i = x,\ S_{i+1} = x + 1\}, \qquad S^n \in \Omega^n, \tag{1.2}$$

$$m_n^-(x, S^n) = \sum_{i=0}^{n-1} 1\{S_i = x,\ S_{i+1} = x - 1\}, \qquad S^n \in \Omega^n. \tag{1.3}$$

Let P_θ and E_θ denote probability and expectation for the nearest-neighbor random walk with transition probabilities $\frac{1}{2}(1 \pm \theta)$ to the right (resp. left). Let

$$f_x(i, j) = i \log(1 - p_x) + j \log p_x, \qquad i, j \in \mathbb{N}, \tag{1.4}$$

$$C_n(\theta) = \left[\tfrac{1}{2}(1 - \theta)\right]^{(n - \lfloor \theta n \rfloor)/2} \left[\tfrac{1}{2}(1 + \theta)\right]^{(n + \lfloor \theta n \rfloor)/2}. \tag{1.5}$$

PROPOSITION 1.

$$\begin{aligned} &P_\omega\big(X_n = \lfloor \theta n \rfloor\big) \\ &\quad = C_n^{-1}(\theta) E_\theta\left(\exp\left[\sum_{x \in \mathbb{Z}} f_x\big(m_n^-(x, S^n), m_n^+(x, S^n)\big)\right] 1\{S_n = \lfloor \theta n \rfloor\}\right). \end{aligned} \tag{1.6}$$

PROOF. The probability that $X_i = S_i$ for $0 \le i \le n$ under P_ω is

$$\prod_{x \in \mathbb{Z}} (1 - p_x)^{m_n^-(x, S^n)} p_x^{m_n^+(x, S^n)}. \tag{1.7}$$

The same path under P_θ has probability

$$\prod_{x \in \mathbb{Z}} \left[\frac{1}{2}(1 - \theta)\right]^{m_n^-(x, S^n)} \left[\frac{1}{2}(1 + \theta)\right]^{m_n^+(x, S^n)}. \tag{1.8}$$

The claim follows by noting that the latter equals $C_n(\theta)$ on the event $S_n = \lfloor \theta n \rfloor$. $\square$

1.2. *Total bond crossing numbers.* In this subsection we extend the random walk with drift θ to a doubly infinite process $(S_i)_{i\in\mathbb{Z}}$ by adding an independent reflected copy of the random walk conditioned on not returning to the origin. The same notation P_θ, E_θ will be used for this extended process. We show that the r.h.s. of (1.6) can be approximated by an analogous expression involving the *total bond crossing numbers.*

Let

$$\Omega = \{S = (S_i)_{i\in\mathbb{Z}}: S_0 = 0,\ |S_{i+1} - S_i| = 1 \text{ for } i \in \mathbb{Z}\}, \tag{1.9}$$

$$m^+(x,S) = \sum_{i\in\mathbb{Z}} 1\{S_i = x,\ S_{i+1} = x+1\}, \qquad S \in \Omega, \tag{1.10}$$

$$m^-(x,S) = \sum_{i\in\mathbb{Z}} 1\{S_i = x,\ S_{i+1} = x-1\}, \qquad S \in \Omega. \tag{1.11}$$

View S^n, Ω^n as the projection of S, Ω on the time coordinates $0, 1, \dots, n$. The following proposition will serve as the starting point for the large deviation analysis in Section 2. Its role is to replace the *time* dependence by a *space* dependence.

PROPOSITION 2.

$$\begin{aligned} &E_\theta\left(\exp\left[\sum_{x\in\mathbb{Z}} f_x\big(m_n^-(x,S^n), m_n^+(x,S^n)\big)\right] 1\{S_n = \lfloor\theta n\rfloor\}\right) \\ &\quad = \exp\big(o(n)\big)E_\theta\left(\exp\left[\sum_{x=0}^{\lfloor\theta n\rfloor} f_x\big(m^-(x,S), m^+(x,S)\big)\right]\right. \\ &\qquad\qquad \times 1\left\{\sum_{x=0}^{\lfloor\theta n\rfloor}\big[m^-(x,S) + m^+(x,S)\big] = n\right\} \\ &\qquad\qquad \left.\times 1\left\{m^-(0,S) = m^-\big(\lfloor\theta n\rfloor, S\big) = 0, m^+(0,S) = m^+\big(\lfloor\theta n\rfloor, S\big) = 1\right\}\right). \end{aligned} \tag{1.12}$$

PROOF. The reader should ignore the last indicator, as it has a purely technical function.

The proof follows from Section 4 in Greven and den Hollander (1992). The only difference is that there, instead of (1.4), we had a function of the form $f_x(i,j) = c_x(i+j)$, but this does not affect the argument. Here we repeat the strategy of proof in order to give the reader some guidance.

Pick $\delta\colon \mathbb{N} \to \mathbb{N}$ such that $\delta(n) \to \infty$ and $\delta(n) = o(\log n)$ as $n \to \infty$. Define

$$A_n^1 = \{S \in \Omega: S_n = \lfloor\theta n\rfloor\}, \tag{1.13}$$

$$A_n^2 = \left\{S \in A_n^1: S_i \in \big(-\delta(n), \lfloor\theta n\rfloor + \delta(n)\big) \text{ for } 0 < i < n\right\}, \tag{1.14}$$

$$\begin{aligned} A_n^3 = \{S \in A_n^2: S_i &< -\delta(n) \text{ for } i < -\delta(n), \\ S_i &> \lfloor\theta n\rfloor + \delta(n) \text{ for } i > n + \delta(n)\}. \end{aligned} \tag{1.15}$$

The two main steps are

$$
\begin{aligned}
&E_\theta\left(\exp\left[\sum_{x\in\mathbb{Z}} f_x\big(m_n^-(x,S), m_n^+(x,S)\big)\right]1\{S\in A_n^1\}\right)\\
&\qquad = \exp\big(o(n)\big)E_\theta\left(\exp\left[\sum_{x\in\mathbb{Z}} f_x\big(m_n^-(x,S), m_n^+(x,S)\big)\right]1\{S\in A_n^2\}\right),
\end{aligned} \tag{1.16}
$$

$$
\begin{aligned}
&E_\theta\left(\exp\left[\sum_{x\in\mathbb{Z}} f_x\big(m_n^-(x,S), m_n^+(x,S)\big)\right]1\{S\in A_n^2\}\right)\\
&\qquad = \exp\big(o(n)\big)E_\theta\left(\exp\left[\sum_{x\in\mathbb{Z}} f_x\big(m_n^-(x,S), m_n^+(x,S)\big)\right]1\{S\in A_n^3\}\right),
\end{aligned} \tag{1.17}
$$

These are the analogues of Lemmas 11 and 12 in Greven and den Hollander (1992).

The idea behind (1.16) is to construct a map from $A_n^1\backslash A_n^2$ to A_n^2 which preserves the exponential functional and which maps only $\exp(o(n))$ paths onto a single image. The existence of such a map shows that the contributions of $A_n^1\backslash A_n^2$ and A_n^2 are comparable on an exponential scale. The construction uses the fact that all local configurations of length $\delta(n)$ in the environment appear in the strip $[0, \lfloor\theta n\rfloor]$ with probability tending to 1 as $n\to\infty$. The point in (1.17) is that the exponential functional does not depend on the path at times $i\notin[0,n]$. By using the Markov property at times $i=0$ and $i=n$, we get that the ratio of the two expectations in (1.17) equals $P_\theta(A_n^3 \mid A_n^2) = \theta[\frac{1}{2}(1+\theta)]^{2\delta(n)} = \exp(o(n))$.

Continuing from (1.16) and (1.17), on the set A_n^3 we have

$$
\begin{aligned}
m_n^+(x,S) &= \begin{cases} m^+(x,S), & \text{for } x\in\big[0,\lfloor\theta n\rfloor\big),\\ m^+(x,S)-1, & \text{for } x\in\big[-\delta(n),0\big)\cup\big[\lfloor\theta n\rfloor,\lfloor\theta n\rfloor+\delta(n)\big],\end{cases}\\
m_n^-(x,S) &= m^-(x,S), \qquad \text{for } x\in\big[-\delta(n),\lfloor\theta n\rfloor+\delta(n)\big],\\
m_n^-(x,S) &= m_n^+(x,S) = 0, \qquad \text{for } x\notin\big(-\delta(n),\lfloor\theta n\rfloor+\delta(n)\big).
\end{aligned} \tag{1.18}
$$

Substitution into the exponent in (1.17) gives

$$
\sum_{x=-\delta(n)}^{\lfloor\theta n\rfloor+\delta(n)} f_x\big(m^-(x,S), m^+(x,S)\big) + O\big(\delta(n)\big), \tag{1.19}
$$

where the error term comes from the fact that $f_x(i,j)$ is linear in both coordinates with bounded coefficients. Moreover, with the help of (1.18) the set A_n^3 can be rewritten as

$$
\begin{aligned}
A_n^3 = \Bigg\{S\in\Omega\colon &\sum_{x=-\delta(n)}^{\lfloor\theta n\rfloor+\delta(n)} \big[m^-(x,S)+m^+(x,S)\big] = n+2\delta(n)+1,\\
& m^-\big(-\delta(n),S\big) = m^-\big(\lfloor\theta n\rfloor+\delta(n),S\big) = 0,\\
& m^+\big(-\delta(n),S\big) = m^+\big(\lfloor\theta n\rfloor+\delta(n),S\big) = 1\Bigg\}.
\end{aligned} \tag{1.20}
$$

The l.h.s. of (1.16) is the l.h.s. of (1.12). The r.h.s. of (1.17), after inserting (1.19) and (1.20), becomes the r.h.s. of (1.12) but with 0, $\lfloor \theta n \rfloor$ and n *perturbed* by $\delta(n)$. However, in subsection 2.3 we shall give a perturbation estimate (Lemma 5) showing that this has no effect on an exponential scale because $\delta(n) = o(n)$. □

1.3. *The associated Markov chain.* Since $\theta > 0$ implies $\lim_{n\to\infty} S_n = \infty$ P_θ-a.s., and similarly at the negative end, we have

$$m^-(x,S) = m^+(x-1,S) - 1 \quad P_\theta\text{-a.s.} \tag{1.21}$$

Hence the r.h.s. of (1.12) can be rewritten in terms of the $m^+(x,S)$'s alone. This is important because the latter form a Markov chain with x playing the role of time. Abbreviate $m^+(x,S) = m(x)$.

PROPOSITION 3. *The process $\{m(x)\}_{x\in\mathbb{Z}}$ is stationary Markov with transition kernel and invariant measure given by*

$$P_\theta(i,j) = \binom{i+j-2}{i-1}\left[\frac{1}{2}(1+\theta)\right]^i \left[\frac{1}{2}(1-\theta)\right]^{j-1}, \qquad i,j \in \mathbb{N}, \tag{1.22}$$

$$\pi_\theta(i) = \frac{2\theta}{1+\theta}\left(\frac{1-\theta}{1+\theta}\right)^{i-1}, \qquad i \in \mathbb{N}. \tag{1.23}$$

PROOF. Greven and den Hollander (1992), Lemma 4. □

Introduce the *empirical pair distribution*

$$\nu_N^\omega = \frac{1}{N}\sum_{x=0}^{N-1} \delta_{((m(x-1),p_{x-1}),(m(x),p_x))}, \tag{1.24}$$

with *periodic boundary conditions* [i.e., $m(-1) = m(N-1)$ and $p_{-1} = p_{N-1}$].

PROPOSITION 4.

$$\begin{aligned} P_\omega\big(X_n = \lfloor \theta n\rfloor\big) = C_n^{-1}(\theta)\exp\big(o(n)\big)E_\theta\Big(&\exp\Big[K_n\langle f, \nu_{K_n}^\omega\rangle\Big]1\{\nu_{K_n}^\omega \in A_{L_n}\}\\ &\times 1\{m(-1) = m(0) = m(K_n-2) = m(K_n-1) = 1\}\Big), \end{aligned} \tag{1.25}$$

where

$$K_n = \lfloor \theta n\rfloor + 1, \qquad L_n = \frac{n+1}{K_n}, \tag{1.26}$$

$$f\big((i,p),(j,q)\big) = (i-1)\log(1-q) + j\log q, \tag{1.27}$$

$$a\big((i,p),(j,q)\big) = i+j-1, \tag{1.28}$$

$$A_{L_n} = \Big\{\nu \in \mathcal{P}\Big(\big[\mathbb{N}\times \operatorname{supp}(\alpha)\big]^2\Big)\colon \nu^1 = \nu^2,\ \langle a,\nu\rangle = L_n\Big\}, \tag{1.29}$$

ν^1 and ν^2 are the first and second marginals of ν on $\mathbb{N}\times\text{supp}(\alpha)$, and $\langle\cdot,\cdot\rangle$ denotes inner product over $[\mathbb{N}\times\text{supp}(\alpha)]^2$.

PROOF. Combine Propositions 1 and 2 and rewrite the r.h.s. of (1.12) using (1.4), (1.21) and (1.24). □

By combining Propositions 3 and 4 we see that at this stage we can forget about the underlying random walk: (1.25) is *a large deviation problem for the associated Markov chain* $\{m(x)\}_{x\in\mathbb{Z}}$ defined by (1.22) and (1.23). The same symbols P_θ and E_θ will be used to denote probability and expectation for this process.

2. Large deviations for ν_N^ω: derivation of the variational formula. The asymptotic analysis of the expectation in (1.25) looks like a standard large deviation problem for Markov chains. However, there are several obstacles blocking the way:

I. The kernel P_θ in (1.22) is not uniformly ergodic [see, e.g., Deuschel and Stroock (1989), page 95].

II. The environment ω is fixed, hence in $\nu_{K_n}^\omega$ of (1.24) only the $m(x)$'s are random.

III. The set A_{L_n} in (1.25) is n-dependent and is not closed (in the weak topology).

IV. The function $\nu\to\langle f,\nu\rangle$ is not continuous (in the weak topology).

Techniques to circumvent I–IV are described in Greven and den Hollander (1992), Sections 3 and 5. This involves a sequence of steps, all of which carry over but one, namely the truncation procedure discussed in subsection 2.2. Again we repeat the strategy of the whole proof in order to give the reader some guidance. Subsections 2.2, 2.6 and 2.7 contain new material. Subsections 2.1 and 2.3 to 2.5 are based on Greven and den Hollander (1992) and therefore we omit details. The main result of Section 2, and the key to Theorem 1, is the following.

THEOREM 3. *For every $\theta>0$,*

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log E_\theta\Big(&\exp\big[K_n\langle f,\nu_{K_n}^\omega\rangle\big]1\{\nu_{K_n}^\omega\in A_{L_n}\}\\ &\times 1\{m(-1)=m(0)=m(K_n-2)=m(K_n-1)=1\}\Big)=J(\theta),\end{aligned}\tag{2.1}$$

where

$$J(\theta)=\theta\sup_{Q\in M_{\theta,\alpha}}\big[\langle f,\pi^2Q\rangle-I(Q\mid R_{\theta,\alpha})\big]\tag{2.2}$$

and $R_{\theta,\alpha}$ is the stationary Markov process with transition kernel

$$R_{\theta,\alpha}\big((i,p)\to(j,q)\big)=P_\theta(i,j)\alpha(q). \tag{2.3}$$

The same limit is obtained when in K_n, L_n of (1.26) the θ is replaced by $\theta_n\to\theta$.

2.1. *Passing to i.i.d. random variables.* In this subsection we rewrite the expectation w.r.t. $\{m(x)\}_{x\in\mathbb{Z}}$ appearing in (2.1) as the expectation of a new functional w.r.t. to an *auxiliary* i.i.d. process.

Fix n. Define

$$\mathcal{Y}_n=\bigg\{y=(1,1,y_1,\dots,y_{K_n-3},1,1)\in\mathbb{N}^{K_n+1}:\ \frac{1}{K_n}\sum_{k=0}^{K_n-1}(y_{k-1}+y_k-1)=\frac{n+1}{K_n}=L_n\bigg\}, \tag{2.4}$$

$$F_\omega(y)=\sum_{k=0}^{K_n-1}f\big((y_{k-1},p_{k-1}),(y_k,p_k)\big),\qquad y\in\mathcal{Y}_n, \tag{2.5}$$

$$P(y)=\sum_{k=0}^{K_n-1}\log P_\theta(y_{k-1},y_k),\qquad y\in\mathcal{Y}_n. \tag{2.6}$$

Introduce i.i.d. random variables $(Y_k)_{k\geq 1}$ given by

$$P(Y_k=l)=(1-c)c^{l-1},\qquad l\in\mathbb{N},\ c\in(0,1). \tag{2.7}$$

Let their law and expectation be denoted by P, E. Abbreviate

$$Y=(1,1,Y_1,\dots,Y_{K_n-3},1,1). \tag{2.8}$$

In terms of these auxiliary objects we may rewrite the expectation in (2.1) as follows.

PROPOSITION 5.

$$\begin{aligned}E_\theta\Big(&\exp\big[K_n\langle f,\nu^\omega_{K_n}\rangle\big]1\{\nu^\omega_{K_n}\in A_{L_n}\}\\&\times 1\{m(-1)=m(0)=m(K_n-2)=m(K_n-1)=1\}\Big)\\&=\pi_\theta(1)(1-c)^{-K_n-1}c^{-\frac12(n-K_n+1)}S_\omega(K_n,L_n),\end{aligned} \tag{2.9}$$

where

$$S_\omega(K_n,L_n)=E\Big(\exp\big[F_\omega(Y)+P(Y)\big]1\{Y\in\mathcal{Y}_n\}\Big). \tag{2.10}$$

PROOF. First note that the event defined by the two indicators in (1.25) is the same as $\{m\in\mathcal{Y}_n\}$ with $m=\{m(x)\}_{x=-1}^{K_n-1}$ [recall (1.24) and (1.29)]. Next note that $K_n\langle f,\nu^\omega_{K_n}\rangle=F_\omega(m)$ by (1.24) and (2.5).

The probability that $m = y$ under the law of the stationary Markov chain given by (1.22) and (1.23) is

$$\begin{aligned}(2.11)\qquad \pi_\theta(1)\prod_{k=0}^{K_n-1} P_\theta(y_{k-1},y_k) &= \pi_\theta(1)\exp\left[\sum_{k=0}^{K_n-1}\log P_\theta(y_{k-1},y_k)\right]\\ &= \pi_\theta(1)\exp\big[P(y)\big].\end{aligned}$$

The probability that $Y = y$ under the law P is

$$(2.12)\qquad \prod_{k=-1}^{K_n-1}\left\{(1-c)c^{y_k-1}\right\} = (1-c)^{K_n+1}c^{\frac{1}{2}(n-K_n+1)}. \qquad \square$$

2.2. *Truncation.* In this subsection we show that the components of Y may be truncated to stay on a finite state space. The proof is somewhat involved, but the truncation will be very important for the large deviation analysis in subsection 2.5 and the variational analysis in subsection 2.6.

Let $R \in \mathbb{N}$ be the truncation level and define

$$(2.13)\qquad \mathcal{Y}_n^R = \mathcal{Y}_n \cap [1,R]^{K_n+1},$$

$$(2.14)\qquad S_\omega^R(K_n,L_n) = E\Big(\exp\big[F_\omega(Y)+P(Y)\big]1\{Y\in\mathcal{Y}_n^R\}\Big).$$

The rest of this section is devoted to the proof of the following proposition.

PROPOSITION 6.

$$(2.15)\qquad \lim_{R\to\infty}\limsup_{n\to\infty}\frac{1}{n}\big|\log S_\omega(K_n,L_n)-\log S_\omega^R(K_n,L_n)\big| = 0 \quad \omega\text{-a.s.}$$

PROOF. The idea is to construct a map that associates with each configuration $y \in \mathcal{Y}_n$ such that $\sup_k y_k > R$ a configuration $y' \in \mathcal{Y}_n$ such that $\sup_k y'_k \le R$, in such a way that y' contributes about as much to the exponential as y. For that purpose a configuration y will be viewed as a collection of $K_n - 3$ piles of units, of sizes $y_1,\dots,y_{K_n-3}$ [recall (2.5)], and y' will be built out of y by moving units around in blocks of piles. Since all y have the same probability, because $y \in \mathcal{Y}_n$ fixes $\Sigma_k y_k$ [recall (2.4) and (2.7)], all units may be moved around freely. The problem will be to control the effect on the exponential. This has two aspects: $F_\omega(y)$ that was induced by the environment and $P(y)$ that was induced by the Markov dependence of $(m(x))_{x\ge 0}$. The hard part is to deal with both at the same time.

It suffices to show that there exists $c(R)$, with $c(R)\to 0$ as $R\to\infty$, such that, for all $n \ge n_0(\omega,R)$,

$$\begin{aligned}(2.16)\qquad &E\Big(\exp\big[F_\omega(Y)+P(Y)\big]1\{Y\in\mathcal{Y}_n\backslash\mathcal{Y}_n^R\}\Big)\\ &\quad\le \exp\big(nc(R)\big)E\Big(\exp\big[F_\omega(Y)+P(Y)\big]1\{Y\in\mathcal{Y}_n^R\}\Big).\end{aligned}$$

The proof of (2.16) will be achieved by showing the following result.

LEMMA 1. *For every $R \in \mathbb{N}$ and $n \geq n_0(\omega, R)$, there exists a map*

$$T: \mathcal{Y}_n \backslash \mathcal{Y}_n^R \to \mathcal{Y}_n^R, \tag{2.17}$$

with the properties:

(a) $P(Y = y) = P(Y = Ty)$ *for* $y \in \mathcal{Y}_n \backslash \mathcal{Y}_n^R$,
(b) $F_\omega(y) = F_\omega(Ty)$ *for* $y \in \mathcal{Y}_n \backslash \mathcal{Y}_n^R$,
(c) $P(y) \leq P(Ty) + nc'(R)$ *for* $y \in \mathcal{Y}_n \backslash \mathcal{Y}_n^R$,
(d) $|\{y \in \mathcal{Y}_n \backslash \mathcal{Y}_n^R: Ty = y'\}| \leq \exp(nc''(R))$ *for* $y' \in \mathcal{Y}_n^R$,

where $c'(R), c''(R) \to 0$ *as* $R \to \infty$.

Lemma 1 gives (2.16) as follows. First apply (b) and (c) to get that the l.h.s. of (2.16) is less than or equal to

$$\exp\big(nc'(R)\big)E\Big(\exp\big[F_\omega(TY) + P(TY)\big]1\{Y \in \mathcal{Y}_n \backslash \mathcal{Y}_n^R\}\Big). \tag{2.18}$$

Then apply (a) and (d) to get that the r.h.s. of (2.18) is less than or equal to

$$\exp\Big(n\big[c'(R) + c''(R)\big]\Big)E\Big(\exp\big[F_\omega(Y') + P(Y')\big]1\{Y' \in \mathcal{Y}_n^R\}\Big). \tag{2.19}$$

Combination of (2.18) and (2.19) proves (2.16) with $c(R) = c'(R) + c''(R)$. The rest of this subsection is devoted to the proof of Lemma 1.

PROOF OF LEMMA 1. In order to construct a map T with the desired properties, we shall need some preparation. This comes in the form of Lemmas 2 to 4 below.

Pick $s, t, u \in \mathbb{N}$. Later we shall put $R = 2st + 1$ and let

$$\begin{aligned} &\text{(i)} \quad s, t, u \to \infty, \\ &\text{(ii)} \quad \frac{1}{t}\log s \to \infty, \\ &\text{(iii)} \quad u/s \to 0. \end{aligned} \tag{2.20}$$

Define the following sets:

$$\begin{aligned} A^1 &= \{k: y_k > st\}, \\ A^2 &= \text{smallest set containing } A^1 \text{ such that } y_k \leq ut \\ &\quad\ \text{when } |k - A^2| = 1, 2. \end{aligned} \tag{2.21}$$

Let $\partial A^2 = \{k: |k - A^2| = 1\}$. The set $A^2 \cup \partial A^2$ can be decomposed into clusters (= maximal connected components)

$$A^2 \cup \partial A^2 = \sum_i [a_i, b_i] \tag{2.22}$$

with $b_i + 2 < a_{i+1}$. Abbreviate

$$z_k = y_{k-1} + y_k - 2 \tag{2.23}$$

and note that

$$y \in \mathcal{Y}_n \quad \Rightarrow \quad \sum_k y_k = \frac{1}{2}(n + K_n - 1) = \frac{1}{2}(n + \lfloor \theta n \rfloor) \le n. \tag{2.24}$$

LEMMA 2.

(a) $|A^2| \le n\left(\frac{1}{st} + \frac{2}{ut}\right)$.

(b) $|\partial A^2| \le n\left(\frac{2}{st}\right)$.

(c) $\sum_i z_{a_i} + \sum_i z_{b_i+1} \le 2ut|\partial A^2|$.

(d) *For every i such that $b_i - a_i > 2t$ there exists a sequence of "cutting points" $(c_i^j)_{j=1}^{i'(i)}$ with $a_i < c_i^1 < \cdots < c_i^{i'(i)} < b_i$ such that*:

$$\begin{aligned} &\text{(i)} \quad c_i^{j+1} - c_i^j \le 2t \qquad \textit{for } 1 \le j < i'(i),\\ &\qquad c_i^1 - a_i \le 2t,\\ &\qquad b_i - c_i^{i'(i)} \le 2t;\\ &\text{(ii)} \quad \sum_j z_{c_i^j} \le \frac{1}{K} \sum_{k \in [a_i, b_i+1] \setminus (c_i^j)} z_k\\ &\qquad \textit{with } K = K(t) \textit{ given by } t - 2 = \log K / \log\left(1 + \frac{1}{K}\right). \end{aligned} \tag{2.25}$$

PROOF. (a) Use (2.21) and (2.24) to get $|A^1| \le n/st$ and $|A^2 \backslash A^1| \le 2n/ut$.

(b) $\partial A^2 = \sum_i \{a_i, b_i\}$ and every $[a_i, b_i]$ contains at least one site of A^1.

(c) Obvious from (2.21) and (2.23).

(d) Pick i such that $b_i - a_i > 2t$. A set of cutting points (c_i^j) may be constructed recursively as follows. First find a site $c \in (a_i + t, b_i - t]$ such that

$$z_c = \sup_{k \in (a_i, +t, b_i - t]} z_k. \tag{2.26}$$

Next let

$$c' = \max\left\{ l < c \colon z_l \le \frac{1}{K} \sum_{k \in (l, c]} z_k \right\}. \tag{2.27}$$

We claim that $c - c' \le t$. Indeed, since

$$z_l > \frac{1}{K} \sum_{k \in (l, c]} z_k \quad \text{for } l \in (c', c), \tag{2.28}$$

it follows by iteration that

$$z_l > z_c \frac{1}{K}\left(1 + \frac{1}{K}\right)^{c-l-1} \quad \text{for } l \in (c', c). \tag{2.29}$$

Since z_c is a maximum we must have

$$\frac{1}{K}\left(1 + \frac{1}{K}\right)^{c-c'-2} < 1. \tag{2.30}$$

This implies $c - c' \le t$ when K is such that $(1 + 1/K)^{t-2}/K \ge 1$, which identifies $K = K(t)$ as in (2.25)(ii). The point c' is a first cutting point in the construction, that is, $c' = c_i^j$ for some j.

The same argument works on the right of c and gives a second cutting point $c'' > c$ such that $c'' - c \le t$. Clearly, $c'' - c' \le 2t$ and so we have found two successive cutting points in the interval (a_i, b_i). If $c = b_i - t$, then it may happen that $c'' = b_i$. In that case c'' can be dropped because $b_i - c' \le 2t$ already.

Repeat the procedure on the two remaining pieces $[a_i, c')$ and $(c'', b_i]$, and so on, until what is left over has length $\le 2t$. By construction (2.25) (i) and (ii) hold.

Define the following sets:

$$\begin{aligned} B^1 &= \{k: y_k \le st, |k - A^2| \ge 3\}, \\ B^2 &= \text{subset of } B^1 \text{ obtained from } B^1 \text{ by successively chopping off} \\ &\qquad \text{intervals of length } 2t+1 \\ &= \sum_j J_j. \end{aligned} \tag{2.31}$$

An element $\tau \in [\mathrm{supp}(\alpha)]^{2t+2}$ will be called a *type*. Each interval J_j carries a type in the environment ω, namely $\tau = (p_k)_{k \in J_j}$. Now define

$$C_\tau = \{j: J_j \text{ has type } \tau\}. \tag{2.32}$$

LEMMA 3.

(a) $|B^2| \ge K_n - n\dfrac{2t+2}{t}\left(\dfrac{2}{s} + \dfrac{6}{u}\right)$.

(b) $|C_\tau| \ge \dfrac{1}{2}\dfrac{|B^2|}{2t+2}[\min_p \alpha(p)]^{2t+2}$ *for all* τ *and* $n \ge n_0(\omega, s, u, t)$.

PROOF. (a) Since no more than $2t + 1$ sites can be deleted in a row while building B^2 from B^1, we have $|B^1 \backslash B^2| \leq (2t+1)|B^{1c}|$. Since $|B^{1c}| \leq |A^1| + |A^2| + 2|\partial A^2|$ and $|A^1| \leq n/st$, it follows from Lemmas 2(a) and (b) that $|B^{1c}| \leq n(2/st + 6/ut)$. Now note that $|B^2| = |B^1| - |B^1 \backslash B^2| = K_n - |B^{1c}| - |B^1 \backslash B^2| \geq K_n - (2t+2)|B^{1c}|$.

(b) Since $\sum_\tau |C_\tau| = |B^2|/(2t+2)$ and the environment $\omega = (p_x)_{x \in \mathbb{Z}}$ is i.i.d. with marginal α, the claim follows from (a) and the ergodic theorem when s and u are large enough [recall (2.20)(i) and $K_n \sim \theta n$ as $n \to \infty$].

We are now in a position to define a map T which satisfies the requirements of Lemma 1. Put

$$y_k = 1 + \Delta_k, \qquad \Delta_k \geq 0. \tag{2.33}$$

According to (2.22) and Lemma 2(d), we have a decomposition of $A^2 \cup \partial A^2$ into intervals of length $\leq 2t$:

$$A^2 \cup \partial A^2 = \sum_j I_j. \tag{2.34}$$

Recall that B^2 has a similar decomposition, but with all intervals of length $2t+1$.

1. Take I_1 and record its type τ_1 defined by $\tau_1 = (p_k)_{k=i_1}^{i_1+2t+1}$ with $i_1 = \min\{k\colon k \in I_1\}$ (note that τ_1 reads the environment over a length $2t + 1$, which exceeds at least one over the right edge of I_1). Let

$$M_1 = \sup_{k \in I_1} \Delta_k, \tag{2.35}$$

$$\Delta_k^l = \lceil M_1 \backslash st \rceil^{-1} \Delta_k + \varepsilon_k^l, \qquad k \in I_1,\ 1 \leq l \leq \lceil M_1/st \rceil, \tag{2.36}$$

where ε_k^l is picked such that

$$|\varepsilon_k^l| \leq 1, \qquad \sum_l \varepsilon_k^l = 0 \text{ for each } k \in I_1. \tag{2.37}$$

Think of $(\Delta_k^l)_{k \in I_1}$ as one of $\lceil M_1/st \rceil$ versions of $(\Delta_k)_{k \in I_1}$, each reduced by a factor $\lfloor M_1/st \rfloor^{-1}$. Think of $(\varepsilon_k^l)_{k \in I_1}$ as round off errors needed to make each of the *reduced versions* integer-valued.

2. Pick $\lfloor M_1/st \rfloor$ intervals in C_{τ_1}. Transport $(\Delta_k^l)_{k \in I_1}$ to the lth interval picked in C_{τ_1}, in such a way that $\Delta_{i_1}^l$ (with $i_1 = \min\{k{:}k \in I_1\}$) comes on top of the leftmost site in the interval. Only transport if $M_1 \geq st$, otherwise not.

Repeat steps 1 and 2 for I_2, that is, record its type τ_2, reduce $(\Delta_k)_{k \in I_1}$ by $\lceil M_2/st \rceil^{-1}$ with $M_2 = \sup_{k \in I_2} \Delta_2$, and if $M_2 \geq st$ then transport the $\lceil M_2/st \rceil$ reduced versions to $\lceil M_2/st \rceil$ intervals in C_{τ_2} disjoint from all the intervals picked earlier, and so on.

The resulting configuration, after all I_j's in $A^2 \cup \partial A^2$ have been thus transported to some J_j's in B^2, makes up the image configuration Ty.

LEMMA 4. (a) *T is well defined, in the sense that $|C_\tau|$ is large enough to accommodate all reduced versions for all τ and $n \geq n_0(\omega, s, t, u)$.*

(b) *$Ty \in \mathcal{Y}_n^R$ with $R = 2st + 1$.*

PROOF. (a) The total number of reduced versions moved is

$$\sum_i \left\lceil \frac{M_i}{st} \right\rceil \leq \sum_i \frac{2M_i}{st} \leq \frac{2}{st} \sum_k y_k \leq \frac{2n}{st}$$

because of (2.24), (2.33) and (2.35). From Lemma 3(a) we have $|B^2| > \frac{1}{2}K_n > \frac{1}{2}\theta n$ for s and u large enough by (2.20)(i). From Lemma 3(b) it therefore follows that $|C_\tau| \geq n/s$ provided

$$\frac{\theta}{4} \frac{t\left[\min_p \alpha(p)\right]^{2t+2}}{2t+2} > \frac{1}{s}, \tag{2.38}$$

which holds eventually by (2.20)(ii).

(b) At $k \in A^2 \cup \partial A^2$ remains $(Ty)_k = 1$ after applying T. At $k \in B^2$ we have $y_k \leq st$, and not more than $st + 1$ is added by the map T because $\Delta_k^l \leq st + 1$ for all k, l.

We are now finally ready to verify (a)–(d) in Lemma 1.

Proof of (a): Obvious.

Proof of (b): All reduced versions have the same environment before and after moving, including the site at the right boundary which contributes to $F_\omega(y)$ in (2.5).

Proof of (c): This is the hardest estimate. By substituting P_θ of (1.22) into (2.6) we have

$$P(y) - P(Ty) = \log \prod_{k=0}^{K_n - 1} \binom{y_{k-1} + y_k - 2}{y_{k-1} - 1} \binom{(Ty)_{k-1} + (Ty)_k - 2}{(Ty)_{k-1} - 1}^{-1}. \tag{2.39}$$

Note that here the θ-dependent part of P_θ drops out because y, $Ty \in \mathcal{Y}_n$, implies

$$\sum_{k=0}^{K_n - 1} y_{k-1} = \sum_{k=0}^{K_n - 1} y_k = \sum_{k=0}^{K_n - 1} (Ty)_{k-1} = \sum_{k=0}^{K_n - 1} (Ty)_k.$$

Let B^3 be the subset of B^2 consisting of the sites where a reduced version is moved to by the map T. Then

$$(Ty)_k = \begin{cases} y_k, & \text{for } k \notin A^2 \cup \partial A^2 \cup B^3, \\ 1, & \text{for } k \in A^2 \cup \partial A^2, \\ y_k + \Delta_{k'}^{l'}, & \text{for } k \in B^3 \text{ and some } k' = k'(k),\ l' = l'(k). \end{cases} \tag{2.40}$$

The set B^3 has a decomposition into intervals of length $\leq 2t$:

$$B^3 = \sum_j J'_j = \sum_j [a'_j, b'_j]. \tag{2.41}$$

Recall (2.22). The sets $A^2 \cup \partial A^2$ and B^3 are everywhere separated by at least one site. Hence the product in the r.h.s. of (2.39) has two contributions, namely

$$\begin{aligned} \Pi_1 &= \prod_i \prod_{k \in [a_i, b_i + 1]}, \\ \Pi_2 &= \prod_j \prod_{k \in [a'_j, b'_j + 1]}. \end{aligned} \tag{2.42}$$

We shall estimate these products separately, for which we need the following fact which is immediate from Stirling's formula:

$$\binom{\Delta + \Delta'}{\Delta} = C(\Delta, \Delta') \exp\left[\Delta I\left(\frac{\Delta'}{\Delta}\right)\right], \qquad \Delta, \Delta' \geq 1, \tag{2.43}$$

with $I(z) = (1+z)\log(1+z) - z\log z$ and $\delta(1/\Delta + 1/\Delta')^{1/2} \leq C(\Delta, \Delta') \leq 1$ for some $\delta > 0$.

First consider Π_1. We get an upper bound for Π_1 by dropping the second binomial coefficient in (2.39). Now return to Lemma 2. The contribution of the boundary points and the cutting points can be estimated by using the inequality

$$\binom{a+b}{a} \leq 2^{a+b}.$$

This gives, via (2.23)–(2.25) and Lemma 2(b) and (c), that

$$\begin{aligned} &\prod_i \prod_{k \in \{a_i c_i^1, \dots, c_i^{i'(i)}, b_i + 1\}} \binom{y_{k-1} + y_k - 2}{y_{k-1} - 1} \\ &\leq 2^{\sum_i (z_{a_i} + z_{b_i + 1}) + \sum_i \sum_j z_{c_i^j}} \\ &\leq 2^{2ut|\partial A^2| + (1/K)\sum_k z_k} \\ &\leq 2^{n(4u/s + 2/K)}. \end{aligned} \tag{2.44}$$

Adding the contribution of the remaining points, we get, after substitution of (2.33) [recall also (2.22) versus (2.34)], that

$$\begin{aligned} \Pi_1 &\leq 2^{n(4u/s + 2/K)} \prod_i \prod_{k \in [a_i, b_i + 1] \setminus \{a_i, c_i^1, \dots, c_i^{i'(i)}, b_i + 1\}} \binom{y_{k-1} + y_k - 2}{y_{k-1} - 1} \\ &= \prod_j \prod_{k: k-1,\, k \in I_j} \binom{\Delta_{k-1} + \Delta_k}{\Delta_{k-1}}. \end{aligned} \tag{2.45}$$

Next consider Π_2. By using the inequality

$$\binom{a}{b}\binom{c}{d} \le \binom{a+c}{b+d},$$

we have, after dropping the boundary terms $k \in \sum_j \{a'_j, b'_j + 1\}$ in (2.42), that

$$\Pi_2 \le \prod_j \prod_{k \in (a'_j, b'_j]} \binom{\Delta^{l'(k-1)}_{k'(k-1)} + \Delta^{l'(k)}_{k'(k)}}{\Delta^{l'(k-1)}_{k'(k-1)}}^{-1}$$

$$(2.46) \qquad = \prod_j \prod_{k:k-1,\, k \in I_j} \prod_{l=1}^{\lceil M_j/st \rceil} \binom{\lceil M_j/st \rceil^{-1}(\Delta_{k-1} + \Delta_k) + \varepsilon^l_{k-1} + \varepsilon^l_k}{\lceil M_j/st \rceil^{-1} + \Delta_{k-1} + \varepsilon^l_{k-1}}^{-1}$$

$$\le \prod_j \prod_{k:k-1,\, k \in I_j} \binom{\lceil M_j/st \rceil^{-1}(\Delta_{k-1} + \Delta_k)}{\lceil M_j/st \rceil^{-1} + \Delta_{k-1}}^{-\lceil M_j/st \rceil}.$$

The last inequality follows from $\sum_l \varepsilon^l_k = 0$ because

$$x \to \log \binom{a+b+x}{a}$$

is convex.

Combine (2.44)–(2.46) with (2.39) to get

$$\begin{aligned} P(y) - P(Ty) &= \log \Pi_1 + \log \Pi_2 \\ &\le n\left(\frac{4u}{s} + \frac{2}{K}\right)\log 2 + \sum_j \sum_{k:k-1,\, k \in I_j} \left[\log \binom{\Delta_{k-1} + \Delta_k}{\Delta_{k-1}} \right. \\ &\qquad \left. - \lceil M_j/st \rceil \log \binom{\lceil M_j/st \rceil^{-1}(\Delta_{k-1} + \Delta_k)}{\lceil M_j/st \rceil^{-1}\Delta_{k-1}} \right]. \end{aligned} \tag{2.47}$$

Now substitute (2.43) into (2.47) and use that the exponential terms cancel because the two binomials have coefficients with the same ratio. It follows that the sum in the r.h.s. of (2.47) is bounded above by

$$\begin{aligned} &\sum_j \sum_{k:k-1,\, k \in I_j} \left\lceil \frac{M_j}{st} \right\rceil \left\{ \log \frac{1}{\delta} + \frac{1}{2} \log\left(\frac{\Delta_{k-1}\Delta_k}{\lceil \frac{M_j}{st} \rceil (\Delta_{k-1} + \Delta_k)} \right) \right\} \\ &\le \sum_j \left\lceil \frac{M_j}{st} \right\rceil \left\{ \log \frac{1}{\delta} + \frac{1}{2} \log(st) \right\} 2t, \end{aligned} \tag{2.48}$$

where the inequality uses that each I_j has length $\le 2t$ and that $\Delta_k \le M_j$ for

$k \in I_j$. Finally, use that $\sum_j M_j \leq \sum_k y_k \leq n$ to get

$$P(y) - P(Ty) \leq n\left(\frac{4u}{s} + \frac{2}{K}\right)\log 2 + n\frac{4}{s}\left\{\log\frac{1}{\delta} + \frac{1}{2}\log(st)\right\}. \tag{2.49}$$

The r.h.s. is $no(1)$ under (2.20).

Proof of (d): The total number of reduced versions moved by the map T does not exceed $2n/st$, as we remarked in the proof of Lemma 4(a). Since each reduced version has length $\leq 2t$, the total number of piles in all of the reduced versions does not exceed $4n/s$. Since a reduced version can come from not more than K_n different sites, we have

$$|\{y \in \mathcal{Y}_n \backslash \mathcal{Y}_n^{2st+1}: Ty = y'\}| \leq \binom{n}{4n/s}\binom{K_n}{2n/st}, \tag{2.50}$$

where we again use (2.24). The bound is $\exp(no(1))$ under (2.20)(i).

2.3. *Perturbation: going to the slab.* The aim of this section is to relax the restriction

$$\frac{1}{K_n}\sum_{k=0}^{K_n-1}(y_{k-1} + y_k - 1) = L_n$$

in $\mathcal{Y}_n$ of (2.4) by allowing the sum to vary in a thin slab $[\theta^{-1}-\varepsilon, \theta^{-1}+\varepsilon]$ around L_n [note that $L_n \sim \theta^{-1}$ as $n \to \infty$ by (1.26)]. This will be needed later in subsection 2.5 in order to be able to apply standard large deviation arguments.

For $\varepsilon > 0$ define

$$\mathcal{Y}_n^{\varepsilon,R} = \left\{ y = (1,1,y_1,\ldots,y_{K_n-3},1,1) \in [1,R]^{K_N+1}: \frac{1}{K_n}\sum_{k=0}^{K_n-1}(y_{k-1}+y_k-1) \in \left[\theta^{-1}-\varepsilon, \theta^{-1}+\varepsilon\right] \right\}, \tag{2.51}$$

$$S_\omega^{\varepsilon,R}(K_n) = E\Big(\exp\left[F_\omega(Y) + P(Y)\right]1\{Y \in \mathcal{Y}_n^{\varepsilon,R}\}\Big). \tag{2.52}$$

PROPOSITION 7.

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\left|\log S_\omega^R(K_n,L_n) - \log S_\omega^{\varepsilon,R}(K_n)\right| = 0 \quad \omega\text{-a.s.} \tag{2.53}$$

PROOF. The assertion follows from the following lemma.

LEMMA 5. *For every $\theta \in (0,1]$ there exists $C(\theta)$ such that for every $\varepsilon: \mathbb{N} \to \mathbb{Z}$ satisfying $|\varepsilon(n)| \leq \varepsilon n$ for ε sufficiently small,*

$$\frac{1}{n}\left|\log S_\omega^R\big(K_n, L_n + \varepsilon(n)\big) - \log S_\omega^R(K_n,L_n)\right| \leq -C(\theta)\left|\frac{\varepsilon(n)}{n}\right|\log\left|\frac{\varepsilon(n)}{n}\right|. \tag{2.54}$$

PROOF. This is the analogue of Lemmas 7 and 16 in Greven and den Hollander (1992). The proof carries over even though we there had a function of the form $f((i,p),(j,q)) = (i+j-1)\log q$ instead of $f((i,p),(j,q)) = (i-1)\log(1-q)+j\log q$ appearing in $F_\omega(y)$ [recall (1.27) and (2.5)]. All that was used about f is that there exist $0 < m < M < \infty$ such that $(i+j-1)\log m \le f((i,p),(j,q)) \le (i+j-1)\log M$, which holds here with $m = \min_{q\in\mathrm{supp}(\alpha)}(q, 1-q)$ and $M = \max_{q\in\mathrm{supp}(\alpha)}(q, 1-q)$.

2.4. *Eliminating the ω-dependence.* In this section we make two important steps: (i) the growth rate is ω-a.s. constant; (ii) integrate over ω w.r.t $\alpha^{\mathbb{Z}}$.

PROPOSITION 8.

$$\lim_{n\to\infty}\frac{1}{n}\left|\log S_\omega^{\varepsilon,R}(K_n) - \log S_{\omega'}^{\varepsilon,R}(K_n)\right| = 0 \quad \textit{for a.s. all } \omega \textit{ and } \omega'. \tag{2.55}$$

PROOF. This is the analogue of Lemma 17 in Greven and den Hollander (1992). Again the proof carries over. The idea is to divide space into blocks of size N and to construct a map T acting on $y \in [1,R]^{\mathbb{Z}}$ which permutes the blocks in such a way that (y,ω) imitates (Ty,ω') as much as possible; that is, Ty sees the same environment in ω' as y in ω except at the boundaries of the blocks. For $N\to\infty$ the boundary effects vanish. An important fact that is used in the proof is that the block statistics of ω and ω' are the same, that is,

$$\tilde{\mu}_{K_n}^N(\omega) \in B^{\delta,N} \quad \omega\text{-a.s. for } n \ge n_0(\omega,\delta,N) \text{ and all } \delta > 0 \text{ and } N \in \mathbb{N}, \tag{2.56}$$

where

$$\tilde{\mu}_{K_n}^N(\omega) = \frac{N}{K_n}\sum_{m=0}^{K_n/N-1} \delta_{(p_{mN+k})_{k=0}^{N-1}}, \tag{2.57}$$

$$B^{\delta,N} = \left\{\tilde{\mu} \in \mathcal{P}\left(\left[\mathrm{supp}(\alpha)\right]^N\right) : \left\|\tilde{\mu} - \alpha^N\right\| \le \delta\right\}. \tag{2.58}$$

In (2.57) we use *periodic boundary conditions* of the same type as in (1.24) (i.e., $p_1 = p_{K_n}$). For the asymptotic statements below it suffices to consider K_n a multiple of N.

Define

$$S^{\delta,N,\varepsilon,R}(K_n) = \int S_\omega^{\varepsilon,R}(K_n)1\{\tilde{\mu}_{K_n}^N(\omega) \in B^{\delta,N}\}\alpha^{\mathbb{Z}}(d\omega). \tag{2.59}$$

PROPOSITION 9.

$$\lim_{\substack{N\to\infty\\ \delta\to 0}}\lim_{n\to\infty}\frac{1}{n}\left|\log S^{\delta,N,\varepsilon,R}(K_n) - \log S_\omega^{\varepsilon,R}(K_n)\right| = 0 \quad \omega\text{-a.s.} \tag{2.60}$$

PROOF. This is the analogue of Lemma 18 in Greven and den Hollander (1992). The proof follows from Proposition 8 and (2.56), with the additional observation that the l.h.s. of (2.55) is uniformly bounded for all ω, ω' such that $\widetilde{\mu}^N_{K_n}(\omega), \widetilde{\mu}^N_{K_n}(\omega') \in B^{\delta,N}$, and the bound goes to zero as $N \to \infty$, $\delta \to 0$ [see the proof of Lemma 17 in Greven and den Hollander (1992)].

2.5. *Large deviation analysis.* The nice fact about the quantity defined in (2.59) is that it can be viewed as the expectation of a functional of the *empirical N-block distribution*

$$\mu^N_{K_n}(y,\omega) = \frac{N}{K_n} \sum_{m=0}^{K_n/N-1} \delta_{(y_{mN+k}, p_{mN+k})_{k=0}^{N-1}} \tag{2.61}$$

w.r.t. the *double layer* process (Y,ω). Let

$$\nu^N_{K_n}(y,\omega) = \frac{1}{N}\sum_{l=0}^{N-1} \pi^2\sigma^l\big(\mu^N_{K_n}(y,\omega)\big), \tag{2.62}$$

$$\begin{aligned} \widehat{M}^{\delta,N,\varepsilon,R}_{\theta,\alpha} = \Bigg\{ &\mu \in \mathcal{P}\Big(\big([1,R]\times \mathrm{supp}(\alpha)\big)^N\Big): \\ &\left\langle \frac{1}{N}\sum_{l=0}^{N-1} \pi^2\sigma^l(\mu), a\right\rangle \in \big[\theta^{-1}-\varepsilon, \theta^{-1}+\varepsilon\big],\ \widetilde{\pi}^N\mu \in B^{\delta,N}\Bigg\}, \end{aligned} \tag{2.63}$$

where σ is the cyclic shift, π^2 is the projection on $([1,R]\times \mathrm{supp}(\alpha))^2$, $\langle\cdot,\cdot\rangle$ is the inner product on $([1,R]\times\mathrm{supp}(\alpha))^2$, a is defined in (1.28) and $\widetilde{\pi}^N\mu$ denotes the projection of μ on $[\mathrm{supp}(\alpha)]^N$.

PROPOSITION 10.

$$S^{\delta,N,\varepsilon,R}(K_n) = \widehat{E}\left(\exp\Big[K_n\Big\{\widehat{F}\big(\nu^N_{K_n}\big) + \widehat{P}\big(\nu^N_{K_n}\big)\Big\}\Big] 1\{\mu^N_{K_n} \in \widehat{M}^{\delta,N,\varepsilon,R}_{\theta,\alpha}\}\right), \tag{2.64}$$

with $\widehat{E}$ expectation w.r.t. (Y,ω) and

$$\widehat{F}(\nu) = \langle f, \nu\rangle, \tag{2.65}$$

$$\widehat{P}(\nu) = \langle \log P_\theta, \nu\rangle, \tag{2.66}$$

where $\log P_\theta$ *is viewed as a function on* $[\mathbb{N}\times\mathrm{supp}(\alpha)]^2$ *and* f *is defined in* (1.27).

PROOF. Use (2.5), (2.6), (2.51), (2.52) and (2.59). Note that (2.57) is the projection of (2.61) on $[\mathrm{supp}(\alpha)]^N$.

The r.h.s. of (2.64) is ideally suited to a large deviation analysis of the classical type, as it involves the empirical distribution of an i.i.d. process with *finite* state space $([1,R] \times \mathrm{supp}(\alpha))^N$. Therefore we can now immediately write down the following result.

PROPOSITION 11.

$$\lim_{n\to\infty} \frac{1}{n} \log S^{\delta,N,\varepsilon,R}(K_n) = \theta \sup_{\mu \in M_{\theta,\alpha}^{\delta,N,\varepsilon,R}} \left[\widehat{F}(\pi^2\mu) + \widehat{P}(\pi^2\mu) - \frac{1}{N}\widehat{I}_{c,\alpha}^N(\mu) \right], \tag{2.67}$$

where

$$M_{\theta,\alpha}^{\delta,N,\varepsilon,R} = \Big\{ \mu \in \mathcal{P}\Big(([1,R] \times \mathrm{supp}(\alpha))^N\Big): \sigma\mu = \mu,\ \langle \pi^2\mu, a\rangle \in [\theta^{-1} - \varepsilon, \theta^{-1} + \varepsilon],\ \widetilde{\pi}^N\mu \in B^{\delta,N} \Big\}, \tag{2.68}$$

$$\widehat{I}_{c,\alpha}^N(\mu) = \textit{relative entropy of } \mu \textit{ w.r.t. } \pi^N G_{c,\alpha}, \tag{2.69}$$

where $G_{c,\alpha}$ *is the i.i.d. process with one-dimensional marginal* $(1-c)c^{j-1}\alpha(q)$ *and* π^N *denotes the N-dimensional marginal.*

PROOF. This follows by applying Varadhan's theorem to (2.64), because the family $(\nu_{K_n}^N)$ (with K_n a multiple of N) satisfies the large deviation principle on $\mathcal{P}(([1,R] \times \mathrm{supp}(\alpha))^N)$ with rate function $\widehat{I}_{c,\alpha}^N(\mu)$ [see Deuschel and Stroock (1989), Theorem 3.2.17]. It is important, in order for Varadhan's theorem to be applicable, that $\mu \to \widehat{F}(\pi^2\mu), \widehat{P}(\pi^2\mu), \widehat{I}_{c,\alpha}^N(\mu)$ are bounded and continuous [see Deuschel and Stroock (1989), Theorem 2.1.10]. It is also important that the large deviation principle carries over to the subset $\widehat{M}_{\theta,\alpha}^{\delta,N,\varepsilon,R}$ (by a standard argument). We may finally use that all quantities are σ-invariant in order to reduce the supremum to $M_{\theta,\alpha}^{\delta,N,\varepsilon,R} = \widehat{M}_{\theta,\alpha}^{\delta,N,\varepsilon,R} \cap \{\mu\colon \sigma\mu = \mu\}$ [compare (2.63) and (2.68); see Greven and den Hollander (1992), Section 3.2].

2.6. *Proof of Theorem* 3. Recall that in (2.9) we had expressed the expectation in (2.1) via the function $S_\omega(K_n, L_n)$ in (2.10). The latter has a growth rate that is the limit of the growth rate of the function $S^{\delta,N,\varepsilon,R}(K_n)$ in (2.64) as $\delta \to 0$, $N \to \infty$, $\varepsilon \to 0$, $R \to \infty$, as can be seen by tracing back Propositions 6, 7, 9 and 11. The proof of Theorem 3 will now proceed in two steps: Step 1: calculate the limit of the variational expression in (2.67) of Proposition 11; Step 2: rewrite the result and combine with (2.9) to obtain the variational expression in the r.h.s. of (2.2).

Step 1. The specific relative entropy of a stationary process Q on state space $\mathbb{N} \times \mathrm{supp}(\alpha)$ w.r.t. the i.i.d. process $G_{c,\alpha}$ is defined by [see Georgii (1988), Sections 15.1 and 15.2]

$$I(Q \mid G_{c,\alpha}) = \lim_{N\to\infty} \frac{1}{N} \widehat{I}_{c,\alpha}^N(\pi^N Q). \tag{2.70}$$

PROPOSITION 12.

$$\lim_{N\to\infty}\lim_{\delta\to 0}\sup_{\mu\in M_{\theta,\alpha}^{\delta,N,\varepsilon,R}}\left[\widehat{F}(\pi^2\mu)+\widehat{P}(\pi^2\mu)-\frac{1}{N}\widehat{I}_{c,\alpha}^N(\mu)\right]$$
$$(2.71)\qquad = \sup_{Q\in M_{\theta,\alpha}^{\varepsilon,R}}\left[\widehat{F}(\pi^2Q)+\widehat{P}(\pi^2Q)-I(Q\mid G_{c,\alpha})\right]$$

where

$$(2.72)\qquad M_{\theta,\alpha}^{\varepsilon,R}=\Big\{Q\in\mathcal{P}\Big(([1,R]\times\mathrm{supp}(\alpha))^{\mathbb{Z}}\Big)\colon \sigma Q=Q,$$
$$\langle\pi^2Q,a\rangle\in\left[\theta^{-1}-\varepsilon,\theta^{-1}+\varepsilon\right],\ \widetilde{\pi}^{\mathbb{Z}}Q=\alpha^{\mathbb{Z}}\Big\}.$$

PROOF. Since $\mu\to\widehat{F}(\pi^2\mu),\widehat{P}(\pi^2\mu),\widehat{I}_{c,\alpha}^N(\mu)$ are continuous on the set $\cup_{\delta>0} M_{\theta,\alpha}^{\delta,N,\varepsilon,R}$ (because $[1,R]\times\mathrm{supp}(\alpha)$ is finite), we can first take $\delta\to 0$ and let the supremum run over the set $M_{\theta,\alpha}^{N,\varepsilon,R}=\cap_{\delta>0}M_{\theta,\alpha}^{\delta,N,\varepsilon,R}$. Next we can use (2.70) and the fact that $\widehat{F}(\pi^2\mu)$ and $\widehat{P}(\pi^2\mu)$ only depend on the two-dimensional marginal of μ. We then obtain the r.h.s. of (2.71) as $N\to\infty$ noting that $M_{\theta,\alpha}^{\varepsilon,R}$ is the closure of $\cup_{N<\infty}\phi_N M_{\theta,\alpha}^{N,\varepsilon,R}$, where ϕ_N is the periodic extension operator.

PROPOSITION 13.

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\sup_{Q\in M_{\theta,\alpha}^{\varepsilon,R}}\left[\widehat{F}(\pi^2Q)+\widehat{P}(\pi^2Q)-I(Q\mid G_{c,\alpha})\right]$$
$$(2.73)\qquad = \sup_{Q\in M_{\theta,\alpha}}\left[\widehat{F}(\pi^2Q)+\widehat{P}(\pi^2Q)-I(Q\mid G_{c,\alpha})\right],$$

where $M_{\theta,\alpha}$ is the set defined in (0.12).

PROOF. Since $Q\to\widehat{F}(\pi^2Q),\widehat{P}(\pi^2Q)$ are continuous and $Q\to I(Q\mid G_{c,\alpha})$ is lower semicontinuous on the set $\cup_{\varepsilon>0}M_{\theta,\alpha}^{\varepsilon,R}$, we can first take the limit $\varepsilon\to 0$ and let the supremum run over the set $M_{\theta,\alpha}^R=\cap_{\varepsilon>0}M_{\theta,\alpha}^{\varepsilon,R}$. Next we observe that $Q\to\widehat{F}(\pi^2Q),\widehat{P}(\pi^2\mu)$ are continuous on $M_{\theta,\alpha}$ [see the proof of Lemma 10 in Greven and den Hollander (1992)], which is the closure of $\cup_{R<\infty}M_{\theta,\alpha}^R$. Therefore the claim follows by letting $R\to\infty$ and using the following property: For every Q there exists a sequence (Q_R) such that

$$(2.74)\qquad Q_R\Big(([1,R]\times\mathrm{supp}(\alpha))^{\mathbb{Z}}\Big)=1,$$

$$(2.75)\qquad \limsup_{R\to\infty} I(Q_R\mid G_{c,\alpha})\le I(Q\mid G_{c,\alpha}).$$

The proof of this property is similar to that of Proposition 16.34 in Georgii (1988). The idea is to pick $R=R(n)$ and to pick for $Q_{R(n)}$ the $[-n,n]$-block marginal of the process Q conditioned to stay below $R(n)$, that is,

$$\pi^{[-n,n]}Q\Big(\cdot \,\Big|\, \big([1,R(n)] \times \mathrm{supp}(\alpha)\big)^{[-n,n]}\Big), \tag{2.76}$$

periodically repeated to form a stationary process. One easily shows that (2.74) and (2.75) hold when $R(n) \to \infty$ sufficiently fast as $n \to \infty$.

Step 2. We shall need the following representation for the specific relative entropy of a stationary process Q w.r.t. a stationary Markov process R with kernel P_R:

$$I(Q \mid R) = -\langle \pi^2 Q, \log P_R\rangle - H(Q), \tag{2.77}$$

with $H(Q)$ the Kolmogorov–Sinai entropy of Q [see Ellis (1985), page 24, and Georgii (1988), Theorem 15.12].

Combine (2.9) of Proposition 5 with Propositions 6, 7, 9 and 11–13, use (2.77) and note that $K_n \sim \theta n$, to get that $J(\theta)$ in Theorem 3 (2.2) is given by

$$\begin{aligned} J(\theta) &= -\theta\log(1-c) - \tfrac{1}{2}(1-\theta)\log c \\ &\quad + \theta \sup_{Q \in M_{\theta,\alpha}} \Big[\widehat{F}(\pi^2 Q) + \widehat{P}(\pi^2 Q) + \langle \pi^2 Q, \log P_{G_{c,\alpha}}\rangle + H(Q)\Big], \end{aligned} \tag{2.78}$$

where $P_{G_{c,\alpha}}((i,p) \to (j,q)) = (1-c)c^{j-1}\alpha(q)$ is the kernel associated with the i.i.d. process $G_{c,\alpha}$. Next note that $\langle \pi^2 Q, a\rangle = \theta^{-1}$ and $\sigma Q = Q$ imply [use (0.11)]

$$\begin{aligned} \sum_{i,p}\sum_{j,q} i\pi^2 Q\big((i,p),(j,q)\big) &= \frac{1+\theta}{2\theta}, \\ \sum_{i,p}\sum_{j,q} (j-1)\pi^2 Q\big((i,p),(j,q)\big) &= \frac{1-\theta}{2\theta}. \end{aligned} \tag{2.79}$$

From the latter identity together with (2.66) we have

$$\begin{aligned} &\widehat{P}(\pi^2 Q) + \langle \pi^2 Q, \log P_{G_{c,\alpha}}\rangle \\ &\quad = \sum_{i,p}\sum_{j,q}\Big\{\log P_\theta(i,j) + \log\big[(1-c)c^{j-1}\alpha(q)\big]\Big\}\pi^2 Q\big((i,p),(j,q)\big) \\ &\quad = \log(1-c) + \frac{1-\theta}{2\theta}\log c + \langle \pi^2 Q, \log P_{R_{\theta,\alpha}}\rangle, \end{aligned} \tag{2.80}$$

where $P_{R_{\theta,\alpha}}$ denotes the kernel $P_\theta(i,j)\alpha(q)$ of the Markov process $R_{\theta,\alpha}$ defined in (2.3). After substitution of (2.80) into (2.78) the c-dependent terms cancel [recall that (2.7) and (2.8) were auxiliary objects]. Insert now (2.65) and use (2.77) to obtain (2.2) in Theorem 3.

The last claim in Theorem 3 is an immediate consequence of (2.54) in Lemma 5, since this shows that perturbations of K_n, L_n of order $o(n)$ do not affect the limit [see also the remark below (1.20)].

2.7. *Proof of Theorem* 1. Combine (1.25) in Proposition 4 with (2.1)–(2.3) in Theorem 3, using (1.5), to get for the rate function I

$$(2.81)\quad \begin{aligned} I(\theta) &= \lim_{n\to\infty}\frac{1}{n}\log C_n^{-1}(\theta)+J(\theta)\\ &= -\frac{1}{2}(1-\theta)\log\left[\frac{1}{2}(1-\theta)\right]-\frac{1}{2}(1+\theta)\log\left[\frac{1}{2}(1+\theta)\right]+J(\theta).\end{aligned}$$

Next use (2.77) and (2.79) and substitute (1.22) and (1.27), to obtain [recall the definition of $A_q(i,j)$ in (0.14)]

$$(2.82)\quad \begin{aligned} &\langle f,\pi^2Q\rangle - I(Q\,|\,R_{\theta,\alpha})\\ &= \sum_{i,p}\sum_{j,q}\Bigg\{\log[(1-q)^{i-1}q^j]\\ &\qquad + \log\left[\binom{i+j-2}{i-1}\left[\frac{1}{2}(1+\theta)\right]^i\left[\frac{1}{2}(1-\theta)\right]^{j-1}\alpha(q)\right]\Bigg\}\\ &\qquad \times \pi^2Q\big((i,p),(j,q)\big)+H(Q)\\ &= \frac{1+\theta}{2\theta}\log\left[\frac{1}{2}(1+\theta)\right]+\frac{1-\theta}{2\theta}\log\left[\frac{1}{2}(1-\theta)\right]+H(Q)\\ &\quad + \sum_{i,p}\sum_{j,q}\Big\{\log[A_q(j,i)\alpha(q)]\Big\}\pi^2Q\big((i,p),(j,q)\big).\end{aligned}$$

After substitution of (2.82) into (2.2), and the resulting expression for $J(\theta)$ into (2.81), we arrive at

$$(2.83)\quad I(\theta)=\theta\sup_{Q\in M_{\theta,\alpha}}\left[\sum_{i,p}\sum_{j,q}\Big\{\log[A_q(j,i)\alpha(q)]\Big\}\pi^2Q\big((i,p),(j,q)\big)+H(Q)\right].$$

The last step is to observe that $M_{\theta,\alpha}$ is invariant under time reversal of the top layer; that is, if $Q\in M_{\theta,\alpha}$, then $Q^{\mathrm{rev}}\in M_{\theta,\alpha}$ where Q^{rev} is the law of the process with the projection on $\mathbb{N}^{\mathbb{Z}}$ reversed [see (0.11)]. Since $H(Q)=H(Q^{\mathrm{rev}})$ when $\tilde{\pi}^{\mathbb{Z}}Q=\alpha^{\mathbb{Z}}$ (because $\alpha^{\mathbb{Z}}$ is time-reversible), we may replace Q by Q^{rev} under the sum in (2.83), which is the same as replacing $A_q(j,i)\alpha(q)$ by $A_q(i,j)\alpha(q)$. The latter is a transition kernel on $\mathbb{N}\times\mathrm{supp}(\alpha)$ and is the one appearing in (0.14). Finally, again apply (2.77) to obtain (0.17).

This completess the proof of Theorem 1 for $\theta\in(0,1]$.

Reversing space, we have that $I(-\theta)$ equals $I(\theta)$ but with A_q replaced by A_{1-q}. Since

$$A_{1-q}(j,i)=\frac{1-q}{q}A_q(i,j),$$

we get from (2.83)

$$(2.84)\quad I(-\theta)=I(\theta)-\theta\langle\log\rho\rangle,\qquad \theta\in(0,1],$$

using that $\sum_{i,j,p}\pi^2 Q((i,p),(j,q)) = \alpha(q)$ and $\rho = \rho(q) = (1-q)/q$. This identifies $I(\theta)$ for $\theta \in [-1,0)$ as in (0.18).

The value $I(0)$ is obtained by continuity via a perturbation argument applied directly to the random walk. Indeed, since supp(α) is bounded away from 0 and 1 (see the remark at the end of subsection 0.3), it is trivial to see, via a Radon-Nikodym argument, that $I(0) = \lim_{\theta\to 0} I(\theta)$.

3. Solution of the variational formula. Recall (0.22)–(0.26). In this section we show that for every $\theta \in [\theta_c, 1)$ the infimum in Theorem 1 (0.17) is attained at a $\overline{Q}_\theta \in M_{\theta,\alpha}$. It will turn out that $\overline{Q}_\theta$ is a Gibbs measure with a nearest-neighbor potential and with an external field. The potential is random and the external field is θ-dependent. The boundary case $\theta = 1$ is degenerate because $M_{1,\alpha} = \{[\delta_1 \times \alpha]^{\mathbb{Z}}\}$ is a singleton.

We proceed in two steps. In subsection 3.2 we define a one-parameter family of processes $(Q_r)_{r\geq r_c}$ and show that there exists an invertible function $\theta \to r(\theta)$ from $[\theta_c, 1)$ to $[r_c,\infty)$ such that $Q_{r(\theta)} \in M_{\theta,\alpha}$. In subsection 3.3 we show that $Q_{r(\theta)}$ is a minimizer of (0.17) and we evaluate the minimum. In subsection 3.1 we do some preparations. The proof of Theorem 2 comes in subsection 3.4.

3.1. *Maps and continued fractions.* The key to the construction in subsection 3.2 is the following property of the matrices $A_q(i,j)$, $q \in (0,1)$, defined in (0.14).

LEMMA 6. *For $\xi \in [0,\infty)$ let x_ξ be the vector with components $x_\xi(i) = \xi^{i-1}$, $i \in \mathbb{N}$. Let $\rho(q) = (1-q)/q$. Then*

$$A_q x_\xi = S_{\rho(q)}(\xi) x_{S_{\rho(q)}(\xi)}, \tag{3.1}$$

$$x_\xi A_q = \rho^{-1}(q)\widehat{S}_{\rho(q)}(\xi) x_{\widehat{S}_{\rho(q)}(\xi)}, \tag{3.2}$$

where

$$S_\rho(\xi) = \frac{1}{1+\rho-\rho\xi}, \tag{3.3}$$

$$\widehat{S}_\rho(\xi) = \frac{1}{1+\rho^{-1}-\rho^{-1}\xi}, \tag{3.4}$$

$$\xi^{-1} = S_\rho(\widehat{S}_\rho^{-1}(\xi)) = \widehat{S}_\rho(S_\rho^{-1}(\xi)). \tag{3.5}$$

[$S_\rho^{-1}(\xi) = 1/S_\rho(\xi)$ *and* $\rho^{-1}(q) = 1/\rho(q)$, *not the inverse.*]

PROOF. Use (0.14) and the identity

$$\sum_{j\in\mathbb{N}} \binom{i+j-2}{i-1} a^{j-1} = (1-a)^{-i} \quad \text{for } a \in [0,1). \qquad \square$$

Define the matrices

$$A_{q,r}(i,j) = e^{-r(i+j-1)} A_q(i,j), \qquad q \in (0,1),\ r \in \mathbb{R}. \tag{3.6}$$

Crucial is the action of $A_{q,r}$ on x_ξ as we shall see in the following result.

LEMMA 7.

$$A_{q,r}x_\xi = T_{\rho(q),r}(\xi)x_{T_{\rho(q),r}(\xi)}, \tag{3.7}$$

$$x_\xi A_{q,r} = \rho^{-1}(q)\widehat{T}_{\rho(q),r}(\xi)x_{\widehat{T}_{\rho(q),r}(\xi)}, \tag{3.8}$$

with

$$T_{\rho,r}(\xi) = \frac{1}{e^r(1+\rho) - \rho\xi}, \tag{3.9}$$

$$\widehat{T}_{\rho,r}(\xi) = \frac{1}{e^r(1+\rho^{-1}) - \rho^{-1}\xi}, \tag{3.10}$$

$$\xi^{-1} = T_{\rho,r}\big(\widehat{T}^{-1}_{\rho,r}(\xi)\big) = \widehat{T}_{\rho,r}\big(T^{-1}_{\rho,r}(\xi)\big). \tag{3.11}$$

PROOF. Same as in Lemma 6 with $q, 1-q$ replaced by $e^{-r}q, e^{-r}(1-q)$.

Lemma 7 says that the family of matrices $A_{q,r}, q \in (0,1), r \in \mathbb{R}$, has the cone of vectors $\{cx_\xi: c > 0, \xi \geq 0\}$ as an invariant set both from the left and from the right. On this cone therefore the actions of $A_{q,r}$ can be represented by the maps $\xi \to T_{\rho,r}(\xi)$ and $\xi \to \widehat{T}_{\rho,r}(\xi)$. The latter satisfy a duality relation given by (3.11).

Let $\omega = (p_x)_{x\in\mathbb{Z}}$ and $\rho_x = (1-p_x)/p_x$. Define the forward and backward continued fractions

$$f(r,\omega) = \frac{1|}{|e^r(1+\rho_0)} - \frac{\rho_0|}{|e^r(1+\rho_1)} - \frac{\rho_1|}{|\cdots}, \tag{3.12}$$

$$\widehat{f}(r,\omega) = \frac{1|}{|e^r\big(1+(\rho_{-1})^{-1}\big)} - \frac{(\rho_{-1})^{-1}|}{|e^r\big(1+(\rho_{-2})^{-1}\big)} - \frac{(\rho_{-2})^{-1}|}{|\cdots}. \tag{3.13}$$

The basic properties of these objects will be derived in subsection 4.1.

PROPOSITION 14. *Let σ be the shift defined by $(\sigma\omega)_x = \omega_{x+1}$. Then*

$$A_{p_0,r}x_{f(r,\sigma\omega)} = f(r,\omega)x_{f(r,\omega)}, \tag{3.14}$$

$$x_{\widehat{f}(r,\omega)}A_{p_0,r} = \rho_0^{-1}\widehat{f}(r,\sigma\omega)x_{\widehat{f}(r,\sigma\omega)}. \tag{3.15}$$

PROOF. Use (3.7) and (3.8), together with the observation

$$T_{\rho_0,r}\big(f(r,\sigma\omega)\big) = f(r,\omega), \tag{3.16}$$

$$\widehat{T}_{\rho_0,r}\big(\widehat{f}(r,\omega)\big) = \widehat{f}(r,\sigma\omega), \tag{3.17}$$

following from (3.9), (3.10) and (3.12).

3.2. *Ansatz for a minimizer.* In this subsection we construct the process Q_r that will later turn out to be a minimizer for a value of r depending on θ.

In view of the restriction $\widetilde{\pi}^{\mathbb{Z}} Q_r = \alpha^{\mathbb{Z}}$, we consider the law of Q_r given ω, written as

$$\begin{aligned} Q_{r,\omega}(\cdot) &= Q_r(\cdot \mid \omega), \\ Q_{r,\omega} &\in \mathcal{P}(\mathbb{N}^{\mathbb{Z}}) \quad \text{for all } \omega. \end{aligned} \tag{3.18}$$

Next, we define the process $Q_{r,\omega}$ for fixed r, ω by specifying its marginals on blocks $[-M, N]$ as follows:

$$\begin{aligned} &Q_{r,\omega}\big((i_x)_{x\in[-M,N]}\big) \\ &\quad = \frac{1}{Z_{r,\omega}^{[-M,N]}} x_{\hat{f}(r,\sigma^{-M}\omega)}(i_{-M}) \left\{ \prod_{x\in(-M,N]} A_{p_{x-1},r}(i_{x-1}, i_x) \right\} x_{f(r,\sigma^N\omega)}(i_N), \end{aligned} \tag{3.19}$$

with $Z_{r,\omega}^{[-M,N]}$ the normalizing constant.

LEMMA 8.

$$\begin{aligned} Z_{r,\omega}^{[-M,N]} &= \frac{1}{1 - f(r,\sigma^{-M}\omega)\widehat{f}(r,\sigma^{-M}\omega)} \prod_{x\in[-M,N)} f(r,\sigma^x\omega) \\ &= \frac{1}{1 - f(r,\sigma^{N}\omega)\widehat{f}(r,\sigma^{N}\omega)} \prod_{x\in(-M,N]} \rho_{x-1}^{-1} \widehat{f}(r,\sigma^x\omega). \end{aligned} \tag{3.20}$$

PROOF. The two expressions for $Z_{r,\omega}^{[-M,N]}$ are obtained by summing out the coordinates from the left resp. the right, applying (3.14) resp. (3.15).

From the structure of (3.20) together with the rules (3.14) and (3.15), we immediately see that (3.19) defines a *consistent* family of finite-dimensional distributions. Hence, by the Kolmogorov extension theorem, $Q_{r,\omega}$ is uniquely defined as a process on $\mathbb{N}^{\mathbb{Z}}$. Inspection of (3.19) shows that $Q_{r,\omega}$ is a *Gibbs measure* of the type introduced in subsection 0.3, (0.25) and (0.26).

PROPOSITION 15. (a) $\widetilde{\pi}^{\mathbb{Z}} Q_r = \alpha^{\mathbb{Z}}$.

(b) Q_r *is stationary.*

(c)

$$\begin{aligned} E_{\pi^2 Q_r}(a) &= \int \alpha^{\mathbb{Z}}(d\omega) \left[\frac{1 + \widehat{f}(r,\omega) f(r,\omega)}{1 - \widehat{f}(r,\omega) f(r,\omega)} \right] \\ &= \int \alpha^{\mathbb{Z}}(d\omega) \left[-\frac{(\partial/\partial r) f(r,\omega)}{f(r,\omega)} \right] = \int \alpha^{\mathbb{Z}}(d\omega) \left[-\frac{(\partial/\partial r) \widehat{f}(r,\omega)}{\widehat{f}(r,\omega)} \right]. \end{aligned} \tag{3.21}$$

(d) $Q_0 = R_\alpha$ *defined in* (0.14) *and* (0.15).

PROOF. (a) Obvious from (3.18).

(b) From (3.19) we see that $Q_{r,\sigma\omega}(\sigma\cdot) = Q_{r,\omega}(\cdot)$. Use (a) to get the claim.

(c) Pick $M = N = 0$ in (3.19) to see that

$$Q_{r,\omega}(i_0) = \left[1 - \widehat{f}(r,\omega)f(r,\omega)\right] x_{\widehat{f}(r,\omega)}(i_0)\, x_{f(r,\omega)}(i_0) \tag{3.22}$$

which is a geometric distribution with parameter $\widehat{f}(r,\omega)f(r,\omega)$. Now use (b) and recall (0.11) to compute

$$\begin{aligned} E_{\pi^2 Q_r}(a) &= \int \alpha^{\mathbb{Z}}(d\omega) \sum_{i_0,i_1} (i_0 + i_1 - 1) Q_{r,\omega}(i_0,i_1) \\ &= 2\left\{ \int \alpha^{\mathbb{Z}}(d\omega) \sum_{i_0} i_0 Q_{r,\omega}(i_0) \right\} - 1 \end{aligned} \tag{3.23}$$

and

$$\sum_{i_0} i_0 Q_{r,\omega}(i_0) = \frac{1}{1 - \widehat{f}(r,\omega)f(r,\omega)}. \tag{3.24}$$

This gives the first expression in the r.h.s. of (3.21). Alternatively, pick $M = 0$ and $N = 1$ in (3.19) to see that

$$Q_{r,\omega}(i_0,i_1) = \left[\frac{1}{f(r,\omega)} - \widehat{f}(r,\omega)\right] x_{\widehat{f}(r,\omega)}(i_0) A_{p_0,r}(i_0,i_1) x_{f(r,\sigma\omega)}(i_1). \tag{3.25}$$

By direct computation, using the explicit form of x_ξ and $A_{q,r}$ [see (3.6)], we get

$$\begin{aligned} -\frac{\partial}{\partial r} Q_{r,\omega}(i_0,i_1) = \Bigg\{ &(i_0 + i_1 - 1) - (i_0 - 1)\frac{(\partial/\partial r)\widehat{f}(r,\omega)}{\widehat{f}(r,\omega)} \\ &- (i_1 - 1)\frac{(\partial/\partial r) f(r,\sigma\omega)}{f(r,\sigma\omega)} \\ &+ \frac{\left[(\partial/\partial r) f(r,\omega) + f^2(r,\omega)(\partial/\partial r)\widehat{f}(r,\omega)\right]}{f(r,\omega)\left[1 - \widehat{f}(r,\omega)f(r,\omega)\right]} \Bigg\} Q_{r,\omega}(i_0,i_1). \end{aligned} \tag{3.26}$$

Using (3.24), it follows that

$$\begin{aligned} \sum_{i_0,i_1} -\frac{\partial}{\partial r} Q_{r,\omega}(i_0,i_1) = &\sum_{i_0,i_1} (i_0 + i_1 - 1) Q_{r,\omega}(i_0,i_1) + \frac{(\partial/\partial r) f(r,\sigma\omega)}{f(r,\sigma\omega)} \\ &+ \left\{ \frac{(\partial/\partial r) f(r,\omega)}{f(r,\omega)\left[1 - \widehat{f}(r,\omega)f(r,\omega)\right]} \right. \\ &\left. - \frac{(\partial/\partial r) f(r,\sigma\omega)}{f(r,\sigma\omega)\left[1 - \widehat{f}(r,\sigma\omega)f(r,\sigma\omega)\right]} \right\}. \end{aligned} \tag{3.27}$$

Since $Q_{r,\omega}$ is a probability measure for all r, the l.h.s. of (3.27) is 0. Substitution into the first integral in (3.23) gives the second expression in the r.h.s. of (3.21). [The contribution of the term between braces in (3.27) cancels by shift invariance.] The third expression is derived analogously after picking $M = -1$ and $N = 0$ in (3.19).

(d) Compare (0.14) and (0.15) with (3.18) and (3.19). Recall (3.6).

PROPOSITION 16. *For every $\theta \in [\theta_c, 1)$ there is a unique solution $r = r(\theta)$ of the equation*

$$E_{\pi^2 Q_r}(a) = \theta^{-1}. \tag{3.28}$$

The function $\theta \to r(\theta)$ is strictly increasing and continuous on $[\theta_c, 1)$, with

$$\lim_{\theta \downarrow \theta_c} r(\theta) = r_c, \qquad \lim_{\theta \uparrow 1} r(\theta) = \infty. \tag{3.29}$$

PROOF. From (3.12) and (3.13) we see that $r \to f(r,\omega)$ and $r \to \widehat{f}(r,\omega)$ are strictly decreasing and continuous for all ω. Hence the first integrand in the r.h.s. of (3.21) has the same property. This proves that $r \to E_{\pi^2 Q_r}(a)$ is strictly decreasing and continuous, which implies the claim in (3.28). To get the first limit in (3.29), see (3.42) and (3.43) below. The second limit uses that $f(r,\omega), \widehat{f}(r,\omega) \to 0$ as $r \to \infty$ for all ω [see Lemma 10(c) in Section 4.1].

3.3. *Identification of minimizer and minimum.* The key to the solution of the minimization problem in Theorem 1 (0.17) is the following property. Remember that $R_\alpha = Q_0$.

PROPOSITION 17. *For every $Q \in M_{\theta,\alpha}$ and $r \geq r_c$,*

$$I(Q \mid R_\alpha) - I(Q \mid Q_r) = -\frac{r}{\theta} - \log\frac{\lambda(r)}{\lambda(0)}, \tag{3.30}$$

with $\lambda(r)$ given by

$$\log\lambda(r) = \int \alpha^{\mathbb{Z}}(d\omega)\log f(r,\omega). \tag{3.31}$$

PROOF. Pick $Q \in M_{\theta,\alpha}$ and $Q_r \in M_{\theta',\alpha}$ for some θ, θ'. Let $Q_\omega, Q_{r,\omega} \in \mathcal{P}(\mathbb{N}^{\mathbb{Z}})$ denote the corresponding laws given ω. We start with the following property. Define

$$I(Q_\omega \mid Q_{r,\omega}) = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{z\in\mathbb{N}^{2N+1}} Q_\omega^{[-N,N]}(z)\log\left(\frac{Q_\omega^{[-N,N]}(z)}{Q_{r,\omega}^{[-N,N]}(z)}\right), \tag{3.32}$$

that is, the specific relative entropy of Q_ω w.r.t. $Q_{r,\omega}$. The limit in (3.32) exists ω-a.s. and in $L^1(\alpha^{\mathbb{Z}})$, and moreover

$$I(Q \mid Q_r) = \int \alpha^{\mathbb{Z}}(d\omega) I(Q_\omega \mid Q_{r,\omega}). \tag{3.33}$$

The proof of this property is straightforward. Indeed, the analogous statement holds for $H(Q)$, the Kolmogorov–Sinai entropy of Q, by applying the subadditive ergodic theorem together with the fact that the θ-restriction in $M_{\theta,\alpha}$ implies that $\pi^1 Q$ has finite entropy [see the proof of Lemma 10 in Greven and den Hollander (1992)]. For $r = 0$ one has that $Q_0 = R_\alpha$ is Markov and then the claim follows almost instantly from (2.77). For $r \neq 0$, on the other hand, Q_r is not Markov but one can appeal to its Gibbs structure [see (3.19)] and get the claim after inserting a Radon-Nikodym factor. The reader can easily work out the details.

We want to compute

$$I(Q \mid R_\alpha) - I(Q \mid Q_r) = \int \alpha^{\mathbb{Z}}(d\omega)\Big[I(Q_\omega \mid R_{\alpha,\omega}) - I(Q_\omega \mid Q_{r,\omega})\Big]. \tag{3.34}$$

Now, by (3.32) and (3.33) we have

$$\begin{aligned} &I(Q_\omega \mid R_{\alpha,\omega}) - I(Q_\omega \mid Q_{r,\omega}) \\ &\quad = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{z \in \mathbb{N}^{2N+1}} Q_\omega^{[-N,N]}(z) \log\left(\frac{Q_{r,\omega}^{[-N,N]}(z)}{R_{\alpha,\omega}^{[-N,N]}(z)}\right). \end{aligned} \tag{3.35}$$

From (3.19)

$$\begin{aligned} \frac{Q_{r,\omega}^{[-N,N]}(z)}{R_{\alpha,\omega}^{[-N,N]}(z)} &= \frac{Z_{0,\omega}^{[-N,N]}}{Z_{r,\omega}^{[-N,N]}} \left(\frac{\hat{f}(r,\sigma^{-N}\omega)}{\hat{f}(0,\sigma^{-N}\omega)}\right)^{z_{-N}-1} \\ &\quad \times \left(\prod_{x \in (-N,N]} e^{-r(z_{x-1}+z_x-1)}\right)\left(\frac{f(r,\sigma^N\omega)}{f(0,\sigma^N\omega)}\right)^{z_N-1}. \end{aligned} \tag{3.36}$$

Substitution into (3.35) gives four terms. Two of these, coming from the second and the fourth factor in (3.36), are boundary terms and vanish in the limit as $N \to \infty$ [use (3.24)]. The remaining two terms can be computed as follows.

By (3.23) and stationarity,

$$\begin{aligned} &\int \alpha^{\mathbb{Z}}(d\omega) \sum_{z \in \mathbb{N}^{2N+1}} Q_\omega^{[-N,N]}(z) \log\left(\prod_{x \in (-N,N]} e^{-r(z_{x-1}+z_x-1)}\right) \\ &\quad = 2N\big(-rE_{\pi^2 Q}(a)\big), \end{aligned} \tag{3.37}$$

which accounts for the first term in the r.h.s. of (3.30) because $E_{\pi^2 Q}(a) = \theta^{-1}$. Finally,

$$\begin{aligned} &\int \alpha^{\mathbb{Z}}(d\omega) \sum_{z \in \mathbb{N}^{2N+1}} Q_\omega^{[-N,N]}(z) \log\left(\frac{Z_{0,\omega}^{[-N,N]}}{Z_{r,\omega}^{[-N,N]}}\right) \\ &\quad = \int \alpha^{\mathbb{Z}}(d\omega)\big[\log Z_{0,\omega}^{[-N,N]} - \log Z_{r,\omega}^{[-N,N]}\big] \end{aligned} \tag{3.38}$$

and by (3.20)

$$\lim_{N\to\infty} \frac{1}{2N+1} \log Z_{r,\omega}^{[-N,N]} = \log \lambda(r) \quad \omega\text{-a.s.,} \tag{3.39}$$

which accounts for the second term in the r.h.s. of (3.30).

PROPOSITION 18. *Fix $\theta \in (0,1)$. If there exists $r(\theta)$ such that $Q_{r(\theta)} \in M_{\theta,\alpha}$, then*

$$I(\theta) = -r(\theta) - \theta \log \frac{\lambda(r(\theta))}{\lambda(0)} \tag{3.40}$$

and $Q_{r(\theta)}$ is a minimizer of (0.17).

PROOF. By substituting (3.30) into (0.17), we get the identity

$$\begin{aligned} I(\theta) &= \theta \inf_{Q\in M_{\theta,\alpha}} I(Q \mid R_\alpha) \\ &= -r - \theta \log \frac{\lambda(r)}{\lambda(0)} + \theta \inf_{Q\in M_{\theta,\alpha}} I(Q \mid Q_r). \end{aligned} \tag{3.41}$$

Since this identity holds for every r, we may pick $r = r(\theta)$ so that $Q_r \in M_{\theta,\alpha}$. In that case the infimum is taken at the point $Q = Q_{r(\theta)}$ because $I(Q_{r(\theta)} \mid Q_{r(\theta)}) = 0$.

Suppose that $Q' \in M_{\theta,\alpha}$ is another minimizer of (0.17). Then by (3.41) we have $I(Q' \mid Q_{r(\theta)}) = 0$. The solution of this equation is the class of Gibbs measures $\mathcal{G}_{r(\theta),\alpha} \cap M_{\theta,\alpha}$ as was mentioned in subsection 0.5.

3.4. *Proof of Theorem* 2. In subsection 4.2 we shall see that $\lambda(0) = 1$. Proposition 18 therefore proves Theorem 2 for all $\theta \in (0,1)$ such that there exists $r = r(\theta)$ with $Q_r \in M_{\theta,\alpha}$. From Proposition 15 and the definition of $M_{\theta,\alpha}$ in (0.12), we see that the latter amounts to the condition

$$\theta^{-1} = E_{\pi^2 Q_r(a)} = \int \alpha^{\mathbb{Z}}(d\omega) \left[-\frac{(\partial/\partial r) f(r,\omega)}{f(r,\omega)} \right], \tag{3.42}$$

where we pick the middle one of the three expressions in (3.21). The r.h.s. of (3.42) equals $-(d/dr)\int \alpha^{\mathbb{Z}}(d\omega) \log f(r,\omega)$, which is $-(d/dr)\log\lambda(r)$ by (0.21). This identifies $r(\theta)$ as the solution of

$$\theta^{-1} = -\frac{\lambda'(r)}{\lambda(r)}. \tag{3.43}$$

In subsection 4.2 we shall see that $r \to -\lambda'(r)/\lambda(r)$ is strictly decreasing. This means that (3.43) has a unique solution when $\theta \geq \theta_c$, with θ_c defined by (0.22).

The case $\theta \in (0, \theta_c)$ is different. We can no longer use the minimizer $Q_{r(\theta)}$ because (3.43) fails to have a solution [see (0.22)]. What in fact happens is that $r(\theta)$ sticks at the value r_c [see (0.23)] because the variational problem in Theorem 1 (0.17) does not achieve a minimum. More precisely, one can show that for every $\theta \in (0, \theta_c)$ there exists a sequence (Q_n) in $M_{\theta,\alpha}$ such that Q_n converges to $Q_{r_c} \in M_{\theta_c,\alpha}$ and $I(Q_n \mid R_\alpha)$ converges to $I(Q_{r_c} \mid R_\alpha)$ as $n \to \infty$ [see Greven and den Hollander (1994), Section 2.4]. Rather than giving an explicit construction of this sequence (which turns out to be rather tedious), we prefer to cut short the proof by appealing to the argument that was given below Corollary 1 in subsection 0.4. Here we argued that the flat piece in Figure 2 (corresponding to case A) can be easily understood in terms of the random walk losing time in long stretches where the local drifts point to the center. All that is needed to make this argument rigorous is a proof that the probability for the random walk to spend a time of order n inside such a stretch of length N equals $O(\exp(-c_N n))$ with $c_N \to 0$ as $N \to \infty$. But this is straightforward [cf. Sinai (1982)].

To get the linear piece in Figure 3 (corresponding to case B), the trick is to do a *transformation of the environment*. Let us define a new process $\widehat{R}_\alpha$ as follows. For $\widehat{R}_{\alpha,\omega}$, the law given ω, we specify the marginals on blocks $[-M,N]$ to be the same as for $Q_{0,\omega} = R_{\alpha,\omega}$ in (3.19) except that the matrix $A_{p_x,0} = A_{p_x}$ [recall (0.14) and (3.6)] is replaced by $A_{\widehat{p}(\sigma^x\omega)}$ (σ is the shift) with

$$\widehat{p}(\omega) = \frac{e^{-r_c}p_0}{f(r_c,\omega)}. \tag{3.44}$$

In other words, the local step probability p_x is replaced by $\widehat{p}(\sigma^x\omega)$ [which depends on the forward environment]. Note that from (0.20) we have [recall that $\rho_0 = (1-p_0)/p_0$]

$$\frac{e^{-r_c}p_0}{f(r_c,\omega)} = 1 - e^{-r_c}(1-p_0)f(r_c,\sigma\omega) \tag{3.45}$$

so that $\widehat{p}_0(\omega) \in (0,1)$ for all ω. Comparing $\widehat{R}_{\alpha,\omega}$ with $R_{\alpha,\omega}$ we find, for $z \in \mathbb{N}^{2N+1}$,

$$\begin{aligned}\frac{\widehat{R}^{[-N,N]}_{\alpha,\omega}(z)}{R^{[-N,N]}_{\alpha,\omega}(z)} &= \left(\frac{1}{f\left(r_c,\sigma^{-N}\omega\right)}\right)^{i_{-N}}\\ &\quad\times\left(\prod_{x\in(-N,N]}\frac{e^{-r_c(z_{x-1}+z_x-1)}}{f(r_c,\sigma^x\omega)}\right)\left(f\left(r_c,\sigma^N\omega\right)\right)^{i_N-1},\end{aligned} \tag{3.46}$$

where we use (3.45) to cancel telescoping factors. Now, (3.46) is the same kind of expression as (3.36). Therefore, doing the same kind of computation as in (3.34)–(3.39) we get the analogue of Proposition 17, namely, for every $Q \in M_{\theta,\alpha}$,

$$I\left(Q \mid R_\alpha\right) - I\left(Q \mid \widehat{R}_\alpha\right) = -\frac{r_c}{\theta} - \log\lambda(r_c). \tag{3.47}$$

Substitution of (3.47) into (0.17) yields [compare with (3.41)]

$$\begin{aligned}I(\theta) &= \theta \inf_{Q\in M_{\theta,\alpha}} I\left(Q \mid R_\alpha\right)\\ &= -r_c - \theta\log\lambda(r_c) + \theta \inf_{Q\in M_{\theta,\alpha}} I\left(Q \mid \widehat{R}_\alpha\right).\end{aligned} \tag{3.48}$$

The first part in the r.h.s. of (3.48) is precisely the linear piece in Figure 3. The second part is the same as (0.17) but with R_α replaced by $\widehat{R}_\alpha$.

Now, $\widehat{R}_\alpha$ corresponds to the random walk with local step probabilities $\widehat{p}(\sigma^x\omega)$. In case B the maximal value of $\rho_0 = (1-p_0)/p_0$ is strictly smaller than 1. However, the maximal value of

$$\widehat{\rho}(\omega) = \frac{1-\widehat{p}(\omega)}{\widehat{p}(\omega)} = \rho_0 f(r_c,\omega) f(r_c,\sigma\omega) \tag{3.49}$$

is precisely 1. Indeed, by picking ω such that $\rho_x = \rho_{\max}$ for $x \in [0, N]$ and substituting the definition $e^{r_c} = 2\rho_{\max}^{1/2}/(1+\rho_{\max})$ into the formula for $f(r_c,\omega)$ [see (0.19) and (0.20)], one easily checks that $f(r_c,\omega)$ can be made arbitrarily close to its maximal value $1/\rho_{\max}^{1/2}$ by picking N large [see also (4.3) and (4.4) in subsection 4.1] and hence $\widehat{\rho}(\omega)$ arbitrarily close to 1. What this says is that the transformed environment just falls under case A. We can therefore repeat the argument of the random walk losing time in long stretches (consisting now of points where $\rho_x = \rho_{\max}$), even though the transformed environment is no longer i.i.d. The result is that $\inf_{Q\in M_{\theta,\alpha}} I(Q \mid \widehat{R}_\alpha) = 0$ for $\theta < \theta_c$.

The boundary cases $\theta = 0, 1$ are treated in subsection 4.3.

4. Proof of Corollaries 1 and 2. In subsections 4.1 and 4.2 we first collect some elementary properties of $f(r,\omega)$, $\lambda(r)$ and $r(\theta)$ defined in (0.20), (0.21) and (0.23). These will be used in subsections 4.3 and 4.4 to prove Corollaries 1 and 2.

4.1. *Elementary properties of $f(r,\omega)$.* The first question to be addressed is the convergence of the continued fractions in (3.12) and (3.13). The answer depends on r and α.

LEMMA 9. *Let r_c be given by* (0.19). *All statements below hold ω-a.s.*

(a) *$f(z,\omega)$ converges if* $\mathrm{Re}\, z \geq r_c$.
(b) *$z \to f(z,\omega)$ is analytic on $\{z\colon \mathrm{Re}\, z > r_c\}$ if* $\langle \log \rho \rangle \leq 0$.
(c) *$f(r,\omega) \in \mathbb{R}^+$ if $r \geq r_c$.*

PROOF. (a) Sufficient criteria for convergence of continued fractions are given in Perron (1913), Chapter 7. In particular, according to Theorems 7.53.24 and 7.53.29 in that volume,

$$\frac{a_0|}{|b_0} + \frac{a_1|}{|b_1} + \cdots, a_i, b_i \in \mathbb{C}, \tag{4.1}$$

(i) converges if $|b_i| \geq |a_i| + 1$ for all i,

(ii) converges uniformly on any domain where $\sum_{i=0}^{\infty} \prod_{j=0}^{i} (|b_j| - 1)$ diverges uniformly [under (i)].

To apply (4.1), rewrite (3.12) as

$$f(z,\omega) = \frac{\rho_0^{-1}|}{|e^z(1+\rho_0^{-1})} - \frac{\rho_1^{-1}|}{|e^z(1+\rho_1^{-1})} - \cdots. \tag{4.2}$$

Then from (4.1) (i) we see that $f(z,\omega)$ converges if $\mathrm{Re}\, z \geq 0$. This settles case A [see (0.19)] for which $r_c = 0$. Case B can be handled by the same criterion after the following transformation of (4.2):

$$cf(z,\omega) = \frac{c^2\rho_0^{-1}|}{|ce^z(1+\rho_0^{-1})} - \frac{c^2\rho_1^{-1}|}{|ce^z(1+\rho_1^{-1})} - \cdots, \qquad c > 0. \tag{4.3}$$

Indeed, from (4.1) (i) we now get the condition

$$|e^z| \geq \frac{\rho_i + c^2}{c(\rho_i + 1)} \quad \text{for all } i. \tag{4.4}$$

The r.h.s. of (4.4) is minimal when $c = \rho_i$ and maximal when $c = \rho_{\max}^{1/2}$ or $\rho_{\min}^{1/2}$. If $\rho_{\max} < 1$, then $(\rho_i + \rho_{\max})/\rho_{\max}^{1/2}(\rho_i + 1)$ is maximal when $\rho_i = \rho_{\max}$. Hence z should satisfy $\mathrm{Re}\, z \geq \log[2\rho_{\max}^{1/2}/(\rho_{\max}+1)]$. The r.h.s. is r_c in case B.

(b) Let $f^{(n)}(z,\omega)$ be the continued fraction in (4.2) or (4.3) truncated after the nth term. Note that $z \to f^{(n)}(z,\omega)$ is analytic on $\{z\colon \mathrm{Re}\, z \geq r_c\}$ for all $n \geq n_0(\omega)$ (because convergence excludes the occurrence of infinitely many zero denominators). It follows from the Weierstrass theorem for normal families of analytic functions [Behnke and Sommer (1955), Theorem 2.7.42] that $z \to f(z,\omega)$ is analytic on $\{z\colon \mathrm{Re}\, z > r_c\}$ if $f^{(n)}(z,\omega)$ converges uniformly. But the latter follows by applying (4.1)(ii) to (4.2) (case A) or to (4.3) (case B). In case A the criterion (4.1)(ii) requires $\langle \log \rho \rangle \leq 0$.

(c) According to Theorem 7.52.22 in Perron (1913), the continued fraction in (4.1) is strictly positive when $a_i, b_i \in \mathbb{R}$, $a_0 > 0$ and $b_i \geq |a_i| + 1$ for all i.

REMARK. Note that $f^{(n)}(r,\omega) = T_{\rho_0,r}\dots T_{\rho_n,r}(0)$ with $T_{\rho,r}$ the map defined in (3.9). By a closer analysis of the fixed points and the poles of $T_{\rho,r}$, it can be shown that $\alpha^{\mathbb{Z}}(\omega\colon \limsup_{n\to\infty} f^{(n)}(r,\omega) < 0) > 0$ when $r < r_c$. This means that Lemma 9(c) fails when $r < r_c$, so that the Ansatz in (3.19) becomes meaningless.

The following lemma lists some relevant properties of $f(r,\omega)$.

LEMMA 10. *All statements below hold ω-a.s.*

(a) *$r \to f(r,\omega)$ is strictly decreasing and continuous on $[r_c,\infty)$.*
(b) *Case* A: $f(0,\omega) = 1$ *if* $\langle \log \rho \rangle \leq 0$,
$f(0,\omega) < 1$ *if* $\langle \log \rho \rangle > 0$.
(c) $\lim_{r\to\infty} f(r,\omega) = 0$.
(d) *$r \to \log f(r,\omega)$ is strictly convex.*

(e) *Case* A: $\lim_{r\downarrow 0} -\frac{\partial}{\partial r}\log f(r,\omega) = 1 + 2\sum_{i=0}^{\infty}\prod_{j=0}^{i}\rho_j$, *which is finite if* $\langle\log\rho\rangle < 0$ *and infinite if* $\langle\log\rho\rangle = 0$.

(f) $\lim_{r\to\infty} -(\partial/\partial r)\log f(r,\omega) = 1$.

(g) *Case* B: $f(r_c,\omega) \le 1/\rho_{\max}^{1/2}$,

$$\lim_{r\downarrow r_c} -\frac{\partial}{\partial r}\log f(r,\omega) \le 1 + 2\sum_{i=0}^{\infty}\prod_{j=0}^{i}\frac{\rho_j}{\rho_{\max}} < \infty.$$

PROOF. (a) Straightforward.

(b) From the definition of $f(r,\omega)$ in (3.12) we have

$$f(r,\omega) = \frac{1}{e^r(1+\rho_0) - \rho_0 f(r,\sigma\omega)}, \tag{4.5}$$

with σ the shift. Pick $r = 0$ and rewrite as

$$[1 - f(0,\omega)]^{-1} = 1 + \rho_0^{-1}[1 - f(0,\sigma\omega)]^{-1} \tag{4.6}$$

to get

$$[1 - f(0,\omega)]^{-1} = \sum_{i=0}^{\infty}\prod_{j=0}^{i}\rho_j^{-1}. \tag{4.7}$$

The r.h.s. diverges iff $\langle\log\rho\rangle \le 0$. (Note that $\log\rho_j - \langle\log\rho\rangle \ge 0$ infinitely often.)

(c) Obvious from (4.5).

(d) From (4.5) we have

$$\frac{\partial}{\partial r}f(r,\omega) = -f^2(r,\omega)\left[e^r(1+\rho_0) - \rho_0\frac{\partial}{\partial r}f(r,\sigma\omega)\right]. \tag{4.8}$$

Use (4.5) once more to write $e^r(1+\rho_0) = 1/f(r,\omega) + \rho_0 f(r,\sigma\omega)$. Substitution into (4.8) gives

$$-\frac{(\partial/\partial r)f(r,\omega)}{f(r,\omega)} = 1 + \rho_0 f(r,\omega)f(r,\sigma\omega)\left[1 - \frac{(\partial/\partial r)f(r,\sigma\omega)}{f(r,\sigma\omega)}\right]. \tag{4.9}$$

Iteration of (4.9) yields

$$-\frac{(\partial/\partial r)f(r,\omega)}{f(r,\omega)} = 1 + 2\sum_{i=0}^{\infty}\prod_{j=0}^{i}\rho_j f(r,\sigma^j\omega)f(r,\sigma^{j+1}\omega). \tag{4.10}$$

Now use (a) to get the claim.

(e) Let $r\downarrow 0$ in (4.10) and use (b).

(f) Let $r \to \infty$ in (4.10) and use (c).

(g) According to Theorem 7.53.24 in Perron (1913), the continued fraction in (4.1) is bounded by 1 in absolute value under (4.1)(i). Now use (4.3) with $c = \rho_{\max}^{1/2}$ to obtain the first claim. The second claim follows after substitution of the first claim into (4.10).

4.2. *Elementary properties of* $\lambda(r)$ *and* $r(\theta)$. From Lemmas 9 and 10 follow some necessary properties for the quantity $\lambda(r)$ defined in (0.21), which plays the key role in the solution of $I(\theta)$ in Theorem 2 (0.24).

LEMMA 11.

(a) $r \to \lambda(r)$ *is strictly decreasing and continuous on* $[r_c, \infty)$, *analytic on* (r_c, ∞).

(b) *Case* A: $\lambda(0) = 1$ *if* $\langle \log \rho \rangle \le 0$,
$\lambda(0) < 1$ *if* $\langle \log \rho \rangle > 0$.

(c) $\lim_{r \to \infty} \lambda(r) = 0$.

(d) $r \to \log \lambda(r)$ *is strictly convex.*

(e) *Case* A: $$\lim_{r \downarrow 0} -\frac{d}{dr} \log \lambda(r) = \begin{cases} \dfrac{1 + \langle \rho \rangle}{1 - \langle \rho \rangle}, & \text{if } \langle \rho \rangle < 1, \\ \infty, & \text{if } \langle \rho \rangle \ge 1,\ \langle \log \rho \rangle \le 0. \end{cases}$$

(f) $$\lim_{r \to \infty} -\frac{d}{dr} \log \lambda(r) = 1.$$

(g) *Case* B: $\lambda(r_c) \le 1/\rho_{\max}^{1/2}$,
$$\lim_{r \downarrow r_c} -\frac{d}{dr} \log \lambda(r) \le \frac{1 + \langle \rho \rangle / \rho_{\max}}{1 - \langle \rho \rangle / \rho_{\max}} < \infty.$$

PROOF. Immediate from Lemmas 9 and 10. Note that in Lemma 10(e) the limit is finite iff $\langle \log \rho \rangle < 0$ but that in Lemma 11(e) we need integrability of the limit, which requires the stronger restriction $\langle \rho \rangle < 1$.

The analyticity of $r \to \lambda(r)$ is proved as follows. By Lemma 9(b) and (c), $z \to \log f(z, \omega)$ is ω-a.s. analytic on $\{z\colon \operatorname{Re} z > r_c\}$. It follows from Fatou's Lemma and Lemma 10(d) that

$$\int \alpha^{\mathbb{Z}}(d\omega) \left[-\frac{\partial}{\partial r} \log f(r, \omega) \right] \le -\left(\frac{d}{dr} \right)^{+} \log \lambda(r), \tag{4.11}$$

where $(d/dr)^{+}$ denotes the right derivative. The r.h.s. of (4.11) exists and is finite for all $r > r_c$, because $\lambda(r) \le \lambda(r_c) < \infty$ and $r \to \log \lambda(r)$ is convex [Lemma 11(a) and (c)], and so the same is true for the l.h.s. of (4.11). Now, from (4.5), with r replaced by z, we have by a straightforward induction argument

$$\operatorname{Re} f(z, \omega) \le f\big(\log \operatorname{Re}(e^{z}), \omega\big). \tag{4.12}$$

Using (4.5) and (4.12), we can deduce

$$|f(z, \omega)| \le f\big(\log \operatorname{Re}(e^{z}), \omega\big). \tag{4.13}$$

It follows from (4.13) and Lemma 10(g) that (4.10), with r replaced by z, is integrable on

$$D = \{z\colon \mathrm{Re}(e^{z}) > e^{r_c}\}. \tag{4.14}$$

This in turn implies that $\log\lambda(z)$ is complex differentiable on D [cf. Behnke and Sommer (1955), Theorem I.11.62], that is, analytic. Since $(r_c, \infty) \subset D$ the claim follows.

Lemma 11(e) identifies $\lim_{r\downarrow 0} -(\partial/\partial r)\log\lambda(r)$ as $1/\theta^*$ with θ^* defined in (0.27). Lemma 11(d) shows that $r \to -\lambda'(r)/\lambda(r)$ is strictly increasing, with limits θ_c^{-1} and 1 as $r \downarrow r_c$ resp. $r \to \infty$. Hence the equation $\theta^{-1} = -\lambda'(r)/\lambda(r)$ has a unique solution $r(\theta)$ for $\theta \geq \theta_c$, as claimed in (0.23).

LEMMA 12. *$\theta \to r(\theta)$ is strictly increasing and analytic on $(\theta_c, 1)$.*

PROOF. Obvious by the implicit function theorem. Use Lemmas 11(a) and (d).

4.3. *Proof of Corollary* 1. To get (a)–(d), combine (0.22)–(0.24) with Lemmas 11 and 12 and the identities

$$I'(\theta) = -\log\lambda\big(r(\theta)\big), \tag{4.15}$$

$$I''(\theta) = \frac{r'(\theta)}{\theta}. \tag{4.16}$$

The latter follow from (0.24) by twice differentiating w.r.t. θ and using (0.23) to cancel terms. The continuity of $\theta \to I(\theta)$ has been explained in the last paragraph of subsection 2.7. The value $I(0) = -r_c$ follows from (0.24) by continuity at $\theta = 0$. The value $I(1) = \langle\log(1+\rho)\rangle$ follows straight from (0.17) because $M_{1,\alpha} = \{[\delta_1 \times \alpha]^{\mathbb{Z}}\}$. To get (e), let $\theta \downarrow \theta_c$ in (4.15) and use the continuity of $\lambda(r)$ at $r = r_c = r(\theta_c)$. To get (f), note that by (a) we only have to check that I is convex in $\theta = 0$ [recall (0.18)]. This is obvious in case A. In case B, on the other hand, the right slope at $\theta = 0$ is $-\log\lambda(r_c)$ and the left slope is $\log\lambda(r_c) + \langle\log\rho\rangle$. Now use Lemma 11(g), which implies that $2\log\lambda(r_c) \leq -\log\rho_{\max} \leq -\langle\log\rho\rangle$.

4.4. *Proof of Corollary* 2.

LEMMA 13. *Let $\eta = \exp\langle\log\rho\rangle$. Then, for all $r > r_c$,*

$$-\frac{\lambda'(r)}{\lambda(r)} \geq \frac{1+\eta\lambda^2(r)}{1-\eta\lambda^2(r)}, \tag{4.17}$$

with equality iff α is a point mass.

PROOF. Use (4.10), Jensen's inequality and (0.21) to compute

$$
\begin{aligned}
-\frac{\lambda'(r)}{\lambda(r)} &= 1+2\sum_{i=0}^{\infty}\int \alpha^{\mathbb{Z}}(d\omega)\prod_{j=0}^{i}\rho_j f(r,\sigma^j\omega)f\left(r,\sigma^{j+1}\omega\right)\\
&\geq 1+2\sum_{i=0}^{\infty}\exp\left\{\sum_{j=0}^{i}\int \alpha^{\mathbb{Z}}(d\omega)\log\left[\rho_j f(r,\sigma^j\omega)f\left(r,\sigma^{j+1}\omega\right)\right]\right\}\\
&= 1+2\sum_{i=0}^{\infty}\exp\left\{(i+1)\left[\langle\log\rho\rangle+2\log\lambda(r)\right]\right\} \qquad (4.18)\\
&= 1+2\sum_{i=0}^{\infty}\left\{\eta\lambda^2(r)\right\}^{i+1}\\
&= \frac{1+\eta\lambda^2(r)}{1-\eta\lambda^2(r)}. \qquad \square
\end{aligned}
$$

Let $\lambda_\eta(r)$, $r_\eta(\theta)$ and $I_\eta(\theta)$ be the quantities corresponding to the homogeneous medium with $\rho_x \equiv \eta$.

LEMMA 14. *Let $\eta = \exp\langle\log\rho\rangle$. Then, for all $\theta > \theta_c$,*

$$\lambda\left(r(\theta)\right) \leq \lambda_\eta\left(r_\eta(\theta)\right), \tag{4.19}$$

$$I'(\theta) \geq I'_\eta(\theta), \tag{4.20}$$

with equality iff α is a point mass.

PROOF. Use (4.17) and (0.23) twice to write

$$\frac{1+\eta\lambda_\eta^2\left(r_\eta(\theta)\right)}{1-\eta\lambda_\eta^2\left(r_\eta(\theta)\right)} = -\frac{\lambda'_\eta\left(r_\eta(\theta)\right)}{\lambda_\eta\left(r_\eta(\theta)\right)} = \theta^{-1} = -\frac{\lambda'\left(r(\theta)\right)}{\lambda\left(r(\theta)\right)} \geq \frac{1+\eta\lambda^2\left(r(\theta)\right)}{1-\eta\lambda^2\left(r(\theta)\right)}. \tag{4.21}$$

This proves (4.19). Substitution of (4.19) into (4.15) proves (4.20).

Corollary 2(a) follows from (4.20). Indeed, in the recurrent case where $\langle\log\rho\rangle = 0$ we have $\eta = 1$ and $\theta_c = \theta^* = 0$, so $I'(\theta) > I'_1(\theta)$ for $\theta > 0$. Since $I(0) = I_1(0) = 0$ it follows that $I(\theta) > I_1(\theta)$ for $\theta > 0$.

Corollary 2(b) is nontrivial only for case B where $\rho_{\max} < 1$ and $r_c < 0$.

LEMMA 15. *For all $r_c \leq r < 0$,*

$$-\frac{\lambda'(r)}{\lambda(r)} \geq \frac{1+\langle\rho\rangle\lambda^2(r)}{1-\langle\rho\rangle\lambda^2(r)}, \tag{4.22}$$

with equality iff α is a point mass.

PROOF. First we observe that $e^r - f(r,\omega) < 0$ for $r < 0$, because $r \to e^r - f(r,\omega)$ is strictly increasing by Lemma 10(a) and crosses the value 0 at $r = 0$ by

Lemma 10(b). Next, by differentiating (4.5) w.r.t. ρ_i, we get

$$\frac{\partial}{\partial \rho_0} f(r,\omega) = f^2(r,\omega)\Big\{-\big(e^r - f(r,\sigma\omega)\big)\Big\}, \tag{4.23}$$

$$\frac{\partial}{\partial \rho_i} f(r,\omega) = f^2(r,\omega)\Big\{\rho_0 \frac{\partial}{\partial \rho_i} f(r,\sigma\omega)\Big\}, \qquad i > 0. \tag{4.24}$$

It follows from (4.23) and (4.24) that $\omega \to f(r,\omega)$ is increasing in ρ_i for all $i \geq 0$ when $r < 0$. Finally, we use (4.10) and (0.21) to compute

$$\begin{aligned} -\frac{\lambda'(r)}{\lambda(r)} &= 1 + 2\sum_{i=0}^{\infty} \int \alpha^{\mathbb{Z}}(d\omega) \prod_{j=0}^{i} \rho_j f(r,\sigma^j\omega) f\big(r,\sigma^{j+1}\omega\big) \\ &\geq 1 + 2\sum_{i=0}^{\infty} \prod_{j=0}^{i} \Big\{\int \alpha^{\mathbb{Z}}(d\omega)\rho_j\Big\}\Big\{\int \alpha^{\mathbb{Z}}(d\omega) f(r,\sigma^j\omega) f\big(r,\sigma^{j+1}\omega\big)\Big\} \qquad (4.25) \\ &= 1 + 2\sum_{i=0}^{\infty} \Big[\langle\rho\rangle\Big\{\int \alpha^{\mathbb{Z}}(d\omega) f(r,\omega) f(r,\sigma\omega)\Big\}\Big]^{i+1}, \end{aligned}$$

where we apply the FKG inequality [see Georgii (1988), pages 445 and 446] for correlations of increasing functions of $(\rho_i)_{i\geq 0}$. Since by Jensen's inequality

$$\int \alpha^{\mathbb{Z}}(d\omega) f(r,\omega) f(r,\sigma\omega) \geq \lambda^2(r), \tag{4.26}$$

the claim follows.

LEMMA 16. *For all* $\theta_c \leq \theta < \theta^*$,

$$\lambda\big(r(\theta)\big) \leq \lambda_{\langle\rho\rangle}\big(r_{\langle\rho\rangle}(\theta)\big), \tag{4.27}$$

$$I'(\theta) \geq I'_{\langle\rho\rangle}(\theta), \tag{4.28}$$

with equality iff α *is a point mass.*

PROOF. Same as that of Lemma 14.

Corollary 2(b) follows from (4.28) because $I(\theta^*) = I_{\langle\rho\rangle}(\theta^*) = 0$ (θ^* being, by (0.5) and (0.27), the typical speed of the random walk and a function of $\langle\rho\rangle$) and $\theta \to I_{\langle\rho\rangle}(\theta)$ is strictly convex.

In subsection 0.5 we raised the question whether the reverse of (4.22) holds for $r > 0$ (note that equality holds at $r = 0$), as this would imply the conjectured upper bound in Figures 5 and 6.

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