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Large deviations for biorthogonal ensembles and variational formulation for the Dykema-Haagerup distribution

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Abstract

This note provides a large deviation principle for a class of biorthogonal ensembles. We extend the results of Eichelsbacher, Sommerauer and Stolz to a more general type of interactions. In particular, our result covers the case of the singular values of lower triangular random matrices with independent entries introduced by Cheliotis and implies a variational formulation for the Dykema–Haagerup distribution.

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1 Introduction and results

The aim of this note is to study the limiting distribution of the n-particle system on $\mathbb{R}_+ := (0, +\infty)$ with joint distribution

$$\frac{1}{Z_n}\prod_{i<j}|x_i - x_j|\prod_{i<j}|g(x_i) - g(x_j)|e^{-n\sum_{i=1}^n V(x_i)}\prod_{i=1}^n x_i^{b-1} d\ell_{\mathbb{R}_+}(x_1)\dots d\ell_{\mathbb{R}_+}(x_n) \tag{1.1}$$

where g is a C^1 function such that $g' > 0$ on $\mathbb{R}_+$, $b \geq 1$, V is a continuous function satisfying (1.3), $\ell_{\mathbb{R}_+}$ is the Lebesgue measure on $\mathbb{R}_+$, and Z_n is a normalizing constant. This is a generalization of the biorthogonal ensembles introduced by Muttalib [16] in physics in the context of disordered systems and by Borodin [6] in mathematics. This model covers the classical random matrix ensembles, biorthogonal Laguerre ensembles or the matrix model of Lueck, Sommers and Zirnbauer [15] for disordered bosons. In equation (1.1), the two first products are interpreted as a repulsion between the particles while the exponential term represents a confining potential preventing the particles from going to infinity. The last product term pushes the particles away from 0, which appears in some model such a Wishart matrices. The main focus of our study will be the empirical measure of the system of points $\{x_1, \dots, x_n\}$ defined by

$$\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}.$$

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In this note, the weak topology is associated to the bounded Lipschitz metric d. It is defined on $\mathcal{M}_1(\mathbb{R}_+)$, the set of probability measures on $\mathbb{R}_+$, by

$$\forall \mu, \nu \in \mathcal{M}_1(\mathbb{R}_+) \quad d(\mu,\nu) = \sup_f \left| \int f\mu - \int f d\nu \right|$$

where the supremum runs over all functions f satisfying $\|f\|_\infty \leq 1$ and which are 1-Lipschitz. This metric makes $\mathcal{M}_1(\mathbb{R}_+)$ a complete space, see [5, Section 8.3].

Definition 1.1 (Logarithmic energy). *The logarithmic energy is the functional*

$$\begin{array}{rccl} \mathcal{E}: & \mathcal{M}_1(\mathbb{R}_+) & \longrightarrow & \mathbb{R}\cup\{+\infty\} \\ & \mu & \longmapsto & \begin{cases} \iint -\log|x-y| d\mu(x)d\mu(y) & \text{if } \int \log(1+|x|)d\mu(x) < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \end{array}$$

We also define the off-diagonal logarithmic energy

$$\begin{array}{rccl} \mathcal{E}_{\neq}: & \mathcal{M}_1(\mathbb{R}_+) & \longrightarrow & \mathbb{R}\cup\{+\infty\} \\ & \mu & \longmapsto & \begin{cases} \iint_{\neq} -\log|x-y| d\mu(x)d\mu(y) & \text{if } \int \log(1+|x|)d\mu(x) < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \end{array}$$

where we integrate over the complement of the diagonal of $\mathbb{R}_+^2$.

The distribution (1.1) can be written in the form:

$$\frac{1}{Z_n}\exp\left[-n^2\left(\frac{1}{2}\mathcal{E}_{\neq}(\mu_n)+\frac{1}{2}\mathcal{E}_{\neq}(g_*\mu_n)+\int V(x)d\mu_n(x)\right)\right]\prod_{j=1}^n x_j^{b-1} d\ell_{\mathbb{R}_+}(x_1)\dots d\ell_{\mathbb{R}_+}(x_n) \tag{1.2}$$

where $g_*\mu$ is the push-forward of the measure μ by the function g.

Theorem 1.2 (Large deviation principle for μ_n). *Let μ_n be the empirical measure of* (1.1)*. Let g be a C^1 function on $\mathbb{R}_+$, such that its derivative is positive. Let V be a continuous function on $\mathbb{R}_+$, bounded from below, such that there exists a constant $\beta > \max(b,1)$ such that*

$$\varliminf_{x\to\infty} \frac{V(x)}{\beta\log|x| + \beta\log|g(x)|} > 1. \tag{1.3}$$

Let $I : \mathcal{M}_1(\mathbb{R}_+) \to \mathbb{R}\cup\{+\infty\}$ as

$$I(\mu) = \begin{cases} \frac{1}{2}\mathcal{E}(\mu) + \frac{1}{2}\mathcal{E}(g_*\mu) + \int V(x)d\mu(x) & \text{if } \int V(x)d\mu_(x) < +\infty \\ +\infty & \text{otherwise} \end{cases}$$

The random sequence $(\mu_n)_{n\in\mathbb{N}}$ satisfies a large deviation principle in $\mathcal{M}_1(\mathbb{R}_+)$, for the weak topology, with speed n^2, and good rate function $\tilde{I} = I - \inf I$. In other words, for any Borel set $A \in \mathcal{M}_1(\mathbb{R}_+)$ we have:

$$\text{Lower Bound:} \quad -\inf_{\mathrm{Int}A} \tilde{I} \leq \varliminf_{n\to\infty} \frac{1}{n^2}\log\mathbb{P}(\mu_n\in A) \tag{1.4}$$

$$\text{Upper Bound:} \quad \varlimsup_{n\to\infty} \frac{1}{n^2}\log\mathbb{P}(\mu_n\in A) \leq -\inf_{\mathrm{Clo}A} \tilde{I} \tag{1.5}$$

Moreover, the rate function $\tilde{I}$ is lower semi-continuous and strictly convex on the set of measures on which it is finite.

This theorem is proved in Section 3.

Remark 1.3 (Assumptions on g and V)**.** The assumptions on V are standard in large deviations for Coulomb gases. They ensure that the distribution (1.1) is well defined and that the particles cannot escape to infinity. The assumptions on g mean that the two interaction terms have the same short range repulsion effect.

Corollary 1.4 (Almost sure convergence towards the minimizer)**.** *Under the assumptions of Theorem 1.2, $\tilde{I}$ has a unique minimizer ν, and almost surely*

$$\lim_{n\to\infty} d(\mu_n, \nu) = 0.$$

This corollary is proved at the end of Section 3.

Large deviations for Coulomb gases and random matrices started with the article of Ben Arous and Guionnet [3] for GUE matrices, Ben Arous and Zeitouni [2] for real Ginibre matrices and Hiai and Petz [14] for Ginibre and Wishart random matrices. In [12], Eichelsbacher, Sommerauer and Stolz proved a large deviation principle for biorthogonal ensembles with $g(x) = x^\theta$, θ being a positive integer. Their proof of the large deviations lower bound cannot be adapted to cover the case considered in this note.

The article [7] from Chafaï, Gozlan and Zitt provides a general framework to establish a large deviation principle for particle systems with two points interaction in any dimension. Surprisingly, their model covers the biorthogonal ensembles we consider. Using the results from [7] requires to verify that the technical assumptions are fulfilled. In the case we consider, it is equally difficult to prove the large deviations directly or to prove that these hypotheses are satisfied, so we decided not to refer to their result.

Theorem 1.2 allows us to study large deviations for models such as the original model of Muttalib from [16] with $g(x) = \mathrm{Argsh}^2(\sqrt{x})$ or the model from [9] with $g(x) = \exp(x)$. Finally, the matrix model introduced by Cheliotis in [8] corresponds to $g(x) = x^\theta$ or $\log x$, $\theta > 0$. We will give more details about the consequences of our theorem on Cheliotis's model in the Application section. The key point of this article is how we deal with the lower bound. Instead of following the proof of the lower bound originally given by Ben Arous and Guionnet in [3], we adapt the proof of Hiai and Petz from [14].

In [4], a similar model to (1.1) is studied where g is holomorphic but the density is integrated with respect to general measure on compact sets $K \subset \mathbb{C}$.

Remark 1.5 (Large deviations for the largest particle)**.** Let $(x_1, \dots, x_n)$ be distributed according to (1.1) and let $x_n^* = \max_{1\le i\le n} x_i$. Suppose that the assumptions of Theorem 1.2 are satisfied. Let ν be the limit measure of $(\mu_n)_{n\in\mathbb{N}}$ and let M be the right endpoint of its support. Assume that

$$\varliminf_{n\to\infty} \frac{1}{n} \log \frac{Z^*_{n-1}}{Z_n} = \inf I$$

where Z^*_{n-1} is the normalizing constant of the gas (1.1) with $n-1$ particles and confining potential $\frac{n}{n-1}V$. It follows from the proof of the equivalent result in [1, Theorem 2.6.6] that $(x_n^*)_{n\in\mathbb{N}}$ satisfies a large deviation principle in $\mathbb{R}_+$ with speed n and good rate function

$$J(x) = \begin{cases} -\frac{1}{2}\displaystyle\int \log|x-y| + \log|g(x) - g(y)| d\nu(y) + V(x) - \inf I & \text{if } x \ge M \\ +\infty & \text{if } x < M. \end{cases}$$

This is an adaptation to our model of the work of Credner and Eichelsbacher [10] who proved the same result for the model from [12].

The rest of the note is organized as follows: in Section 2 we apply Theorem 1.2 to a model of triangular random matrices introduced by Cheliotis in [8] and obtain results about the Dykema–Haagerup distribution. In Section 3, we prove Theorem 1.2 and Corollary 1.4. In Section 4, we suggest some extentions to the results presented in this note.

2 Application to triangular matrices

In this section, we show that Theorem 1.2 can be used to obtain new results for a recent model of random triangular matrices. Cheliotis in [8] considers the setting where T_n is a lower triangular matrix with independent entries with distribution:

$$X_{i,j} \sim \begin{cases} \mathcal{N}_{\mathbb{C}}(0,1) \text{ if } i > j, \\ \frac{1}{\pi\Gamma(c_j)} e^{-|z|^2} |z|^{2(c_j-1)} d\ell_{\mathbb{C}}(z) \text{ if } i = j \end{cases}$$

where $c_j = \theta(j-1) + b$ and $d\ell_{\mathbb{C}}$ is the Lebesgue measure on the complex plane. The distribution of the eigenvalues of $\frac{1}{n} T_n T_n^*$ is given by (1.1), with $g(x) = x^\theta$ or $\log(x)$ if $\theta > 0$ or $\theta = 0$ and $V(x) = x$. Notice that for good choices of θ and b, we can recover many classical ensembles, such as the Laguerre ensembles.

The special case where $\theta = 0$ and $b = 1$ corresponds to the case where all the coefficients $X_{i,j}$ are independent complex Gaussian variables with variance 1. This particular case was studied using free probability theory by Dykema and Haagerup in [11]. They proved that the random sequence $(\mu_n)_{n\in\mathbb{N}}$ converges almost surely in probability towards a deterministic measure μ_{DH}, known as the Dykema-Haagerup distribution. In [8], the same result is proved using the moments method and path counting. The distribution μ_{DH} is supported on $[0, e]$ and is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$ with density

$$f_{DH}(x) = \frac{1}{\pi} \mathrm{Im} \left[-\frac{1}{x W_0(x)} \right] 1_{[0,e]}$$

where W_0 is the Lambert function, see figure 1.

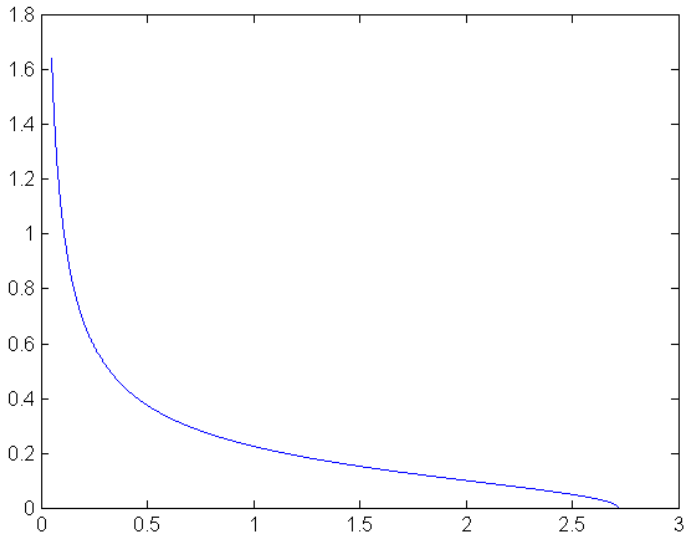

Figure 1: Density of the Dykema-Haagerup distribution.

As a corollary of our Theorem 1.2, we get the following result on the Dykema-Haagerup distribution in the setting of [8]:

Corollary 2.1 (Variational formulation for the Dykema-Haagerup distribution)**.** *The empirical measure of* $\frac{1}{n} T_n T_n^*$ *converges almost surely for the bounded-Lipschitz metric* *towards the Dykema-Haagerup distribution μ_{DH}. In addition, the Dykema–Haagerup distribution is the unique minimizer on $\mathcal{M}_1(\mathbb{R}_+)$ of*

$$\frac{1}{2}\mathcal{E}(\mu) + \frac{1}{2}\mathcal{E}(\log_* \mu) + \int x d\mu_{DH}(x).$$

Cheliotis proved that the spectral radius of $\frac{1}{n}T_nT_n^*$ converges to the right end point of the support of μ_{DH}. Like in Remark 1.5, a large deviation principle holds for the maximal eigenvalue of $\frac{1}{n}T_nT_n^*$ which is a consequence of [1, Theorem 2.6.6].

3 Proof of Theorem 1.2 and Corollary 1.4

The proof of Theorem 1.2 follows the classical pattern for large deviations for log-gases in $\mathbb{R}$, which can be found in [1, p. 71-80] or [7]. We will focus on what differs from the usual proofs, as some steps are just a reformulation of the classical proofs. We start by giving the counterpart of inequality (2.6.13) from [1, p. 73]. This inequality is the key to prove that $\tilde{I}$ is well defined, has compact level sets and that $(\mu_n)_{n\in\mathbb{N}}$ is exponentially tight. Knowing the properties of the logarithmic energy (see [1]), we prove that the rate function is strictly convex.

The last steps of the proof of Theorem 1.2 is to prove the following inequalities for any $\sigma \in \mathcal{M}_1(\mathbb{R}_+)$:

$$\text{Weak Upper Bound:} \quad \lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n^2} \log Z_n\mathbb{P}(\mu_n \in B(\sigma,\delta)) \le -I(\sigma)$$

and

$$\text{Weak Lower Bound:} \quad \lim_{\delta\to 0} \underline{\lim_{n\to\infty}} \frac{1}{n^2} \log Z_n\mathbb{P}(\mu_n \in B(\sigma,\delta)) \ge -I(\sigma).$$

These inequalities combined with the exponential tightness of the sequence of measures $Z_n\mathbb{P}(\mu_n \in .)$ imply that for any Borel set A in $\mathcal{M}_1(\mathbb{R}_+)$:

$$-\inf_{\mathrm{int}A} I \le \underline{\lim_{n\to\infty}} \frac{1}{n^2} \log Z_n\mathbb{P}(\mu_n \in A) \le \overline{\lim_{n\to\infty}} \frac{1}{n^2} \log Z_n\mathbb{P}(\mu_n \in A) \le -\inf_{\mathrm{clo}A} I. \tag{3.1}$$

This is not exactly the large deviation principle we want to prove. To obtain the bounds (1.4) and (1.5), it is sufficient to prove that

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z_n = \inf_{\mathcal{M}_1(\mathbb{R}_+)} I.$$

Fortunately, this is an immediate consequence of (3.1) using $A = \mathcal{M}_1(\mathbb{R}_+)$.

The rest of the proof is organized as follows: first, we briefly explain how to obtain the properties of the rate function. Then we focus on the proof of (3.1). We only deal with the lower bound because the proof for the upper-boun is exactly the classical one. The proof of the weak lower bound relies on the approach of Hiai and Petz [14].

3.1 Study of the rate function

Definition 3.1. *We set, for any x and y in $\mathbb{R}_+$,*

$$f(x,y) = -\frac{1}{2}\log|x-y| - \frac{1}{2}\log|g(x)-g(y)| + \frac{1}{2}\left(V(x)+V(y)\right).$$

As, for any x and y in $\mathbb{R}$,

$$\log|x-y| \le \log(1+|x|) + \log(1+|y|)$$

we obtain

$$f(x,y) \geq \left(-\frac{1}{2}\log(1+|x|) - \frac{1}{2}\log(1+|g(x)|) + \frac{1}{2}V(x)\right) \tag{3.2}$$

$$+ \left(-\frac{1}{2}\log(1+|y|) - \frac{1}{2}\log(1+|g(y)|) + \frac{1}{2}V(y)\right). \tag{3.3}$$

This inequality implies that f is bounded from below. Hence the function I is well defined and takes its values in $\mathbb{R} \cup \{+\infty\}$. This inequality is the key to prove that I is a good rate function. All the details are given in the reference book [1, Lemma 2.6.2 p. 72].

To prove that the rate function I is strictly convex where it is finite, we observe that the logarithmic energy $\mu \mapsto \mathcal{E}(\mu)$ is known to be a strictly convex function where it is finite, see [1, Lemma 2.6.2 p72]. As the function $\mu \mapsto g_*\mu$ is linear, the function $\mu \mapsto \mathcal{E}(g_*\mu)$ is strictly convex where it is finite. The rate function I is the sum of two strictly convex functions and a linear function, hence it is strictly convex where it is finite.

3.2 Proof of the weak lower bound

We want to prove that we have, for any $\sigma \in \mathcal{M}_1(\mathbb{R}_+)$

$$\lim_{\delta\to 0} \varliminf_{n\to\infty} \frac{1}{n^2} \log Z_n \mathbb{P}(\mu_n \in B(\sigma,\delta)) \geq -I(\sigma). \tag{3.4}$$

First step: reduction to “nice” measures

First, we show that it is sufficient to prove the weak lower bound with additional assumptions on σ. Notice that we can assume that $I(\sigma) < +\infty$, as (3.4) is trivial if $I(\sigma) = +\infty$. We introduce the function $\phi : \mathcal{M}_1(\mathbb{R}_+) \to \mathbb{R} \cup \{-\infty\}$ given by:

$$\phi(\sigma) = \inf\left\{ \varliminf_{n\to\infty} \frac{1}{n^2} \log Z_n \mathbb{P}(\mu_n \in G), G \text{ neighborhood of } \sigma \right\}.$$

Using this notation, the weak lower bound becomes

$$\phi(\sigma) \geq -I(\sigma). \tag{3.5}$$

We claim that ϕ is upper semicontinuous. Let $(\sigma_k)_{k\in\mathbb{N}}$ be a sequence of measures such that $\sigma_k \to \sigma$ in $\mathcal{M}_1(\mathbb{R}_+)$. Let G be a neighborhood of σ, then there exists an integer K such that for all $k \geq K$, $\sigma_k \in G$. As G is a neighborhood of σ_k, this implies that for any $k \geq K$

$$\phi(\sigma_k) \leq \varliminf_{n\to\infty} \frac{1}{n^2} \log Z_n \mathbb{P}(\mu_n \in G)$$

Then if we take the limit superior in k of this inequality and the infimum over all neighborhoods G of σ, we obtain the upper semi-continuity of ϕ. If we prove (3.5) for a dense set of measures, then for any measure $\sigma \in \mathcal{M}_1(\mathbb{R}_+)$, there exist measures σ_k such that (3.5) holds and $\sigma_k \to \sigma$ we get

$$\phi(\sigma) \geq \limsup_k \phi(\sigma_k) \geq \limsup_k -I(\sigma_k).$$

We will consider a specific sequence of measures σ_k such that for any k, σ_k is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$, with compact support and density bounded from above and below by positive constants and such that

$$\sigma_k \xrightarrow[k\to\infty]{} \sigma \text{ and } \lim_{k\to\infty} I(\sigma_k) = I(\sigma). \tag{3.6}$$

Once we have obtained this sequence, we will only have to prove the lower bound for the measures satisfying the regularity conditions given above.

We can assume that σ has compact support in $\mathbb{R}_+$ because if we set

$$\sigma_k = \frac{1_{1/k \le x \le k}}{\sigma([1/k, k])}\sigma,$$

then, as f is bounded from below, by the monotone convergence theorem

$$\lim_{k\to\infty} I(\sigma_k) = \lim_{k\to\infty} \iint f(x,y)d\sigma_k(x)d\sigma_k(y) = \iint f(x,y)d\sigma(x)d\sigma(y) = I(\sigma).$$

To show that we can assume that σ has a continuous density with respect to the Lebesgue measure, we find a sequence σ_k converging to σ such that

$$-\mathcal{E}(\sigma_k) \ge -\mathcal{E}(\sigma) \tag{3.7}$$

$$-\mathcal{E}(g_*\sigma_k) \ge -\mathcal{E}(g_*\sigma) \tag{3.8}$$

$$\lim_{k\to\infty} \int V(x)d\sigma_k(x) = \int V(x)d\sigma(x). \tag{3.9}$$

The inequalities above along with the lower semicontinuity of I imply that $I(\sigma_k)$ converges to $I(\sigma)$. Now let ϕ_ε be a $\mathcal{C}^\infty$ probability density with support in $[0,\varepsilon]$, then we set $\sigma_\varepsilon = \phi_\varepsilon * \sigma$. The measures σ_ε have compact support in $\mathbb{R}_+$ with continuous density and converge towards σ as ε goes to zero.

Since it is easy to check that $\int V(x)d\sigma_\varepsilon(x) \xrightarrow[\varepsilon\to 0]{} \int V(x)d\sigma(x)$, we only have to prove that for any ε

$$-\mathcal{E}(\phi_\varepsilon * \sigma) \ge -\mathcal{E}(\sigma) \text{ and } -\mathcal{E}(g_*\phi_\varepsilon * \sigma) \ge -\mathcal{E}(g_*\sigma).$$

Recall that the functions $-\mathcal{E}$ and $-\mathcal{E}(g_*.)$ are concave, so if we notice that

$$\phi_\varepsilon * \sigma = \int \phi_\varepsilon(y)\sigma(\cdot - y)dy$$

then, thanks to Jensen's inequality and the fact that the logarithmic energy is invariant under translation, we obtained the desired inequalities.

The last thing we want for our "nice" measures is that the density is bounded from above and from below. As the densities of the measures σ_ε are continuous with compact support, those densities are already bounded from above. Changing σ_ε to $\delta m + (1-\delta)\sigma_\varepsilon$ where m is the uniform measure on the support of σ_ε allows us to deal with measures with continuous density bounded from above and from below.

Second step: weak lower bound for "nice" measures

From now on, σ will be a measure with compact support $[a, A] \subset \mathbb{R}_+$, with density h with respect to the Lebesgue measure on $\mathbb{R}_+$ for which there exists a constant $C > 0$ such that

$$\forall x \in [a, A] \quad , \quad \frac{1}{C} \le h(x) \le C.$$

Let $a_0, \dots, a_n$ be the $\frac{1}{n}$-quantiles of σ, with $a_0 = a$ and $a_n = A$. For each k, we have

$$\frac{1}{Cn} \le a_{k+1} - a_k \le \frac{C}{n}. \tag{3.10}$$

Now divide each interval $[a_{k-1}, a_k]$ in 3 equal parts and let $[c_k, d_k]$ be the central interval. If we set $\Delta_n = \prod_{i=1}^n [c_i, d_i]$, then for any $(z_1, \dots, z_n) \in \Delta_n$, we have:

$$d\left(\frac{1}{n}\sum_{i=1}^n \delta_{z_i}, \sigma\right) \le \max_k |a_{k+1} - a_k| \le \frac{C}{n}$$

where d is the bounded-Lipschitz distance. We are now ready to prove the lower bound. Let ρ_1 be the finite measure on $\mathbb{R}_+$ $x^{b-1}e^{-V(x)}dx$ and $\rho_n = \rho_1 \otimes \cdots \otimes \rho_1$ the finite n-th product measure on $\mathbb{R}_+^n$. Recalling (1.2) we have

$$\begin{aligned}
&Z_n\mathbb{P}(\mu_n \in B(\sigma,\delta)) \\
&\quad= \int 1_{\mu_n\in B(\sigma,\delta)} \exp\left[-n^2\left(\frac{1}{2}\mathcal{E}_{\neq}(\mu_n) + \frac{1}{2}\mathcal{E}_{\neq}(g_*\mu_n) + \frac{n-1}{n}\int V(x)d\mu_n(x)\right)\right] d\rho_n(x) \\
&\quad\geq \int 1_{\Delta_n} \exp\left[-n^2\left(\frac{1}{2}\mathcal{E}_{\neq}(\mu_n) + \frac{1}{2}\mathcal{E}_{\neq}(g_*\mu_n) + \frac{n-1}{n}\int V(x)d\mu_n(x)\right)\right] d\rho_n(x) \\
&\quad\geq \exp\left(-n^2\left[\frac{n-1}{n^2}\sum_{i=1}^{n}\max_{[c_i,d_i]} V(x)\right]\right) \times \exp\left(-n^2\left[-\frac{1}{n^2}\sum_{i<j}\min_{[c_i,d_i]\times[c_j,d_j]}\log|x-y|\right]\right) \\
&\qquad\times \exp\left(-n^2\left[-\frac{1}{n^2}\sum_{i<j}\min_{[c_i,d_i]\times[c_j,d_j]}\log|g(x)-g(y)|\right]\right)\int 1_{\Delta_n} d\rho_n(x).
\end{aligned}$$

We notice that:

$$\frac{1}{n^2}\log\int 1_{\Delta_n} d\rho_n(x) \xrightarrow[n\to\infty]{} 0.$$

Hence, to obtain the lower bound, it is sufficient to prove that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\max_{[c_i,d_i]} V = \int V(x)d\sigma(x), \tag{3.11}$$

and, using the fact that g and the logarithm are increasing functions,

$$\varliminf_{n\to\infty}\frac{1}{n^2}\sum_{i<j} -\log(d_j-c_i) \geq \frac{1}{2}\iint -\log|x-y|d\sigma(x)d\sigma(y) = \frac{1}{2}\mathcal{E}(\sigma) \tag{3.12}$$

and also

$$\varliminf_{n\to\infty}\frac{1}{n^2}\sum_{i<j}\log(g(d_j)-g(c_i)) \geq \frac{1}{2}\iint \log|g(x)-g(y)|d\sigma(x)d\sigma(y) = \frac{1}{2}\mathcal{E}(g_*\sigma). \tag{3.13}$$

If (3.11), (3.12) and (3.13) hold then the proof of the lower bound for regular measures is completed. The last step will consist in proving that these three inequalities indeed hold.

Last step: proof of the inequalities

First, (3.11) is easy to check as we approximate a continuous integrable function on $[a, A]$ by simple functions.

We now prove (3.12) following the proof of [14]. For the moment, let's assume that there exists a constant $A_1 > 0$ such that for $i < j$

$$A_1(d_j - c_i) \geq (a_j - a_{i-1}) \tag{3.14}$$

and also that

$$\lim_{n\to\infty}\frac{2}{n^2}\#\left\{(i,j) : i<j \mid \frac{(a_j-a_{i-1})}{(d_j-c_i)} \leq 1+\varepsilon\right\} = 1. \tag{3.15}$$

We postpone the proof of the inequalities (3.14) and (3.15) to prove (3.12). We call

$$B_n = \mathcal{E}(\sigma) - \frac{2}{n^2}\sum_{i\neq j}\min_{[c_i,d_i]\times[c_j,d_j]}\log|x-y|$$

and we want to prove that

$$\varlimsup_{n\to\infty} B_n \le 0$$

Since

$$\iint \log|z-w|d\sigma(z)d\sigma(w) \le \frac{2}{n^2}\sum_{i<j}\log|a_j-a_{i-1}| + \frac{1}{n^2}\sum_{i=1}^{n}\log|a_i-a_{i-1}|$$

then for every $\varepsilon>0$ we get

$$\begin{aligned}
B_n \le& \frac{2}{n^2}\sum_{i<j}\log|a_j-a_{i-1}| - \frac{2}{n^2}\sum_{i<j}\log|d_j-c_i| + \frac{1}{n^2}\sum_{i=1}^{n}\log|a_i-a_{i-1}| \\
\le& \frac{2}{n^2}\sum_{i<j}\log\frac{(a_j-a_{i-1})}{(d_j-c_i)} + \frac{1}{n^2}\sum_{i=1}^{n}\log|a_i-a_{i-1}| \\
\le& \frac{2}{n^2}\#\{i<j \mid \frac{(a_j-a_{i-1})}{(d_j-c_i)} \le 1+\varepsilon\}\log(1+\varepsilon) \\
&+ \frac{1}{n^2}\left[n(n-1) - 2\#\{i<j \mid \frac{(a_j-a_{i-1})}{(d_j-c_i)} \le 1+\varepsilon\}\right]\log A_1 + \frac{1}{n^2}\sum_{i=1}^{n}\log|a_i-a_{i-1}|.
\end{aligned}$$

Then we take the limit superior of both sides, and the limit when $\varepsilon\to 0$

$$\mathcal{E}(\sigma) - \varliminf_{n\to\infty}\frac{2}{n^2}\sum_{i<j}\log|d_j-c_i| \le 0$$

which proves (3.12).

We prove now inequality (3.14). From inequality (3.10), we get for any $k>0$

$$\frac{a_{i+k}-a_{i-1}}{d_{i+k}-c_i} \le \frac{(k+1)C/n}{(k+2/3)/Cn} \le \frac{(k+1)C^2}{k+2/3} \le \frac{3C^2}{2}$$

We deduce from this inequality that its left hand side is bounded by a constant independent of k and n, which proves (3.14). In order to prove (3.15), we start from

$$\frac{a_{i+k}-a_{i-1}}{d_{i+k}-c_i} = 1 + \frac{a_{i+k}-d_{i+k}}{d_{i+k}-c_i} + \frac{c_i-a_{i-1}}{d_{i+k}-c_i}.$$

Using (3.10) we get

$$\frac{a_{i+k}-d_{i+k}}{d_{i+k}-c_{i-1}} \le \frac{C/3n}{k/Cn} \le \frac{C^2}{3k}$$

and

$$\frac{c_i-a_{i-1}}{d_{i+k}-c_i} \le \frac{C/3n}{k/Cn} \le \frac{C^2}{3k}.$$

Those two terms can be made as small as desired is k is sufficiently large, independently of n, which proves (3.15).

The proof of the inequality (3.13) mimics the proof of inequality (3.12). Like in the previous case, it is sufficient to find a constant A' such that for any $i<j$

$$A'(g(d_j)-g(c_{i-1})) \ge g(a_j)-g(a_{i-1}) \tag{3.16}$$

and to prove that

$$\lim_{n\to\infty}\frac{2}{n^2}\#\left\{i<j \mid \frac{g(a_j)-g(a_{i-1})}{g(d_j)-g(c_i)} \le 1+\varepsilon\right\} = 1. \tag{3.17}$$

As the support of σ is a compact subset of $\mathbb{R}_+$, there exist two constants m and M such that for all $x \in [a, A]$
$$m \leq g'(x) \leq M.$$

The inequality (3.16) is a consequence of (3.14), using the mean value theorem for g and the fact that its derivative is bounded from above and from below

$$\frac{g(a_{i+k}) - g(a_{i-1})}{g(d_{i+k}) - g(c_i)} \leq \frac{M}{m}\frac{a_{i+k} - a_{i-1}}{d_{i+k} - c_i} \leq \frac{3MC^2}{2m}.$$

To prove (3.17) it is sufficient to prove that the quantities

$$\frac{g(a_{i+k}) - g(d_{i+k})}{g(d_{i+k}) - g(c_i)} \quad \text{and} \quad \frac{g(c_i) - g(a_i)}{g(d_{i+k}) - g(c_i)}$$

can be made as small as desired when k is sufficiently large. Using the mean value theorem we get
$$\frac{g(a_{i+k}) - g(d_{i+k})}{g(d_{i+k}) - g(c_i)} \leq \frac{M}{m}\frac{a_{i+k} - d_{i+k}}{d_{i+k} - c_i} \leq \frac{M}{m}\frac{C^2}{3k}.$$

The other term is treated in the same way. Now that we have proved (3.16) and (3.17), the proof of (3.13) is the exactly the same as the proof of (3.12).

Proof of Corollary 1.4. As the function I is lower semi-continuous and strictly convex, it has a unique minimizer ν. Consider the sets

$$A_\varepsilon = \mathcal{M}_1(\mathbb{R}_+) \setminus B(\nu, \varepsilon).$$

As I is lower semi-continuous, $\inf\{\tilde{I}(\mu), \mu \in A_\varepsilon\} > 0$. We use the large deviations upper bound with the set A_ε to prove that $\sum_{n\in\mathbb{N}} \mathbb{P}(\mu_n \in A_\varepsilon)$ converges. The Borel-Cantelli lemma ensures that, for any fixed $\varepsilon > 0$, for sufficiently large n, μ_n almost surely belongs to $B(\nu, \varepsilon)$. Restricting this result to rational ε implies that, almost surely,

$$d(\mu_n, \nu) \xrightarrow[n\to\infty]{} 0.$$ □

4 Perspectives and comments

Theorem 1.2 can easily be extended in several directions. The first direction would be to consider not only two interactions but any finite number of them, with different exponents:

$$\prod_{i<j} |f_1(x_i) - f_1(x_j)|^{\beta_1} \prod_{i<j} |f_2(x_i) - f_2(x_j)|^{\beta_2} \cdots \prod_{i<j} |f_p(x_i) - f_p(x_j)|^{\beta_p}$$

where each of the f_k is locally a C^1 diffeomorphism and the β_k's are positive numbers. Large deviations will be valid if the confining potential V dominates all the functions $\log f_k$ at infinity. One could prove the same theorem as we stated on $\mathbb{R}$ with the same assumptions. We stated the theorem on $\mathbb{R}_+$ because biorthogonal ensembles were originally defined on the positive axis.

The result of this note can be proved to hold in any dimension, considering the Lebesgue measure on $\mathbb{R}^d$ but it would require stronger assumptions on the function g, and stronger behavior at infinity for V. One could assume that g is continuously differentiable and that on any compact $K \subset \mathbb{R}^d$, there exist two constants m_K and M_K such that for any $x, y \in K$

$$m_K\|x - y\|| \leq \|g(x) - g(y)\| \leq M_K\|x - y\|.$$

The model studied by Götze and Venker in [13] is not covered by this note, as they deal with a double interaction term of the type $\prod_{i<j} |x_i - x_j|^2 \phi(x_i - x_j)$. This is really the combination of two different interactions whereas our model deals with the usual logarithmic interaction at two different scales. As this model is covered by the study [7], one could try to find the optimal conditions of ϕ so that a large deviation principle is valid.

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