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LARGE DEVIATIONS FOR COMBINATORIAL DISTRIBUTIONS. I: CENTRAL LIMIT THEOREMS[1]

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We prove a general central limit theorem for probabilities of large deviations for sequences of random variables satisfying certain analytic conditions. This theorem has wide applications to combinatorial structures and to the distribution of additive arithmetical functions. The method of proof is an extension of Kubilius' version of Cramér's classical method based on analytic moment generating functions. We thus generalize Cramér's and Kubilius's theorems on large deviations.

1. Introduction. Given a sequence of random variables $\{\Omega_n\}_{n\geq 1}$ with means μ_n and variances σ_n^2, if $\sigma_n \to \infty$, as $n \to \infty$, then probabilities of the forms

$$\Pr\{\Omega_n - \mu_n > x_n\sigma_n\} \quad \text{and} \quad \Pr\{\Omega_n - \mu_n < -x_n\sigma_n\}$$

when $x_n \to \infty$ $(n \to \infty)$ are called *probabilities of large deviations* of the random variable Ω_n. Defining the (centered and normalized) random variable $\Omega_n^* = (\Omega_n - \mu_n)/\sigma_n$, we consider its distribution function $F_n(x)$. If $F_n(x)$ has a Gaussian limit, that is,

$$F_n(x) \to \Phi(x), \tag{1}$$

where (here and throughout this paper)

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt,$$

then we have

$$\frac{1-F_n(x)}{1-\Phi(x)} \to 1 \quad \text{and} \quad \frac{F_n(-x)}{\Phi(-x)} \to 1, \qquad n \to \infty, \tag{2}$$

whenever $x = O(1)$. If (1) holds in the interval $0 \leq x \leq X(n)$, where $X(n) \to \infty$, we call, as in [33], Section 8.1, the interval a *zone of normal convergence*.

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Note that, from relation (1), we merely have

$$\lim_{x\to\infty}\lim_{n\to\infty} F_n(x) = 1,$$
$$\lim_{x\to-\infty}\lim_{n\to\infty} F_n(x) = 0.$$

It is well known that the analytic properties of a characteristic function are connected to the rate of decrease of the tails of the distribution function $F(x)$. That is, the functions $1 - F(x)$ and $F(-x)$, as $x \to \infty$. To be more precise, we consider two frequent cases. If $\int_{-\infty}^{\infty} |x|^k \, dF(x) < \infty$, k being a positive integer and F being a distribution function, then an argument similar to the proof of Chebyshev's inequality leads to

$$1 - F(x) = O(x^{-k}) \quad \text{and} \quad F(-x) = O(x^{-k}), \qquad x \to \infty.$$

In other words, if the kth absolute moment of the random variable X exists, where X is distributed according to F, then the tail probabilities of X decay *polynomially* like x^{-k} as x increases or decreases without bound. On the other hand, if

$$\int_{-\infty}^{\infty} e^{sx} \, dF(x) < \infty \quad \text{for some } s > 0, \tag{3}$$

then a similar argument leads to (cf. [27], Chapter 7)

$$1 - F(x) = O(e^{-sx}) \quad \text{and} \quad F(-x) = O(e^{-sx}), \qquad x \to \infty,$$

the condition (3) being not only sufficient but also necessary (cf. [27], Chapter 7). Thus, in this case, the tails of $F(x)$ decay *exponentially* as the parameter $x \to \pm\infty$. These two standard facts, well known in integral transforms, depict the general relationship between the tail probabilities of a random variable and the analytic properties of its integral transforms (or generating functions).

It should be pointed out that these two crude estimates, often useful in applications due to their generality, are not very precise. More results of a similar nature can be found in [33], Section 3.4.

This paper is concerned with Cramér-type large deviations in which the dependence of the rate that $x \to \pm\infty$ on another large parameter n is explicitly specified by asymptotic expressions. Our objective is to derive a general central limit of this sort which applies to both combinatorial and number-theoretic problems. This versatility is due to the fact that we principally work at the level of moment generating functions, the source of which is not specified.

Before going into the details, let us briefly mention some relevant results in the probability literature. Cramér, in his pioneering paper [7], first establishes general theorems for probabilities of large deviations in the case of sums of independent and identically distributed random variables. His powerful analytic method, which is widely used, consists of two major steps: the technique of associated distribution [10] (whose effect is to "shift" the mean) and the implicit use of the saddle-point method; see the next section for more details. His results together with his method have since been generalized in many directions by many authors; for detailed information, applications and references, consult [33], [35], [3], [37] and [25].

1.1. *Main result*. Let $\{\Omega_n\}_n$ be a sequence of random variables with distribution functions $W_n(x)$. A number of such sequences related to combinatorial structures and arithmetical functions have moment generating functions satisfying the same algebraic schemes. This fact allows then a systematic treatment of their limiting properties and we first state the general conditions under which we are developing our arguments [20, 19]. Let us assume that the moment generating functions $M_n(s)$ of Ω_n satisfy, as $n \to \infty$,

$$M_n(s) = \mathbf{E}e^{\Omega_n s} = \int_{-\infty}^{\infty} e^{sy}\, dW_n(y) = e^{\phi(n)u(s)+v(s)}\bigl(1 + O(\kappa_n^{-1})\bigr), \tag{4}$$

uniformly for $|s| \le \rho$, $s \in \mathbb{C}$, $\rho > 0$, where:

1. $\{\phi(n)\}$ is a sequence of positive numbers such that $\phi(n) \to \infty$ as $n \to \infty$;
2. $u(s)$ and $v(s)$ are functions of s independent of n and are analytic for $|s| \le \rho$; furthermore, $u''(0) \ne 0$;
3. $\kappa_n \to \infty$.

Let us introduce the following notation:

$$u_m := u^{(m)}(0), \qquad m = 1, 2, 3, \dots,$$
$$\mu_n := u_1 \phi(n), \qquad \sigma_n^2 := u_2 \phi(n),$$

and

$$F_n(x) := W_n(\mu_n + x\sigma_n) = \Pr\left\{\frac{\Omega_n - \mu_n}{\sigma_n} < x\right\}, \qquad x \in \mathbb{R}.$$

The notation $[z^n]f(z)$ denotes the coefficient of z^n in the Taylor expansion of $f(z)$, $n \in \mathbb{N}$. Then we have the following result.

THEOREM 1 (Central limit theorem for large deviations). *For* $x > 0$, $x = o(\min\{\kappa_n, \sqrt{\phi(n)}\})$, *we have*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp(\phi(n) Q(\xi)) \left(1 + O\left(\frac{x}{\kappa_n} + \frac{x}{\sqrt{\phi(n)}}\right)\right), \qquad \xi := \frac{x}{\sigma_n}, \tag{5}$$

and

$$\frac{F_n(-x)}{\Phi(-x)} = \exp(\phi(n) Q(-\xi)) \left(1 + O\left(\frac{x}{\kappa_n} + \frac{x}{\sqrt{\phi(n)}}\right)\right), \tag{6}$$

where $Q(\xi) = Q(\xi; u)$ depends only on $u(s)$ and is defined by

$$
\begin{aligned}
Q(\xi) &= \sum_{m\ge 3} q_m \xi^m \\
&= \frac{u_3}{6}\xi^3 + \frac{1}{24}\left(u_4 - \frac{u_3^2}{u_2}\right)\xi^4 + \frac{1}{120}\left(u_5 - \frac{10u_3u_4}{u_2} + \frac{15u_3^3}{u_2^2}\right)\xi^5 \\
&\quad + \frac{1}{720}\left(u_6 - \frac{10u_4^2}{u_2} - \frac{15u_3u_5}{u_2} + \frac{105u_3^2u_4}{u_2^2} - \frac{105u_3^4}{u_2^3}\right)\xi^6 + \cdots,
\end{aligned} \tag{7}
$$

the series being convergent for small $|\xi|$ with

$$
q_m = \frac{-1}{m}[w^{m-2}]u''(w)\left(\frac{u'(w)-u_1}{u_2 w}\right)^{-m}, \qquad m = 3,4,5,\dots. \tag{8}
$$

Obviously, Theorem 1 generalizes Cramér's classical results [7] on large deviations for sums of independent, identically distributed random variables; cf. Section 5.

We shall prove Theorem 1 in the next section. Many immediate consequences of this result will be given in Section 3. We then apply formulas (5) and (6) to the combinatorial distributions studied by Flajolet and Soria [15, 16] in Section 4. Finally, we shall briefly discuss some examples from many different applications.

Throughout this paper, all generating functions (ordinary, exponential, probability, characteristic function, moment, etc.) will denote functions *analytic at* 0 with *nonnegative coefficients*. All limits (including O, o and $\sim$), whenever unspecified, will be taken as $n \to \infty$.

2. Proof of Theorem 1. In this section we prove Theorem 1. Conceptually, the proof utilizes Cramér's method [7] which consists of two main steps (cf. [10]): *the technique of associated distributions* and (implicitly) *the saddle-point method*. Actually, we shall follow Kubilius's method [26], Chapter 9, which is more suitable for our purposes. We thus generalize his statements.

Proof of Theorem 1. Let us fix for the moment a small number s, $s \in \mathbb{R}$, $-\rho \le s \le \rho$. Recall that W_n is the distribution function of Ω_n. Define the auxiliary function

$$\tilde{M}_n(w) := M_n^{-1}(s)\int_{-\infty}^{w} e^{ys}\, dW_n(y).$$

Since $-\rho \le s \le \rho$, $\tilde{M}_n(w)$ is well defined for any fixed w and it is easily seen to satisfy the common properties of a distribution function. It follows that

$$W_n(y) = M_n(s)\int_{-\infty}^{y} e^{-sw}\, d\tilde{M}_n(w)$$

and

$$1 - W_n(y) = M_n(s)\int_{y}^{\infty} e^{-sw}\, d\tilde{M}_n(w). \tag{9}$$

We then consider

$$\begin{aligned} W_n(u'(s)\phi(n)) &= M_n(s)\int_{-\infty}^{u'(s)\phi(n)} e^{-sw}\, d\tilde{M}_n(w) \\ &= M_n(s)e^{-su'(s)\phi(n)}\int_{-\infty}^{0} \exp\Big(-sw\sqrt{u''(s)\phi(n)}\,\Big)d\tilde{M}_n \\ &\qquad \times\Big(u'(s)\phi(n) + w\sqrt{u''(s)\phi(n)}\,\Big) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &1 - W_n(u'(s)\phi(n)) \\ &\quad = M_n(s)e^{-su'(s)\phi(n)}\int_{0}^{\infty} \exp\Big(-sw\sqrt{u''(s)\phi(n)}\,\Big)\, d\tilde{M}_n\Big(u'(s)\phi(n) \\ &\qquad\qquad + w\sqrt{u''(s)\phi(n)}\,\Big). \end{aligned} \tag{11}$$

In writing the two formulas above, we need that $u''(s) > 0$. To see this, let $L_n(s) = \log M_n(s)$, where the principal value of the logarithm is taken so that $L_n(0) = 0$. Now $M_n(s) > 0$ for $-\rho \le s \le \rho$. Hence

$$\begin{aligned} L''_n(s) &= \frac{M''_n(s)M_n(s) - M'^2_n(s)}{M_n^2(s)} \\ &= \frac{1}{M_n(s)}\left(M''_n(s) - \frac{2}{M_n(s)}M'^2_n(s) + \left(\frac{M'_n(s)}{M_n(s)}\right)^2 M_n(s)\right) \\ &= \frac{1}{M_n(s)}\int_{-\infty}^{\infty}\left(y - \frac{M'_n(s)}{M_n(s)}\right)^2 e^{ys}\, dW_n(y) > 0 \end{aligned}$$

for $-\rho \le s \le \rho$. As $L_n(s) \sim u(s)\phi(n)$, the assertion follows. In other words, the function $L_n(s)$ is convex.

To arrive at results (5) and (6), we decompose the derivations into three main steps:

1. We prove that

$$\tilde{M}_n\Big(u'(s)\phi(n) + w\sqrt{u''(s)\phi(n)}\,\Big) = \Phi(w) + O\Big(\kappa_n^{-1} + \phi(n)^{-1/2}\Big), \tag{12}$$

   uniformly for any fixed w, by Esseen's inequality (cf. [9], page 32, and [33], page 109).
2. We then derive an asymptotic formula for the leading factor $M_n(s)\exp(-su'(s)\phi(n))$ on the right-hand side of (10) and (11), by choosing $s = z$ as the unique solution of the equation

$$u'(z)\phi(n) = u_1\phi(n) + x\sqrt{u_2\phi(n)}\,, \qquad x \ne 0,\ x = o\Big(\sqrt{\phi(n)}\,\Big). \tag{13}$$

3. We evaluate the integrals on the right-hand side of (10) and (11) with approximations (12) and (13).

Let us start with the proof of (12). The characteristic function of the distribution function $\tilde{M}_n(u'(s)\phi(n) + w\sqrt{u''(s)\phi(n)})$ satisfies

$$\begin{aligned}\lambda_n(t) &= \int_{-\infty}^{\infty} e^{itw}\, d\tilde{M}_n\left(u'(s)\phi(n) + w\sqrt{u''(s)\phi(n)}\right)\\ &= \exp\left(-\frac{u'(s)\phi(n)}{\sqrt{u''(s)\phi(n)}}it\right) M_n^{-1}(s) M_n\left(s + \frac{it}{\sqrt{u''(s)\phi(n)}}\right).\end{aligned}$$

Now, by assumption, $M_n(0) = 1$ and $M_n(s)$ is analytic in a neighborhood of the origin; we can write $\log M_n(s) = u(s)\phi(n) + v(s) + \varepsilon_n(s)$, for s small enough. Here $\varepsilon_n(s)$ satisfies $\varepsilon_n(0) = 0$ and $\varepsilon_n(z) = O(|z|\kappa_n^{-1})$, $z \sim 0$. Thus we have, by (4) and Taylor's expansion,

$$\log \lambda_n(t) = -\frac{t^2}{2} + O\left(\frac{|t|}{\kappa_n} + \frac{|t| + |t|^3}{\sqrt{\phi(n)}}\right), \qquad |t| \le T_n,$$

where $T_n = c_1\sqrt{u''(s)\phi(n)}$, $c_1 > 0$ being a sufficiently small constant. Consequently, there exist two constants $C_1, C_2 > 0$ such that

$$|\lambda_n(t)| \le \exp\left(-\frac{t^2}{2} + \frac{C_1|t|}{\kappa_n} + \frac{C_2(|t| + |t|^3)}{\sqrt{\phi(n)}}\right), \qquad |t| \le T_n.$$

We now apply Esseen's inequality [9, 33] to establish (12). Thus the estimate of the difference of two distribution functions is reduced to the corresponding problem for associated characteristic functions. It is sufficient to show that

$$J_n = \int_{-T_n}^{T_n} \left|\frac{\lambda_n(t) - \exp(-t^2/2)}{t}\right| dt = O\left(\kappa_n^{-1} + \phi(n)^{-1/2}\right).$$

Using the inequality $|e^w - 1| \le |w|e^{|w|}$, we obtain

$$\begin{aligned}\left|\lambda_n(t) - \exp\left(\frac{-t^2}{2}\right)\right| &\le \left(\frac{C_1|t|}{\kappa_n} + \frac{C_2(|t| + |t|^3)}{\sqrt{\phi(n)}}\right)\exp\left(-\frac{t^2}{2} + \frac{C_1|t|}{\kappa_n} + \frac{C_2(|t| + |t|^3)}{\sqrt{\phi(n)}}\right)\\ &= O\left(\left(\frac{|t|}{\kappa_n} + \frac{|t| + |t|^3}{\sqrt{\phi(n)}}\right)\exp\left(-\frac{t^2}{4}\right)\right), \qquad n \to \infty,\end{aligned}$$

for c_1 sufficiently small. Hence

$$\begin{aligned}J_n &= \int_{-T_n}^{T_n} \left|\frac{\lambda_n(t) - \exp(-t^2/2)}{t}\right| dt\\ &= O\left(\kappa_n^{-1}\int_{-T_n}^{T_n} \exp\left(-\frac{t^2}{4}\right) dt + \frac{1}{\sqrt{\phi(n)}}\int_{-T_n}^{T_n} (1 + t^2)\exp\left(\frac{t^2}{4}\right) dt\right)\\ &= O\left(\kappa_n^{-1} + \phi(n)^{-1/2}\right).\end{aligned}$$

Consider now (13), which can be written as

$$\xi = \frac{x}{\sqrt{u_2 \phi(n)}} = \frac{u'(z) - u_1}{u_2}.$$

As $u''(z) > 0$ for z near 0 (z real), the right-hand side increases with z and thus has a unique positive solution when $x > 0$, $x = o\left(\sqrt{\phi(n)}\right)$, and a unique negative solution when $x < 0$, $x = o\left(\sqrt{\phi(n)}\right)$. When $x = 0$, the solution z is obviously 0. By Lagrange's inversion formula ([39], Section 7.32), we obtain

$$\begin{aligned} z &= \xi - \frac{u_3}{2u_2}\xi^2 - \frac{1}{6}\left(\frac{u_4}{u_2} - \frac{3u_3^2}{u_2^2}\right)\xi^3 \\ &\quad - \frac{1}{24}\left(\frac{u_5}{u_2} - \frac{10u_3u_4}{u_2^2} + \frac{15u_3^3}{u_2^3}\right)\xi^4 + \cdots \\ &= \sum_{m \geq 1} \zeta_m \xi^m, \end{aligned} \tag{14}$$

the series being convergent for small $|\xi|$, where

$$\zeta_m = \frac{1}{m}[w^{m-1}]\left(\frac{u'(w) - u_1}{u_2 w}\right)^{-m}. \tag{15}$$

From now on, $s = z$ *will denote this function of* ξ. This choice of s does not violate any result established until now. Note that s and ξ have the same sign.

Now

$$\begin{aligned} \log M_n(s) - su'(s)\phi(n) &= u(s)\phi(n) + v(s) - su'(s)\phi(n) + O\left(\frac{1}{\kappa_n}\right) \\ &= (u(s) - su'(s))\phi(n) + O\left(s + \frac{1}{\kappa_n}\right), \end{aligned}$$

since $v(0) = 0$ and $v(s) = O(s)$. Again, by Lagrange's inversion formula [39], we expand the function $u(s) - su'(s)$ in powers of ξ:

$$u(s) - su'(s) = -\frac{u_2}{2}\xi^2 + \sum_{m \geq 3} q_m \xi^m = -\frac{x^2}{2\phi(n)} + Q(\xi), \tag{16}$$

where q_m is given by

$$q_m = \frac{-1}{m}[w^{m-2}]u''(w)\left(\frac{u'(w) - u_1}{u_2 w}\right)^{-m}, \qquad m = 3, 4, 5, \ldots .$$

It follows that, with this choice of s,

$$M_n(s)\exp(-su'(s)\phi(n)) = \exp\left(-\frac{x^2}{2} + \phi(n)Q(\xi)\right) \times \left(1 + O\left(\frac{x}{\sqrt{\phi(n)}} + \kappa_n^{-1}\right)\right) \tag{17}$$

as $n \to \infty$.

To estimate the asymptotic behavior of the integral

$$I(n) = \int_0^\infty \exp\left(-sw\sqrt{u''(s)\phi(n)}\right) d\tilde{M}_n\left(u'(s)\phi(n) + w\sqrt{u''(s)\phi(n)}\right),$$

we write, by (12),

$$\tilde{M}_n\left(u'(s)\phi(n) + w\sqrt{u''(s)\phi(n)}\right) = \Phi(w) + R_n(w),$$

where $R_n(w) = O(\kappa_n^{-1} + \phi(n)^{-1/2})$ uniformly with respect to w. The integral $I(n)$ is then decomposed into two parts: $I(n) = I_1 + I_2$, where

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-sw\sqrt{u''(s)\phi(n)} - w^2/2\right) dw$$

and

$$I_2 = \int_0^\infty \exp\left(-sw\sqrt{u''(s)\phi(n)}\right) dR_n(w).$$

Since the estimate $R_n(w) = O(\kappa_n^{-1} + \phi(n)^{-1/2})$ is uniformly valid whenever w is $O(1)$, we can choose a constant $K > 0$ such that $R_n(w)$ satisfies the above estimate for $|w| \le K$ and $R_n(w) < 1$, say, for $|w| > K$, as $n \to \infty$. The integral I_2 then equals, by integration by parts,

$$\begin{aligned} I_2 &= -R_n(0) + s\sqrt{u''(s)\phi(n)} \int_0^\infty \exp\left(-sw\sqrt{u''(s)\phi(n)}\right) R_n(w)\, dw \\ &= O\left(\kappa_n^{-1} + \phi(n)^{-1/2}\right) + O\left(\left(1 + \frac{\sqrt{\phi(n)}}{\kappa_n}\right) \int_0^K \exp\left(-sw\sqrt{u''(s)\phi(n)}\right) dw \right. \\ &\qquad \left. + \sqrt{\phi(n)} \int_K^\infty \exp\left(-sw\sqrt{u''(s)\phi(n)}\right) dw\right) \\ &= O\left(\kappa_n^{-1} + \phi(n)^{-1/2}\right), \end{aligned} \tag{18}$$

since $s\sqrt{u''(s)\phi(n)} > 0$. As a matter of fact, we have

$$\begin{aligned} s\sqrt{u''(s)\phi(n)} &= \sqrt{u_2\phi(n)}\left(s + \frac{u_3}{2u_2}s^2 + \cdots\right) \\ &= \sqrt{u_2\phi(n)}\left(\xi + O(\xi^3)\right) = x + O\left(\frac{x^3}{\phi(n)}\right). \end{aligned} \tag{19}$$

Note that the power ξ^2 was canceled. On the other hand,

$$I_1 = \frac{\exp\big((s^2/2)u''(s)\phi(n)\big)}{\sqrt{2\pi}} \int_0^\infty \exp\Big(-\Big(w + s\sqrt{u''(s)\phi(n)}\Big)^2/2\Big)\,dw$$
$$= \exp\big((s^2/2)u''(s)\phi(n)\big)\Big(1 - \Phi\Big(s\sqrt{u''(s)\phi(n)}\Big)\Big).$$

Let

$$\Psi(y) := \frac{\exp(y^2/2)}{\sqrt{2\pi}} \int_y^\infty \exp(-t^2/2)\,dt, \qquad y \in \mathbb{R},$$

so that $I_1 = \Psi(s\sqrt{u''(s)\phi(n)})$. Since $|\Psi'(y)| = |y\Psi(y) - (2\pi)^{-1/2}| \le y^{-2}$ for $y > 0$ and

$$\frac{1}{y} - \frac{1}{y^3} < \Psi(y) < \frac{1}{y}, \qquad y > 0,$$

we obtain, for $x > 0$ and any $\varepsilon = o(x)$, $\varepsilon > 0$,

$$\Psi(x + \varepsilon) = \Psi(x) + \frac{c_2}{x^2}\varepsilon = \Psi(x)\left(1 + O\left(\frac{\varepsilon}{x}\right)\right) \tag{20}$$

for some positive constant $c_2 \le 1$. When $x = o(\sqrt{\phi(n)})$, $x^3\phi(n)^{-1} = o(x)$. Hence, by (19) and (20), we get

$$I_1 = \exp\left(\frac{x^2}{2}\right)(1 - \Phi(x))\left(1 + O\left(\frac{x^2}{\phi(n)}\right)\right). \tag{21}$$

Using (18) and (21), $I(n)$ can be written in the form

$$\begin{aligned} I(n) &= \exp\left(\frac{x^2}{2}\right)(1 - \Phi(x))\left(1 + O\left(\frac{x^2}{\phi(n)}\right)\right) + O\left(\kappa_n^{-1} + \phi(n)^{-1/2}\right) \\ &= \exp\left(\frac{x^2}{2}\right)(1 - \Phi(x))\left(1 + O\left(\frac{x}{\kappa_n} + \frac{x}{\sqrt{\phi(n)}}\right)\right), \end{aligned} \tag{22}$$

since $e^{-x^2/2}(1 - \Phi(x)) = O(x^{-1})$. Formula (5) follows from (11), (17) and (22). Formula (6) can be established in an entirely analogous manner. □

REMARK. From (9), it follows that

$$\Pr\{\Omega_n \ge y\} = M_n(s)\int_y^\infty e^{-sw}\,d\tilde{M}_n(w) \le M_n(s)e^{-ys}$$

for $y > \mathbf{E}\Omega_n$, where s is chosen to satisfy (13) or the saddle point of the right-hand side. This estimate is often referred to as Chernoff's bound [6]; see also [34]. We remark that this technique frequently proves useful in different contexts under different names: for power series, this is known as the saddle-point estimate; for Dirichlet series, this is known as Rankin's technique. All these can be formulated in the form of a Laplace–Stieltjes transform [32].

3. Some corollaries of Theorem 1. From the results established in the previous section, namely, (5) and (6), we can deduce some interesting corollaries as in [7], Theorems 2–4, and [33], pages 228–230. For convenience, we may suppose that $\kappa_n^{-1} = O(\phi(n)^{-1/2})$.

COROLLARY 1. *For $x > 0$ and $x = O(\phi(n)^{1/6})$, we have*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp\left(\frac{u_3 x^3}{6u_2^{3/2}\sqrt{\phi(n)}}\right) + O\left(\frac{x}{\sqrt{\phi(n)}}\right), \qquad n \to \infty, \tag{23}$$

and

$$\frac{F_n(-x)}{\Phi(-x)} = \exp\left(-\frac{u_3 x^3}{6u_2^{3/2}\sqrt{\phi(n)}}\right) + O\left(\frac{x}{\sqrt{\phi(n)}}\right), \qquad n \to \infty. \tag{24}$$

Hence, when $x = c_3\phi(n)^{1/6}$, $c_3 > 0$, the two ratios on the left-hand side of (23) and (24) tend to a finite limit as $n \to \infty$. When $x = o(\phi(n)^{1/6})$, the two limits are both 1 which means that the zone of normal convergence contains at least the range $0 \le x \le X(n)$, where $X(n) = o(\phi(n)^{1/6})$. In the case $u_3 = u_4 = \cdots = u_k = 0$, the zone of normal convergence of Ω_n is even larger, as the following result states.

COROLLARY 2. *If $u_3 = u_4 = \cdots = u_k = 0$, $k = 3, 4, \ldots,$ then, for $x > 0$, $x = o(\phi(n)^{k/(2(k+2))})$, the asymptotic equivalents*

$$1 - F_n(x) \sim 1 - \Phi(x) \quad \textit{and} \quad F_n(-x) \sim \Phi(-x)$$

hold.

Now, using the inequality

$$1 - \Phi(x) = \Phi(-x) < \frac{\exp(-x^2/2)}{x\sqrt{2\pi}}, \qquad x > 0,$$

we can write Corollary 1 as follows.

COROLLARY 3. *For $x > 0$ and $x = O(\phi(n)^{1/6})$, we have*

$$1 - F_n(x) = (1 - \Phi(x))\exp\left(\frac{u_3 x^3}{6u_2^{3/2}\sqrt{\phi(n)}}\right) + O\left(\frac{\exp(-x^2/2)}{\sqrt{\phi(n)}}\right)$$

and

$$F_n(-x) = \Phi(-x)\exp\left(-\frac{u_3 x^3}{6u_2^{3/2}\sqrt{\phi(n)}}\right) + O\left(\frac{\exp(-x^2/2)}{\sqrt{\phi(n)}}\right).$$

More generally, we have the following result.

COROLLARY 4. *For* $x > 0$ *and* $x = O(\phi(n)^{k/(2(k+2))})$, $k = 3, 4, \dots,$

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp(\phi(n) Q^{[k]}(\xi)) \left(1 + O\left(\frac{x}{\sqrt{\phi(n)}}\right)\right)$$

and

$$\frac{F_n(-x)}{\Phi(-x)} = \exp(\phi(n) Q^{[k]}(-\xi)) \left(1 + O\left(\frac{x}{\sqrt{\phi(n)}}\right)\right),$$

where $Q^{[k]}(\xi) = \sum_{m=3}^{k} q_m \xi^m$, $k = 3, 4, 5, \dots$.

It is worthwhile noting that the preceding corollaries show that the asymptotic equivalence of $1 - F_n(x)$ and $1 - \Phi(x)$ holds only when $x = o(\phi(n)^{k/(2(k+2))})$, where k is the smallest integer such that $u_{k+1} \neq 0$, since, for x outside this range and an $o\left(\sqrt{\phi(n)}\right)$ error term, the right-hand sides of (5) and (6) can tend to any limit between 1 and ∞.

With some simple computations, we obtain the following result.

COROLLARY 5. *If* $x \to \infty$ *and* $x = o\left(\sqrt{\phi(n)}\right)$, *then*

$$\lim_{n\to\infty} \frac{F_n(x + c/x) - F_n(x)}{1 - F_n(x)} = \lim_{n\to\infty} \frac{\Phi(x + c/x) - \Phi(x)}{1 - \Phi(x)} = 1 - e^{-c}$$

for $c > 0$.

Define the real function Λ of y, $y > -1$, by

$$\Lambda(y) := (1 + y)\log(1 + y) - y - \frac{y^2}{2} = \sum_{k\geq 3} \frac{(-1)^k}{k(k-1)} y^k,$$

the latter equality being true only for $-1 < y \leq 1$. The following theorem is very useful for many applications.

THEOREM 2. *Let* $u(s) = e^s - 1$ *in Theorem* 1 *and* $\kappa_n^{-1} = O(\phi(n)^{-1/2})$. *Then, for* $x > 0$, $x = o(\sqrt{\phi(n)})$, *we have*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp\left(-\phi(n)\Lambda\left(\frac{x}{\sqrt{\phi(n)}}\right)\right)\left(1 + O\left(\frac{x}{\sqrt{\phi(n)}}\right)\right)$$

and

$$\frac{F_n(-x)}{\Phi(-x)} = \exp\left(-\phi(n)\Lambda\left(-\frac{x}{\sqrt{\phi(n)}}\right)\right)\left(1 + O\left(\frac{x}{\sqrt{\phi(n)}}\right)\right).$$

PROOF. From Theorem 1, it suffices to evaluate the coefficients q_k of $\Lambda(\xi) = Q(\xi)$:

$$\begin{aligned} q_k &:= \frac{-1}{k}[w^{k-2}]e^w\left(\frac{e^w - 1}{w}\right)^{-k} \\ &= \frac{-1}{2k\pi i}\int_{|w|=\delta} e^w w(e^w - 1)^{-k}\,dw, \qquad 0 < \delta < 2\pi, \\ &= \frac{-1}{2k\pi i}\int_{|z|=\delta'} \frac{\log(1+z)}{z^k}\,dz, \qquad 0 < \delta' < 1,\ z = e^w - 1, \\ &= \frac{(-1)^{k-1}}{k(k-1)}, \qquad k = 3, 4, \dots . \end{aligned}$$

This completes the proof. □

REMARK. Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent random variables with a common Poisson(τ) distribution, $\tau > 0$. Then the moment generating function of the sum $X_1 + X_2 + \cdots + X_n$ is $\mathbf{E}\exp(s(X_1 + X_2 + \cdots + X_n)) = \exp(\tau(e^s - 1)n)$. Theorem 2 applies.

4. Analytic combinatorial schemes of Flajolet and Soria. It is well recognized, since the original work of Bender [1], that the statistical properties (moments, limit distribution, local behavior, etc.) of parameters in a large class of combinatorial structures are well reflected by the (dominant) singularity type of the associated generating functions; see [4], [5], [15], [16] and [17]. The classification according to the latter leads to the study of (*analytic*) *combinatorial schemes* on which important progress has recently been made by Flajolet and Soria [15, 16]. In their papers only results concerning convergence in distribution to the normal laws were discussed. We have refined their results in [20] and [19] by making explicit the convergence rates and the asymptotic expansions in both central and local limit theorems.

In this section we shall apply Theorems 1 and 2 to establish central limit theorems for large deviations for their schemes.

4.1. *Exp–log scheme*. Recall that a generating function $C(z)$ with only nonnegative coefficients analytic at 0 is called *logarithmic* [15] if there exists a constant $a > 0$, such that, for $z \sim \zeta$, $\zeta > 0$ being the radius of convergence of C,

$$C(z) = a\log\frac{1}{1 - z/\zeta} + H(z), \qquad z \in \Delta,$$

where the function $H(z)$ is analytic inside a domain of Δ:

$$\Delta := \{z : |z| \leq \zeta + \varepsilon \text{ and } |\arg(z - \zeta)| \geq \delta\}, \qquad \varepsilon > 0, 0 < \delta < \frac{\pi}{2},$$

and satisfies there

$$H(z) = K + O\left(\log^{-1/2} \frac{1}{1 - z/\zeta}\right)$$

as $z \to \zeta$, uniformly in z, where K is some constant. For brevity, we shall say that $C(z)$ *is logarithmic with parameters* (ζ, a, K).

Now consider generating functions of the form

$$P(w, z) = \sum_{n, m \geq 0} p_{nm} w^m z^n = e^{wC(z)} Q(w, z),$$

where $C(z)$ is logarithmic with parameters (ζ, a, K) and $Q(w, z)$ satisfies the following two conditions:

1. As a function of z, $Q(w, z)$ is analytic for $|z| \leq \zeta$; namely, it has a larger radius of convergence than C.
2. As a function of w, $Q(w, \zeta)$ is analytic for $|w| \leq \eta$, where $\eta > 1$.

Roughly, these assumptions imply that, for any fixed w, $|w| \leq \eta$, $P(w, z)$ satisfies

$$P(w, z) = e^{Kw} Q(w, \zeta) \left(1 - \frac{z}{\zeta}\right)^{-aw} \left(1 + O\left(\log^{-1/2} \frac{1}{1 - z/\zeta}\right)\right), \qquad z \sim \zeta,\ z \notin [\zeta, \infty),$$

and $P(w, z)$ is analytically continuable to a Δ-region. We can then apply the singularity analysis of Flajolet and Odlyzko [12] to deduce the asymptotic formula

$$\begin{aligned} P_n(w) &:= [z^n] P(w, z) \\ &= \frac{\zeta^{-n} n^{aw-1}}{\Gamma(aw)} e^{Kw} Q(w, \zeta) \left(1 + O\left((\log n)^{-1/2}\right)\right), \end{aligned} \tag{25}$$

the O-term being uniform with respect to w, $|w| \leq \eta$. For details, see [16].

Since $\eta > 1$, $P_n(1)$ is well defined. Thus, for the moment generating functions $M_n(s)$ of the random variables Ω_n defined by $M_n(s) := \mathbf{E} e^{\Omega_n s} = P_n(e^s)/P_n(1)$, we have

$$\begin{aligned} M_n(s) &= \exp((e^s - 1) a \log n) \frac{\exp(K(e^s - 1)) \Gamma(a) Q(e^s, \zeta)}{\Gamma(a e^s) Q(1, \zeta)} \\ &\quad \times \left(1 + O\left((\log n)^{-1/2}\right)\right), \end{aligned} \tag{26}$$

uniformly for $-\log \eta \leq \mathscr{R} s \leq \log \eta$ and $|\mathscr{I} s| \leq \pi$. Note that $\log \eta > 0$. The application of Theorem 2 is straightforward.

THEOREM 3. *Let* $F_n(x) = \Pr\{\Omega_n < a\log n + x\sqrt{a\log n}\}$, *where the random variables* Ω_n *are defined by* (26). *Then, for* $x > 0$, $x = o(\sqrt{\log n})$, *we have*

$$(27)\qquad \frac{1-F_n(x)}{1-\Phi(x)} = \exp\left(-a\log n\cdot\Lambda\left(\frac{x}{\sqrt{a\log n}}\right)\right)\left(1+O\left(\frac{x}{\sqrt{\log n}}\right)\right)$$

and

$$(28)\qquad \frac{F_n(-x)}{\Phi(-x)} = \exp\left(-a\log n\cdot\Lambda\left(\frac{x}{\sqrt{a\log n}}\right)\right)\left(1+O\left(\frac{x}{\sqrt{\log n}}\right)\right).$$

PROOF. Take $\phi(n) = a\log n$ in Theorem 2. □

For a given sequence $\{c_n\}_{n\geq 1}$, let us denote its ordinary generating function by $C(z) := \sum_{n\geq 1} c_n z^n$ and its exponential generating function by $\hat{C}(z) := \sum_{n\geq 1} c_n z^n/n!$. Then it is easily seen that Theorem 3 applies to the following four classes of generating functions issuing from standard combinatorial schemes when $C(z)$ or $\hat{C}(z)$ is logarithmic with parameter (ζ, a, K); cf. [15].

1. *Partitional complex construction*: $P(w,z) = \exp(w\hat{C}(z))$.
2. *Partitional complex construction in which no two components are order-isomorphic*:

$$P(w,z) = \prod_{k\geq 1}\left(1+\frac{wz^k}{k!}\right)^{c_k}.$$

3. *Multiset construction*: (i) total number of components,

$$P(w,z) = \exp\left(\sum_{k\geq 1}\frac{w^k}{k}C(z^k)\right),\qquad \zeta < 1;$$

(ii) number of distinct components,

$$P(w,z) = \prod_{k\geq 1}\left(1+\frac{wz^k}{1-z^k}\right)^{c_k},\qquad \zeta < 1.$$

4. *Set construction*:

$$P(w,z) = \exp\left(\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}w^k C(z^k)\right),\qquad \zeta < 1.$$

In the last two cases, the radius of convergence ζ of C is supposed to satisfy $\zeta < 1$.

A result parallel to Theorem 3 can be derived by replacing $a\log n$ with $a\log\log n$. This will have important applications to some additive arithmetical functions; cf. [26].

4.2. *Algebraico–logarithmic scheme.* Next, let us consider generating functions of the form

$$P(w,z) = \sum_{n,m\geq 0} p_{nm} w^m z^n = \frac{1}{(1-wG(z))^{\alpha}}\left(\log\frac{1}{1-wG(z)}\right)^{\beta}, \qquad G(0)=0,$$

where $\beta \in \mathbb{N}$, $\alpha \geq 0$ and $\alpha+\beta>0$. Define a random variable Ω_n by its moment generating function

$$M_n(s) = \sum_{m\geq 0} \Pr\{\Omega_n = m\} e^{ms} = \frac{[z^n]P(e^s,z)}{[z^n]P(1,z)}. \tag{29}$$

Definition (1-regular function [16]). A generating function $G(z) \not\equiv z^q$, $q=0,1,2,\ldots,$ analytic at $z=0$ is called 1-*regular* if (i) its Taylor expansion at $z=0$ involves only nonnegative coefficients, and (ii) there exists a positive number $\rho' < \rho$, ρ being the radius of convergence of $G(z)$, such that $G(\rho')=1$.

Assume that $G(z)$ is 1-regular and, without loss of generality, that $G(z)$ is aperiodic, that is, $G(z) \not\equiv z^e \sum_{n\geq 0} c_n z^{nd}$ for some integers e and $d \geq 2$.

Let $\rho(w)$ be the smallest positive real solution of the equation $1 = wG(z)$ for $w \sim 1$ and $\rho = \rho(1)$. Then we showed in [20], by singularity analysis, that the formula

$$M_n(s) = \left(\frac{e^s\rho(e^s)G'(\rho(e^s))}{\rho G'(\rho)}\right)^{-\alpha} \frac{\rho(e^s)^{-n}}{\rho^{-n}}\left(1+\varepsilon_{\alpha,\beta}(n)\right) \tag{30}$$

holds, where

$$\varepsilon_{\alpha,\beta}(n) = \begin{cases} O(n^{-1}), & \text{if } (\alpha>0 \text{ and } \beta=0) \text{ or } (\alpha=0 \text{ and } \beta=1),\\ O\left((\log n)^{-1}\right), & \text{if } (\alpha=0 \text{ and } \beta\geq 2) \text{ or } (\alpha>0 \text{ and } \beta>0),\end{cases}$$

uniformly for small $|s|$. Define two constants

$$\alpha_1 = \frac{1}{\rho G'(\rho)} \quad \text{and} \quad \alpha_2 = \frac{1}{\rho^2 G'(\rho)^2} + \frac{G''(\rho)}{\rho G'(\rho)^3} - \frac{1}{\rho G'(\rho)}.$$

We state without proof the following simple lemma.

Lemma 1. *For any function $a(z)$ satisfying $a(0)=0$, the coefficient*

$$[z^n]\left(\frac{1}{1-wa(z)}\right)^{\alpha}\left(\log\frac{1}{1-wa(z)}\right)^{k}, \qquad \alpha\in\mathbb{R},\ k=0,1,2,\ldots,$$

formally defines a polynomial in w of degree at most n, for $n \geq lk$, where l is the least integer satisfying $a^{(l)}(0) \neq 0$.

From this lemma, we infer, by partial summation, that $\alpha_1 = \rho^{-1}G'(\rho)^{-1} < 1$ for any generating function $G(z)$ satisfying $G(0) = 0$. Obviously, $G'(\rho) \leq 0$.

From (30) and Theorem 1, we readily obtain the following result.

THEOREM 4. *Let* $F_n(x) = \Pr\{\Omega_n < \alpha_1 n + x\sqrt{\alpha_2 n}\}$. *Then, for* $x > 0$, $x\varepsilon_{\alpha,\beta}(n) = o(1)$, *we have*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = e^{nQ(\xi)}\left(1 + O\left(x\varepsilon_{\alpha,\beta}(n)\right)\right)$$

and

$$\frac{F_n(-x)}{\Phi(-x)} = e^{nQ(-\xi)}\left(1 + O\left(x\varepsilon_{\alpha,\beta}(n)\right)\right),$$

where $\xi = x(\alpha_2 n)^{-1/2}$ *and* $Q(\xi) = \sum_{m \geq 3} q_m \xi^m$ *whose coefficients* q_m *are given by*

$$q_m = \frac{-1}{m}[s^{m-2}]\frac{\rho'^2(e^s)e^{2s} - \rho(e^s)\varphi'(e^s)e^s - \rho(e^s)\rho''(e^s)e^{2s}}{\rho^2(e^s)}$$
$$\times\left(\frac{-\rho^2\rho'(e^s)\rho^{-1}(e^s)e^s + \rho\rho'(1)}{s\left(\rho'^2(1) - \rho\rho'(1) - \rho\rho''(1)\right)}\right)^{-m}$$

for $m = 3, 4, \ldots$.

REMARK. The expression above for q_m is written in terms of $\rho(e^s)$ and its derivatives; we can also express it in terms of $\rho(1)$ and the derivatives of $G(z)$. The following recursive formula is useful for this purpose (cf. [33], page 135):

$$\rho^{(m)}(y) = \frac{(-1)^m m!}{G'(\rho(y))y^{m+1}}$$
$$- \frac{m!}{G'(\rho(y))}\sum G^{(l)}(\rho(y))\prod_{1 \leq j \leq m-1}\frac{1}{k_j!}\left(\frac{1}{j!}\rho^{(j)}(y)\right)^{k_j}$$

for $m = 1, 2, \ldots,$ where the sum extends over all nonnegative integers $(k_1, k_2, \ldots, k_{m-1})$ such that

$$k_1 + 2k_2 + \cdots + (m-1)k_{m-1} = m, \qquad k_1 + k_2 + \cdots + k_{m-1} = l.$$

For example,

$$q_3 = \frac{3\rho\rho_1^2 + 3\rho\rho_1\rho_2 - 3\rho^2\rho_2 - \rho^2\rho_1 - \rho^2\rho_3 - 2\rho_1^3}{6\rho^3}$$
$$= \frac{-3\rho G_1^3 + 3\rho G_1 G_2 + \rho^2 G_1^4 - 3\rho^2 G_1^2 G_2 - \rho^2 G_3 G_1 + 3\rho^2 G_2^2 + 2G_1^2}{6\rho^3 G_1^5},$$

where we put $\rho = \rho(1)$ and $\rho_j = \rho^{(j)}(1)$ and $G_j = G^{(j)}(\rho)$ for $j = 1, 2, 3$.

5. Examples. Let us indicate some typical examples.

EXAMPLE 1 (Sums of independent random variables). Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent random variables with a common discrete distribution. Suppose that the moment generating function $M(s)$ of X_n is analytic at 0. Then there exists an $\varepsilon > 0$ such that $\log M(s) \neq 0$ for $|s| \leq \varepsilon$, since $M(0) = 1$. In this case, Theorem 1 reduces to Cramér's theorem [7], since $M_n(s) := \mathbf{E} \exp(s(X_1 + X_2 + \cdots + X_n)) = M^n(s)$. For other generalizations, cf. [33], Chapter 8, and [35]. An interesting special case is when $M(s) = (1-p) + pe^s$, $0 < p < 1$, namely, X_1 is a Bernoulli random variable. Then we have in this case $u(s) = \log(1 - p + pe^s)$ and

$$\begin{aligned} q_m &:= -\frac{p(1-p)}{m}[s^{m-2}]\frac{e^s s^m (1-p+pe^s)^{m-2}}{(e^s-1)^m} \\ &= -\frac{p(1-p)}{m} \cdot \frac{1}{2\pi i}\oint_{|s|=r} \frac{e^s s(1-p+pe^s)^{m-2}}{(e^s-1)^m}\,ds, \qquad 0 < r < 2\pi, \end{aligned}$$

which, by the change variable $w = (e^s - 1)/(1 - p + pe^s)$, becomes

$$\begin{aligned} q_m &= -\frac{p(1-p)}{m} \cdot \frac{1}{2\pi i}\oint_{|w|=r'} w^{-m} \log\frac{1+w(1-p)}{1-wp}\,dw, \\ &\qquad\qquad 0 < r' < \min\left\{p^{-1}, (1-p)^{-1}\right\}, \\ &= \frac{p(1-p)}{m(m-1)}\left(p^{m-1} - (p-1)^{m-1}\right), \qquad m = 3, 4, 5, \dots; \end{aligned}$$

cf. [22].

EXAMPLE 2 (Exp–log scheme). The number of cycles in permutations in which no two cycles are order-isomorphic has the bivariate generating function [15]:

$$P(z,w) = \prod_{n \geq 1}\left(1 + \frac{wz^n}{n!}\right)^{(n-1)!} = \exp\left(w \log\frac{1}{1-z} - S(z,w)\right), \qquad |z| < 1,$$

where w marks the number of cycles and

$$S(z,w) = \sum_{k \geq 2} \frac{(-w)^k}{k} \sum_{n \geq 1} \frac{z^{kn}}{n(n!)^{k-1}}$$

is an entire function of z for any finite w. Theorem 3 applies with $a = 1$.

We list some examples belonging to the same class, the description of which can be found in the cited reference. Many other examples to which Theorem 3 applies can be found in [36], subsection 6.2.6.3, [24] and [18], Chapter 5.

1. The number of connected components in a random mapping, $a = \frac{1}{2}$ [15].
2. The number of monic irreducible factors in a random monic polynomial of $F_q[z]$, a finite field with q elements, $a = 1$ [15].

3. The number of connected components in a random mapping pattern, $a = \frac{1}{2}$ [15].
4. The number of successful (or unsuccessful) searches in a random binary search tree, $a = 2$ [29], Sections 2.4 and 2.5.
5. The depth of a random node in a random increasing tree in a polynomial variety of degree d, $d \geq 2$, $a = d/(d-1)$ [2].
6. "Prime-divisor" functions in (additive) arithmetical semigroups under Axiom $A^{\#}$ [24].

EXAMPLE 3 (Algebraico–logarithmic scheme). Let $G(z)$ be the generating function of a subset of positive integers. If $G(z)$ is aperiodic and not of the form cz^q, $q = 1, 2, 3, \ldots,$ then we can apply Theorem 4 to the bivariate generating functions for integer compositions and cyclic compositions:

$$\frac{1}{1 - wG(z)} \quad \text{and} \quad \sum_{k \geq 1} \frac{\varphi(k)}{k} \log \frac{1}{1 - w^k G(z^k)},$$

since all such $G(z)$ are 1-regular, where $\varphi(n)$ denotes Euler's totient function, namely, the number of integers greater than or equal to 1, less than or equal to n, and relatively prime to n. For example, if $G(z) = z/(1-z)$, we obtain $u(s) = -\log(2/(e^s + 1))$ and so

$$Q(\xi) = \sum_{k \geq 3} q_k \xi^k = -\sum_{k \geq 2} \frac{1}{2k(2k-1)} \left(\frac{x}{\sqrt{n}} \right)^{2k}, \qquad \xi := \frac{2x}{\sqrt{n}}.$$

Hence, by (8), we have

$$\begin{aligned} q_k &= \frac{-1}{k} [w^{k-2}] \frac{e^w}{(e^w+1)^2} \frac{w^k (e^w+1)^k}{w^{k-1} 2^k (e^w - 1)^k} \\ &= \frac{-1}{k 2^{k+2}} [z^{k-1}] \log\left(\frac{1+z}{1-z} \right) \\ &= \begin{cases} 0, & k = 3, 5, 7, \ldots, \\ -\left(k(k-1)2^{k+1}\right)^{-1}, & k = 4, 6, 8, \ldots. \end{cases} \end{aligned}$$

This example is interesting since the zone of normal convergence contains the interval $0 \leq x \leq X(n)$, where $X(n) = o(n^{1/3})$.

Other examples include:

1. The number of blocks in a random ordered set partition and cyclic set partition [16, 17]:

$$\frac{1}{1 - w(e^z - 1)} \quad \text{and} \quad \log \frac{1}{1 - w(e^z - 1)}.$$

2. The number of connected components (or cycles) in a random alignment [14], Section 2.3, and its cyclic counterpart:

$$\frac{1}{1 + w\log(1-z)} \quad \text{and} \quad \log\frac{1}{1 + w\log(1-z)}.$$

3. Other "ordered" and "cyclic" structures of partitional complex constructions in the exp–log class, like random mappings, 2-regular graphs, children's yards, etc. [15].
4. Nodes of given out-degree in a random increasing tree in the polynomial variety [2].
5. Nodes of given out-degree in a random tree in the simply generated family of trees [30, 31].
6. "Factorisatio numerorum" in (additive) arithmetical semigroups under Axiom A$^{\#}$ [23].
7. Branching compositions of integers introduced in [18], Chapter 8.

EXAMPLE 4 (Arithmetical functions). Let $f_{n,k}$ denote the number of factorizations of n into k integer factors greater than 1, $n \ge 2$, $k \ge 1$, where the multiplicity of each factor is counted. Define $f_{1,k} = \delta_{0,k}$, where $\delta_{a,b}$ is Kronecker's symbol. Consider the Dirichlet series:

$$P(w,s) := 1 + \sum_{n\ge 2} n^{-s} \sum_{m\ge 1} f_{n,m} w^m,$$

where w marks the number of integral factors. It is easily shown that P satisfies

$$P(w,s) = \prod_{n\ge 2}\left(1 - \frac{w}{n^s}\right)^{-1}, \qquad \mathscr{R}s > 1,\ |w| < 2.$$

Define a polynomial $P_n(w) := \sum_{1\le k\le n}[k^{-s}]P(w,s)$, with which we can associate a random variable ξ_n with distribution:

$$\Pr\{\xi_n = m\} = \frac{[w^m]P_n(w)}{P_n(1)},$$

ξ_n being the number of integral factors (> 1) of a randomly chosen factorization among those of $\{1, 2, \dots, n\}$, each being assigned with the same probability. We show in [18], Chapter 10, that

$$P_n(w) = \frac{w^{1/4} n \exp\left(2\sqrt{w \log m}\,\right)}{2\sqrt{\pi}\,(\log n)^{3/4}}\left(\Gamma(2-w) + O\left((\log n)^{-1/2}\right)\right),$$

the error term being uniform with respect to $|w| \le 2 - \varepsilon$, $|\arg w| \le \pi - \delta$, $\varepsilon, \delta > 0$. Thus we obtain

$$\begin{aligned} M_n(z) := \mathbf{E}e^{\xi_n z} &= \exp\left(2\sqrt{\log n}\,(e^{z/2} - 1) + z/4\right) \\ &\quad \times\left(\Gamma(2 - e^z) + O\left((\log n)^{-1/2}\right)\right), \end{aligned}$$

uniformly for $|z|$ small. Writing $F_n(x) = \Pr\{\xi_n < \sqrt{\log n} + x\sqrt[4]{\frac{1}{4}\log n}\}$, Theorem 1 gives, for example,

$$\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp\left(-2\sqrt{\log n}\,\Lambda\left(\frac{y}{2}\right)\right)\left(1 + O\left(\frac{x}{\sqrt[4]{\log n}}\right)\right)$$

for all $x > 0$, $x = o(\sqrt[4]{\log n})$, where Λ is as in Theorem 2.

For many other examples, cf. [18], Chapters 9 and 10, [8] and [26].

EXAMPLE 5 (Meromorphic functions). The Eulerian numbers $A(n,k)$ are defined by the generating function

$$P(z,w) = \sum_{n,k\ge 0} A(n,k) w^k \frac{z^n}{n!} = \frac{w(1-w)}{e^{(w-1)z} - w},$$

and they enumerate the number of permutations of size n having k rises. Let Ω_n denote the number of rises in a random permutation of size n, where each permutation of size n has probability $1/n!$. Then $\mathbf{E}\Omega_n = \frac{1}{2}(n+1)$ and $\operatorname{Var}\Omega_n = \frac{1}{12}(n+1)$ for $n \ge 2$. In addition, $M_n(s) = \mathbf{E}e^{\Omega_n s} = [z^n]P(z, e^s)$ can be expressed in the form

$$M_n(s) = \left(\frac{e^s - 1}{s}\right)^{n+1} + \sum_{k\ne 0} \frac{(e^s-1)^{n+1}}{(s + 2k\pi i)^{n+1}}, \qquad n \ge 1, |s| < 2\pi,$$

by standard techniques for expanding meromorphic functions. When n becomes large, the sum on the right-hand side gives only exponentially smaller terms. Theorem 1 applies with $u(s) = -\log(s/(e^s-1))$ and $\phi(n) = n+1$. Local limit theorems for Ω_n can be found in [1].

Another example of the same class is the level number sequence of Flajolet and Prodinger [13]. There they consider the number h_n of level number sequences for binary trees, which is equivalent to the cardinality of the set $H_n = \bigcup_k H_{nk}$, where

$$H_k = \{(n_1, n_2, \dots, n_k) : n_1 = 1; 1 \le n_j \le 2n_{j-1},\\ 2 \le j \le k; n_1 + n_2 + \cdots + n_k = n\};$$

k is called the height of the level number sequence. We can consider the distribution of the height. Let $h_{nk} = \#H_{nk}$ with generating function $P(w,z) = \sum_{n,k} h_{nk} z^n w^k$. Then similar derivations as in [13] lead to $P(w,z) = a(w,z)/(1 - b(w,z))$, where

$$a(w,z) = \sum_{k\ge 1} \frac{(-1)^{k+1} z^{2^{k+1}-k-2}}{(1-z)(1-z^3)(1-z^7)\cdots(1-z^{2^{k-1}-1})} w^k$$

and

$$b(w,z) = \sum_{k\ge 1} \frac{(-1)^{k+1} z^{2^{k+1}-k-2}}{(1-z)(1-z^3)(1-z^7)\cdots(1-z^{2^k-1})} w^k,$$

the two functions being analytic for $|z| < 1$ and all w. Following similar arguments used in [13], it is not hard to show that our Theorem 1 is applicable to $\sum_k h_{nk} e^{ks} / \sum_k h_{nk}$. In particular, *the height of a random level number sequence is asymptotically normally distributed.*

EXAMPLE 6 (Orthogonal polynomials). Many interesting combinatorial interpretations have been proposed for classical orthogonal polynomials. For example, the number of fixed points (cycles of length 1), marked by w, in the involutions is enumerated by

$$\exp\left(wz + \frac{z^2}{2}\right) = \sum_{n \geq 0} \frac{h_n(w)}{n!} z^n,$$

where the $h_n(w)$ are Hermite polynomials. The asymptotic behavior of $h_n(w)$ is well known (cf. [38], page 200):

$$h_n(w) = 2^{-1/2} n^{n/2} \exp\left(-\frac{n}{2} + \sqrt{n+1}\, w - \frac{w^2}{4}\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), \qquad n \to \infty,$$

uniformly for all finite w. Defining Ω_n as the number of fixed points in a random involution of size n, where each involution of size n has the same probability, we obtain

$$M_n(s) = \mathbf{E} e^{\Omega_n s} = \exp\left(\sqrt{n+1}\,(e^s - 1) - (e^{2s} - 1)/4\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right),$$

uniformly for small $|s|$. Theorem 2 applies with $\phi(n) = \sqrt{n+1}$. Note that Ω_n is of maximal span 2 for $n \geq 2$.

In this way, we can study distributional aspects of the coefficients of classical polynomials (not necessarily orthogonal), like Laguerre, Gegenbauer (which includes Chebyshev and Legendre), Charlier, Meixner (first kind), Lerch and Humbert polynomials; cf. [38]. Our theorems apply to all these polynomials which also appear in the combinatorial study of histories of a number of data structures: stacks, priority queues, dictionaries and so on; cf. [11]. In the other direction, probabilistic methods have been used to derive classical asymptotic expansions of certain orthogonal polynomials; cf. [28].

6. Concluding remarks. We have hitherto applied our Theorem 1 to discrete random variables. It might be possible that it finds application in other fields.

It seems of some interest to replace the error terms in (5) and (6) by asymptotic expansions under suitable conditions.

Local limit theorems for large deviations applicable to combinatorial distributions are treated in [21].

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