Large deviations for compound Markov renewal processes with dependent jump sizes and jump waiting times *

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Abstract

In [17] the author considered a compound Markov renewal process $(\widetilde{S}_{N_t})$ where $((J_n, S_n))$ and $((\widetilde{J}_n, \widetilde{S}_n))$ are suitable independent Markov additive processes such that $(S_n - S_{n-1})$ are positive random variables, and $N_t = \sum_{n\geq 1} 1_{S_n \leq t}$. In this paper we present the analogous results for a more general situation where we consider a unique Markov additive process $((J_n, Z_n))$ in place of $((J_n, S_n))$ and $((\widetilde{J}_n, \widetilde{S}_n))$, and $Z_n = (\widetilde{S}_n, S_n)$. Some further results are also presented; in particular we relate in terms of large deviations the sequence $((\widetilde{S}_n, S_n))$ and the process $((\widetilde{S}_{N_t}, N_t))$.

1 Introduction

In [17] the author proves large deviation results concerning compound Markov renewal processes $(\widetilde{S}_{N_t})$ (see (5)), where $((J_n, S_n))$ and $((\widetilde{J}_n, \widetilde{S}_n))$ are suitable independent Markov additive processes; thus the jump sizes $(\widetilde{S}_n - \widetilde{S}_{n-1})$ and the jump waiting times $(S_n - S_{n-1})$ of $(\widetilde{S}_{N_t})$ are independent. In particular some results concern some level crossing probabilities and their estimation by importance sampling. There are several works which relate large deviations and importance sampling for the estimation of rare events: some of them are cited in this paper (here the rare event is a level crossing) and further references can be found in the Introduction of [17].

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In this paper we consider a more general model with a unique Markov additive process $((J_n, Z_n))$, where $Z_n = (\widetilde{S}_n, S_n)$. Roughly speaking $((\widetilde{S}_n - \widetilde{S}_{n-1}, S_n - S_{n-1}))$ is a sequence of conditionally independent random variables given $J = (J_n)$ and, for each $n \geq 1$, the conditional distribution of $(\widetilde{S}_n - \widetilde{S}_{n-1}, S_n - S_{n-1})$ given J depends on (J_{n-1}, J_n) only; for a more detailed definition see subsection 2.1. We present the extension of the results in [17] adapted to this more general model. We also illustrate the relationship in terms of large deviations between the sequence $((\widetilde{S}_n, S_n))$ and the process $((\widetilde{S}_{N_t}, N_t))$, where $N_t = \sum_{n\geq 1} 1_{S_n \leq t}$ (see (5)).

The interest of this generalization is motivated by some recent works in the literature. Here we cite the following papers: [1] with a ruin model in insurance with dependence between claim sizes and claim intervals (see also [6] for a related model in queueing theory); [2] where $((\widetilde{S}_n - \widetilde{S}_{n-1}, S_n - S_{n-1}))$ is a sequence of i.i.d. random variables, and the joint distribution of $(\widetilde{S}_1, S_1)$ is modeled by a copula (see e.g. [14] and [18] as references on copulas).

If the jumps are upwards (namely each random variable $\widetilde{S}_n - \widetilde{S}_{n-1}$ is positive) there is a natural interpretation in insurance. Indeed, if the insurance company has an initial reserve b and receives premiums linearly at rate $c > 0$, $(\widetilde{S}_{N_t} - ct)$ and $(b + ct - \widetilde{S}_{N_t})$ are called claim surplus process and risk reserve process respectively (see e.g. [4], page 1); moreover $(\widetilde{S}_n - \widetilde{S}_{n-1})$ are the claim sizes, $(S_n - S_{n-1})$ are the claim waiting times and the ruin of the company occurs when $(\widetilde{S}_{N_t} - ct)$ crosses the positive level b. The model presented in this paper generalizes other known models in the literature, as the renewal model and the Cramér-Lundberg model (see e.g. [12], page 22); the Cramér-Lundberg model is also known in the literature as the compound Poisson model (see e.g. [3], pages 111 and 280).

The outline of the paper is the following. Section 2 is devoted to recall preliminaries. Large deviation principles and the results on level crossing probabilities are presented in sections 3 and 4 respectively. Finally concluding remarks and further minor results are presented in section 5.

2 Preliminaries

2.1 Generalities on Markov additive processes

Set $E = \{1, \ldots, m\}$, let $J = (J_n)$ be a E valued Markov chain and let $(p_{ij})_{i,j\in E}$ be the transition matrix of J. Furthermore let (Z_n) be a $\mathbb{R}^2$ valued sequence of random variables such that $Z_0 = (0, 0)$ and $(Z_n - Z_{n-1})$ is a sequence of conditionally independent random variables given J; moreover, in general, the conditional distribution of $Z_k - Z_{k-1}$ given J depends on (J_{k-1}, J_k) only. In view of what follows we use the notation $Z_n = (\widetilde{S}_n, S_n)$; moreover we denote the conditional distribution of $Z_k - Z_{k-1}$ given $(J_{k-1}, J_k) = (i, j)$ by H_{ij} and the corresponding moment generating function by $\widehat{H}_{ij}$.

As far as the terminology is concerned, $((J_n, Z_n))$ is a Markov additive process, (J_n) is the environment and (Z_n) is the additive part. A reference for the presentation above of Markov additive processes is [4] (chapter 2, section 5, page 40, discrete time case) where the additive part is real valued; anyway the presentation can be easily adapted to more general situations.

Assume that J is irreducible. Then, for each fixed $(\widetilde{\alpha},\alpha)\in\mathbb{R}^2$, we can consider the matrices $F(\widetilde{\alpha},\alpha)=(F_{ij}(\widetilde{\alpha},\alpha))_{i,j\in E}$ defined by

$$F_{ij}(\widetilde{\alpha},\alpha)=\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_1+\alpha S_1}1_{J_1=j}|J_0=i]\ \ (\forall i,j\in E).$$

The entries of these matrices could be infinite if $(\widetilde{\alpha},\alpha)\neq(0,0)$ (if $(\widetilde{\alpha},\alpha)=(0,0)$ the entries are finite, indeed the matrix $F(0,0)$ coincides with the transition matrix of J) and the latter is equivalent to

$$F_{ij}(\widetilde{\alpha},\alpha)=\widehat{H}_{ij}(\widetilde{\alpha},\alpha)p_{ij}\ \ (\forall i,j\in E).$$

Thus, as far as the family of the moment generating functions $(\widehat{H}_{ij})$ is concerned, we remark that some pairs (i,j) can be neglected; indeed we can restrict our attention on the pairs (i,j) such that $p_{ij}>0$. Moreover, if we consider the n-th power $F^n(\widetilde{\alpha},\alpha)=((F^n)_{ij}(\widetilde{\alpha},\alpha))_{i,j\in E}$ of $F(\widetilde{\alpha},\alpha)$, we have

$$(F^n)_{ij}(\widetilde{\alpha},\alpha)=\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}1_{J_n=j}|J_0=i]\ \ (\forall i,j\in E).$$

Finally let us consider the set

$$\mathcal{D}:=\{(\widetilde{\alpha},\alpha)\in\mathbb{R}^2:F_{ij}(\widetilde{\alpha},\alpha)<\infty\ (\forall i,j\in E)\}$$

and let $n\geq 1$ and $(\widetilde{\alpha},\alpha)\in\mathcal{D}$ be arbitrarily fixed (we point out that $\mathcal{D}$ is not empty since $(0,0)\in\mathcal{D}$). Then we can say what follows.

• By the conditional independence of the increments of (Z_n) given J we have

$$\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}]=\mathbb{E}[\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}|J]]=\mathbb{E}\Big[\prod_{k=1}^{n}\widehat{H}_{J_{k-1}J_k}(\widetilde{\alpha},\alpha)\Big];\tag{1}$$

we point out that (1) holds for all $(\widetilde{\alpha},\alpha)\in\mathbb{R}^2$.

• Perron Frobenius Theorem provides the existence of a simple and positive eigenvalue $e^{\Lambda(\widetilde{\alpha},\alpha)}$ of the matrix $F(\widetilde{\alpha},\alpha)$ equal to its spectral radius; moreover

$$\sum_{j\in E}(F^n)_{ij}(\widetilde{\alpha},\alpha)h_j(\widetilde{\alpha},\alpha)=e^{n\Lambda(\widetilde{\alpha},\alpha)}h_i(\widetilde{\alpha},\alpha)\ \ (\forall i\in E)$$

where $(h_i(\widetilde{\alpha},\alpha))_{i\in E}$ is an eigenvector with positive components, unique up to a positive constant.

• Let us consider an arbitrarily fixed initial distribution of J (namely an arbitrarily fixed distribution of J_0). Then we have

$$\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}]=\sum_{i\in E}\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}|J_0=i]P(J_0=i)=$$

$$=\sum_{i\in E}\Big(\sum_{j\in E}\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}1_{J_n=j}|J_0=i]\Big)P(J_0=i)=\sum_{i\in E}\Big(\sum_{j\in E}(F^n)_{ij}(\widetilde{\alpha},\alpha)\Big)P(J_0=i),$$

whence we obtain

$$\frac{\sum_{i\in E}h_i(\widetilde{\alpha},\alpha)P(J_0=i)}{\max_{j\in E}h_j(\widetilde{\alpha},\alpha)}e^{n\Lambda(\widetilde{\alpha},\alpha)}\leq\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}]\leq\frac{\sum_{i\in E}h_i(\widetilde{\alpha},\alpha)P(J_0=i)}{\min_{j\in E}h_j(\widetilde{\alpha},\alpha)}e^{n\Lambda(\widetilde{\alpha},\alpha)}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}]=\Lambda(\widetilde{\alpha},\alpha).\tag{2}$$

We point out that, if we set $\Lambda(\widetilde{\alpha},\alpha)=\infty$ for $(\widetilde{\alpha},\alpha)\notin\mathcal{D}$, (2) holds for all $(\widetilde{\alpha},\alpha)\in\mathbb{R}^2$; moreover we have

$$\mathcal{D}=\{(\widetilde{\alpha},\alpha)\in\mathbb{R}^2:\Lambda(\widetilde{\alpha},\alpha)<\infty\}.$$

2.2 A class of conjugate laws for Markov additive processes

In this subsection we present a class of conjugate laws $(P_{\widetilde{\alpha},\alpha})_{\widetilde{\alpha},\alpha:\Lambda(\widetilde{\alpha},\alpha)<\infty}$ defined as follows. For each fixed $(\widetilde{\alpha},\alpha) \in \mathcal{D}$ we have that P is absolutely continuous with respect to $P_{\widetilde{\alpha},\alpha}$ on each finite time interval $\{0,1,\dots,n\}$ and the corresponding density $\ell_n^{P,P_{\widetilde{\alpha},\alpha}}$ is

$$\ell_n^{P,P_{\widetilde{\alpha},\alpha}} = e^{-(\widetilde{\alpha}\widetilde{S}_n+\alpha S_n)+n\Lambda(\widetilde{\alpha},\alpha)}\frac{h_{J_0}(\widetilde{\alpha},\alpha)}{h_{J_n}(\widetilde{\alpha},\alpha)}. \tag{3}$$

Then (see e.g. Proposition 5.10 and Theorem 5.11 in [4], pages 44-45) $((J_n,S_n))$ is again a Markov additive process under each $P_{\widetilde{\alpha},\alpha}$ with transition matrix $(p_{ij}^{(\widetilde{\alpha},\alpha)})_{i,j\in E}$ for (J_n) defined by

$$p_{ij}^{(\widetilde{\alpha},\alpha)} := \frac{F_{ij}(\widetilde{\alpha},\alpha)h_j(\widetilde{\alpha},\alpha)}{e^{\Lambda(\widetilde{\alpha},\alpha)}h_i(\widetilde{\alpha},\alpha)}$$

and distributions $(H_{ij}^{(\widetilde{\alpha},\alpha)})$ (which play the role of (H_{ij}) under P) defined by

$$H_{ij}^{(\widetilde{\alpha},\alpha)}(d\widetilde{y},dy) := \frac{e^{\widetilde{\alpha}\widetilde{y}+\alpha y}}{\widehat{H}_{ij}(\widetilde{\alpha},\alpha)}H_{ij}(d\widetilde{y},dy).$$

Furthermore, by Proposition 4.1 and by the first part of Lemma 4.2 in [13] applied with respect to each $P_{\widetilde{\alpha},\alpha}$, we have

$$\lim_{n\to\infty}\frac{Z_n}{n} = \nabla\Lambda(\widetilde{\alpha},\alpha) \quad P_{\widetilde{\alpha},\alpha} \text{ a.s.}; \tag{4}$$

see e.g. [19] (Theorem 3.6, page 123) for the first part of Lemma 4.2 in [13]. Finally we point out that $P_{0,0} = P$.

2.3 Compound Markov renewal processes and main hypotheses

The following hypothesis plays a crucial role in the construction of our process of interest:
(A1): *the distributions* (H_{ij}) *are concentrated on* $\mathbb{R}\times]0,\infty[$.
Now let us point out some consequences of **(A1)**. The sequence (S_n) is (almost surely) increasing and the random variables (S_n) can be seen as the jump times of a counting process (N_t). More precisely from now on we set

$$N_t := \sum_{n\geq 1} 1_{S_n\leq t} \text{ and } \widetilde{S}_{N_t} := \sum_{n=1}^{N_t}\widetilde{S}_n, \tag{5}$$

and $(\widetilde{S}_{N_t})$ is called compound Markov renewal process.

We have a generalization of the renewal model. Indeed we have the renewal model if the distributions H_{ij} are all the same product measure $\widetilde{H}\otimes H$ concentrated on $]0,\infty[\times]0,\infty[$ (as pointed out above we can restrict our attention on $i,j \in E$ such that $p_{ij} > 0$): in such a case $\widetilde{H}$ is the common distribution of jump sizes and H is the common distribution of jump waiting times. In particular we have the Cramér-Lundberg model (or the compound Poisson model) if H is an exponential distribution.

For each fixed $\widetilde{\alpha} \in \mathbb{R}$ such that

$$\mathcal{D}_{\widetilde{\alpha}} := \{\alpha \in \mathbb{R} : \Lambda(\widetilde{\alpha}, \alpha) < \infty\} \tag{6}$$

is nonempty, the restriction of $\Lambda(\widetilde{\alpha}, \cdot)$ on $\mathcal{D}_{\widetilde{\alpha}}$ is an increasing function and we can consider the inverse function $[\Lambda(\widetilde{\alpha}, \cdot)]^{-1}$ of $\Lambda(\widetilde{\alpha}, \cdot)$.

Then we can present some further hypotheses which play crucial role in this paper; in particular we refer to the concept of essentially smooth function (see e.g. [10], Definition 2.3.5, page 44).

(A2): **(A1)** *holds*; *for each* $\widetilde{\alpha} \in \mathbb{R}$ *we have* $\mathcal{D}_{\widetilde{\alpha}} \neq \emptyset$ *and* $\Lambda(\widetilde{\alpha}, \cdot) : \mathcal{D}_{\widetilde{\alpha}} \to \mathbb{R}$ *is surjective*; *moreover let us consider the function* Ψ_Λ *defined as follows*

$$\Psi_\Lambda(\widetilde{\alpha}, \alpha) := -[\Lambda(\widetilde{\alpha}, \cdot)]^{-1}(-\alpha).$$

(A3): **(A2)** *holds*; *we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_{N_t}}] = \Psi_\Lambda(\widetilde{\alpha}, 0) \quad (\forall \widetilde{\alpha} \in \mathbb{R});$$

$\Psi_\Lambda(\widetilde{\alpha}, 0)$ *is finite in a neighbourhood of the origin* $\widetilde{\alpha} = 0$, *essentially smooth and lower semicontinuous.*

(A4): **(A2)** *holds*; *we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_{N_t} + \alpha N_t}] = \Psi_\Lambda(\widetilde{\alpha}, \alpha) \quad (\forall (\widetilde{\alpha}, \alpha) \in \mathbb{R}^2);$$

$\Psi_\Lambda(\widetilde{\alpha}, \alpha)$ *is finite in a neighbourhood of the origin* $(\widetilde{\alpha}, \alpha) = (0, 0)$, *essentially smooth and lower semicontinuous.*

It is useful to point out what follows. By the identity $\Lambda(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha)) = -\alpha$ we have

$$\begin{cases} \frac{\partial \Lambda}{\partial \widetilde{\alpha}}(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha)) + \frac{\partial \Lambda}{\partial \alpha}(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha))(-\frac{\partial \Psi_\Lambda}{\partial \widetilde{\alpha}}(\widetilde{\alpha}, \alpha)) = 0 \\ \frac{\partial \Lambda}{\partial \alpha}(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha))(-\frac{\partial \Psi_\Lambda}{\partial \alpha}(\widetilde{\alpha}, \alpha)) = -1 \end{cases},$$

whence we obtain

$$\begin{cases} \frac{\partial \Psi_\Lambda}{\partial \widetilde{\alpha}}(\widetilde{\alpha}, \alpha) = \frac{\frac{\partial \Lambda}{\partial \widetilde{\alpha}}(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha))}{\frac{\partial \Lambda}{\partial \alpha}(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha))} \\ \frac{\partial \Psi_\Lambda}{\partial \alpha}(\widetilde{\alpha}, \alpha) = \frac{1}{\frac{\partial \Lambda}{\partial \alpha}(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha))} \end{cases}. \tag{7}$$

Then the next (almost sure) limit holds as a consequence of (4) and the motivations below:

$$\lim_{t\to\infty} \left(\frac{\widetilde{S}_{N_t}}{t}, \frac{N_t}{t} \right) = \left(\frac{\frac{\partial \Lambda}{\partial \widetilde{\alpha}}(\widetilde{\alpha}, \alpha)}{\frac{\partial \Lambda}{\partial \alpha}(\widetilde{\alpha}, \alpha)}, \frac{1}{\frac{\partial \Lambda}{\partial \alpha}(\widetilde{\alpha}, \alpha)} \right) =^{(7)} \nabla \Psi_\Lambda(\widetilde{\alpha}, \alpha) \ P_{\widetilde{\alpha}, \alpha} \text{ a.s..} \tag{8}$$

The limit of the second component follows from (4) together with the definition of (N_t) in (5) and from a well known result in renewal theory (see e.g. [5], Ex. 2.3.10, page 310), while the limit of the first component follows from the limit of the second component and from $\frac{\widetilde{S}_{N_t}}{t} = \frac{\widetilde{S}_{N_t}}{N_t} \frac{N_t}{t}$ (by taking into account that N_t diverges to ∞ as $t \to \infty$ and by (4)).

3 Large deviation principles

In this section we refer to the concept of large deviation principle (see e.g. [10], page 5). Throughout this paper we use shorthand LDP for large deviation principle.

The two following Propositions 3.1 and 3.2 are an immediate consequence of the hypotheses ((**A3**) and (**A4**) respectively) and of Gärtner Ellis Theorem.

Proposition 3.1. *Assume* (**A3**) *holds. Then* $\left(\frac{\widetilde{S}_{N_t}}{t}\right)$ *satisfies the LDP with rate function* $[\Psi_\Lambda(\cdot,0)]^*$ *defined by*

$$[\Psi_\Lambda(\cdot,0)]^*(\widetilde{x}) = \sup_{\widetilde{\alpha}\in\mathbb{R}}[\widetilde{\alpha}\widetilde{x} - \Psi_\Lambda(\widetilde{\alpha},0)] \quad (\forall \widetilde{x}\in\mathbb{R}).$$

This means that $[\Psi_\Lambda(\cdot,0)]^* : \mathbb{R} \to [0,\infty]$ *is a lower semicontinuous function and we have*

$$\liminf_{t\to\infty} \frac{1}{t}\log P\left(\frac{\widetilde{S}_{N_t}}{t}\in O\right) \geq -\inf_{\widetilde{x}\in O}[\Psi_\Lambda(\cdot,0)]^*(\widetilde{x}) \quad (\forall O \text{ open}) \tag{9}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log P\left(\frac{\widetilde{S}_{N_t}}{t}\in C\right) \leq -\inf_{\widetilde{x}\in C}[\Psi_\Lambda(\cdot,0)]^*(\widetilde{x}) \quad (\forall C \text{ closed}).$$

Proposition 3.2. *Assume* (**A4**) *holds. Then* $\left(\left(\frac{\widetilde{S}_{N_t}}{t},\frac{N_t}{t}\right)\right)$ *satisfies the LDP with rate function* Ψ_Λ^* *defined by*

$$\Psi_\Lambda^*(\widetilde{x},x) = \sup_{(\widetilde{\alpha},\alpha)\in\mathbb{R}^2}[\widetilde{\alpha}\widetilde{x} + \alpha x - \Psi_\Lambda(\widetilde{\alpha},\alpha)] \quad (\forall (\widetilde{x},x)\in\mathbb{R}^2).$$

This means that $\Psi_\Lambda^* : \mathbb{R}^2 \to [0,\infty]$ *is a lower semicontinuous function and we have*

$$\liminf_{t\to\infty} \frac{1}{t}\log P\left(\left(\frac{\widetilde{S}_{N_t}}{t},\frac{N_t}{t}\right)\in O\right) \geq -\inf_{(\widetilde{x},x)\in O}\Psi_\Lambda^*(\widetilde{x},x) \quad (\forall O \text{ open})$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log P\left(\left(\frac{\widetilde{S}_{N_t}}{t},\frac{N_t}{t}\right)\in C\right) \leq -\inf_{(\widetilde{x},x)\in C}\Psi_\Lambda^*(\widetilde{x},x) \quad (\forall C \text{ closed}).$$

In the next Proposition 3.3 we assume (**A4**) holds and we provide an explicit expression of $\Psi_\Lambda^*(\widetilde{x},x)$. If $x>0$ this expression agrees with the expression in [20] proved in a different way for a wide class of cases; the interest of the next proposition is motivated by the expression of $\Psi_\Lambda^*(\widetilde{x},x)$ for $x=0$.

Proposition 3.3. *Assume* (**A4**) *holds. Let* Ψ_Λ^* *be the rate function in Proposition 3.2 and, for any* $\widetilde{\alpha}\in\mathbb{R}$, *let* $\mathcal{D}_{\widetilde{\alpha}}$ *be the set in (6). Then we have*

$$\Psi_\Lambda^*(\widetilde{x},x) = \begin{cases} x\Lambda^*\left(\frac{\widetilde{x}}{x},\frac{1}{x}\right) & \text{if } x>0 \\ \sup_{\widetilde{\alpha}\in\mathbb{R}}[\widetilde{\alpha}\widetilde{x} + \sup\mathcal{D}_{\widetilde{\alpha}}] & \text{if } x=0 \\ \infty & \text{if } x<0 \end{cases}. \tag{10}$$

Proof. Let $\widetilde{x} \in \mathbb{R}$ be arbitrarily fixed and let us distinguish three cases concerning $x \in \mathbb{R}$.
Case $x > 0$. We check two complementary inequalities. First of all we have

$$\Psi_\Lambda^*(\widetilde{x}, x) = \sup_{(\widetilde{\alpha},\alpha)\in\mathbb{R}^2} [\widetilde{\alpha}\widetilde{x} + \alpha x - \Psi_\Lambda(\widetilde{\alpha}, \alpha)] =$$

$$= x \sup_{(\widetilde{\alpha},\alpha)\in\mathbb{R}^2} \left[\widetilde{\alpha}\frac{\widetilde{x}}{x} - \Psi_\Lambda(\widetilde{\alpha}, \alpha) \cdot \frac{1}{x} + \alpha\right] = x \sup_{(\widetilde{\alpha},\alpha)\in\mathbb{R}^2} \left[\widetilde{\alpha}\frac{\widetilde{x}}{x} - \Psi_\Lambda(\widetilde{\alpha}, \alpha) \cdot \frac{1}{x} - \Lambda(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha))\right]$$

where the latter equality follows from the identity $\Lambda(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha}, \alpha)) = -\alpha$ (which holds by construction). Thus the first inequality $\Psi_\Lambda^*(\widetilde{x}, x) \leq x\Lambda^*(\frac{\widetilde{x}}{x}, \frac{1}{x})$ holds. In order to obtain the complementary inequality, let us consider a sequence $((\widetilde{\alpha}_n, \beta_n))$ such that

$$\lim_{n\to\infty} \widetilde{\alpha}_n \frac{\widetilde{x}}{x} + \beta_n \cdot \frac{1}{x} - \Lambda(\widetilde{\alpha}_n, \beta_n) = \Lambda^*\left(\frac{\widetilde{x}}{x}, \frac{1}{x}\right);$$

moreover let us set $\alpha_n = -\Lambda(\widetilde{\alpha}_n, \beta_n)$, which is equivalent to $\Psi_\Lambda(\widetilde{\alpha}_n, \alpha_n) = -\beta_n$. Then

$$\Psi_\Lambda^*(\widetilde{x}, x) \geq \widetilde{\alpha}_n \widetilde{x} + \alpha_n x - \Psi_\Lambda(\widetilde{\alpha}_n, \alpha_n) =$$

$$= x\left[\widetilde{\alpha}_n \frac{\widetilde{x}}{x} - \Psi_\Lambda(\widetilde{\alpha}_n, \alpha_n) \cdot \frac{1}{x} + \alpha_n\right] = x\left[\widetilde{\alpha}_n \frac{\widetilde{x}}{x} + \beta_n \cdot \frac{1}{x} - \Lambda(\widetilde{\alpha}_n, \beta_n)\right] \ (\forall n \geq 1)$$

and the second inequality $\Psi_\Lambda^*(\widetilde{x}, x) \geq x\Lambda^*(\frac{\widetilde{x}}{x}, \frac{1}{x})$ holds by taking the limit as $n \to \infty$ in the right hand side.
Case $x = 0$. We check two complementary inequalities. Since **(A4)** implies **(A2)**, for any $\widetilde{\alpha} \in \mathbb{R}$ it is defined the function $[\Lambda(\widetilde{\alpha}, \cdot)]^{-1} : \mathbb{R} \to \mathcal{D}_{\widetilde{\alpha}}$, and the following limit holds for the increasing function $\Psi_\Lambda(\widetilde{\alpha}, \cdot)$:

$$\Psi_\Lambda(\widetilde{\alpha}, \alpha) = -[\Lambda(\widetilde{\alpha}, \cdot)]^{-1}(-\alpha) \downarrow -\sup \mathcal{D}_{\widetilde{\alpha}} \text{ as } \alpha \downarrow -\infty.$$

Thus $-\Psi_\Lambda(\widetilde{\alpha}, \alpha) \leq \sup \mathcal{D}_{\widetilde{\alpha}}$ for any $(\widetilde{\alpha}, \alpha) \in \mathbb{R}^2$ and, by the definition of Ψ_Λ^*, we get

$$\Psi_\Lambda^*(\widetilde{x}, 0) \leq \sup_{\widetilde{\alpha}\in\mathbb{R}} [\widetilde{\alpha}\widetilde{x} + \sup \mathcal{D}_{\widetilde{\alpha}}].$$

Furthermore, by the definition of Ψ_Λ^*, we have $\Psi_\Lambda^*(\widetilde{x}, 0) \geq \widetilde{\alpha}\widetilde{x} - \Psi_\Lambda(\widetilde{\alpha}, \alpha)$ for all $(\widetilde{\alpha}, \alpha) \in \mathbb{R}^2$ and, by taking the limit as $\alpha \downarrow -\infty$ in the right hand side, we obtain $\Psi_\Lambda^*(\widetilde{x}, 0) \geq \widetilde{\alpha}\widetilde{x} + \sup \mathcal{D}_{\widetilde{\alpha}}$ for any $\widetilde{\alpha} \in \mathbb{R}$; thus

$$\Psi_\Lambda^*(\widetilde{x}, 0) \geq \sup_{\widetilde{\alpha}\in\mathbb{R}} [\widetilde{\alpha}\widetilde{x} + \sup \mathcal{D}_{\widetilde{\alpha}}]$$

holds by taking the supremum with respect to $\widetilde{\alpha} \in \mathbb{R}$.
Case $x < 0$. It is trivial since we have $P\left(\frac{N_t}{t} \geq 0\right) = 1$ for all $t > 0$ and, by well known properties of large deviation principles, the rate function $\Psi_\Lambda^*(\widetilde{x}, x)$ has to be equal to ∞ outside the closed set $\mathbb{R} \times [0, \infty[$. ∎

4 Results on level crossing probabilities

4.1 Preliminaries and exponential decay of $p(b)$ as $b \to \infty$

The results in this section concern the level crossing probabilities $(p(b))_{b>0}$ of $(\widetilde{S}_{N_t} - ct)$, where $c > 0$ is a suitable constant as before. In detail we set

$$p(b) = P(T_b < \infty) \text{ where } T_b = \inf\{t \geq 0 : \widetilde{S}_{N_t} - ct \geq b\}.$$

In order to avoid the trivial case $p(b) = 1$ for all $b > 0$, we need to consider c large enough; indeed, if c is large enough, we have $\widetilde{S}_{N_t} - ct \to -\infty$ and therefore the level crossing can fail with positive probability. We shall precise below that we have to consider $c > \frac{\partial \Psi_\Lambda}{\partial \widetilde{\alpha}}(0,0)$.

In view of the presentation of the results in this section some preliminaries are needed.

Let us consider the class of conjugate laws $(P_{\widetilde{\alpha},\alpha})_{\widetilde{\alpha},\alpha:\Lambda(\widetilde{\alpha},\alpha)<\infty}$ presented in subsection 2.2. This class of conjugate laws is described on a pair of parameters $(\widetilde{\alpha}, \alpha)$; on the other hand, in order to use the methods in the proofs of similar results in the literature, we need to consider a family of densities which depends on only one parameter. Then we shall consider the family of conjugate laws $(Q_{\widetilde{\alpha}})_{\widetilde{\alpha}:[\Lambda(\widetilde{\alpha},\cdot)]^{-1}(0)<\infty}$ defined as follows:

$$Q_{\widetilde{\alpha}} := P_{\widetilde{\alpha},[\Lambda(\widetilde{\alpha},\cdot)]^{-1}(0)} = P_{\widetilde{\alpha},-\Psi_\Lambda(\widetilde{\alpha},0)}. \tag{11}$$

For each law of the family $(Q_{\widetilde{\alpha}})_{\widetilde{\alpha}:[\Lambda(\widetilde{\alpha},\cdot)]^{-1}(0)<\infty}$ in (11), the density $\ell_n^{P,P_{\widetilde{\alpha},\alpha}}$ in (3) will be denoted by $\ell_n^{P,Q_{\widetilde{\alpha}}}$ and, by taking into account the identity $\Lambda(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha},0)) = 0$, we have

$$\ell_n^{P,Q_{\widetilde{\alpha}}} = e^{-(\widetilde{\alpha}\widetilde{S}_n - \Psi_\Lambda(\widetilde{\alpha},0)S_n)} \frac{h_{J_0}(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha},0))}{h_{J_n}(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha},0))}.$$

There are some (almost sure) inequalities which are useful in what follows. If we set

$$M(\widetilde{\alpha}) = \max\left\{\frac{h_i(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha},0))}{h_j(\widetilde{\alpha}, -\Psi_\Lambda(\widetilde{\alpha},0))} : i,j \in E\right\},$$

we have

$$\ell_n^{P,Q_{\widetilde{\alpha}}} \leq M(\widetilde{\alpha}) e^{-(\widetilde{\alpha}\widetilde{S}_n - \Psi_\Lambda(\widetilde{\alpha},0)S_n)}. \tag{12}$$

Moreover, by construction, we have $S_{N_t} \leq t$ which implies

$$-(\widetilde{S}_{N_t} - cS_{N_t}) \leq -(\widetilde{S}_{N_t} - ct) \text{ (for any } c > 0). \tag{13}$$

The value w in the next condition **(A5)** is called Lundberg's parameter and plays a crucial role in this section. The condition **(A5)** is the analogous of (A3) in [16]:

(A5): **(A2)** *holds and there exists* $w > 0$ *such that* $\Psi_\Lambda(w,0) - cw = 0$ *and* $\frac{\partial \Psi_\Lambda}{\partial \widetilde{\alpha}}(w,0) - c > 0$.

Remark. We point out that **(A5)** implies the existence of a positive zero w for the convex function

$$f_c(\widetilde{\alpha}) = \Psi_\Lambda(\widetilde{\alpha}, 0) - c\widetilde{\alpha}$$

and moreover $f_c'(w) > 0$. Thus, since $f_c(0) = 0$ follows from $\Psi_\Lambda(0,0) = 0$, the derivative of f_c at the origin is negative; thus we have $0 > f_c'(0) = \frac{\partial \Psi_\Lambda}{\partial \alpha}(0,0) - c$, which is equivalent to $c > \frac{\partial \Psi_\Lambda}{\partial \alpha}(0,0)$. We also remark that, if **(A5)** holds, the level crossing can fail with positive probability; indeed we have $\widetilde{S}_{N_t} - ct \to -\infty$ since $\frac{\widetilde{S}_{N_t}}{t} - c$ converges to $f_c'(0) < 0$. Finally we point out that $Q_w = P_{w,-cw}$.

In the next Proposition 4.1 we prove the exponential decay of $p(b)$ as $b \to \infty$ in the large deviations fashion; such exponential decay can be expressed in terms of the Lundberg's parameter.

Proposition 4.1. *Assume* **(A3)** *and* **(A5)** *hold. Then* $\lim_{b\to\infty} \frac{1}{b} \log p(b) = -w$.

Proof. The proof has some analogies with other proofs of similar results in the literature (see e.g. Theorem 1 in [15] and Theorem 3.1 in [16]). The limit follows from the lower bound

$$\liminf_{b\to\infty} \frac{1}{b} \log p(b) \geq -w$$

and the upper bound

$$\limsup_{b\to\infty} \frac{1}{b} \log p(b) \leq -w.$$

For the lower bound we use a standard procedure based on the lower bound for the open sets (9) for the LDP of $(\frac{\widetilde{S}_{N_t}}{t})$; for instance see e.g. Lemma 2.1 in [11], which deals with a more general situation. For the upper bound let $b > 0$ be arbitrarily fixed. Then

$$p(b) = \mathbb{E}_P[1_{T_b<\infty}] = \mathbb{E}_{Q_w}[\ell^{P,Q_w}_{N_{T_b}} 1_{T_b<\infty}] \leq^{(12)} \mathbb{E}_{Q_w}[M(w) e^{-(w\widetilde{S}_{N_{T_b}} - \Psi_\Lambda(w,0) S_{N_{T_b}})} 1_{T_b<\infty}] =$$

$$= M(w)\mathbb{E}_{Q_w}[e^{-w(\widetilde{S}_{N_{T_b}} - cS_{N_{T_b}}) + (\Psi_\Lambda(w,0) - cw) S_{N_{T_b}}} 1_{T_b<\infty}] \leq^{\textbf{(A5)} \text{ and } (13)}$$

$$\leq M(w)\mathbb{E}_{Q_w}[e^{-w(\widetilde{S}_{N_{T_b}} - cT_b)} 1_{T_b<\infty}] \leq M(w)\mathbb{E}_{Q_w}[e^{-wb} 1_{T_b<\infty}] = M(w) e^{-wb} Q_w(T_b < \infty).$$

In conclusion we have $p(b) \leq M(w)e^{-wb}$ for all $b > 0$ and the upper bound holds. ■

4.2 Importance sampling

Let us start with some preliminaries on the technique called importance sampling used for the estimation of $p(b)$ by Monte Carlo simulations. A reference with large deviations and importance sampling used for the estimation of rare events by simulation is [9].

Let us suppose we want to estimate $p(b)$ (for a fixed $b > 0$) by Monte Carlo simulation. Thus let us consider R independent replications of $(\widetilde{S}_{N_t} - ct)$ under the law P; then an unbiased estimator of $p(b)$ is the relative frequency $\widehat{p(b)}$ of the level crossings

$$\widehat{p(b)} = \frac{1}{R}\sum_{i=1}^{R} 1_{T_b^{(i)}<\infty},$$

where $T_b^{(1)}, \ldots, T_b^{(R)}$ are the values of T_b in the replications. In such a case we have two difficulties: the simulation time under P is not finite if the level crossing does not occur; by Proposition 4.1 this Monte Carlo approach needs R growing exponentially with b to keep a fixed relative precision, indeed the relative precision of $\widehat{p(b)}$ is

$$\frac{1}{p(b)}\sqrt{\frac{p(b)(1-p(b))}{R}}.$$

Thus, in order to overcome these two difficulties, we consider R independent replications under another law Q chosen in a suitable way. First of all Q is such that P is absolutely continuous with respect to Q locally on the event $\{T_b < \infty\}$; moreover an unbiased estimator of $p(b)$ is

$$[\widehat{p(b)}]_Q = \frac{1}{R}\sum_{i=1}^{R} \ell^{P,Q}_{N_{T_b^{(i)}}} 1_{T_b^{(i)}<\infty},$$

where in general $\ell^{P,Q}_{N_{T_b}}$ is the local density of P with respect to Q.

Furthermore we choose Q in a class of admissible laws; this means that $Q(T_b < \infty) = 1$ for all $b > 0$, so that we have finite time in simulation almost surely. The choice of Q will be done in order to minimize $\mathrm{Var}_Q[[\widehat{p(b)}]_Q]$ in some sense. We point out that

$$\mathrm{Var}_Q[[\widehat{p(b)}]_Q] = \frac{\mathbb{E}_Q[(\ell^{P,Q}_{N_{T_b}})^2 1_{T_b<\infty}] - p^2(b)}{R} =_{Q(T_b<\infty)=1} \frac{\mathbb{E}_Q[(\ell^{P,Q}_{N_{T_b}})^2] - p^2(b)}{R}$$

and the only part which depends on Q is the second moment

$$\eta_Q(b) := \mathbb{E}_Q[(\ell^{P,Q}_{N_{T_b}})^2 1_{T_b<\infty}] = \mathbb{E}_Q[(\ell^{P,Q}_{N_{T_b}})^2].$$

The minimization of this second moment for a fixed b is often intractable and, since in the applications we are interested in large values of b, in order to use standard features on large deviations we concentrate our attention on the asymptotic behaviour of $\frac{1}{b}\log \eta_Q(b)$. Then we have

$$\begin{aligned}\liminf_{b\to\infty} \frac{1}{b}\log \mathbb{E}_Q[(\ell^{P,Q}_{N_{T_b}})^2 1_{T_b<\infty}] &\geq \liminf_{b\to\infty} \frac{1}{b}\log \mathbb{E}^2_Q[\ell^{P,Q}_{N_{T_b}} 1_{T_b<\infty}] \\ &= \liminf_{b\to\infty} \frac{1}{b}\log(p(b))^2 = -2w \quad (14)\end{aligned}$$

by Jensen's inequality and Proposition 4.1. Thus an admissible law Q is said to be an asymptotically efficient simulation law if

$$\lim_{b\to\infty} \frac{1}{b}\log \mathbb{E}_Q[(\ell^{P,Q}_{N_{T_b}})^2 1_{T_b<\infty}] = -2w; \quad (15)$$

indeed, if R is chosen to guarantee a fixed relative precision

$$\frac{1}{p(b)}\sqrt{\frac{\mathbb{E}_Q[(\ell_{N_{T_b}}^{P,Q})^2 1_{T_b<\infty}] - p^2(b)}{R}}$$

of $[\widehat{p(b)}]_Q$, R has chance of growing less than exponentially if and only if (15) holds.

The aim of this subsection is to show that Q_w is an asymptotically efficient simulation law (Proposition 4.3). The first step consists to prove the admissibility of Q_w (Proposition 4.2).

Proposition 4.2. *Assume* **(A3)** *and* **(A5)** *hold. Then* $Q_w(T_b < \infty) = 1$ *for all* $b > 0$.

Proof. We prove this Proposition showing that, Q_w a.s., $\frac{\widetilde{S}_{N_t}}{t}$ converges to some positive limit, so that we can say that $\widetilde{S}_{N_t} - ct \to \infty$ as $t \to \infty$, with probability 1 with respect to Q_w. This fact is immediate; indeed (11) and (8) provide

$$\lim_{t\to\infty} \frac{\widetilde{S}_{N_t}}{t} = \frac{\frac{\partial\Lambda}{\partial\widetilde{\alpha}}(w, -\Psi_\Lambda(w, 0))}{\frac{\partial\Lambda}{\partial\alpha}(w, -\Psi_\Lambda(w, 0))} = \frac{\partial\Psi_\Lambda}{\partial\widetilde{\alpha}}(w, 0) \quad Q_w \text{ a.s.},$$

where $\frac{\partial\Psi_\Lambda}{\partial\widetilde{\alpha}}(w, 0) > c$ by **(A5)**. ∎

Proposition 4.3. *Assume* **(A3)** *and* **(A5)** *hold. Then* Q_w *is an asymptotically efficient simulation law, namely (15) holds with* $Q = Q_w$.

Proof. First of all Q_w is an admissible law by Proposition 4.2. Then we have to check the lower bound

$$\liminf_{b\to\infty} \frac{1}{b} \log \mathbb{E}_{Q_w}[(\ell_{N_{T_b}}^{P,Q_w})^2 1_{T_b<\infty}] \geq -2w$$

and the upper bound

$$\limsup_{b\to\infty} \frac{1}{b} \log \mathbb{E}_{Q_w}[(\ell_{N_{T_b}}^{P,Q_w})^2 1_{T_b<\infty}] \leq -2w.$$

The lower bound holds by (14) with $Q = Q_w$. For the upper bound we can follow the lines of the proof of the upper bound in Proposition 4.1. Thus let $b > 0$ be arbitrarily fixed; then we have

$$\mathbb{E}_{Q_w}[(\ell_{N_{T_b}}^{P,Q_w})^2 1_{T_b<\infty}] \leq (M(w))^2 e^{-2wb} Q_w(T_b < \infty),$$

whence we obtain $\mathbb{E}_{Q_w}[(\ell_{N_{T_b}}^{P,Q_w})^2 1_{T_b<\infty}] \leq (M(w))^2 e^{-2wb}$ and the upper bound holds. ∎

5 Concluding remarks and minor results

5.1 The independence case and two counterexamples

In this subsection we investigate in detail the case in which jump sizes and jump waiting times are independent. A particular consequence is that $\mathcal{D}_{\widetilde{\alpha}}$ does not depend on $\widetilde{\alpha}$ (see (16) below) but this is not true in general, as shown in the counterexample presented below.

If the jump sizes and jump waiting times are independent we have

$$\Lambda(\widetilde{\alpha},\alpha)=\widetilde{\kappa}(\widetilde{\alpha})+\kappa(\alpha)\quad(\forall(\widetilde{\alpha},\alpha)\in\mathbb{R}^2)$$

for suitable functions $\widetilde{\kappa}(\cdot)=\Lambda(\cdot,0)$ and $\kappa(\cdot)=\Lambda(0,\cdot)$. Throughout this section we always assume that **(A2)** holds. As an immediate consequence we have $\widetilde{\kappa}(\widetilde{\alpha})<\infty$ for all $\widetilde{\alpha}\in\mathbb{R}$ and $\mathcal{D}_{\widetilde{\alpha}}$ does not depend on $\widetilde{\alpha}$, indeed we have

$$\mathcal{D}_{\widetilde{\alpha}}=\{\alpha\in\mathbb{R}:\kappa(\alpha)<\infty\}. \tag{16}$$

Moreover there exists the inverse κ^{-1} of κ and we can consider the function $\Gamma_\kappa(\cdot)=-\kappa^{-1}(-(\cdot))$. In conclusion we have

$$\Psi_\Lambda(\widetilde{\alpha},\alpha)=-\kappa^{-1}(-\widetilde{\kappa}(\widetilde{\alpha})-\alpha)=\Gamma_\kappa(\widetilde{\kappa}(\widetilde{\alpha})+\alpha)$$

and the rate function $[\Psi_\Lambda(\cdot,0)]^*$ coincides with the rate function I in Lemma 3.1 in [17].

Furthermore it is easy to check that

$$\Lambda^*(\widetilde{x},x)=\widetilde{\kappa}^*(\widetilde{x})+\kappa^*(x)\quad(\forall(\widetilde{x},x)\in\mathbb{R}^2)$$

where $\widetilde{\kappa}^*(\widetilde{x})=\sup_{\widetilde{\alpha}\in\mathbb{R}}[\widetilde{\alpha}\widetilde{x}-\widetilde{\kappa}(\widetilde{\alpha})]$ and $\kappa^*(x)=\sup_{\alpha\in\mathbb{R}}[\alpha x-\kappa(\alpha)]$.

Finally assume **(A4)** holds and set

$$\gamma=\sup\{\alpha\in\mathbb{R}:\kappa(\alpha)<\infty\};$$

we point out that $\gamma\in[0,\infty]$. Then, by Proposition 3.3 and by (16), we have

$$\Psi_\Lambda^*(\widetilde{x},x)=\begin{cases} x\left[\widetilde{\kappa}^*\left(\frac{\widetilde{x}}{x}\right)+\kappa^*\left(\frac{1}{x}\right)\right] & \text{if } x>0\\ \gamma & \text{if } (\widetilde{x},x)=(0,0)\\ \infty & \text{otherwise}\end{cases}.$$

We pointed out above that, if jump sizes and jump waiting times are independent, under condition **(A2)** we have:
(C1): $\mathcal{D}_{\widetilde{\alpha}}$ does not depend on $\widetilde{\alpha}$;
(C2): $\Lambda(\widetilde{\alpha},0)<\infty$ for all $\widetilde{\alpha}\in\mathbb{R}$.
Obviously we can have **(C1)** and **(C2)** even if there is no independence; it is enough to have $\Lambda(\widetilde{\alpha},\alpha)<\infty$ for all $(\widetilde{\alpha},\alpha)\in\mathbb{R}^2$, and in particular **(C1)** holds with $\mathcal{D}_{\widetilde{\alpha}}=\mathbb{R}$ for all $\widetilde{\alpha}\in\mathbb{R}$. Moreover, if $((\widetilde{S}_n,S_n))$ is a bivariate random walk independent of (J_n), we have $\Lambda(\widetilde{\alpha},\alpha)=\log\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_1+\alpha S_1}]$ for all $(\widetilde{\alpha},\alpha)\in\mathbb{R}^2$; thus, if we also have $\Lambda(\widetilde{\alpha},\alpha)<\infty$ for all $(\widetilde{\alpha},\alpha)\in\mathbb{R}^2$, **(A2)** holds since $\lim_{\alpha\to\pm\infty}\Lambda(\widetilde{\alpha},\alpha)=\pm\infty$ for each fixed $\widetilde{\alpha}\in\mathbb{R}$.

Counterexample. Let $\lambda, \beta > 0$ be arbitrarily fixed. Again let $((\widetilde{S}_n, S_n))$ be a bivariate random walk independent of (J_n), and therefore we have the identity $\Lambda(\widetilde{\alpha}, \alpha) = \log \mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_1 + \alpha S_1}]$. We assume $\widetilde{S}_1$ exponentially distributed with failure rate $\lambda > 0$ and, given $\{\widetilde{S}_1 = x\}$, S_1 is Gamma distributed with parameters (x, β). One can check that

$$\Lambda(\widetilde{\alpha}, \alpha) = \begin{cases} \log\left(\frac{\lambda}{\lambda - \left(\widetilde{\alpha} + \log(\frac{\beta}{\beta - \alpha})\right)}\right) & \text{if } \alpha < \beta \text{ and } \widetilde{\alpha} + \log(\frac{\beta}{\beta-\alpha}) < \lambda \\ \infty & \text{otherwise} \end{cases};$$

in such a case **(C1)** fails since $\mathcal{D}_{\widetilde{\alpha}} = \{\alpha \in \mathbb{R} : \alpha < \beta(1 - e^{\widetilde{\alpha} - \lambda})\}$ for all $\widetilde{\alpha} \in \mathbb{R}$ and **(C2)** fails since $\{\widetilde{\alpha} \in \mathbb{R} : \Lambda(\widetilde{\alpha}, 0) < \infty\} = \{\widetilde{\alpha} \in \mathbb{R} : \widetilde{\alpha} < \lambda\} \neq \mathbb{R}$. Finally we check that **(A2)** holds; indeed, for each fixed $\widetilde{\alpha} \in \mathbb{R}$, $\mathcal{D}_{\widetilde{\alpha}}$ is trivially nonempty and $\Lambda(\widetilde{\alpha}, \cdot) : \mathcal{D}_{\widetilde{\alpha}} \to \mathbb{R}$ is surjective by noting that, for all $y \in \mathbb{R}$, we have $\Lambda(\widetilde{\alpha}, \alpha(y)) = y$, with $\alpha(y) = \beta(1 - e^{\widetilde{\alpha} - \lambda(1 - e^{-y})}) \in \mathcal{D}_{\widetilde{\alpha}}$.

Remark. This contrasts situations where, conversely, one does expect dependence but where the latter dissolves completely. Although the following example is based on embedding, and hence not directly comparable, it displays this contrast clearly. Let (N_t) be a Poisson process with intensity (Λ_t) and arrival times (S_n). Put independent marks $X_1, X_2, \ldots$ on $S_1, S_2, \ldots$ and select the sub-process of arrival times $(\widetilde{S}_n)$ with records marks (see e.g. [7] and [8]). Since each X_n is a record with probability $\frac{1}{n}$, the record counting process $(\widetilde{N}_t)$ should completely depend on the history of (N_t). However, if (Λ_t) is randomized exponentially, then $(\widetilde{N}_t)$ is an independent Poisson process.

5.2 Inequalities and comparisons between convergences

Let $((J_n, Z_n))$ be a Markov additive process according to the presentation in the previous sections. Let $((\overline{J}_n, \overline{Z}_n))$ be another Markov additive process of the same kind and in particular we assume that $(\overline{J}_n)$ takes values on E as (J_n). We shall use the notation $\overline{Z} = (\widetilde{U}_n, U_n)$; for all the other items concerning $((\overline{J}_n, \overline{Z}_n))$ we use overlined symbols (for instance we write $\overline{\Lambda}$, $\overline{N}_t$, $\overline{p}(b)$ and $\overline{w}$ in place of Λ, N_t, $p(b)$ and w respectively).

The following statements relate some inequalities with some comparisons between convergences.

• Assume there exist the limits

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\left(\frac{\widetilde{S}_n}{n}, \frac{S_n}{n}\right) \in B\right) =: \Lambda^*(B) \text{ and } \lim_{n\to\infty} \frac{1}{n} \log P\left(\left(\frac{\widetilde{U}_n}{n}, \frac{U_n}{n}\right) \in B\right) =: \overline{\Lambda}^*(B);$$

then

$$\Lambda^*(B) > \overline{\Lambda}^*(B) \text{ implies } \lim_{n\to\infty} \frac{P\left(\left(\frac{\widetilde{S}_n}{n}, \frac{S_n}{n}\right) \in B\right)}{P\left(\left(\frac{\widetilde{U}_n}{n}, \frac{U_n}{n}\right) \in B\right)} = 0.$$

• Assume there exist the limits

$$\lim_{t\to\infty} \frac{1}{t} \log P\left(\left(\frac{\widetilde{S}_{N_t}}{t}, \frac{N_t}{t}\right) \in B\right) =: \Psi^*_\Lambda(B) \text{ and } \lim_{t\to\infty} \frac{1}{t} \log P\left(\left(\frac{\widetilde{U}_{\overline{N}_t}}{t}, \frac{\overline{N}_t}{t}\right) \in B\right) =: \Psi^*_{\overline{\Lambda}}(B);$$

then

$$\Psi^*_\Lambda(B) > \Psi^*_{\overline{\Lambda}}(B) \text{ implies } \lim_{t\to\infty} \frac{P\left(\left(\frac{\widetilde{S}_{N_t}}{t}, \frac{N_t}{t}\right) \in B\right)}{P\left(\left(\frac{\widetilde{U}_{\overline{N}_t}}{t}, \frac{\overline{N}_t}{t}\right) \in B\right)} = 0.$$

• Let $(p(b))$ be the level crossing probabilities as in section 4 and let $(\overline{p}(b))$ be the analogous of $(p(b))$ if we have $((\overline{J}_n, \overline{Z}_n))$ in place of $((J_n, Z_n))$. Assume there exist the limits

$$\lim_{b\to\infty} \frac{1}{b} \log p(b) = -w \text{ and } \lim_{b\to\infty} \frac{1}{b} \log \overline{p}(b) = -\overline{w};$$

then

$$w > \overline{w} \text{ implies } \lim_{b\to\infty} \frac{p(b)}{\overline{p}(b)} = 0.$$

In the statements above we require strict inequalities in order to have that some ratios goes to zero. The next Lemma 5.1 provides necessary conditions for these strict inequalities.

Lemma 5.1. *Let $((J_n, Z_n))$ and $((\overline{J}_n, \overline{Z}_n))$ be as above. Assume that*

$$\Lambda(\widetilde{\alpha}, \alpha) \leq \overline{\Lambda}(\widetilde{\alpha}, \alpha) \ (\forall(\widetilde{\alpha}, \alpha) \in \mathbb{R}^2). \tag{17}$$

Then we have

$$\Lambda^*(\widetilde{x}, x) \geq \overline{\Lambda}^*(\widetilde{x}, x) \ (\forall(\widetilde{x}, x) \in \mathbb{R}^2); \tag{18}$$

if **(A4)** *holds together with its version adapted to $((\overline{J}_n, \overline{Z}_n))$, then we have*

$$\Psi^*_\Lambda(\widetilde{x}, x) \geq \Psi^*_{\overline{\Lambda}}(\widetilde{x}, x) \ (\forall(\widetilde{x}, x) \in \mathbb{R}^2); \tag{19}$$

if **(A5)** *holds together with its version adapted to $((\overline{J}_n, \overline{Z}_n))$, then we have $w \geq \overline{w}$.*

Proof. First of all (18) is an immediate consequence of (17). Moreover, by (17), we have $\overline{\mathcal{D}}_{\widetilde{\alpha}} \subset \mathcal{D}_{\widetilde{\alpha}}$ and then $\sup \overline{\mathcal{D}}_{\widetilde{\alpha}} \leq \sup \mathcal{D}_{\widetilde{\alpha}}$ for all $\widetilde{\alpha} \in \mathbb{R}$; in conclusion (19) follows from Proposition 3.3. Finally let $(\widetilde{\alpha}, \alpha) \in \mathbb{R}^2$ be arbitrarily fixed; then we have

$$\sup_{(\widetilde{x},x)\in\mathbb{R}^2} [\widetilde{\alpha}\widetilde{x} + \alpha x - \Psi^*_\Lambda(\widetilde{x}, x)] = \Psi_\Lambda(\widetilde{\alpha}, \alpha) \text{ and } \sup_{(\widetilde{x},x)\in\mathbb{R}^2} [\widetilde{\alpha}\widetilde{x} + \alpha x - \Psi^*_{\overline{\Lambda}}(\widetilde{x}, x)] = \Psi_{\overline{\Lambda}}(\widetilde{\alpha}, \alpha),$$

whence we have $\Psi_\Lambda(\widetilde{\alpha}, \alpha) \leq \Psi_{\overline{\Lambda}}(\widetilde{\alpha}, \alpha)$ by (19). Thus we have $w \geq \overline{w}$ since $\Psi_\Lambda(\widetilde{\alpha}, 0) - c\widetilde{\alpha} \leq \Psi_{\overline{\Lambda}}(\widetilde{\alpha}, 0) - c\widetilde{\alpha}$ for all $\widetilde{\alpha} \in \mathbb{R}$. ■

Thus Lemma 5.1 motivates our interest in conditions which guarantee that (17) holds. The next Proposition 5.2 provides a condition in terms of some inequalities between moment generating functions, under the assumption that (J_n) and $(\overline{J}_n)$ have the same transition matrix.

Proposition 5.2. *Let $((J_n, Z_n))$ and $((\overline{J}_n, \overline{Z}_n))$ be as above. Assume that (J_n) and $(\overline{J}_n)$ have the same transition matrix $(p_{ij})_{i,j\in E}$ and*

$$\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_1 + \alpha S_1} | J_0 = i, J_1 = j] \leq \mathbb{E}[e^{\widetilde{\alpha}\widetilde{U}_1 + \alpha U_1} | J_0 = i, J_1 = j] \ (\forall(\widetilde{\alpha}, \alpha) \in \mathbb{R}^2), \tag{20}$$

for all $i, j \in E$ such that $p_{ij} > 0$. Then (17) holds.

Proof. Let $(\widetilde{\alpha},\alpha)\in\mathbb{R}^2$ be arbitrarily fixed. Then we have

$$\mathbb{E}[e^{\widetilde{\alpha}\widetilde{S}_n+\alpha S_n}]\leq\mathbb{E}[e^{\widetilde{\alpha}\widetilde{U}_n+\alpha U_n}]\quad(\forall n\geq 1)$$

by (1) together its version adapted to $((\overline{J}_n,\overline{Z}_n))$ and by (20); thus (17) follows from (2) together its version adapted to $((\overline{J}_n,\overline{Z}_n))$. ∎

We remark that, since the function $(\widetilde{x},x)\mapsto e^{\widetilde{\alpha}\widetilde{x}+\alpha x}$ is a convex function for each fixed $(\widetilde{\alpha},\alpha)\in\mathbb{R}^2$, $H_{ij}\leq_{cx}\overline{H}_{ij}$ implies (20), where $\leq_{cx}$ is the convex order (namely we mean the multivariate convex order; see e.g. [21], eq. (5.A.4), page 154).

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References

[1] Albrecher, H. and Boxma, O.J., A ruin model with dependence between claim sizes and claim intervals. *Insurance Math. Econom.* 35 (2004) 245–254.

[2] Albrecher, H. and Teugels, J.L., Exponential behavior in the presence of dependence in risk theory. *J. Appl. Prob.* 43 (2006) 257–273.

[3] Asmussen, S., *Applied Probability and Queues.* Wiley and Sons, New York, 1987.

[4] Asmussen, S., *Ruin Probabilities.* World Scientific, London, 2000.

[5] Billingsley, P. *Probability and measure.* Third edition. John Wiley & Sons, Inc., New York, 1995.

[6] Boxma, O. and Perry, D., A queueing model with dependence between service and interarrival times. *European J. Oper. Res.* 128 (2001) 611–624.

[7] Bruss, F.T., Invariant record processes and applications to best choice modelling. *Stochastic Process. Appl.* 30 (1988) 303–316.

[8] Bruss, F.T. and Rogers, L.C.G., Pascal processes and their characterization. *Stochastic Process. Appl.* 37 (1991) 331–338.

[9] Bucklew, J.A., *Large Deviation Techniques in Decision, Simulation, and Estimation.* Wiley, New York, 1990.

[10] Dembo, A. and Zeitouni, O., *Large Deviations Techniques and Applications.* Jones and Bartlett, Boston, 1993.

[11] Duffield, N.G. and O'Connell, N., Large deviations and overflow probabilities for a general single server queue, with applications. *Math. Proc. Camb. Phil. Soc.* 118 (1995) 363–374.

[12] Embrechts, P., Klüppelberg, C. and Mikosch T., *Modelling Extremal Events*. Springer, Berlin Heidelberg, 1997.

[13] Iscoe, I., Ney, P. and Nummelin, E., Large deviations of uniformly recurrent Markov additive processes. *Adv. Appl. Math.* 6 (1985) 373–412.

[14] Joe, H., *Multivariate Models and Dependence Concepts*. Chapman & Hall CRC Press Company, Boca Raton, 1997.

[15] Lehtonen, T. and Nyrhinen, H., Simulating level crossing probabilities by importance sampling. *Adv. Appl. Prob.* 24 (1992) 858–874.

[16] Lehtonen, T. and Nyrhinen, H., On Asymptotically Efficient Simulation of Ruin Probabilities in a Markovian Environment. *Scand. Actuarial J.* (1992) 60–75.

[17] Macci, C., Large deviation results for compound Markov renewal processes. *Braz. J. Probab. Stat.* 19 (2005) 1–12.

[18] Nelsen, R.B., *An Introduction to Copulas*. Springer, New York, 1999.

[19] Revuz, D., *Markov Chains*. North-Holland, Amsterdam, 1975.

[20] Russell, R., *The Large Deviations of Random Time-Changes*. Ph.D. Thesis, School of Mathematics, Trinity College, University of Dublin, 1997.

[21] Shaked, M. and Shanthikumar, J.G., *Stochastic Orders and their Applications*. Academic Press, San Diego, 1994.

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