LARGE ELASTIC DEFORMATIONS OF ISOTROPIC MATERIALS IV. FURTHER DEVELOPMENTS OF THE GENERAL THEORY

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The equations of motion, boundary conditions and stress-strain relations for a highly elastic material can be expressed in terms of the stored-energy function. This has been done in part I of this series (Rivlin 1948*a*), for both the cases of compressible and incompressible materials, following the methods given by E. & F. Cosserat for compressible materials.

The stored-energy function may be defined for a particular material in terms of the invariants of strain. The form in which the equations of motion, etc., are deduced, in the previous paper, does not permit the evaluation of the forces necessary to produce a specified deformation unless the actual expression for the stored-energy function in terms of the scalar invariants of the strain is introduced. In the present paper, the equations are transformed into forms more suitable for carrying out such an explicit evaluation. As examples, the surface forces necessary to produce simple shear in a cuboid of either compressible or incompressible material and those required to produce simple torsion in a right-circular cylinder of incompressible material are derived.

1. Introduction

In part I of this series of papers (Rivlin 1948*a*) the equations of motion and boundary conditions for a highly elastic material, which is isotropic in its undeformed state, are derived both for the case when the material is compressible and when it is incompressible. These were given in terms of an arbitrary stored-energy function W, which was considered to be completely determined by the principal extensions of the material at the point considered. The notion of an incompressible, neo-Hookean material was introduced as one in which the stored-energy function and the corresponding stress-strain relationships take a particularly simple form. The equations of motion and boundary conditions for an incompressible material were particularized for this case. In parts II and III (Rivlin 1948*b*, *c*) certain implications of these equations of motion and boundary conditions for an incompressible, neo-Hookean material are derived.

In the present paper, we first discuss, in § 2, the definition of components of strain somewhat more critically than has been done hitherto and derive a relationship between the components of large strain as defined by Coker & Filon (1931) and those based on the original definition of large strain given by Cauchy (1827) and employed in the previous papers of this series.

In §§ 3 to 6 the expression of the stored-energy function in terms of the strain invariants is discussed and the stress-strain relations corresponding to any choice of the stored-energy function are derived in a form more suitable for application to particular problems than those given in part I (Rivlin 1948*a*, §§ 7, 8). In § 7 the application of these results to the problem of determining the stored-energy function, from experiments on a highly elastic material, are discussed.

In §§ 8 to 11, the equations of motion and boundary conditions obtained in part I are also expressed in a form more suitable for application to particular problems and, in §§ 12 to 14, these are applied to determine the system of forces required to produce simple shear in a cuboid of either compressible or incompressible material and pure torsion in a right-circular cylinder of incompressible, highly elastic material.

A. THE STRESS-STRAIN RELATIONSHIPS

2. The definition of strain

The original complete definition of a large strain was, it appears, given by Cauchy (1827). The strain at a point of a body which, in the undeformed state of the body, lies at (x, y, z), is defined by means of six components ϵ_{xx}, ϵ_{yy}, ϵ_{zz}, ϵ_{yz}, ϵ_{zx} and ϵ_{xy}, in a rectangular, Cartesian co-ordinate system (x, y, z). Suppose, in the deformation, the point (x, y, z) moves to $(x+u, y+v, z+w) = (\xi, \eta, \zeta)$, where u, v and w and hence ξ, η and ζ are functions of x, y and z. Suppose, too, that a linear element of the material, which in the undeformed state is situated at (x, y, z), has length ds and direction-cosines (l, m, n) and moves, in the deformation, to the point (ξ, η, ζ), its length changing to ds' and its direction-cosines to (l', m', n').

Then
$$l' = d\xi/ds', \quad m' = d\eta/ds' \quad \text{and} \quad n' = d\zeta/ds', \tag{2·1}$$
where $d\xi$, $d\eta$ and $d\zeta$ are the components of length of the element ds' parallel to the axes x, y and z respectively.

Since ξ, η and ζ are functions of x, y and z, we have
$$d\xi = \xi_x dx + \xi_y dy + \xi_z dz, \tag{2·2}$$
together with similar expressions for $d\eta$ and $d\zeta$. Thus, combining (2·1) and (2·2), and bearing in mind that
$$l = dx/ds, \quad m = dy/ds \quad \text{and} \quad n = dz/ds, \tag{2·3}$$
we have
$$\left.\begin{aligned} l' &= (ds/ds')\,(\xi_x l + \xi_y m + \xi_z n), \quad m' = (ds/ds')\,(\eta_x l + \eta_y m + \eta_z n) \\ \text{and} \quad n' &= (ds/ds')\,(\zeta_x l + \zeta_y m + \zeta_z n). \end{aligned}\right\} \tag{2·4}$$

Since
$$l'^2 + m'^2 + n'^2 = 1,$$
equations (2·4) yield
$$\begin{aligned} (ds'/ds)^2 = {} & (\xi_x^2 + \eta_x^2 + \zeta_x^2)\, l^2 + (\xi_y^2 + \eta_y^2 + \zeta_y^2)\, m^2 \\ & + (\xi_z^2 + \eta_z^2 + \zeta_z^2)\, n^2 + 2(\xi_y \xi_z + \eta_y \eta_z + \zeta_y \zeta_z)\, mn \\ & + 2(\xi_z \xi_x + \eta_z \eta_x + \zeta_z \zeta_x)\, nl + 2(\xi_x \xi_y + \eta_x \eta_y + \zeta_x \zeta_y)\, lm. \end{aligned} \tag{2·5}$$

Since
$$l^2 + m^2 + n^2 = 1,$$
$$\begin{aligned} \tfrac{1}{2}\{(ds'/ds)^2 - 1\} = {} & \tfrac{1}{2}(\xi_x^2 + \eta_x^2 + \zeta_x^2 - 1)\, l^2 + \tfrac{1}{2}(\xi_y^2 + \eta_y^2 + \zeta_y^2 - 1)\, m^2 \\ & + \tfrac{1}{2}(\xi_z^2 + \eta_z^2 + \zeta_z^2 - 1)\, n^2 + (\xi_y \xi_z + \eta_y \eta_z + \zeta_y \zeta_z)\, mn \\ & + (\xi_z \xi_x + \eta_z \eta_x + \zeta_z \zeta_x)\, nl + (\xi_x \xi_y + \eta_x \eta_y + \zeta_x \zeta_y)\, lm. \end{aligned} \tag{2·6}$$

The six components of strain ϵ_{xx}, ϵ_{yy}, ϵ_{zz}, ϵ_{yz}, ϵ_{zx} and ϵ_{xy} are defined by Cauchy as the coefficients of l^2, m^2, n^2, mn, nl and lm respectively, in the expression on the right-hand side of equation (2·6).

Thus, writing $\xi = x+u, \quad \eta = y+v \quad \text{and} \quad \zeta = z+w,$

we have
$$\epsilon_{xx} = u_x + \tfrac{1}{2}(u_x^2+v_x^2+w_x^2), \text{ etc.}$$
and
$$\epsilon_{yz} = w_y + v_z + u_y u_z + v_y v_z + w_y w_z, \text{ etc.} \tag{2·7}$$

If we know the six components of strain at a point of specified position, in the undeformed state of the body, then equation (2·6) can be used to calculate the extension of any linear element at that point, whose direction-cosines in the undeformed state of the body are given. It is just in this sense that the components of strain as defined above describe the deformation in the neighbourhood of the point to which they refer. It is clear that the deformation could equally be described by any six independent functions of the six components already defined. However, such a procedure and the description of these functions as the components of strain would not add anything to our knowledge of the deformation in the neighbourhood of the point considered and could only be justified on the grounds of mathematical convenience.

It is noted that in the definition of strain based on equation (2·6), the point at which the strain is specified is defined by its position in the undeformed state of the body, and the strain components are given by the variation of the displacement components (u, v, w) as functions of position (x, y, z), measured in the undeformed state. It has been suggested by Coker & Filon (1931)—and this suggestion has been taken up by others, notably by Seth (1935) and by Murnaghan (1937)—that the strain should be defined in terms of the variations of the displacement components (u, v, w) as functions of position (ξ, η, ζ) measured in the *deformed* state of the body.

Since (x, y, z), the co-ordinates of a point of the body before deformation, can be considered as functions of its co-ordinates (ξ, η, ζ) after deformation, we have

$$dx = x_\xi d\xi + x_\eta d\eta + x_\zeta d\zeta, \tag{2·8}$$

together with similar expressions for dy and dz. From these equations and the relations (2·1) and (2·3), we have

$$\left.\begin{aligned} l = (ds'/ds)\,(x_\xi l' + x_\eta m' + x_\zeta n'), \quad m = (ds'/ds)\,(y_\xi l' + y_\eta m' + y_\zeta n') \\ \text{and} \qquad n = (ds'/ds)\,(z_\xi l' + z_\eta m' + z_\zeta n'). \end{aligned}\right\} \tag{2·9}$$

Equations (2·9) yield

$$\begin{aligned} -\tfrac{1}{2}\{(ds/ds')^2-1\} = & -\tfrac{1}{2}(x_\xi^2+y_\xi^2+z_\xi^2-1)\,l'^2 - \tfrac{1}{2}(x_\eta^2+y_\eta^2+z_\eta^2-1)\,m'^2 \\ & -\tfrac{1}{2}(x_\zeta^2+y_\zeta^2+z_\zeta^2-1)\,n'^2 - (x_\eta x_\zeta + y_\eta y_\zeta + z_\eta z_\zeta)\,m'n' \\ & -(x_\zeta x_\xi + y_\zeta y_\xi + z_\zeta z_\xi)\,n'l' - (x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta)\,l'm'. \end{aligned} \tag{2·10}$$

The six components of strain, which we shall denote by $\epsilon_{\xi\xi}, \epsilon_{\eta\eta}, \epsilon_{\zeta\zeta}, \epsilon_{\eta\zeta}, \epsilon_{\zeta\xi}$ and $\epsilon_{\xi\eta}$, are defined as the coefficients of l'^2, m'^2, n'^2, $m'n'$, $n'l'$ and $l'm'$ respectively in the expression on the right-hand side of (2·10). Thus, writing

$$x = \xi - u, \quad y = \eta - v \quad \text{and} \quad z = \zeta - w,$$

$$\left.\begin{aligned} \text{we have} \qquad & \epsilon_{\xi\xi} = u_\xi - \tfrac{1}{2}(u_\xi^2+v_\xi^2+w_\xi^2), \text{ etc.} \\ \text{and} \qquad & \epsilon_{\eta\zeta} = v_\zeta + w_\eta - u_\eta u_\zeta - v_\eta v_\zeta - w_\eta w_\zeta, \text{ etc.} \end{aligned}\right\} \tag{2·11}$$

Now, a knowledge of these six components of strain, appertaining to a specified point of the deformed material, enables us to calculate, from equation (2·10), the extension which has been undergone by any linear element at the point considered, whose directions in the deformed state are given.

Alternatively, instead of using the expression (2·9) for obtaining (l, m, n) in terms of (l', m', n') and deriving the result (2·10), we find, from (2·4), that

$$\left.\begin{aligned} l &= \frac{ds'}{ds}\frac{1}{\tau}\left(\frac{\partial\tau}{\partial\xi_x}l' + \frac{\partial\tau}{\partial\eta_x}m' + \frac{\partial\tau}{\partial\zeta_x}n'\right), \quad m = \frac{ds'}{ds}\frac{1}{\tau}\left(\frac{\partial\tau}{\partial\xi_y}l' + \frac{\partial\tau}{\partial\eta_y}m' + \frac{\partial\tau}{\partial\zeta_y}n'\right) \\ \text{and} \qquad n &= \frac{ds'}{ds}\frac{1}{\tau}\left(\frac{\partial\tau}{\partial\xi_z}l' + \frac{\partial\tau}{\partial\eta_z}m' + \frac{\partial\tau}{\partial\zeta_z}n'\right), \end{aligned}\right\} \tag{2·12}$$

where

$$\tau = \begin{vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{vmatrix}.$$

Since

$$l^2 + m^2 + n^2 = 1,$$

equations (2·12) yield

$$\begin{aligned} \left(\frac{ds}{ds'}\right)^2 \tau^2 = {} & \left[\left(\frac{\partial\tau}{\partial\xi_x}\right)^2 + \left(\frac{\partial\tau}{\partial\xi_y}\right)^2 + \left(\frac{\partial\tau}{\partial\xi_z}\right)^2\right] l'^2 + \left[\left(\frac{\partial\tau}{\partial\eta_x}\right)^2 + \left(\frac{\partial\tau}{\partial\eta_y}\right)^2 + \left(\frac{\partial\tau}{\partial\eta_z}\right)^2\right] m'^2 \\ & + \left[\left(\frac{\partial\tau}{\partial\zeta_x}\right)^2 + \left(\frac{\partial\tau}{\partial\zeta_y}\right)^2 + \left(\frac{\partial\tau}{\partial\zeta_z}\right)^2\right] n'^2 + 2\left[\frac{\partial\tau}{\partial\eta_x}\frac{\partial\tau}{\partial\zeta_x} + \frac{\partial\tau}{\partial\eta_y}\frac{\partial\tau}{\partial\zeta_y} + \frac{\partial\tau}{\partial\eta_z}\frac{\partial\tau}{\partial\zeta_z}\right] m'n' \\ & + 2\left[\frac{\partial\tau}{\partial\zeta_x}\frac{\partial\tau}{\partial\xi_x} + \frac{\partial\tau}{\partial\zeta_y}\frac{\partial\tau}{\partial\xi_y} + \frac{\partial\tau}{\partial\zeta_z}\frac{\partial\tau}{\partial\xi_z}\right] n'l' + 2\left[\frac{\partial\tau}{\partial\xi_x}\frac{\partial\tau}{\partial\eta_x} + \frac{\partial\tau}{\partial\xi_y}\frac{\partial\tau}{\partial\eta_y} + \frac{\partial\tau}{\partial\xi_z}\frac{\partial\tau}{\partial\eta_z}\right] l'm'. \end{aligned} \tag{2·13}$$

Thus, comparing (2·13) with (2·10), the components of strain given by equations (2·11) are also given by

$$\left.\begin{aligned} 1 - 2\epsilon_{\xi\xi} &= \frac{1}{\tau^2}\left[\left(\frac{\partial\tau}{\partial\xi_x}\right)^2 + \left(\frac{\partial\tau}{\partial\xi_y}\right)^2 + \left(\frac{\partial\tau}{\partial\xi_z}\right)^2\right], \text{ etc.} \\ \text{and} \qquad -\epsilon_{\eta\zeta} &= \frac{1}{\tau^2}\left[\frac{\partial\tau}{\partial\eta_x}\frac{\partial\tau}{\partial\zeta_x} + \frac{\partial\tau}{\partial\eta_y}\frac{\partial\tau}{\partial\zeta_y} + \frac{\partial\tau}{\partial\eta_z}\frac{\partial\tau}{\partial\zeta_z}\right], \text{ etc.} \end{aligned}\right\} \tag{2·14}$$

Now, if we accept the Cauchy definition of strain, i.e. that given by equations (2·7), it can be shown that the strain in the neighbourhood of any point of the material can be considered to consist of a pure rotation followed by a pure, homogeneous strain. This pure, homogeneous strain has, in the axial system (x, y, z), the six components $\epsilon'_{xx}, \epsilon'_{yy}, \ldots, \epsilon'_{xy}$, given by

$$\left.\begin{aligned} 1 + 2\epsilon'_{xx} &= (1+u_x)^2 + u_y^2 + u_z^2, \text{ etc.} \\ \text{and} \qquad \epsilon'_{yz} &= v_x w_x + (1+v_y)\, w_y + v_z(1+w_z), \text{ etc.} \end{aligned}\right\} \tag{2·15}$$

From the formulae (2·14) and (2·15), we can derive, by simple algebraic manipulation, the six relations

$$\left.\begin{aligned} 1 - 2\epsilon_{\xi\xi} &= [(1+2\epsilon'_{yy})(1+2\epsilon'_{zz}) - \epsilon'^2_{yz}]/\tau^2, \text{ etc.} \\ \text{and} \qquad -\epsilon_{\eta\zeta} &= [\epsilon'_{xy}\epsilon'_{zx} - (1+2\epsilon'_{xx})\,\epsilon'_{yz}]/\tau^2, \text{ etc.} \end{aligned}\right\} \tag{2·16}$$

Bearing in mind that

$$\tau^2 = \begin{vmatrix} 1+2\epsilon'_{xx} & \epsilon'_{xy} & \epsilon'_{zx} \\ \epsilon'_{xy} & 1+2\epsilon'_{yy} & \epsilon'_{yz} \\ \epsilon'_{zx} & \epsilon'_{yz} & 1+2\epsilon'_{zz} \end{vmatrix},$$

it is seen that equations (2·16) express the six strain components $\epsilon_{\xi\xi}, \ldots, \epsilon_{\xi\eta}$ in terms of the six strain components $\epsilon'_{xx}, \ldots, \epsilon'_{xy}$.

From (2·16), the six inverse relations

$$\left.\begin{aligned} 1+2\epsilon'_{xx} &= \tau^2[(1-2\epsilon_{\eta\eta})(1-2\epsilon_{\zeta\zeta})-\epsilon^2_{\eta\zeta}], \text{ etc.} \\ \text{and} \qquad \epsilon'_{yz} &= \tau^2[\epsilon_{\xi\eta}\epsilon_{\zeta\xi}+(1-2\epsilon_{\xi\xi})\,\epsilon_{\eta\zeta}], \text{ etc.} \end{aligned}\right\} \tag{2·17}$$

can be obtained.

3. The principal axes of strain and strain invariants

By a suitable choice of the co-ordinate axes (x, y, z), equation (2·5) can be written in the form

$$(ds'/ds)^2 = \lambda_1^2 l^2+\lambda_2^2 m^2+\lambda_3^2 n^2. \tag{3·1}$$

At each point of the material, there is, for a specified deformation, one and only one choice of the co-ordinate axes which allows $(ds'/ds)^2$ to be expressed in this way. The directions of these axes remain at right angles after the deformation has taken place, but, in general, each direction is changed. Taking axes in these new directions as co-ordinate axes, it can be seen that equation (2·10) becomes

$$\left(\frac{ds}{ds'}\right)^2 = \frac{1}{\lambda_1^2}l'^2+\frac{1}{\lambda_2^2}m'^2+\frac{1}{\lambda_3^2}n'^2, \tag{3·2}$$

where (l', m', n') are the direction-cosines of a linear element in its deformed state, relative to the new system of axes. The directions of the axes of this new system vary, in general, from point to point of the material.

The values of λ_1^2, λ_2^2 and λ_3^2 can be determined in terms of the components of strain in a rectangular, Cartesian co-ordinate system, as the values of λ^2 satisfying

$$\begin{vmatrix} 1+2\epsilon_{xx}-\lambda^2 & \epsilon_{xy} & \epsilon_{zx} \\ \epsilon_{xy} & 1+2\epsilon_{yy}-\lambda^2 & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{yz} & 1+2\epsilon_{zz}-\lambda^2 \end{vmatrix} \equiv \begin{vmatrix} 1+2\epsilon'_{xx}-\lambda^2 & \epsilon'_{xy} & \epsilon'_{zx} \\ \epsilon'_{xy} & 1+2\epsilon'_{yy}-\lambda^2 & \epsilon'_{yz} \\ \epsilon'_{zx} & \epsilon'_{yz} & 1+2\epsilon'_{zz}-\lambda^2 \end{vmatrix} = 0. \tag{3·3}$$

From (3·3), we have

$$\left.\begin{aligned} I_1 &= \lambda_1^2+\lambda_2^2+\lambda_3^2 = 3+2(\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}) \equiv 3+2(\epsilon'_{xx}+\epsilon'_{yy}+\epsilon'_{zz}), \\ I_2 &= \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2 \\ &= (1+2\epsilon_{yy})(1+2\epsilon_{zz})+(1+2\epsilon_{zz})(1+2\epsilon_{xx})+(1+2\epsilon_{xx})(1+2\epsilon_{yy})-\epsilon^2_{yz}-\epsilon^2_{zx}-\epsilon^2_{xy} \\ &\equiv (1+2\epsilon'_{yy})(1+2\epsilon'_{zz})+(1+2\epsilon'_{zz})(1+2\epsilon'_{xx})+(1+2\epsilon'_{xx})(1+2\epsilon'_{yy})-\epsilon'^2_{yz}-\epsilon'^2_{zx}-\epsilon'^2_{xy} \\ \text{and} \quad I_3 &= \tau^2 = \lambda_1^2\lambda_2^2\lambda_3^2 = (1+2\epsilon_{xx})(1+2\epsilon_{yy})(1+2\epsilon_{zz}) \\ &\qquad +2\epsilon_{yz}\epsilon_{zx}\epsilon_{xy}-(1+2\epsilon_{xx})\,\epsilon^2_{yz}-(1+2\epsilon_{yy})\,\epsilon^2_{zx}-(1+2\epsilon_{zz})\,\epsilon^2_{xy} \\ &\equiv (1+2\epsilon'_{xx})(1+2\epsilon'_{yy})(1+2\epsilon'_{zz})+2\epsilon'_{yz}\epsilon'_{zx}\epsilon'_{xy} \\ &\qquad -(1+2\epsilon'_{xx})\,\epsilon'^2_{yz}-(1+2\epsilon'_{yy})\,\epsilon'^2_{zx}-(1+2\epsilon'_{zz})\,\epsilon'^2_{xy}, \end{aligned}\right\} \tag{3·4}$$

where τ is the ratio of a volume element of the material in the deformed state to that in the undeformed state. I_1, I_2 and I_3 are invariants, since λ_1^2, λ_2^2 and λ_3^2 depend only on the state of strain and not on the choice of the reference axes.

Again $1/\lambda_1^2$, $1/\lambda_2^2$ and $1/\lambda_3^2$ can be obtained, in terms of the components of strain $\epsilon_{\xi\xi}$, $\epsilon_{\eta\eta}$, $\epsilon_{\zeta\zeta}$, $\epsilon_{\eta\zeta}$, $\epsilon_{\zeta\xi}$ and $\epsilon_{\xi\eta}$, as the values of $1/\lambda^2$ given by

$$\begin{vmatrix} 1-2\epsilon_{\xi\xi}-(1/\lambda^2) & -\epsilon_{\xi\eta} & -\epsilon_{\zeta\xi} \\ -\epsilon_{\xi\eta} & 1-2\epsilon_{\eta\eta}-(1/\lambda^2) & -\epsilon_{\eta\zeta} \\ -\epsilon_{\zeta\xi} & -\epsilon_{\eta\zeta} & 1-2\epsilon_{\zeta\zeta}-(1/\lambda^2) \end{vmatrix} = 0. \tag{3·5}$$

Whence
$$\left.\begin{aligned} J_1 &= \frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2} = 3-2(\epsilon_{\xi\xi}+\epsilon_{\eta\eta}+\epsilon_{\zeta\zeta}), \\ J_2 &= \frac{1}{\lambda_2^2\lambda_3^2}+\frac{1}{\lambda_3^2\lambda_1^2}+\frac{1}{\lambda_1^2\lambda_2^2} = (1-2\epsilon_{\eta\eta})(1-2\epsilon_{\zeta\zeta}) \\ &\quad +(1-2\epsilon_{\zeta\zeta})(1-2\epsilon_{\xi\xi})+(1-2\epsilon_{\xi\xi})(1-2\epsilon_{\eta\eta})-\epsilon_{\eta\zeta}^2-\epsilon_{\zeta\xi}^2-\epsilon_{\xi\eta}^2 \\ \text{and}\quad J_3 &= 1/\tau^2 = 1/\lambda_1^2\lambda_2^2\lambda_3^2 = (1-2\epsilon_{\xi\xi})(1-2\epsilon_{\eta\eta})(1-2\epsilon_{\zeta\zeta}) \\ &\quad -2\epsilon_{\eta\zeta}\epsilon_{\zeta\xi}\epsilon_{\xi\eta}-(1-2\epsilon_{\xi\xi})\epsilon_{\eta\zeta}^2-(1-2\epsilon_{\eta\eta})\epsilon_{\zeta\xi}^2-(1-2\epsilon_{\zeta\zeta})\epsilon_{\xi\eta}^2, \end{aligned}\right\} \tag{3·6}$$

where J_1, J_2 and J_3 are also strain invariants. It can readily be seen, by comparing equations (3·4) and (3·6), that

$$J_1 = I_2/I_3, \quad J_2 = I_1/I_3 \quad \text{and} \quad J_3 = 1/I_3. \tag{3·7}$$

4. Expression for the stored-energy function in an initially isotropic material

If the highly elastic material considered is isotropic in its undeformed state, then the stored energy W, per unit volume measured in the undeformed state, must be a symmetrical function of λ_1, λ_2 and λ_3. Moreover, the energy stored in an element of the material is unaltered by a pure rotation of the element—for example, by a change of sign, but not of magnitude, of λ_1 and λ_2, leaving λ_3 unaltered. Consequently, the stored energy can be considered to be a symmetrical function of λ_1^2, λ_2^2 and λ_3^2 and can therefore be expressed as a function of the three strain invariants I_1, I_2 and I_3, defined by (3·4). We may then write

$$W = W(I_1, I_2, I_3). \tag{4·1}$$

In view of the relationships (3·7) it is, of course, possible to write

$$W = W(J_1, J_2, J_3).$$

However, nothing further except perhaps, in some cases, formal simplicity can be obtained by writing W in this form and we shall not consider it further.

The corresponding stress-strain relationship can be obtained, for a compressible material, by substituting for W, from (4·1), in the relationships (Rivlin 1948 *a*, equations (7·5))

$$\left.\begin{aligned} t_{xx} &= \frac{1}{\tau}\left[(1+u_x)\frac{\partial W}{\partial u_x}+u_y\frac{\partial W}{\partial u_y}+u_z\frac{\partial W}{\partial u_z}\right], \text{ etc.} \\ \text{and}\quad t_{yz} &= \frac{1}{\tau}\left[w_x\frac{\partial W}{\partial v_x}+w_y\frac{\partial W}{\partial v_y}+(1+w_z)\frac{\partial W}{\partial v_z}\right], \text{ etc.} \end{aligned}\right\} \tag{4·2}$$

Thus, for example,

$$t_{xx} = \frac{1}{\tau}\Big[(1+u_x)\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial u_x}+\frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial u_x}+\frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial u_x}\right)+u_y\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial u_y}+\frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial u_y}+\frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial u_y}\right)$$
$$+u_z\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial u_z}+\frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial u_z}+\frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial u_z}\right)\Big]$$
$$= \frac{1}{\tau}\left[(1+u_x)\frac{\partial I_1}{\partial u_x}+u_y\frac{\partial I_1}{\partial u_y}+u_z\frac{\partial I_1}{\partial u_z}\right]\frac{\partial W}{\partial I_1}+\frac{1}{\tau}\left[(1+u_x)\frac{\partial I_2}{\partial u_x}+u_y\frac{\partial I_2}{\partial u_y}+u_z\frac{\partial I_2}{\partial u_z}\right]\frac{\partial W}{\partial I_2}$$
$$+\frac{1}{\tau}\left[(1+u_x)\frac{\partial I_3}{\partial u_x}+u_y\frac{\partial I_3}{\partial u_y}+u_z\frac{\partial I_3}{\partial u_z}\right]\frac{\partial W}{\partial I_3}. \qquad (4{\cdot}3)$$

Making use of the formulae (3·4) and (2·15), equation (4·3) and similar equations for the remaining stress components become

$$\left.\begin{aligned} t_{xx} &= \frac{2}{\tau}\left[(1+2\epsilon'_{xx})\frac{\partial W}{\partial I_1}-\{(1+2\epsilon'_{yy})(1+2\epsilon'_{zz})-\epsilon'^2_{yz}\}\frac{\partial W}{\partial I_2}+\left(I_3\frac{\partial W}{\partial I_3}+I_2\frac{\partial W}{\partial I_2}\right)\right], \text{ etc.}\\ \text{and}\quad t_{yz} &= \frac{2}{\tau}\left[\epsilon'_{yz}\frac{\partial W}{\partial I_1}-\{\epsilon'_{xy}\epsilon'_{zx}-(1+2\epsilon'_{xx})\epsilon'_{yz}\}\frac{\partial W}{\partial I_2}\right], \text{ etc.}\end{aligned}\right\} \qquad (4{\cdot}4)$$

In view of the relations (2·16), these equations may be written in the alternative form

$$\left.\begin{aligned} t_{xx} &= \frac{2}{\tau}\left[(1+2\epsilon'_{xx})\frac{\partial W}{\partial I_1}-(1-2\epsilon_{\xi\xi})I_3\frac{\partial W}{\partial I_2}+\left(I_3\frac{\partial W}{\partial I_3}+I_2\frac{\partial W}{\partial I_2}\right)\right], \text{ etc.}\\ \text{and}\qquad t_{yz} &= \frac{2}{\tau}\left[\epsilon'_{yz}\frac{\partial W}{\partial I_1}+\epsilon_{\eta\zeta}I_3\frac{\partial W}{\partial I_2}\right], \text{ etc.}\end{aligned}\right\} \qquad (4{\cdot}5)$$

For an incompressible material, the stored energy W is a function of I_1 and I_2 only, I_3 being always unity. The stress-strain relationships then become

$$\left.\begin{aligned} t_{xx} &= 2\left[(1+2\epsilon'_{xx})\frac{\partial W}{\partial I_1}-(1-2\epsilon_{\xi\xi})\frac{\partial W}{\partial I_2}+I_2\frac{\partial W}{\partial I_2}\right]+p, \text{ etc.}\\ \text{and}\qquad t_{yz} &= 2\left[\epsilon'_{yz}\frac{\partial W}{\partial I_1}+\epsilon_{\eta\zeta}\frac{\partial W}{\partial I_2}\right], \text{ etc.,}\end{aligned}\right\} \qquad (4{\cdot}6)$$

where p is a hydrostatic pressure. The terms $I_2(\partial W/\partial I_2)$ in the first three of these equations may be incorporated into p.

5. Limitations on the form of the stress-strain relationships

It has been tacitly assumed by certain workers that if the material is initially isotropic, then any stress-strain relationship, which is unaltered by a cyclic rotation of the axes x, y, z of the rectangular, Cartesian reference system, is allowable. Thus, Seth (1935), for example, has taken as basic stress-strain relationships, six equations which have, with our notation, the form

$$t_{xx} = \lambda(\epsilon_{\xi\xi}+\epsilon_{\eta\eta}+\epsilon_{\zeta\zeta})+2\mu\epsilon_{\xi\xi}, \text{ etc.} \quad \text{and} \quad t_{yz} = \mu\epsilon_{\eta\zeta}, \text{ etc.} \qquad (5{\cdot}1)$$

In these, λ and μ are physical constants for the particular material considered. The material to which the stress-strain relationships (5·1) apply must be compressible, for by means of them the stress components are uniquely determined when the strain components are specified. For an incompressible material they are, of course, undetermined to the extent of an arbitrary hydrostatic pressure.

From (3·6) we have $$\epsilon_{\xi\xi}+\epsilon_{\eta\eta}+\epsilon_{\zeta\zeta}=-\tfrac{1}{2}(J_1-3). \tag{5·2}$$

Employing the relations (3·7), this yields

$$\epsilon_{\xi\xi}+\epsilon_{\eta\eta}+\epsilon_{\zeta\zeta}=-\frac{1}{2}\left(\frac{I_2}{I_3}-3\right). \tag{5·3}$$

Thus, the stress-strain relations (5·1) may be rewritten in the form

$$t_{xx}=-\mu(1-2\epsilon_{\xi\xi})-\tfrac{1}{2}\lambda\left(\frac{I_2}{I_3}-3\right)+\mu, \text{ etc.} \quad \text{and} \quad t_{yz}=\mu\epsilon_{\eta\zeta}, \text{ etc.} \tag{5·4}$$

Comparing equations (5·4) and (4·5), we have

$$\left.\begin{aligned} \frac{\partial W}{\partial I_1}=0, \qquad \mu=\frac{2}{\tau}I_3\frac{\partial W}{\partial I_2}=2I_3^{\frac{1}{2}}\frac{\partial W}{\partial I_2} \\ \text{and} \qquad -\tfrac{1}{2}\lambda\left(\frac{I_2}{I_3}-3\right)+\mu=\frac{2}{\tau}\left[I_3\frac{\partial W}{\partial I_3}+I_2\frac{\partial W}{\partial I_2}\right]. \end{aligned}\right\} \tag{5·5}$$

Whence $$\frac{\partial W}{\partial I_1}=0,\quad \frac{\partial W}{\partial I_2}=\tfrac{1}{2}\mu I_3^{-\frac{1}{2}} \quad \text{and} \quad \frac{\partial W}{\partial I_3}=\tfrac{1}{2}I_3^{-\frac{1}{2}}\left[-(\tfrac{1}{2}\lambda+\mu)\frac{I_2}{I_3}+(\tfrac{3}{2}\lambda+\mu)\right]. \tag{5·6}$$

From the second and third of equations (5·6), we obtain

$$\frac{\partial^2 W}{\partial I_3\partial I_2}=-\tfrac{1}{4}\mu I_3^{-\frac{3}{2}} \quad \text{and} \quad \frac{\partial^2 W}{\partial I_2\partial I_3}=-\tfrac{1}{2}(\tfrac{1}{2}\lambda+\mu)\,I_3^{-\frac{3}{2}}, \tag{5·7}$$

respectively.

Equations (5·7) are compatible only if $\lambda=-\mu$. Otherwise, the stored-energy function W for the material cannot be a function of the strain invariants only. The equations (5·1) are therefore not allowable for the description of the elastic properties of an isotropic material in which the stored energy is determined solely by the state of pure, homogeneous strain at the point considered.

6. Stress-strain relationships for pure, homogeneous strain

For a general pure, homogeneous strain of a highly elastic material, in which a unit cube of the material is deformed into a cuboid of dimensions λ_1, λ_2 and λ_3 parallel to the x, y and z axes respectively, the stress-strain relationships are readily obtained, from equations (4·5) or (4·6), by putting

$$u=(\lambda_1-1)\,x, \quad v=(\lambda_2-1)\,y \quad \text{and} \quad w=(\lambda_3-1)\,z. \tag{6·1}$$

They are, for a compressible material, from (4·5),

$$t_{xx}=\frac{2}{\tau}\left[\lambda_1^2\frac{\partial W}{\partial I_1}-\frac{I_3}{\lambda_1^2}\frac{\partial W}{\partial I_2}+I_3\frac{\partial W}{\partial I_3}+I_2\frac{\partial W}{\partial I_2}\right], \tag{6·2}$$

with similar equations for t_{yy} and t_{zz} obtained by replacing λ_1 by λ_2 and λ_3 respectively.

For an incompressible material the corresponding equations may be obtained from (4·6) in a similar manner. They are

$$t_{xx}=2\left[\lambda_1^2\frac{\partial W}{\partial I_1}-\frac{1}{\lambda_1^2}\frac{\partial W}{\partial I_2}\right]+p, \text{ etc.} \tag{6·3}$$

For a pure, homogeneous strain, in which forces are applied only to the faces of the cube normal to the x and y axes, $t_{zz} = 0$. Then, for a compressible material, we obtain, from (6·2),

$$t_{xx} = \frac{2}{\tau}\left(\lambda_1^2 - \frac{I_3}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2\frac{\partial W}{\partial I_2}\right) \quad \text{and} \quad t_{yy} = \frac{2}{\tau}\left(\lambda_2^2 - \frac{I_3}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2\frac{\partial W}{\partial I_2}\right). \tag{6·4}$$

For an incompressible material we obtain, from (6·3),

$$t_{xx} = 2\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial W}{\partial I_1} + \lambda_2^2\frac{\partial W}{\partial I_2}\right) \quad \text{and} \quad t_{yy} = 2\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial W}{\partial I_1} + \lambda_1^2\frac{\partial W}{\partial I_2}\right). \tag{6·5}$$

Equations (6·2) or (6·3) can be specialized for the case of simple extension by putting

$$\lambda_2^2 = \lambda_3^2 = I_3^{\frac{1}{2}}/\lambda_1,$$

and, for the case of pure shear, by putting

$$\lambda_2 = 1 \quad \text{and} \quad \lambda_3 = I_3^{\frac{1}{2}}/\lambda_1,$$

bearing in mind that $I_3 = 1$ for an incompressible material.

7. Some remarks on the deduction of the elastic laws from experiment

By comparing the results of experiments with the predictions of the theory it should be possible to determine the form of the stored-energy function. The accuracy of this determination is, of course, ultimately limited by the accuracy with which the experiments are carried out.

In the past it has been customary to carry out deformation-load tests for some simple type of deformation, e.g. simple extension or simple shear, subsequently fitting some arbitrary and apparently simple types of stress-strain characteristic to the experimental results obtained. In the light of the theory given above, it is clear that the form of the stress-strain characteristic, which should be fitted to the experimental results, is the specialization of (4·5) or (4·6) to the type of deformation obtaining in the experiment. Various simple forms of W as functions of the strain invariants should be introduced and the physical parameters involved obtained from the experimental results. It is only by such means that it can be hoped to obtain from experiment stress-strain relations for particular materials, which will enable us to correlate the results of experiments on different types of deformation.

The simplest forms of W, which will in turn lead to simple mathematical formulations of the general elasticity theory for the materials concerned, do not necessarily lead to the simplest stress-deformation relationship for particular simple types of deformation.

As an example of this, we can consider the case of an incompressible neo-Hookean material considered in earlier papers of this series. The stress-deformation relationship for simple extension takes the form

$$\text{stress proportional to } \left(\lambda^2 - \frac{1}{\lambda}\right),$$

where λ is the ratio of stretched to unstretched length. The stored-energy function, however, has the particularly simple form

$$W \text{ proportional to } (I_1 - 3).$$

If we considered a material for which stress is proportional to $(\lambda - 1)$ for simple extension, the corresponding stored-energy function would have a relatively complicated form and the general stress-strain relationships would also assume a relatively complicated form.

In deducing the form of the stored-energy function from experiment, care must be taken not to make too general a deduction from any particular experiment. In general, any particular experiment will leave certain elements in the form of the stored-energy function undetermined, and it is important, in interpreting results, to see from an examination of equations (4·5) or (4·6) just how much information we could expect the experiment to yield. Let us consider, as an example, the case of simple shear. It is shown below, in § 12, that the relation between shearing stress and amount of shear K, for a simple shear, is

$$t_{xy} = 2\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) K$$

and

$$I_1 = I_2 = 3 + K^2.$$

If, in an experiment, we find that the relation between t_{xy} and K is linear, this implies that $(\partial W/\partial I_1 + \partial W/\partial I_2)$ is a constant, but does not tell us completely the form of W. Other measurements are necessary to obtain this.

B. The equations of motion and boundary conditions

8. The equations of motion for a compressible material

In part I (Rivlin 1948*a*, § 15) the equations of motion for a body of compressible, elastic material, which is isotropic in its undeformed state, are given as

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial u_z}\right) + \rho X = \rho \frac{\partial^2 u}{\partial t^2}, \text{ etc.,} \tag{8·1}$$

where ρ is the density of the material of the body in its undeformed state and (x, y, z) is the body force per unit mass of material at the point considered. The equations (8·1) refer to a point of the body which is at (x, y, z) in the undeformed state.

These may be transformed into a form more suitable for the discussion of certain problems by introducing the expression (4·1) for W.

Thus, from (4·1), we have

$$\left.\begin{aligned} \frac{\partial W}{\partial u_x} &= \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial u_x} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial u_x} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial u_x}, \\ \frac{\partial W}{\partial u_y} &= \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial u_y} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial u_y} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial u_y}, \text{ etc.} \end{aligned}\right\} \tag{8·2}$$

Introducing the expressions (8·2) into the first of equations (8·1), we obtain

$$\begin{aligned} &\frac{\partial W}{\partial I_1}\left[\frac{\partial}{\partial x}\left(\frac{\partial I_1}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial I_1}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial I_1}{\partial u_z}\right)\right] + \frac{\partial W}{\partial I_2}\left[\frac{\partial}{\partial x}\left(\frac{\partial I_2}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial I_2}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial I_2}{\partial u_z}\right)\right] \\ &+ \frac{\partial W}{\partial I_3}\left[\frac{\partial}{\partial x}\left(\frac{\partial I_3}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial I_3}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial I_3}{\partial u_z}\right)\right] + \frac{\partial I_1}{\partial u_x}\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial I_1}\right) + \frac{\partial I_1}{\partial u_y}\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial I_1}\right) + \frac{\partial I_1}{\partial u_z}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial I_1}\right) \\ &+ \frac{\partial I_2}{\partial u_x}\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial I_2}\right) + \frac{\partial I_2}{\partial u_y}\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial I_2}\right) + \frac{\partial I_2}{\partial u_z}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial I_2}\right) + \frac{\partial I_3}{\partial u_x}\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial I_3}\right) + \frac{\partial I_3}{\partial u_y}\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial I_3}\right) + \frac{\partial I_3}{\partial u_z}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial I_3}\right) = \rho\left(\frac{\partial^2 u}{\partial t^2} - X\right). \end{aligned} \tag{8·3}$$

Employing the expressions (3·4) for I_1, I_2 and I_3 and the relations (2·15), we obtain

$$\frac{\partial I_1}{\partial u_x} = 2(1+u_x), \quad \frac{\partial I_1}{\partial u_y} = 2u_y \quad \text{and} \quad \frac{\partial I_1}{\partial u_z} = 2u_z; \tag{8·4}$$

$$\left.\begin{aligned}
\frac{\partial I_2}{\partial u_x} &= 2\left[(1+v_y)\frac{\partial \tau}{\partial w_z} + (1+w_z)\frac{\partial \tau}{\partial v_y} - v_z\frac{\partial \tau}{\partial w_y} - w_y\frac{\partial \tau}{\partial v_z}\right],\\
\frac{\partial I_2}{\partial u_y} &= 2\left[v_z\frac{\partial \tau}{\partial w_x} + w_x\frac{\partial \tau}{\partial v_z} - v_x\frac{\partial \tau}{\partial w_z} - (1+w_z)\frac{\partial \tau}{\partial v_x}\right]\\
\text{and} \qquad \frac{\partial I_2}{\partial u_z} &= 2\left[v_x\frac{\partial \tau}{\partial w_y} + w_y\frac{\partial \tau}{\partial v_x} - (1+v_y)\frac{\partial \tau}{\partial w_x} - w_x\frac{\partial \tau}{\partial v_y}\right];
\end{aligned}\right\} \tag{8·5}$$

$$\frac{\partial I_3}{\partial u_x} = 2\tau\frac{\partial \tau}{\partial u_x}, \quad \frac{\partial I_3}{\partial u_y} = 2\tau\frac{\partial \tau}{\partial u_y} \quad \text{and} \quad \frac{\partial I_3}{\partial u_z} = 2\tau\frac{\partial \tau}{\partial u_z}. \tag{8·6}$$

τ is given by

$$\tau = \begin{vmatrix} 1+u_x & u_y & u_z \\ v_x & 1+v_y & v_z \\ w_x & w_y & 1+w_z \end{vmatrix}. \tag{8·7}$$

From the relations (8·4), we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial I_1}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial I_1}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial I_1}{\partial u_z}\right) = 2\nabla^2 u = A_1 \text{ (say)}. \tag{8·8}$$

From the relations (8·5), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial I_2}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial I_2}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial I_2}{\partial u_z}\right)\\
&= 2\left\{v_x\left[\frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial w_y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial w_z}\right)\right] + (1+v_y)\left[\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial w_z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial w_x}\right)\right] + v_z\left[\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial w_x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial w_y}\right)\right]\right.\\
&\qquad \left. + w_x\left[\frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial v_z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial v_y}\right)\right] + w_y\left[\frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial v_x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial v_z}\right)\right] + (1+w_z)\left[\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial v_y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial v_x}\right)\right]\right\}\\
&= A_2 \text{ (say)}.
\end{aligned} \tag{8·9}$$

Also, from (8·6), we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial I_3}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial I_3}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial I_3}{\partial u_z}\right) = 2\left[\frac{\partial \tau}{\partial u_x}\frac{\partial \tau}{\partial x} + \frac{\partial \tau}{\partial u_y}\frac{\partial \tau}{\partial y} + \frac{\partial \tau}{\partial u_z}\frac{\partial \tau}{\partial z}\right] = A_3 \text{ (say)}, \tag{8·10}$$

since

$$\frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \tau}{\partial u_z}\right) \equiv 0.$$

Employing the operational relationships

$$\frac{\partial}{\partial x} = \frac{\partial I_1}{\partial x}\frac{\partial}{\partial I_1} + \frac{\partial I_2}{\partial x}\frac{\partial}{\partial I_2} + \frac{\partial I_3}{\partial x}\frac{\partial}{\partial I_3}, \text{ etc.,}$$

we obtain

$$\frac{\partial I_1}{\partial u_x}\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial I_1}\right)+\frac{\partial I_1}{\partial u_y}\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial I_1}\right)+\frac{\partial I_1}{\partial u_z}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial I_1}\right)$$

$$=\frac{\partial^2 W}{\partial I_1^2}\left[\frac{\partial I_1}{\partial u_x}\frac{\partial I_1}{\partial x}+\frac{\partial I_1}{\partial u_y}\frac{\partial I_1}{\partial y}+\frac{\partial I_1}{\partial u_z}\frac{\partial I_1}{\partial z}\right]+\frac{\partial^2 W}{\partial I_1\partial I_2}\left[\frac{\partial I_1}{\partial u_x}\frac{\partial I_2}{\partial x}+\frac{\partial I_1}{\partial u_y}\frac{\partial I_2}{\partial y}+\frac{\partial I_1}{\partial u_z}\frac{\partial I_2}{\partial z}\right]$$

$$+\frac{\partial^2 W}{\partial I_1\partial I_3}\left[\frac{\partial I_1}{\partial u_x}\frac{\partial I_3}{\partial x}+\frac{\partial I_1}{\partial u_y}\frac{\partial I_3}{\partial y}+\frac{\partial I_1}{\partial u_z}\frac{\partial I_3}{\partial z}\right]$$

$$= B_1 \text{ (say)}, \tag{8·11}$$

$$\frac{\partial I_2}{\partial u_x}\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial I_2}\right)+\frac{\partial I_2}{\partial u_y}\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial I_2}\right)+\frac{\partial I_2}{\partial u_z}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial I_2}\right)$$

$$=\frac{\partial^2 W}{\partial I_1\partial I_2}\left[\frac{\partial I_2}{\partial u_x}\frac{\partial I_1}{\partial x}+\frac{\partial I_2}{\partial u_y}\frac{\partial I_1}{\partial y}+\frac{\partial I_2}{\partial u_z}\frac{\partial I_1}{\partial z}\right]+\frac{\partial^2 W}{\partial I_2^2}\left[\frac{\partial I_2}{\partial u_x}\frac{\partial I_2}{\partial x}+\frac{\partial I_2}{\partial u_y}\frac{\partial I_2}{\partial y}+\frac{\partial I_2}{\partial u_z}\frac{\partial I_2}{\partial z}\right]$$

$$+\frac{\partial^2 W}{\partial I_3\partial I_2}\left[\frac{\partial I_2}{\partial u_x}\frac{\partial I_3}{\partial x}+\frac{\partial I_2}{\partial u_y}\frac{\partial I_3}{\partial y}+\frac{\partial I_2}{\partial u_z}\frac{\partial I_3}{\partial z}\right]$$

$$= B_3 \text{ (say)} \tag{8·12}$$

and

$$\frac{\partial I_3}{\partial u_x}\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial I_3}\right)+\frac{\partial I_3}{\partial u_y}\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial I_3}\right)+\frac{\partial I_3}{\partial u_z}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial I_3}\right)$$

$$=\frac{\partial^2 W}{\partial I_1\partial I_3}\left[\frac{\partial I_3}{\partial u_x}\frac{\partial I_1}{\partial x}+\frac{\partial I_3}{\partial u_y}\frac{\partial I_1}{\partial y}+\frac{\partial I_3}{\partial u_z}\frac{\partial I_1}{\partial z}\right]+\frac{\partial^2 W}{\partial I_2\partial I_3}\left[\frac{\partial I_3}{\partial u_x}\frac{\partial I_2}{\partial x}+\frac{\partial I_3}{\partial u_y}\frac{\partial I_2}{\partial y}+\frac{\partial I_3}{\partial u_z}\frac{\partial I_2}{\partial z}\right]$$

$$+\frac{\partial^2 W}{\partial I_3^2}\left[\frac{\partial I_3}{\partial u_x}\frac{\partial I_3}{\partial x}+\frac{\partial I_3}{\partial u_y}\frac{\partial I_3}{\partial y}+\frac{\partial I_3}{\partial u_z}\frac{\partial I_3}{\partial z}\right]$$

$$= B_3 \text{ (say)}. \tag{8·13}$$

The equation of motion (8·3) may therefore be written

$$A_1\frac{\partial W}{\partial I_1}+A_2\frac{\partial W}{\partial I_2}+A_3\frac{\partial W}{\partial I_3}+B_1+B_2+B_3=\rho\left(\frac{\partial^2 u}{\partial t^2}-X\right), \tag{8·14}$$

where A_1, A_2, A_3, B_1, B_2 and B_3 are given by equations (8·8) to (8·13).

9. The equations of motion for an incompressible material

The equations of motion for a body of incompressible, elastic material, which is isotropic in its undeformed state, are given (Rivlin 1948*a*, § 19) as

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial u_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial u_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial u_z}\right)+\frac{\partial p}{\partial x}\frac{\partial \tau}{\partial u_x}+\frac{\partial p}{\partial y}\frac{\partial \tau}{\partial u_y}+\frac{\partial p}{\partial z}\frac{\partial \tau}{\partial u_z}=\rho\left(\frac{\partial^2 u}{\partial t^2}-X\right), \text{ etc.,} \tag{9·1}$$

where p has the nature of a hydrostatic pressure. These equations can be rewritten, in a manner similar to that employed for a compressible material in § 8, but now $I_3 = 1$, throughout the material, and W is a function of I_1 and I_2 only.

Equations (9·1) therefore become

$$A_1\frac{\partial W}{\partial I_1}+A_2\frac{\partial W}{\partial I_2}+B_1+B_2+\frac{\partial p}{\partial x}\frac{\partial \tau}{\partial u_x}+\frac{\partial p}{\partial y}\frac{\partial \tau}{\partial u_y}+\frac{\partial p}{\partial z}\frac{\partial \tau}{\partial u_z}=\rho\left(\frac{\partial^2 u}{\partial t^2}-X\right), \text{ etc.,} \tag{9·2}$$

where A_1 and A_2 are given by equations (8·8) and (8·9) respectively,

$$B_1 = \frac{\partial^2 W}{\partial I_1^2}\left[\frac{\partial I_1}{\partial u_x}\frac{\partial I_1}{\partial x}+\frac{\partial I_1}{\partial u_y}\frac{\partial I_1}{\partial y}+\frac{\partial I_1}{\partial u_z}\frac{\partial I_1}{\partial z}\right]+\frac{\partial^2 W}{\partial I_1 \partial I_2}\left[\frac{\partial I_1}{\partial u_x}\frac{\partial I_2}{\partial x}+\frac{\partial I_1}{\partial u_y}\frac{\partial I_2}{\partial y}+\frac{\partial I_1}{\partial u_z}\frac{\partial I_2}{\partial z}\right] \tag{9·3}$$

and

$$B_2 = \frac{\partial^2 W}{\partial I_1 \partial I_2}\left[\frac{\partial I_2}{\partial u_x}\frac{\partial I_1}{\partial x}+\frac{\partial I_2}{\partial u_y}\frac{\partial I_1}{\partial y}+\frac{\partial I_2}{\partial u_z}\frac{\partial I_1}{\partial z}\right]+\frac{\partial^2 W}{\partial I_2^2}\left[\frac{\partial I_2}{\partial u_x}\frac{\partial I_2}{\partial x}+\frac{\partial I_2}{\partial u_y}\frac{\partial I_2}{\partial y}+\frac{\partial I_2}{\partial u_z}\frac{\partial I_2}{\partial z}\right]. \tag{9·4}$$

10. The boundary conditions for a compressible material

For a compressible material the boundary conditions are (Rivlin 1948 *a*, § 16)

$$\frac{\partial W}{\partial u_x}\cos(x,\nu)+\frac{\partial W}{\partial u_y}\cos(y,\nu)+\frac{\partial W}{\partial u_z}\cos(z,\nu)-X_\nu = 0, \text{ etc.} \tag{10·1}$$

Here (X_ν, Y_ν, Z_ν) is the surface traction, acting on the surface at the point which is at (x, y, z) in its undeformed state, per unit area of surface measured in the undeformed state. (x,ν), (y,ν) and (z,ν) are the angles between the direction of the normal ν to the surface, in its undeformed state, at the point considered, and the x, y and z axes respectively.

Making use of the relations (8·2), the first of equations (10·1) becomes

$$\begin{aligned}\frac{\partial W}{\partial I_1}&\left[\frac{\partial I_1}{\partial u_x}\cos(x,\nu)+\frac{\partial I_1}{\partial u_y}\cos(y,\nu)+\frac{\partial I_1}{\partial u_z}\cos(z,\nu)\right]\\ &+\frac{\partial W}{\partial I_2}\left[\frac{\partial I_2}{\partial u_x}\cos(x,\nu)+\frac{\partial I_2}{\partial u_y}\cos(y,\nu)+\frac{\partial I_2}{\partial u_z}\cos(z,\nu)\right]\\ &+\frac{\partial W}{\partial I_3}\left[\frac{\partial I_3}{\partial u_x}\cos(x,\nu)+\frac{\partial I_3}{\partial u_y}\cos(y,\nu)+\frac{\partial I_3}{\partial u_z}\cos(z,\nu)\right]-X_\nu = 0.\end{aligned} \tag{10·2}$$

The remaining two equations of (10·1) can be cast into a similar form. We obtain equations similar to (10·2) in which X_ν is replaced by Y_ν and Z_ν respectively and u by v and w respectively.

11. The boundary conditions for an incompressible material

For an incompressible material the boundary conditions are (Rivlin 1948 *a*, § 19)

$$\left(\frac{\partial W}{\partial u_x}+p\frac{\partial \tau}{\partial u_x}\right)\cos(x,\nu)+\left(\frac{\partial W}{\partial u_y}+p\frac{\partial \tau}{\partial u_y}\right)\cos(y,\nu)+\left(\frac{\partial W}{\partial u_z}+p\frac{\partial \tau}{\partial u_z}\right)\cos(z,\nu)-X_\nu = 0, \text{ etc.} \tag{11·1}$$

Making use of the relations (8·2) and bearing in mind that W is now independent of I_3, we obtain

$$\begin{aligned}\frac{\partial W}{\partial I_1}&\left[\frac{\partial I_1}{\partial u_x}\cos(x,\nu)+\frac{\partial I_1}{\partial u_y}\cos(y,\nu)+\frac{\partial I_1}{\partial u_z}\cos(z,\nu)\right]\\ &+\frac{\partial W}{\partial I_2}\left[\frac{\partial I_2}{\partial u_x}\cos(x,\nu)+\frac{\partial I_2}{\partial u_y}\cos(y,\nu)+\frac{\partial I_2}{\partial u_z}\cos(z,\nu)\right]\\ &+p\left[\frac{\partial \tau}{\partial u_x}\cos(x,\nu)+\frac{\partial \tau}{\partial u_y}\cos(y,\nu)+\frac{\partial \tau}{\partial u_z}\cos(z,\nu)\right]-X_\nu = 0, \text{ etc.,}\end{aligned} \tag{11·2}$$

where $\partial I_1/\partial u_x$, $\partial I_2/\partial u_x$, etc. are given by equations (8·4) and (8·5).

C. Simple shear and pure torsion

12. Simple shear of a cuboid of incompressible material

Let us consider a block of incompressible material which, in the undeformed state, is a cuboid whose faces are

$$x = \pm a, \quad y = \pm b \quad \text{and} \quad z = \pm c.$$

If this is subject to a simple shearing deformation, in which each point of the material moves parallel to the x-axis by an amount which is proportional to its y-co-ordinate, then the displacement components u, v and w, for a point which is initially at (x, y, z), are given by

$$u = Ky, \quad v = w = 0, \tag{12·1}$$

where K is a constant.

By substituting these expressions for u, v and w in the stress-strain relations (4·6) for an incompressible material, we see that the simple shear described by (12·1) is associated with stress components given by

$$\left.\begin{aligned} t_{xx} &= 2\left[(1+K^2)\frac{\partial W}{\partial I_1} - \frac{\partial W}{\partial I_2}\right] + p, \\ t_{yy} &= 2\left[\frac{\partial W}{\partial I_1} - (1+K^2)\frac{\partial W}{\partial I_2}\right] + p, \\ t_{zz} &= 2\left[\frac{\partial W}{\partial I_1} - \frac{\partial W}{\partial I_2}\right] + p, \\ t_{yz} &= t_{zx} = 0 \quad \text{and} \quad t_{xy} = 2\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) K. \end{aligned}\right\} \tag{12·2}$$

$\partial W/\partial I_1$ and $\partial W/\partial I_2$ are, in general, functions of I_1 and I_2, where, from (12·1), (3·4) and (2·15),

$$I_1 = I_2 = 3 + K^2. \tag{12·3}$$

It is seen that, in general, a simple shear is associated, not only with a simple shearing stress, but with normal stress components t_{xx}, t_{yy} and t_{zz}. Any one of these may be made zero. For example, if

$$\left.\begin{aligned} & t_{zz} = 0, \quad p = -2\left(\frac{\partial W}{\partial I_1} - \frac{\partial W}{\partial I_2}\right) \\ \text{and} \quad & t_{xx} = 2K^2\frac{\partial W}{\partial I_1} \quad \text{and} \quad t_{yy} = -2K^2\frac{\partial W}{\partial I_2}. \end{aligned}\right\} \tag{12·4}$$

The body forces, which must be applied in order to produce the state of deformation described by (12·1), are given by introducing (12·1) into equations (9·2). We obtain

$$\frac{\partial p}{\partial x} + \rho X = 0, \quad \frac{\partial p}{\partial y} + \rho Y = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} + \rho Z = 0. \tag{12·5}$$

Thus, the state of strain (12·1) can be produced without the application of body forces, i.e. when $X = Y = Z = 0$, and then p is constant throughout the body.

The surface forces which must be applied are given by introducing (12·1) into equations (11·2). We obtain, for the surface forces acting on the planes initially at $x = \pm a$, i.e. for

$$\cos(x,\nu) = \pm 1, \quad \cos(y,\nu) = 0 \quad \text{and} \quad \cos(z,\nu) = 0,$$

$$X_\nu = \pm\left\{2\left(\frac{\partial W}{\partial I_1} + 2\frac{\partial W}{\partial I_2}\right) + p\right\}, \quad Y_\nu = \pm\left\{-2K\frac{\partial W}{\partial I_2} - Kp\right\} \quad \text{and} \quad Z_\nu = 0. \tag{12·6}$$

For the surface forces acting on the planes initially at $y = \pm b$, i.e. for

$$\cos(x,\nu) = 0, \quad \cos(y,\nu) = \pm 1 \quad \text{and} \quad \cos(z,\nu) = 0,$$

$$X_\nu = \pm 2K\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right), \quad Y_\nu = \pm\left\{2\left(\frac{\partial W}{\partial I_1} + 2\frac{\partial W}{\partial I_2}\right) + p\right\} \quad \text{and} \quad Z_\nu = 0. \tag{12·7}$$

Similarly, the surface tractions acting on the surfaces initially at $z = \pm c$, i.e. for

$$\cos(x,\nu) = 0, \quad \cos(y,\nu) = 0 \quad \text{and} \quad \cos(z,\nu) = \pm 1,$$

are given by $$X_\nu = Y_\nu = 0 \quad \text{and} \quad Z_\nu = \pm\left\{2\left[\frac{\partial W}{\partial I_1} + (2+K^2)\frac{\partial W}{\partial I_2}\right] + p\right\}. \tag{12·8}$$

Now, if we make the surface tractions on the surfaces $z = \pm c$ zero, so that

$$p = -2\left[\frac{\partial W}{\partial I_1} + (2+K^2)\frac{\partial W}{\partial I_2}\right], \tag{12·9}$$

we obtain, from (12·6), that the surface tractions on the surfaces initially normal to the x-axis are given by

$$X_\nu = \mp 2K^2\frac{\partial W}{\partial I_2}, \quad Y_\nu = \pm 2K\left[\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right) + K^2\frac{\partial W}{\partial I_2}\right] \quad \text{and} \quad Z_\nu = 0. \tag{12·10}$$

Those on the surfaces normal to the y-axis are given, from (12·7) and (12·9), as

$$X_\nu = \pm 2K\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right), \quad Y_\nu = \mp 2K^2\frac{\partial W}{\partial I_2} \quad \text{and} \quad Z_\nu = 0. \tag{12·11}$$

Resolving the surface tractions (12·10) into components tangential and normal to the surfaces on which they act in their deformed state, we find that the tangential components T are given by

$$T = 2K\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}\right)\Big/(1+K^2)^{\frac{1}{2}} \tag{12·12}$$

and the normal components N by

$$N = -2K^2\left[\left(\frac{\partial W}{\partial I_1} + 2\frac{\partial W}{\partial I_2}\right) + K^2\frac{\partial W}{\partial I_2}\right]\Big/(1+K^2)^{\frac{1}{2}}. \tag{12·13}$$

It is thus apparent that the state of pure shear described by equation (12·1) cannot be supported by tangential tractions applied to the surfaces $y = \pm b$ and $x = \pm a$, as in the case of small deformations dealt with in the classical theory of elasticity. However, the system of surface tractions applied to the surfaces $x = \pm a$ and $y = \pm b$ and described by equations (12·11), (12·12) and (12·13) are adequate to support the deformation.

It should be borne in mind that all these components of the surface traction refer to unit area of the surface measured in the undeformed state of the body. The areas of elements of the surfaces $y = \pm b$ and $z = \pm c$ are unaltered by the deformation. However, the areas of elements of the surfaces $x = \pm a$ are multiplied by the factor $(1+K^2)^{\frac{1}{2}}$ in the deformation.

The tangential and normal components T' and N' respectively of the surface tractions, on the surfaces initially at $x = \pm a$, per unit area of surface of the deformed body, are given, from (12·12) and (12·13), as

$$T' = 2K\left(\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)\Big/(1+K^2) \quad \text{and} \quad N' = -2K^2\left(\frac{\partial W}{\partial I_1}+2\frac{\partial W}{\partial I_2}+K^2\frac{\partial W}{\partial I_2}\right)\Big/(1+K^2). \qquad (12·14)$$

Thus, the system of surface tractions described by (12·11) and (12·14) is adequate to support the state of simple shear described by (12·1). An alternative system of surface tractions may be obtained by adding to this system a hydrostatic pressure

$$2K^2\left(\frac{\partial W}{\partial I_1}+2\frac{\partial W}{\partial I_2}+K^2\frac{\partial W}{\partial I_2}\right)\Big/(1+K^2).$$

This gives the system of surface tractions

$$X_\nu = \pm 2K\left(\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right), \quad Y_\nu = \pm\frac{2K^2}{1+K^2}\left(\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right) \quad \text{and} \quad Z_\nu = 0, \qquad (12·15)$$

acting on the surfaces $y = \pm b$,

$$T' = 2K\left(\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right)\Big/(1+K^2), \qquad (12·16)$$

acting tangentially to the surfaces which are initially at $x = \pm a$, and

$$Z_\nu = 2K^2\left(\frac{\partial W}{\partial I_1}+2\frac{\partial W}{\partial I_2}+K^2\frac{\partial W}{\partial I_2}\right)\Big/(1+K^2), \qquad (12·17)$$

acting normally to the surfaces $z = \pm c$.

13. Simple shear of a cuboid of compressible material

Suppose the cuboid of material considered in the last section is composed of a compressible material, for which the stress-strain relationships are given by (4·5), and is subjected to a simple shearing deformation described by equation (12·1). It can readily be seen by introducing (12·1) into equation (8·3) and two similar equations of motion that, for this state of simple shear to be maintained, the applied body forces (X, Y, Z) must be zero.

The surface forces which must be applied to the cuboid are obtained by substituting from (12·1) for u, v and w in equations (10·2). We obtain, for the surfaces $x = \pm a$,

$$X_\nu = \pm 2\left(\frac{\partial W}{\partial I_1}+2\frac{\partial W}{\partial I_2}+\frac{\partial W}{\partial I_3}\right), \quad Y_\nu = \mp 2K\left(\frac{\partial W}{\partial I_2}+\frac{\partial W}{\partial I_3}\right), \quad \text{and} \quad Z_\nu = 0; \qquad (13·1)$$

for the surfaces $y = \pm b$,

$$X_\nu = \pm 2K\left(\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}\right), \quad Y_\nu = \pm 2\left(\frac{\partial W}{\partial I_1}+2\frac{\partial W}{\partial I_2}+\frac{\partial W}{\partial I_3}\right) \quad \text{and} \quad Z_\nu = 0; \qquad (13·2)$$

and for the surfaces $z = \pm c$,

$$X_\nu = Y_\nu = 0 \quad \text{and} \quad Z_\nu = \pm 2\left\{\frac{\partial W}{\partial I_1}+(2+K^2)\frac{\partial W}{\partial I_2}+\frac{\partial W}{\partial I_3}\right\}. \qquad (13·3)$$

These surface tractions are expressed per unit area of the surface measured in its undeformed state. They can readily be expressed per unit area of surface measured in its deformed state by a method similar to that adopted in § 12.

14. The torsion of a right-circular cylinder of incompressible material

In this section we shall calculate what force system is necessary to produce a pure torsion in a right-circular cylinder of incompressible material, of length l and radius a. The problem has already been worked out for an incompressible, neo-Hookean material in a previous paper of this series (Rivlin 1948 *b*, §§ 8 to 10). There the problem was dealt with on the basis of the appropriate equations of motion and boundary conditions expressed with reference to a cylindrical polar co-ordinate system. Here, however, it will be more convenient to use the equations of motion and boundary conditions, referred to a rectangular, Cartesian co-ordinate system, in the forms they have been given in §§ 9 and 11.

The co-ordinate system chosen has its z-axis coincident with the axis of the cylinder and its origin at the mid-point of this axis. The x and y axes may then be chosen in arbitrary directions consistent with the co-ordinate system being rectangular Cartesian.

If, in the pure torsion, each section of the cylinder which is normal to the z-axis is rotated through an angle ψz, then the displacement components u, v and w, parallel to the axes x, y and z respectively, of a point which before the deformation is at (x, y, z), are given by

$$u = (x\cos\psi z - y\sin\psi z) - x, \quad v = (x\sin\psi z + y\cos\psi z) - y \quad \text{and} \quad w = 0. \tag{14·1}$$

We readily see, by substituting from (14·1) in (8·7), that the relation $\tau = 1$, which must be obeyed by an incompressible material, is automatically satisfied.

Substituting from (14·1) in (2·15) and (3·4), we obtain

$$I_1 = I_2 = 3 + \psi^2 r^2 \quad \text{and} \quad I_3 = 1. \tag{14·2}$$

Employing the relations (14·1) and (14·2) in equations (8·8), (8·9), (9·3) and (9·4), we obtain the expressions for A_1, A_2, B_1 and B_2 in the first of the equations of motion (9·2), as

$$\left.\begin{aligned} A_1 &= -2\psi^2(x\cos\psi z - y\sin\psi z), \\ A_2 &= 4\psi^2(x\cos\psi z - y\sin\psi z), \\ B_1 &= 4\psi^2\left(\frac{\partial^2 W}{\partial I_1^2} + \frac{\partial^2 W}{\partial I_1 \partial I_2}\right)(x\cos\psi z - y\sin\psi z) \\ \text{and} \qquad B_2 &= 4\psi^2\left(\frac{\partial^2 W}{\partial I_1 \partial I_2} + \frac{\partial^2 W}{\partial I_2^2}\right)(2+\psi^2 r^2)\,(x\cos\psi z - y\sin\psi z). \end{aligned}\right\} \tag{14·3}$$

Employing these expressions in equation (9·2), using (14·1) to obtain $\partial\tau/\partial u_x$, $\partial\tau/\partial u_y$, etc., and bearing in mind that $\partial^2 u/\partial t^2 = 0$, we obtain

$$\left\{2\psi^2\left(2\frac{\partial W}{\partial I_2} - \frac{\partial W}{\partial I_1}\right) + 4\psi^2\left[\frac{\partial^2 W}{\partial I_1^2} + (3+\psi^2 r^2)\frac{\partial^2 W}{\partial I_1 \partial I_2} + (2+\psi^2 r^2)\frac{\partial^2 W}{\partial I_2^2}\right]\right\}$$
$$\times (x\cos\psi z - y\sin\psi z) + \cos\psi z\frac{\partial p}{\partial x} - \sin\psi z\frac{\partial p}{\partial y} + \rho X = 0. \tag{14·4}$$

In a similar manner, we obtain the remaining two equations of motion as

$$\left.\begin{aligned}\Big\{2\psi^2\Big(2\frac{\partial W}{\partial I_2}-\frac{\partial W}{\partial I_1}\Big)+4\psi^2\Big[\frac{\partial^2 W}{\partial I_1^2}+(3+\psi^2r^2)\frac{\partial^2 W}{\partial I_1\partial I_2}+(2+\psi^2r^2)\frac{\partial^2 W}{\partial I_2^2}\Big]\Big\}&\\ \times(x\sin\psi z+y\cos\psi z)+\sin\psi z\frac{\partial p}{\partial x}+\cos\psi z\frac{\partial p}{\partial y}+\rho Y=0&\\ \text{and}\qquad \rho Z+\frac{\partial p}{\partial z}=0.&\end{aligned}\right\}\tag{14·5}$$

If the body forces are zero, i.e. $X=Y=Z=0$, we have, noting that $\partial p/\partial z=0$, since the strain is independent of z,

$$\left.\begin{aligned}-\frac{\partial p}{\partial x}&=\Big\{2\psi^2\Big(2\frac{\partial W}{\partial I_2}-\frac{\partial W}{\partial I_1}\Big)+4\psi^2\Big[\frac{\partial^2 W}{\partial I_1^2}+(3+\psi^2r^2)\frac{\partial^2 W}{\partial I_1\partial I_2}+(2+\psi^2r^2)\frac{\partial^2 W}{\partial I_2^2}\Big]\Big\}x\\ \text{and}\quad -\frac{\partial p}{\partial y}&=\Big\{2\psi^2\Big(2\frac{\partial W}{\partial I_2}-\frac{\partial W}{\partial I_1}\Big)+4\psi^2\Big[\frac{\partial^2 W}{\partial I_1^2}+(3+\psi^2r^2)\frac{\partial^2 W}{\partial I_1\partial I_2}+(2+\psi^2r^2)\frac{\partial^2 W}{\partial I_2^2}\Big]\Big\}y.\end{aligned}\right\}\tag{14·6}$$

From (14·6), it can be seen that p is a function of r only and

$$-\frac{\partial p}{\partial r}=2\psi^2\Big(2\frac{\partial W}{\partial I_2}-\frac{\partial W}{\partial I_1}\Big)r+4\psi^2\Big[\frac{\partial^2 W}{\partial I_1^2}+(3+\psi^2r^2)\frac{\partial^2 W}{\partial I_1\partial I_2}+(2+\psi^2r^2)\frac{\partial^2 W}{\partial I_2^2}\Big]r.\tag{14·7}$$

If the form of W as a function of I_1 and I_2 is known, then p can be determined, from this equation, throughout the material, except for a constant of integration which must be obtained from the boundary conditions over the curved surface.

The boundary conditions over the curved surface of the cylinder are obtained from (11·2), by substituting

$$\cos(x,\nu)=x/a,\quad \cos(y,\nu)=y/a\quad\text{and}\quad\cos(z,\nu)=0,\tag{14·8}$$

and the relations of the types (8·4), (8·5) and (8·6), in which the expressions (14·1) have been substituted for u, v and w. We obtain

$$\left.\begin{aligned}aX_\nu&=2\Big\{\frac{\partial W}{\partial I_1}+(2+\psi^2a^2)\frac{\partial W}{\partial I_2}+\tfrac{1}{2}p\Big\}(x\cos\psi z-y\sin\psi z),\\ aY_\nu&=2\Big\{\frac{\partial W}{\partial I_1}+(2+\psi^2a^2)\frac{\partial W}{\partial I_2}+\tfrac{1}{2}p\Big\}(x\sin\psi z+y\cos\psi z)\\ \text{and}\qquad Z_\nu&=0.\end{aligned}\right\}\tag{14·9}$$

The surface traction on the curved surface of the cylinder is thus purely radial in the deformed state of the body and has magnitude R_ν given by

$$R_\nu=2\Big(\frac{\partial W}{\partial I_1}\Big)_{r=a}+2(2+\psi^2a^2)\Big(\frac{\partial W}{\partial I_2}\Big)_{r=a}+p.\tag{14·10}$$

If the value of R_ν is specified, then equation (14·10) can be used to determine the value of p when $r=a$ and thus to eliminate the constant of integration from the integrated form of equation (14·7).

In the particular case when the surface traction R_ν over the curved surface vanishes, then on the curved surface

$$p=-2\Big\{\Big(\frac{\partial W}{\partial I_1}\Big)_{r=a}+(2+\psi^2a^2)\Big(\frac{\partial W}{\partial I_2}\Big)_{r=a}\Big\},\tag{14·11}$$

and from (14·7) we have, throughout the material,

$$p = 2\psi^2 \int_a^r \left\{ \left(\frac{\partial W}{\partial I_1} - 2\frac{\partial W}{\partial I_2} \right) - 2\left[\frac{\partial^2 W}{\partial I_1^2} + (3+\psi^2 r^2)\frac{\partial^2 W}{\partial I_1 \partial I_2} + (2+\psi^2 r^2)\frac{\partial^2 W}{\partial I_2^2} \right] \right\} r\,dr - 2\left\{ \left(\frac{\partial W}{\partial I_1} \right)_{r=a} + (2+\psi^2 a^2)\left(\frac{\partial W}{\partial I_2} \right)_{r=a} \right\}. \quad (14·12)$$

The boundary conditions over the plane ends of the cylinder are obtained, from (11·2), by substituting

$$\cos(x,\nu) = \cos(y,\nu) = 0 \quad \text{and} \quad \cos(z,\nu) = \pm 1 \quad (14·13)$$

and employing the relations (8·4), (8·5) and (8·6), in which (14·1) have been substituted for u, v and w.

We obtain

$$\left.\begin{aligned} X_\nu &= \mp 2\psi\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right)(x \sin\psi z + y\cos\psi z), \\ Y_\nu &= \pm 2\psi\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right)(x\cos\psi z - y\sin\psi z) \\ \text{and} \quad Z_\nu &= 2\left(\frac{\partial W}{\partial I_1} + 2\frac{\partial W}{\partial I_2} \right) + p. \end{aligned}\right\} \quad (14·14)$$

The surface traction on the plane ends therefore consists of a component Θ_ν which is azimuthal in the deformed state of the body and another Z_ν which is normal in the deformed state. Noting that p is given by (14·12), we obtain

$$\left.\begin{aligned} \Theta_\nu &= 2\psi r\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right) \\ \text{and} \quad Z_\nu &= -2\psi^2 a^2\left(\frac{\partial W}{\partial I_2} \right)_{r=a} + 2\psi^2\int_a^r \left\{ \left(\frac{\partial W}{\partial I_1} - 2\frac{\partial W}{\partial I_2} \right) - 2\left[\frac{\partial^2 W}{\partial I_1^2} + (3+\psi^2 r^2)\frac{\partial^2 W}{\partial I_1 \partial I_2} \right.\right. \\ &\quad \left.\left. + (2+\psi^2 r^2)\frac{\partial^2 W}{\partial I_2^2} \right] \right\} r\,dr + 2\left\{ \frac{\partial W}{\partial I_1} - \left(\frac{\partial W}{\partial I_1} \right)_{r=a} \right\} + 4\left\{ \frac{\partial W}{\partial I_2} - \left(\frac{\partial W}{\partial I_2} \right)_{r=a} \right\}. \end{aligned}\right\} \quad (14·15)$$

If the expression $W = G(I_1 - 3)$, for the stored-energy function of an incompressible, neo-Hookean material is substituted in (14·15), we obtain the expressions for Θ_ν and Z_ν, which were obtained in part III (Rivlin 1948*c*, §8). For a material whose stored-energy function is that postulated by Mooney (1940), i.e.

$$W = C_1(I_1 - 3) + C_2(I_2 - 3),$$

equations (14·15) become

$$\Theta_\nu = 2\psi r(C_1 + C_2) \quad \text{and} \quad Z_\nu = -\psi^2[(C_1 - 2C_2)(a^2 - r^2) + 2a^2 C_2]. \quad (14·16)$$

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