DOI: 10.1214/EJP.v3-36

Electronic Journal of Probability

Vol. 3 (1998) Paper no. 14, pages 1–31.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol3/paper14.abs.html

LARGE FAVOURITE SITES OF SIMPLE RANDOM WALK AND THE WIENER PROCESS

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Abstract: Let $U(n)$ denote the most visited point by a simple symmetric random walk $\{S_k\}_{k\geq 0}$ in the first n steps. It is known that $U(n)$ and $\max_{0\leq k\leq n} S_k$ satisfy the same law of the iterated logarithm, but have different upper functions (in the sense of P. Lévy). The distance between them however turns out to be transient. In this paper, we establish the exact rate of escape of this distance. The corresponding problem for the Wiener process is also studied.

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Submitted to EJP on March 6, 1998. Final version accepted on September 30, 1998.

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Large favourite sites of simple random walk and the Wiener process

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Summary. Let $U(n)$ denote the most visited point by a simple symmetric random walk $\{S_k\}_{k\geq 0}$ in the first n steps. It is known that $U(n)$ and $\max_{0\leq k\leq n} S_k$ satisfy the same law of the iterated logarithm, but have different upper functions (in the sense of P. Lévy). The distance between them however turns out to be transient. In this paper, we establish the exact rate of escape of this distance. The corresponding problem for the Wiener process is also studied.

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* Research supported by the Hungarian National Foundation for Scientific Research, Grant No. T 016384 and T 019346.

1. Introduction

Let $\{S_k\}_{k\geq 0}$ denote a simple symmetric (Bernoulli) random walk on the line, starting from 0, i.e. at each step, the random walk visits either of its two neighbours with equal probability 1/2. Define, for $n \geq 0$ and $x \in \mathbb{Z}$,

$$\xi_n^x \stackrel{\text{def}}{=} \sum_{k=0}^{n} 1\!\!1_{\{S_k = x\}}, \tag{1.1}$$

which counts the number of visits of the site x by the random walk in the first n steps. Let

$$\mathbb{U}(n) \stackrel{\text{def}}{=} \Big\{ x \in \mathbb{Z} : \, \xi_n^x = \sup_{y\in\mathbb{Z}} \xi_n^y \Big\},$$

which stands for the set of the **most visited sites** or **favourite sites** of the random walk. We (measurably) choose an arbitrary point in $\mathbb{U}(n)$, say,

$$U(n) \stackrel{\text{def}}{=} \max_{x\in\mathbb{U}(n)} x, \tag{1.2}$$

which is referred to by Erdős and Révész [12] as the (largest) favourite site of $\{S_k\}_{0\leq k\leq n}$. We mention that all the results for $U(n)$ stated in this paper remain true if "max" is replaced for example by "min" in (1.2).

The process $U(n)$ has some surprising properties. For example, it is proved by Bass and Griffin [2] that it is transient, in the sense that $\lim_{n\to\infty} |U(n)| = \infty$ almost surely. More precisely, they obtain the following:

Theorem A ([2]). *With probability one,*

$$\liminf_{n\to\infty} \frac{(\log n)^a}{n^{1/2}} |U(n)| = \begin{cases} 0 & \text{if } a < 1, \\ \infty & \text{if } a > 11. \end{cases}$$

Remark. The exact rate of escape of $|U(n)|$ is unknown.

Concerning the upper limits of $U(n)$, the following is established by Erdős and Révész [12] and by Bass and Griffin [2], using totally different methods:

Theorem B ([12], [2]). *We have,*

$$\limsup_{n\to\infty} \frac{U(n)}{(2n\log\log n)^{1/2}} = 1, \qquad \text{a.s.}$$

Theorem B confirms that both $U(n)$ and $\overline{S}_n \stackrel{\text{def}}{=} \max_{0\le k\le n} S_k$ satisfy the same law of the iterated logarithm (LIL). A natural question is: do they have the same upper functions? Of course, for the random walk, the upper functions are characterized by the classical Kolmogorov test (also referred to as the Erdős–Feller–Kolmogorov–Petrowsky or EFKP test, cf. Révész [21, p. 35]).

Theorem C ([12]). *There exists a deterministic sequence $(a_n)_{n\ge 0}$ of non-decreasing positive numbers such that with probability one,*

$$\begin{aligned} U(n) &< a_n, &\quad &\text{for all sufficiently large } n,\\ \overline{S}_n &> a_n, &\quad &\text{for infinitely many } n. \end{aligned}$$

As a consequence, $U(n)$ and $\overline{S}_n$ have different upper functions.

Remark. An example of the sequence (a_n) satisfying Theorem C is explicitly given in [12], cf. also Révész [21, Theorem 11.25]. Whether it is possible to obtain an integral test to characterize the upper functions of $U(n)$ remains an unanswered question. See Révész [21, pp. 130–131] for a list of 10 (ten) other open problems for $U(n)$ and $\mathbb{U}(n)$.

We suggest to study the upper limits of $U(n)$ in this paper. Intuitively, when $U(n)$ reaches some extraordinarily large values, it would be very close to $\overline{S}_n$. The question is: how close can $U(n)$ be to $\overline{S}_n$? The fact that the process $n \mapsto \overline{S}_n - U(n)$ is transient, follows from Révész [21, Theorem 13.25]. Our aim here is to determine the exact escape rate of the process. This problem is communicated to us by Omer Adelman.

Theorem 1.1. *There exists a universal constant $c_0 \in (0,\infty)$ such that*

$$\liminf_{n\to\infty} \frac{(\log\log n)^{3/2}}{n^{1/2}} \big(\overline{S}_n - U(n)\big) = c_0\,, \qquad \text{a.s.}$$

Remark 1.1.1. The rate $n^{1/2}/(\log\log n)^{3/2}$ might somewhat seem surprising. One might have expected to see for example $n^{1/2}/(\log\log n)^{1/2}$ (the rate in Chung's LIL for the random walk), or even something like $n^{1/2}/(\log n)^a$ (for some $a>0$; the rate in Hirsch's LIL). (For these LIL's, cf. Chung [5], Hirsch [14], or Csáki [7] for a unified approach). The correct rate of escape of $\overline{S}_n - U(n)$ is therefore a kind of "compromise" between the rates in the Chung and Hirsch LIL's.

Remark 1.1.2. An immediate consequence of Theorem 1.1 is that almost surely for all large n, if $\overline{S}_n < c\,n^{1/2}/(\log\log n)^{3/2}$ (where $c < c_0$), then all the favourite points are in the negative part of the line.

Theorem 1.1 provides information about the absolute distance between $U(n)$ and $\overline{S}_n$. However, one may wonder how $U(n)$ can be close to $\overline{S}_n$ in the scale of the latter. Our answer to this is a self-normalized LIL stated as follows.

Theorem 1.2. *With probability one,*

$$\liminf_{n\to\infty} (\log\log n)^2 \, \frac{\overline{S}_n - U(n)}{\overline{S}_n} = j_0^2,$$

where $j_0 \approx 2,405$ *is the smallest positive root of the Bessel function*

$$x \mapsto J_0(x) = \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{(k!)^2}.$$

Remark 1.2.1. It follows from Theorems 1.1 and 1.2 that if $(\overline{S}_n - U(n))/\overline{S}_n$ is as small as possible, then $\overline{S}_n$ should be very large. More precisely, the events $\{\overline{S}_n - U(n) < c_1 \overline{S}_n (\log\log n)^{-2}\}$ and $\{\overline{S}_n < c_2 (n \log\log n)^{1/2}\}$, where $c_1 c_2 < c_0$, cannot occur simultaneously for infinitely many n with probability one.

We conclude the introduction part by mentioning that the problem of the favourite sites for random walk is also studied by Tóth and Werner [24]. See also Khoshnevisan and Lewis [17] for the Poisson process, Borodin [4], Eisenbaum [10] and Leuridan [19] for the Wiener process, Eisenbaum [11] for the stable Lévy process, Bertoin and Marsalle [3] for the drifted Wiener process, and Hu and Shi [15] for the Wiener process in space.

The rest of the paper is as follows. Section 2 is devoted to some preliminaries for Brownian local times and Bessel processes. Theorem 1.2 is proved in Section 3, and Theorem 1.1 in Section 4.

In the sequel, c_i $(3 \le i \le 22)$ denote some (finite positive) universal constants, except that when their values depend on ε, they will be written as $c_i(\varepsilon)$. We adopt the usual notation $a(x) \sim b(x)$ $(x \to x_0)$ to denote $\lim_{x\to x_0} a(x)/b(x) = 1$. Since we only deal with (possibly random) indices n and t which ultimately tend to infinity, our statements — sometimes without further mention — are to be understood for the situation when the appropriate index is sufficiently large. We also mention that our use of "almost surely" is not systematic.

2. Preliminaries

In the rest of the paper, $\{W(t);\, t \ge 0\}$ denotes a real-valued Wiener process with $W(0) = 0$. There exists a jointly continuous version of the local time process of W, denoted by $\{L_t^x;\, t \ge 0,\, x \in \mathbb{R}\}$, i.e. for all positive Borel function f,

$$\int_0^t f(W(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x)\, L_t^x\, \mathrm{d}x, \qquad t \ge 0.$$

We shall be working on this jointly continuous version.

Consider the process of the first hitting times for W:

$$T(r) \stackrel{\text{def}}{=} \inf\Big\{t > 0 :\, W(t) > r\Big\}, \qquad r > 0. \tag{2.1}$$

Let us recall the following well-known Ray–Knight theorem, cf. Ray [20], Knight [18], or Rogers and Williams [23, Theorem VI.52.1 (i)]:

Fact 2.1. *The process $\{L_{T(1)}^{1-x};\, x \ge 0\}$ is continuous inhomogeneous Markovian. When $0 \le x \le 1$, it is a squared Bessel process of dimension 2 starting from 0, and becomes a squared Bessel process of dimension 0 for $x \ge 1$.*

Remark 2.1.1. We recall that when $d \ge 1$ is integer, a d-dimensional Bessel process can be realized as the Euclidean norm of an $\mathbb{R}^d$-valued Wiener process. On the other hand, a squared Bessel process of dimension 0 is a diffusion process with generator $2x\, \mathrm{d}^2/\, \mathrm{d}x^2$, absorbed once it hits 0.

Notation. Throughout the paper,

$$\{Z(t);\, t \ge 0\} \stackrel{\text{def}}{=} \text{squared Bessel process of dimension 0, with } Z(0) = 1, \tag{2.2}$$

$$\zeta_Z \stackrel{\text{def}}{=} \inf\Big\{t > 0 :\, Z(t) = 0\Big\}, \tag{2.3}$$

and

$$\{Q(t);\, t \ge 0\} \stackrel{\text{def}}{=} \text{squared Bessel process of dimension 2, with } Q(0) = 0, \tag{2.4}$$

$$\{H(t);\, t \ge 0\} \stackrel{\text{def}}{=} \text{squared Bessel process of dimension 4, with } H(0) = 0, \tag{2.5}$$

$$\mathbb{L}_H \stackrel{\text{def}}{=} \sup\Big\{t > 0 :\, H(t) = 1\Big\}. \tag{2.6}$$

In words, ζ_Z denotes the life-time of Z, and $\mathbb{L}_H$ is the last exit time of H from 1. Since the 4-dimensional squared Bessel process H is transient, the random variable $\mathbb{L}_H$ is well-defined.

The next is a collection of known results on the Bessel processes, which we shall need later. Fact 2.2 is a duality theorem for Bessel processes of dimensions 0 and 4. A more general result can be found in Revuz and Yor [22, Exercise XI.1.23]. Fact 2.3, which gives an absolute continuity relation for the normalized Bessel process, can be found in Yor [25, p. 52]. Fact 2.4 concerns the lower tail probabilities of Q and H. It is borrowed from a celebrated theorem of Ciesielski and Taylor [6]. The probability transition density function for Q in Fact 2.5 is well-known, cf. Revuz and Yor [22, Chap. XI]. Fact 2.6 is a straightforward consequence of Anderson's general inequality for Gaussian measures (cf. [1]).

Fact 2.2. *We have*

$$\Big\{ Z(\zeta_Z - t);\, 0 \le t \le \zeta_Z \Big\} \stackrel{\text{law}}{=} \Big\{ H(t);\, 0 \le t \le \mathbb{L}_H \Big\},$$

where "$\stackrel{\text{law}}{=}$" denotes identity in distribution. In words, a Bessel process of dimension 0*, starting from* 1*, is the time reversal of a Bessel process of dimension* 4*, starting from* 0*, killed when exiting from* 1 *for the last time.*

Fact 2.3. *For any bounded functional F, we have*

$$\mathbb{E}\Big[F\Big(\frac{H(s\,\mathbb{L}_H)}{\mathbb{L}_H};\, 0 \le s \le 1 \Big) \Big] = \mathbb{E}\Big[\frac{2}{H(1)}\, F\Big(H(s);\, 0 \le s \le 1 \Big) \Big].$$

Fact 2.4. *As x goes to* 0*,*

$$\mathbb{P}\Big(\sup_{0\le s\le 1} Q(s) < x \Big) \sim c_3 \exp\Big(-\frac{j_0^2}{2x} \Big),$$

where j_0 is as before the smallest positive root of J_0, and c_3 is an absolute constant whose value is explicitly known. As a consequence, there exists an absolute constant c_4 such that for all $t > 0$ and $x > 0$,

$$\mathbb{P}\Big(\sup_{0\le s\le t} Q(s) < x \Big) \le c_4 \exp\Big(-\frac{j_0^2}{2}\frac{t}{x} \Big). \tag{2.7}$$

Similarly, there exist c_5 and c_6 such that for all positive t and x,

$$\mathbb{P}\Big(\sup_{0\le s\le t} H(s) < x\Big) \le c_5 \exp\Big(-c_6\frac{t}{x}\Big). \tag{2.8}$$

Fact 2.5. *The probability transition density of the (strong) Markov process Q is given by, for $t > 0$,*

$$p_t(x,y) = \frac{1}{2t}\exp\Big(-\frac{x+y}{2t}\Big) I_0\Big(\frac{\sqrt{xy}}{t}\Big), \qquad x \ge 0,\ y > 0, \tag{2.9}$$

where I_0 is the modified Bessel function of index 0.

Fact 2.6. *Let $t > 0$, and let $\{B(s);\ s \ge 0\}$ be an $\mathbb{R}^2$-valued Wiener process starting from 0. If f is a real deterministic function defined on $[0,t]$, then for all $x > 0$,*

$$\mathbb{P}\Big(\sup_{0\le s\le t} \|B(s) + f(s)\| < x\Big) \le \mathbb{P}\Big(\sup_{0\le s\le t} \|B(s)\| < x\Big),$$

where "$\|\cdot\|$" is the Euclidean norm in $\mathbb{R}^2$.

Finally, let us recall three results for local times. The first (Fact 2.7) is Kesten's LIL for the maximum local time, cf. [16]. For an improvement in form of integral criterion, cf. Csáki [8]. The second (Fact 2.8), which concerns the increments of the Wiener local time with respect to the space variable, is due to Bass and Griffin [2]. The third (Fact 2.9) is a joint strong approximation theorem, cf. Révész [21, pp. 105–107].

Fact 2.7. *With probability one,*

$$\limsup_{t\to\infty} (2t\log\log t)^{-1/2} \sup_{x\in\mathbb{R}} L_t^x = 1.$$

Fact 2.8. *For any $\varepsilon > 0$, as t goes to infinity,*

$$\sup_{x\in\mathbb{Z}} \sup_{x\le y\le x+1} |L_t^x - L_t^y| = o(t^{1/4+\varepsilon}), \qquad \text{a.s.} \tag{2.10}$$

Fact 2.9. *(Possibly in an enlarged probability space), there exists a coupling for the Bernoulli walk $\{S_k\}_{k\ge 0}$ and the Wiener process $\{W(t);\ t \ge 0\}$, such that for all $\varepsilon > 0$, as n goes to infinity,*

$$\max_{x\in\mathbb{Z}} |\xi_n^x - L_n^x| = o(n^{1/4+\varepsilon}), \qquad \text{a.s.,} \tag{2.11}$$

$$|S_n - W(n)| = \mathcal{O}(\log n), \qquad \text{a.s.,} \tag{2.12}$$

where ξ_n^x and L_n^x denote the local times of (S_k) and W respectively.

Remark 2.9.1. The approximation rate in (2.11) is not optimal, but is sufficient for our needs. For the best possible rates, cf. Csörgő and Horváth [9].

3. Proof of Theorem 1.2

Without loss of generality, we shall be working in an enlarged probability space where the coupling for $\{S_k\}_{k\geq 0}$ and W in Fact 2.9 is satisfied. Recall that L_t^x is the local time of W. For brevity, write

$$\overline{W}(t) \stackrel{\text{def}}{=} \sup_{0\leq s\leq t} W(s), \qquad t\geq 0. \tag{3.1}$$

The main result in this section is the following theorem.

Theorem 3.1. *Let $\varphi(t) \stackrel{\text{def}}{=} j_0^2/(\log\log t)^2$. There exists $\varepsilon_0 \in (0,1)$ such that for all $0 < \varepsilon < \varepsilon_0$, we have,*

(i) *almost surely for all sufficiently large t,*

$$\sup_{0\leq x\leq (1-(1-\varepsilon)\varphi(t))\overline{W}(t)} L_t^x > \sup_{x>(1-(1-\varepsilon)\varphi(t))\overline{W}(t)} L_t^x + \frac{t^{1/2}}{(\log t)^{1+\varepsilon}}; \tag{3.2}$$

(ii) *almost surely, there exists a sequence $(t_n)\uparrow\infty$, satisfying*

$$\sup_{-\infty<x\leq (1-(1+\varepsilon)\varphi(t_n))\overline{W}(t_n)} L_{t_n}^x < \sup_{x>(1-(1+\varepsilon)\varphi(t_n))\overline{W}(t_n)} L_{t_n}^x - \frac{t_n^{1/2}}{(\log t_n)^{1+\varepsilon}}. \tag{3.3}$$

By admitting Theorem 3.1 for the moment, we can now easily prove Theorem 1.2.

Proof of Theorem 1.2. Fix a small $\varepsilon>0$. Let $\{S_k\}_{k\geq 0}$ and W be the coupling in Fact 2.9. According to (2.12), for all large n,

$$\begin{aligned}(1-(1-\varepsilon)\varphi(n))\,\overline{W}(n) &\leq (1-(1-\varepsilon)\varphi(n))\,\overline{S}_n + \mathcal{O}(\log n)\\ &\leq (1-(1-2\varepsilon)\varphi(n))\,\overline{S}_n.\end{aligned} \tag{3.4}$$

In the last inequality, we have used the following well-known LIL's (cf. for example Révész [21, pp. 35 and 39]): for $a>0$ and almost surely all large n,

$$\frac{n^{1/2}}{(\log n)^{1+a}} \leq \overline{S}_n \leq (1+a)(2n\log\log n)^{1/2}. \tag{3.5}$$

For other applications later, we mention that (3.5) has a continuous-time analogue (Révész [21, p. 53]): for $a > 0$ and almost surely all large t,

$$\frac{t^{1/2}}{(\log t)^{1+a}} \le \overline{W}(t) \le (1+a)(2t \log\log t)^{1/2}, \tag{3.6}$$

or, equivalently, for $a > 0$ and almost surely all large r,

$$\frac{(1-a)r^2}{2\log\log r} \le T(r) \le r^2 (\log r)^{2+a}. \tag{3.7}$$

Applying (3.2), (2.11) and (2.10), and in view of (3.4), we obtain (writing $b_\varepsilon(n) \stackrel{\text{def}}{=} 1-(1-2\varepsilon)\varphi(n)$ for brevity):

$$\begin{aligned} \sup_{0\le x\le b_\varepsilon(n)\,\overline{S}_n;\, x\in\mathbb{Z}} \xi_n^x &> \sup_{x> b_\varepsilon(n)\,\overline{S}_n;\, x\in\mathbb{Z}} \xi_n^x + \frac{n^{1/2}}{(\log n)^{1+\varepsilon}} - n^{1/4+\varepsilon} \\ &> \sup_{x> b_\varepsilon(n)\,\overline{S}_n;\, x\in\mathbb{Z}} \xi_n^x. \end{aligned}$$

By the definition of $U(n)$ (cf. (1.2)), this yields that (almost surely) for all large n, $U(n) \le b_\varepsilon(n)\,\overline{S}_n$. Therefore

$$\liminf_{n\to\infty} (\log\log n)^2 \, \frac{\overline{S}_n - U(n)}{\overline{S}_n} \ge (1-2\varepsilon) j_0^2, \qquad \text{a.s.}$$

This implies the lower bound in Theorem 1.2, as ε can be as close to 0 as possible. The upper bound in the theorem can be proved exactly in the same way, using (3.3) instead of (3.2). $\square$

To prove Theorem 3.1, we need the following two lemmas.

Lemma 3.2. *Recall $r \mapsto T(r)$ from (2.1). For any $y > 0$,*

$$\mathbb{P}\Big(\sup_{x\le 0} L_{T(1)}^x < y \Big) = 1 - \frac{2}{y}\Big(1 - \mathrm{e}^{-y/2}\Big).$$

Consequently, for all $0 < y \le 1$,

$$\mathbb{P}\Big(\sup_{x\le 0} L_{T(1)}^x < y \Big) \ge c_7\, y. \tag{3.8}$$

Proof. Let as before Q and Z be squared Bessel processes of dimensions 2 and 0 respectively, with $Q(0) = 0$ and $Z(0) = 1$. Assume they are independent. By the Ray–Knight theorem (cf. Fact 2.1 in Section 2), $\sup_{x\le 0} L^x_{T(1)}$ has the same law as $Q(1)\sup_{t\ge 0} Z(t)$. Since Z is a linear diffusion process in natural scale (Revuz and Yor [22, Chap. XI]), we have $\mathbb{P}(\sup_{t\ge 0} Z(t) < z) = 1 - z^{-1}$ for all $z > 1$. Accordingly, by conditioning on $Q(1)$,

$$\mathbb{P}\Big(\sup_{x\le 0} L^x_{T(1)} < y\Big) = \mathbb{P}\Big(Q(1)\sup_{t\ge 0} Z(t) < y\Big)$$
$$= \mathbb{E}\Big[\Big(1 - \frac{Q(1)}{y}\Big)\, \mathbb{1}_{\{Q(1)<y\}}\Big].$$

Recall that $Q(1)$ has the exponential distribution, with mean 2, this immediately yields the lemma. □

Lemma 3.3. *Let Q be a 2-dimensional squared Bessel process starting from 0. There exists a universal constant c_8 such that for all $0 < b \le a < 1$,*

$$\mathbb{P}\Big(\sup_{0\le t\le a} Q(t) > \sup_{a\le t\le 1} Q(t) - b\Big) \le \frac{c_8}{a}\, \exp\Big(-j_0\sqrt{a^{-1}-1}\Big). \tag{3.9}$$

Proof. Write Λ_1 for the probability term on the left hand side of (3.9). Since Q can be considered as the squared modulus of a planar Wiener process, by conditioning on $\{Q(t);\, 0\le t\le a\}$ and using Anderson's inequality (Fact 2.6),

$$\Lambda_1 \le \mathbb{P}\Big(\sup_{0\le t\le 1-a} \widetilde{Q}_2(t) < \sup_{0\le t\le a} Q(t) + b\Big),$$

where $\widetilde{Q}_2$ is an independent copy of Q. Now, applying (2.7) yields

$$\begin{aligned}
\Lambda_1 &\le c_4\,\mathbb{E}\exp\Big(-\frac{j_0^2}{2}\frac{1-a}{\sup_{0\le t\le a} Q(t) + b}\Big)\\
&= c_4\,\mathbb{E}\exp\Big(-\frac{j_0^2}{2}\frac{a^{-1}-1}{\sup_{0\le t\le 1} Q(t) + b/a}\Big)\\
&\le c_4\,\mathbb{E}\exp\Big(-\frac{j_0^2}{2}\frac{a^{-1}-1}{\sup_{0\le t\le 1} Q(t) + 1}\Big).\\
&= -c_4\int_1^\infty \exp\Big(-\frac{j_0^2\,(a^{-1}-1)}{2x}\Big)\,\mathrm{d}_x\,\mathbb{P}\Big(\sup_{0\le t\le 1} Q(t) > x-1\Big),\\
&= c_4\,\frac{j_0^2\,(a^{-1}-1)}{2}\int_1^\infty \exp\Big(-\frac{j_0^2\,(a^{-1}-1)}{2x}\Big)\,\mathbb{P}\Big(\sup_{0\le t\le 1} Q(t) > x-1\Big)\,\frac{\mathrm{d}x}{x^2},
\end{aligned}$$

the last identity following from integration by parts. By the usual Gaussian tail estimate, $\mathbb{P}(\sup_{0\le t\le 1} Q(t) > x-1) \le c_9\, x^{3/2}\exp(-x/2)$ for all $x\ge 1$. Accordingly,

$$\begin{aligned}
\Lambda_1 &\le \frac{c_{10}}{a}\int_0^\infty \exp\Big(-\frac{j_0^2\,(a^{-1}-1)}{2x} - \frac{x}{2}\Big)\,\frac{\mathrm{d}x}{\sqrt{x}}\\
&= \frac{c_{10}}{a}\sqrt{2\pi}\,\exp\Big(-j_0\sqrt{a^{-1}-1}\Big).
\end{aligned}$$

We have used the fact that $\int_0^\infty x^{-1/2} \exp(-p/2x - qx/2)\,\mathrm{d}x = \sqrt{2\pi/q}\,\exp(-\sqrt{pq}\,)$ (for positive p and q). This yields (3.9). $\square$

The rest of the section is devoted to the proof of Theorem 3.1. For the sake of clarity, we prove (3.2) and (3.3) separately.

Proof of (3.2). Fix a small $\varepsilon > 0$, and define

$$\begin{aligned} r_n &= r_n(\varepsilon) = \exp(n^{1-\varepsilon}), \\ \delta_n &= \delta_n(\varepsilon) = \frac{(1-7\varepsilon)j_0^2}{(\log\log r_n)^2}. \\ \Theta_n &= \inf\Big\{ t \ge T(r_{n-1}) : \sup_{0\le x\le (1-\delta_n)r_n} L_t^x - \sup_{(1-\delta_n)r_n \le x \le r_n} L_t^x < r_n - r_{n-1} \Big\}. \end{aligned}$$

Clearly, for each n, Θ_n is a stopping time with respect to the natural filtration of W. Moreover, on $\{T(r_{n-1}) < \Theta_n < \infty\}$,

$$\sup_{0\le x\le (1-\delta_n)r_n} L_{\Theta_n}^x = \sup_{(1-\delta_n)r_n\le x\le r_n} L_{\Theta_n}^x + (r_n - r_{n-1}).$$

Consider the events, on $\{\Theta_n < \infty\}$,

$$\begin{aligned} E_n &= \Big\{ \sup_{-\infty < x \le (1-\delta_n)r_n} (L_{T(r_n)}^x - L_{\Theta_n}^x) < r_n - r_{n-1} \Big\}, \\ F_n &= \Big\{ \sup_{x\ge (1-\delta_n)r_n} L_{T(r_n)}^x > \sup_{0\le x\le (1-\delta_n)r_n} L_{T(r_n)}^x - 2(r_n - r_{n-1}) \Big\}. \end{aligned}$$

On the event $\{T(r_{n-1}) < \Theta_n \le T(r_n)\} \cap E_n$,

$$\begin{aligned} \sup_{x\ge(1-\delta_n)r_n} L_{T(r_n)}^x &\ge \sup_{x\ge (1-\delta_n)r_n} L_{\Theta_n}^x \\ &= \sup_{0\le x\le(1-\delta_n)r_n} L_{\Theta_n}^x - (r_n - r_{n-1}) \\ &> \sup_{0\le x\le (1-\delta_n)r_n} L_{T(r_n)}^x - 2(r_n - r_{n-1}). \end{aligned}$$

This means

$$\Big(\{T(r_{n-1}) < \Theta_n \le T(r_n)\} \cap E_n \Big) \subset F_n. \tag{3.10}$$

Consider now the process $\{\widetilde{W}(t) \stackrel{\mathrm{def}}{=} W(t+\Theta_n) - W(\Theta_n);\, t\ge 0\}$, on $\{\Theta_n < \infty\}$. By the strong Markov property, $\widetilde{W}$ is again a Wiener process, independent of $\mathcal{F}_{\Theta_n}$, where

$\{\mathcal{F}_t\}_{t\geq 0}$ denotes the natural filtration of W. We can define the local time $\widetilde{L}$ and first hitting time $\widetilde{T}$ for $\widetilde{W}$ exactly as L and T for W. Clearly, for all $t \geq 0$ and $x \in \mathbb{R}$,

$$\widetilde{L}_t^x = L_{t+\Theta_n}^{x+W(\Theta_n)} - L_{\Theta_n}^{x+W(\Theta_n)}. \tag{3.11}$$

Assume $T(r_{n-1}) < \Theta_n \leq T(r_n)$. Then $W(\Theta_n) \geq (1-\delta_n)r_n$, which implies $T(r_n) - \Theta_n \leq \widetilde{T}(\delta_n r_n)$. In view of (3.11), we have, on $\{T(r_{n-1}) < \Theta_n \leq T(r_n)\}$,

$$E_n \supset \Big\{ \sup_{x\leq 0} \widetilde{L}^x_{\widetilde{T}(\delta_n r_n)} < r_n - r_{n-1} \Big\}.$$

Since $\{T(r_{n-1}) < \Theta_n \leq T(r_n)\}$ is an $\mathcal{F}_{\Theta_n}$–measurable event, combining this with (3.10) gives

$$\mathbb{P}(F_n) \geq \mathbb{P}\Big(T(r_{n-1}) < \Theta_n \leq T(r_n)\Big)\, \mathbb{P}\Big(\sup_{x\leq 0} L^x_{T(\delta_n r_n)} < r_n - r_{n-1}\Big). \tag{3.12}$$

By scaling, the second probability term on the right hand side is

$$= \mathbb{P}\Big(\sup_{x\leq 0} L^x_{T(1)} < \frac{r_n - r_{n-1}}{\delta_n\, r_n}\Big) \geq \mathbb{P}\Big(\sup_{x\leq 0} L^x_{T(1)} < \frac{c_{11}(\varepsilon)}{n^\varepsilon}\Big) \geq \frac{c_{12}(\varepsilon)}{n^\varepsilon},$$

by means of (3.8). It follows that

$$\mathbb{P}\Big(T(r_{n-1}) \leq \Theta_n \leq T(r_n)\Big) \leq \frac{n^\varepsilon}{c_{12}(\varepsilon)} \mathbb{P}(F_n) + \mathbb{P}\Big(\Theta_n = T(r_{n-1})\Big). \tag{3.13}$$

By the scaling property of W,

$$\mathbb{P}(F_n) = \mathbb{P}\Big(\sup_{0\leq t\leq \delta_n} L^{1-t}_{T(1)} > \sup_{\delta_n\leq t\leq 1} L^{1-t}_{T(1)} - \frac{2(r_n - r_{n-1})}{r_n}\Big).$$

According to the Ray–Knight theorem (cf. Fact 2.1),

$$\mathbb{P}(F_n) = \mathbb{P}\Big(\sup_{0\leq t\leq \delta_n} Q(t) > \sup_{\delta_n\leq t\leq 1} Q(t) - \frac{2(r_n - r_{n-1})}{r_n}\Big),$$

where Q is a 2-dimensional squared Bessel process (with $Q(0) = 0$) as in (2.4). Since $2(r_n - r_{n-1})/r_n < \delta_n$ (for large n), we can apply Lemma 3.3 to arrive at

$$\mathbb{P}(F_n) \leq \frac{c_8}{\delta_n} \exp\Big(-j_0\sqrt{\frac{1}{\delta_n} - 1}\Big) \leq n^{-(1+2\varepsilon)}.$$

Moreover,

$$\begin{aligned}
&\mathbb{P}\Big(\Theta_n = T(r_{n-1})\Big) \\
&\quad = \mathbb{P}\Big(\sup_{0\le x\le (1-\delta_n)r_n/r_{n-1}} L^x_{T(1)} < \sup_{x\ge (1-\delta_n)r_n/r_{n-1}} L^x_{T(1)} + \frac{r_n - r_{n-1}}{r_n}\Big) \\
&\quad \le \mathbb{P}\Big(\sup_{0\le x\le 1-\delta_n} L^x_{T(1)} < \sup_{x\ge 1-\delta_n} L^x_{T(1)} + \frac{2(r_n - r_{n-1})}{r_n}\Big) \\
&\quad = \mathbb{P}(F_n) \\
&\quad \le n^{-(1+2\varepsilon)}.
\end{aligned}$$

In view of (3.13), we have $\sum_n \mathbb{P}(T(r_{n-1}) \le \Theta_n \le T(r_n)) < \infty$. By the Borel–Cantelli lemma, almost surely for all large n and $t \in [T(r_{n-1}), T(r_n)]$,

$$\sup_{0\le x\le (1-\delta_n)r_n} L^x_t \ge \sup_{x\ge (1-\delta_n)r_n} L^x_t + (r_n - r_{n-1}).$$

Since for $t \in [T(r_{n-1}), T(r_n)]$,

$$r_n - r_{n-1} \ge \frac{r_n}{(\log r_n)^{2\varepsilon}} \ge \frac{\overline{W}(t)}{(\log \overline{W}(t))^{2\varepsilon}} \ge \frac{t^{1/2}}{(\log t)^{1+3\varepsilon}},$$

(the last inequality following from (3.6)), and we also have

$$\begin{aligned}
(1-\delta_n)r_n &= r_n - \frac{(1-7\varepsilon)j_0^2 r_n}{(\log\log r_n)^2} \\
&\le r_{n-1} - \frac{(1-8\varepsilon)j_0^2 r_n}{(\log\log r_n)^2} \\
&\le \overline{W}(t) - \frac{(1-9\varepsilon)j_0^2\, \overline{W}(t)}{(\log\log t)^2},
\end{aligned}$$

This yields (3.2) (replacing ε by a small constant multiple of ε), hence the first part in Theorem 3.1. □

Proof of (3.3). Fix an $\varepsilon > 0$ and define

$$\begin{aligned}
r_n &= n^{3n}, \\
\delta_n &= \frac{(1+22\varepsilon)^2 j_0^2}{(\log\log r_n)^2}.
\end{aligned}$$

Consider the events

$$\begin{aligned}
G_n = \Big\{ &\sup_{x\ge r_n-(r_n-r_{n-1})\delta_n} (L^x_{T(r_n)} - L^x_{T(r_{n-1})}) \\
&\ge \sup_{x\le r_n-(r_n-r_{n-1})\delta_n} (L^x_{T(r_n)} - L^x_{T(r_{n-1})}) + j_0\, \delta_n^{1/2}\, (r_n - r_{n-1})\Big\}.
\end{aligned}$$

By the strong Markov property,

$$\mathbb{P}(G_n) = \mathbb{P}\Big(\sup_{x\ge r_n - r_{n-1} - (r_n - r_{n-1})\delta_n} L^x_{T(r_n - r_{n-1})}$$
$$\ge \sup_{x\le r_n - r_{n-1} - (r_n - r_{n-1})\delta_n} L^x_{T(r_n - r_{n-1})} + j_0\,\delta_n^{1/2}\,(r_n - r_{n-1})\Big)$$
$$= \mathbb{P}\Big(\sup_{x\ge 1-\delta_n} L^x_{T(1)} \ge \sup_{x\le 1-\delta_n} L^x_{T(1)} + j_0\,\delta_n^{1/2}\Big).$$

According to the Ray–Knight theorem (Fact 2.1), the last probability term equals

$$\mathbb{P}\Big(\sup_{0\le t\le \delta_n} Q(t) \ge \sup_{\delta_n\le t\le 1} Q(t) \vee Q(1) \sup_{t\ge 0} Z(t) + j_0\,\delta_n^{1/2}\Big),$$

where Q is as before a 2-dimensional squared Bessel process starting from 0, and Z is a squared Bessel process of dimension 0, starting from 1, independent of Q. Therefore,

$$\mathbb{P}(G_n) \ge \mathbb{P}(\sup_{t\ge 0} Z(t) \le 1+\varepsilon)\,\mathbb{P}\Big(\sup_{0\le t\le \delta_n} Q(t) \ge (1+\varepsilon) \sup_{\delta_n\le t\le 1} Q(t) + j_0\delta_n^{1/2}\Big).$$

Write $c_{13}(\varepsilon) \overset{\text{def}}{=} \mathbb{P}(\sup_{t\ge 0} Z(t) \le 1+\varepsilon) = \varepsilon/(1+\varepsilon)$. Accordingly,

$$\mathbb{P}(G_n) \ge c_{13}(\varepsilon)\,\mathbb{P}\Big(\sup_{0\le t\le \delta_n} Q(t) \ge (1+2\varepsilon) j_0\delta_n^{1/2},\ \sup_{\delta_n\le t\le 1} Q(t) \le j_0\delta_n^{1/2}\Big)$$
$$\ge c_{13}(\varepsilon)\,\mathbb{P}\Big((1+2\varepsilon) j_0\delta_n^{1/2} < Q((1-\varepsilon)\delta_n) < (1+4\varepsilon) j_0\delta_n^{1/2},$$
$$(1-6\varepsilon) j_0\delta_n^{1/2} < Q(\delta_n) < (1-4\varepsilon) j_0\delta_n^{1/2},$$
$$\text{after time } \delta_n, \text{ the process } Q \text{ hits } \varepsilon^2 j_0\delta_n^{1/2} \text{ before hitting } j_0\delta_n^{1/2},$$
$$\text{and after hitting } \varepsilon^2 j_0\delta_n^{1/2}, \text{ it spends at least time 1 below } j_0\delta_n^{1/2}\Big).$$

Recall that Q is a (strong) Markov process. Write $\mathbb{P}_x$ (for $x \ge 0$) the probability under with Q starts from x (thus $\mathbb{P}_0 = \mathbb{P}$). Define for $r > 0$,

$$\sigma(r) = \inf\Big\{ t > 0 :\ Q(t) = r \Big\}.$$

By virtue of the strong Markov property,

$$\mathbb{P}(G_n) \ge c_{13}(\varepsilon) \times \Lambda_2(n) \times \Lambda_3(n) \times \Lambda_4(n) \times \Lambda_5(n), \tag{3.14}$$

where

$$\Lambda_2(n) \overset{\text{def}}{=} \mathbb{P}\Big((1+2\varepsilon) j_0\delta_n^{1/2} < Q((1-\varepsilon)\delta_n) < (1+4\varepsilon) j_0\delta_n^{1/2}\Big),$$
$$\Lambda_3(n) \overset{\text{def}}{=} \inf_{(1+2\varepsilon) j_0\delta_n^{1/2} < x < (1+4\varepsilon) j_0\delta_n^{1/2}} \mathbb{P}_x\Big((1-6\varepsilon) j_0\delta_n^{1/2} < Q(\varepsilon\delta_n) < (1-4\varepsilon) j_0\delta_n^{1/2}\Big),$$
$$\Lambda_4(n) \overset{\text{def}}{=} \inf_{(1-6\varepsilon) j_0\delta_n^{1/2} < x < (1-4\varepsilon) j_0\delta_n^{1/2}} \mathbb{P}_x\Big(\sigma(\varepsilon^2 j_0\delta_n^{1/2}) < \sigma(j_0\delta_n^{1/2})\Big),$$
$$\Lambda_5(n) \overset{\text{def}}{=} \mathbb{P}_{\varepsilon^2 j_0\delta_n^{1/2}}\Big(\sup_{0\le t\le 1} Q(t) < j_0\delta_n^{1/2}\Big).$$

Let us begin to estimate $\Lambda_2(n)$. By scaling,

$$\begin{aligned}\Lambda_2(n) &= \mathbb{P}\Big(\frac{(1+2\varepsilon)j_0}{(1-\varepsilon)\delta_n^{1/2}} < Q(1) < \frac{(1+4\varepsilon)j_0}{(1-\varepsilon)\delta_n^{1/2}}\Big)\\ &\geq \mathbb{P}\Big(Q(1) > \frac{(1+2\varepsilon)j_0}{(1-\varepsilon)\delta_n^{1/2}}\Big) - \mathbb{P}\Big(Q(1) > \frac{(1+4\varepsilon)j_0}{(1-\varepsilon)\delta_n^{1/2}}\Big).\end{aligned}$$

Since $\log \mathbb{P}(Q(1) > x) \sim -x/2$ for $x \to \infty$, it follows that for large n,

$$\Lambda_2(n) \geq \exp\Big(-\frac{(1+3\varepsilon)j_0}{2(1-\varepsilon)\delta_n^{1/2}}\Big) \geq \exp\Big(-\frac{(1+5\varepsilon)j_0}{2\delta_n^{1/2}}\Big). \tag{3.15}$$

To estimate $\Lambda_3(n)$, we consider the function (for $x > 0$)

$$\begin{aligned}h(x) &\stackrel{\text{def}}{=} \mathbb{P}_x\Big((1-6\varepsilon)j_0\delta_n^{1/2} < Q(\varepsilon\delta_n) < (1-4\varepsilon)j_0\delta_n^{1/2}\Big)\\ &= \frac{1}{2\varepsilon\delta_n}\int_{(1-6\varepsilon)j_0\delta_n^{1/2}}^{(1-4\varepsilon)j_0\delta_n^{1/2}} \exp\Big(-\frac{x+y}{2\varepsilon\delta_n}\Big) I_0\Big(\frac{\sqrt{xy}}{\varepsilon\delta_n}\Big)\,dy,\end{aligned}$$

the last equality following from (2.9). It is known that as z goes to infinity, $I_0(z) \sim \mathrm{e}^z/\sqrt{2\pi z}$ (cf. for example Gradshteyn and Ryzhik [13, p. 962]). Hence $I_0(z) \geq \mathrm{e}^z/z$ for all sufficiently large z. Accordingly, for all $(1+2\varepsilon)j_0\delta_n^{1/2} < x < (1+4\varepsilon)j_0\delta_n^{1/2}$,

$$\begin{aligned}h(x) &\geq \frac{1}{2\varepsilon\delta_n}\int_{(1-6\varepsilon)j_0\delta_n^{1/2}}^{(1-4\varepsilon)j_0\delta_n^{1/2}} \frac{\varepsilon\delta_n}{\sqrt{xy}}\exp\Big(-\frac{(\sqrt{x}-\sqrt{y}\,)^2}{2\varepsilon\delta_n}\Big)\,\mathrm{d}y\\ &\geq \frac{1}{2\varepsilon\delta_n}\frac{\varepsilon\delta_n}{j_0\delta_n^{1/2}}\exp\Big(-\frac{(\sqrt{1+4\varepsilon}-\sqrt{1-6\varepsilon}\,)^2 j_0}{2\varepsilon\delta_n^{1/2}}\Big)\,2\varepsilon j_0\delta_n^{1/2}\\ &\geq \varepsilon\,\exp\Big(-\frac{18\varepsilon j_0}{\delta_n^{1/2}}\Big),\end{aligned}$$

the last inequality following from $\sqrt{1+4\varepsilon}-\sqrt{1-6\varepsilon} \leq 6\varepsilon$. Therefore, for all large n,

$$\Lambda_3(n) = \inf_{(1+2\varepsilon)j_0\delta_n^{1/2} < x < (1+4\varepsilon)j_0\delta_n^{1/2}} h(x) \geq \varepsilon\,\exp\Big(-\frac{18\varepsilon j_0}{\delta_n^{1/2}}\Big). \tag{3.16}$$

The third term $\Lambda_4(n)$ can be explicitly computed. By diffusion (or martingale) theory, for $x \in (\varepsilon^2 j_0\delta_n^{1/2}, j_0\delta_n^{1/2})$,

$$\mathbb{P}_x\Big(\sigma(\varepsilon^2 j_0\delta_n^{1/2}) < \sigma(j_0\delta_n^{1/2})\Big) = \frac{\log(j_0\delta_n^{1/2}) - \log x}{\log(j_0\delta_n^{1/2}) - \log(\varepsilon^2 j_0\delta_n^{1/2})},$$

from which it follows that

$$\Lambda_4(n) \geq c_{14}(\varepsilon). \tag{3.17}$$

Finally, by triangular inequality and Fact 2.4,

$$\Lambda_5(n) \geq \mathbb{P}\Big(\sup_{0\leq t\leq 1} Q(t) < (1-\varepsilon)^2 j_0 \delta_n^{1/2} \Big) \geq \exp\Big(-(1+3\varepsilon) \frac{j_0}{2\delta_n^{1/2}} \Big). \tag{3.18}$$

Assembling (3.14)–(3.18):

$$\mathbb{P}(G_n) \geq c_{15}(\varepsilon)\ \exp\Big(-\frac{(1+22\varepsilon) j_0}{\delta_n^{1/2}} \Big),$$

which implies $\sum_n \mathbb{P}(G_n) = \infty$. By the strong Markov property, G_n are independent events. Therefore, according to the Borel–Cantelli lemma, almost surely there are infinitely many n satisfying

$$\begin{aligned} &\sup_{x\geq r_n-(r_n-r_{n-1})\delta_n} L^x_{T(r_n)} \\ &\qquad \geq \sup_{x\leq r_n-(r_n-r_{n-1})\delta_n} L^x_{T(r_n)} - \sup_{x\in\mathbb{R}} L^x_{T(r_{n-1})} + j_0\, \delta_n^{1/2}\, (r_n - r_{n-1}). \end{aligned}$$

Applying (3.7) and Kesten's LIL for local time (cf. Fact 2.7) yields that for all large n,

$$\sup_{x\in\mathbb{R}} L^x_{T(r_{n-1})} \leq \sqrt{3T(r_{n-1}) \log\log T(r_{n-1})} < r_{n-1} (\log r_{n-1})^2,$$

which is smaller than $j_0\, \delta_n^{1/2}\, (r_n - r_{n-1})/2$. Therefore, infinitely often,

$$\sup_{x\geq r_n-(r_n-r_{n-1})\delta_n} L^x_{T(r_n)} > \sup_{x\leq r_n-(r_n-r_{n-1})\delta_n} L^x_{T(r_n)} + \frac{j_0}{2}\, \delta_n^{1/2}\, (r_n - r_{n-1}).$$

Since by (3.7),

$$\frac{j_0}{2}\, \delta_n^{1/2}\, (r_n - r_{n-1}) \geq \frac{\sqrt{T(r_n)}}{(\log T(r_n))^{1+\varepsilon}},$$

and since

$$\begin{aligned} r_n - (r_n - r_{n-1})\delta_n &\geq r_n - r_n \delta_n \\ &\geq r_n - \frac{(1+23\varepsilon)^2 j_0^2\, r_n}{(\log\log T(r_n))^2} \\ &= \overline{W}(T(r_n)) - \frac{(1+23\varepsilon)^2 j_0^2\, \overline{W}(T(r_n))}{(\log\log T(r_n))^2}, \end{aligned}$$

this yields (3.3) (replacing ε by a constant multiple of ε), and hence completes the proof of Theorem 3.1. □

4. Proof of Theorem 1.1

That the liminf expression in Theorem 1.1 should be a constant (possibly zero or infinite) can be seen by means of a 0–1 argument. Indeed, write $S_n = \sum_{i=1}^n X_i$, so that $\{X_i\}_{i\geq 1}$ are iid Bernoulli variables. Let

$$c_0 \stackrel{\text{def}}{=} \liminf_{n\to\infty} \frac{(\log\log n)^{3/2}}{n^{1/2}} \big(\overline{S}_n - U(n)\big),$$

and we now show that c_0 is almost surely a constant.

By the Hewitt–Savage 0–1 law, it suffices to check that c_0 remains unchanged under any finite permutation of the variables $\{X_i\}_{i\geq 1}$. By induction, we only have to treat the case of permutation between two elements, say X_i and X_j. Without loss of generality, we can assume that $|j-i| = 1$.

For typesetting simplification, we write the proof only for the case $i = 1$ and $j = 2$. Let

$$\widetilde{X}_k \stackrel{\text{def}}{=} \begin{cases} X_2, & \text{if } k = 1, \\ X_1, & \text{if } k = 2, \\ X_k, & \text{if } k \geq 3, \end{cases}$$

and define the corresponding simple random walk $\widetilde{S}_0 = 0$ and

$$\widetilde{S}_n = \sum_{k=1}^n \widetilde{X}_k.$$

There is also a local time process $\widetilde{\xi}_n^x$ associated with $\{\widetilde{S}_n\}_{n\geq 0}$, and the (largest) favourite point is denoted by $\widetilde{U}(n)$. For all $x \in \mathbb{Z}\backslash\{-1, 1\}$, $\widetilde{\xi}_n^x = \xi_n^x$, and $|\widetilde{\xi}_n^y - \xi_n^y| \leq 1$ if $y = \pm 1$.

It is proved by Bass and Griffin [2] that $\xi_n^y \leq \sup_{x\in\mathbb{Z}} \xi_n^x - 2$ (for $y = \pm 1$), almost surely for all large n. Therefore

$$U(n) = \widetilde{U}(n), \qquad \text{eventually}.$$

Since $\max_{0\leq k\leq n} \widetilde{S}_k = \overline{S}_n$ for all large n, this proves that c_0 remains unchanged under the permutation between X_1 and X_2.

Consequently, c_0 is almost surely a constant.

It remains to show that c_0 lies in $(0, \infty)$. The finiteness of c_0 is a straightforward consequence of Theorem 1.2 and the usual LIL for random walk (cf. (3.5)), with $c_0 \leq \sqrt{2}\, j_0^2$.

The positiveness of c_0 follows from the following general result and the strong approximation, exactly in the same way as Theorem 1.2 from Theorem 3.1. For details of the argument, cf. Section 3.

Theorem 4.1. *Let $\psi(t) \stackrel{\mathrm{def}}{=} t^{1/2}/(\log\log t)^{3/2}$. There exist universal constants $c>0$ and $\varepsilon_0 \in (0,1)$ such that for all $0<\varepsilon<\varepsilon_0$ and almost surely all sufficiently large t,*

$$\sup_{x \le \overline{W}(t) - c\,\psi(t)} L_t^x > \sup_{x > \overline{W}(t) - c\,\psi(t)} L_t^x + \frac{t^{1/2}}{(\log t)^{1+\varepsilon}}. \tag{4.1}$$

The rest of the section aims at the proof of Theorem 4.1, which is based on several preliminary estimates. We start with the following estimates for Gaussian tails, which will be frequently used later. Recall that Q, H are squared Bessel processes of dimensions 2 and 4 respectively, both starting from 0 (cf. (2.4) and (2.5)), and that T is the process of first hitting times for W, cf. (2.1). Then for all positive x, t and r,

$$\mathbb{P}\Big(\sup_{0\le s\le t} Q(s) > x\Big) \le c_{16}\, \exp\Big(-\frac{x}{3t}\Big), \tag{4.2}$$

$$\mathbb{P}\Big(\sup_{0\le s\le t} H(s) > x\Big) \le c_{17}\, \exp\Big(-\frac{x}{3t}\Big), \tag{4.3}$$

$$\mathbb{P}\Big(T(r) < t\Big) \le \exp\Big(-\frac{r^2}{2t}\Big), \tag{4.4}$$

$$\mathbb{P}\Big(T(r) > t\Big) \le \frac{r}{\sqrt{t}}. \tag{4.5}$$

Recall that Z is a squared Bessel process of dimension 0, starting from 1, and ζ_Z is its life-time, cf. (2.2)–(2.3).

Lemma 4.2. *There exist universal constants c_{18}, c_{19} and c_{20} such that for all $x>0$,*

$$\mathbb{P}\Big(\frac{1}{\zeta_Z}\sup_{t\ge 0} Z(t) < x\Big) \le c_{18}\, \exp\Big(-\frac{c_{19}}{x}\Big), \tag{4.6}$$

$$\mathbb{P}\Big(\frac{1}{\zeta_Z}\sup_{t\ge 0} Z(t) > x\Big) \le c_{20}\, \exp\Big(-\frac{x}{9}\Big), \tag{4.7}$$

Proof. Let $\mathbb{L}_H$ be the last exit time from 1 of H, cf. (2.6). By Fact 2.2, $\sup_{t\ge 0} Z(t)/\zeta_Z$ has the same law as $\sup_{0\le t\le \mathbb{L}_H} H(t)/\mathbb{L}_H$. Applying Fact 2.3 to the bounded functional $F:\, f \mapsto \mathbb{1}_{\{\sup_{0\le s\le 1} f(s) < x\}}$ gives

$$\begin{aligned}\mathbb{P}\Big(\frac{1}{\zeta_Z}\sup_{t\ge 0} Z(t) < x\Big) &= 2\mathbb{E}\Big[\frac{1}{H(1)}\mathbb{1}_{\{\sup_{0\le s\le 1} H(s) < x\}}\Big] \\ &\le 2\Big[\mathbb{E}\big(H^{-3/2}(1)\big)\Big]^{2/3}\Big[\mathbb{P}\Big(\sup_{0\le s\le 1} H(s) < x\Big)\Big]^{1/3},\end{aligned}$$

by means of the Hölder inequality. Since by a Gaussian calculation, $H^{-3/2}(1)$ has finite expectation, this yields (4.6) by using (2.8) (which, as was recalled in Section 2, goes back to Ciesielski and Taylor [6]). The proof of (4.7) follows exactly from the same lines, using (4.3) instead of (2.8). □

Lemma 4.3. *For any $x > 0$ and $t > 0$,*

$$\mathbb{P}\Big(\sup_{0\le s\le t} Z(s) > x\Big) \le c_{21}\, \exp\Big(-\frac{x-2}{9t}\Big). \tag{4.8}$$

Proof. Recall that Z is a diffusion process, starting from 1, absorbed by 0, with generator $2x\,\mathrm{d}^2/\,\mathrm{d}x^2$. Therefore, it can be realized as, for $t < \zeta_Z$,

$$\sqrt{Z(t)} = 1 + W(t) - \frac{1}{2}\int_0^t \frac{\mathrm{d}s}{\sqrt{Z(s)}},$$

where W is the Wiener process. Hence $\sqrt{Z(t)} \le 1 + W(t)$ for all $t < \zeta_Z$. Accordingly,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le s\le t} Z(s) > x,\ t<\zeta_Z\Big) &\le \mathbb{P}\Big(\sup_{0\le s\le t} \big(1+W(s)\big) > x^{1/2}\Big)\\ &= \mathbb{P}\Big(\sup_{0\le s\le t} W(s) > x^{1/2}-1\Big)\\ &\le \exp\Big(-\frac{(x^{1/2}-1)^2}{2t}\Big),\end{aligned}$$

by virtue of the usual Gaussian tail estimate. Since $(x^{1/2}-1)^2 \ge (x-2)/2$, this gives

$$\mathbb{P}\Big(\sup_{0\le s\le t} Z(s) > x,\ t<\zeta_Z\Big) \le \exp\Big(-\frac{x-2}{4t}\Big). \tag{4.9}$$

On the other hand,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le s\le t} Z(s) > x,\ t\ge\zeta_Z\Big) &\le \mathbb{P}\Big(\frac{1}{\zeta_Z}\sup_{s\ge 0} Z(s) > \frac{x}{t}\Big)\\ &\le c_{20}\, \exp\Big(-\frac{x}{9t}\Big),\end{aligned} \tag{4.10}$$

by means of (4.7). Combining (4.9) and (4.10) yields the lemma. □

Remark 4.3.1. The constant 9 in Lemma 4.3 is clearly not the best possible. Moreover, the lemma can also be proved by writing the probability in terms of the first hitting times of Z and using the Laplace transform of it via Chernoff's method (this Laplace transform has been computed by Ciesielski and Taylor [6]). This remark was suggested to us by an anonymous referee.

Proof of Theorem 4.1. We choose a universal constant c_{22} such that

$$\frac{1}{(18\times 2^6)\, c_{22}} \geq 5, \tag{4.11}$$

$$\frac{c_{19}}{2^{15}\, c_{22}} \geq 7, \tag{4.12}$$

where c_{19} is the absolute constant in (4.6).

We need a good maximal inequality, which bears some similarities with (3.12). To this end, fix a small $\varepsilon \in (0,1)$, and define

$$\begin{aligned} r_n &\stackrel{\text{def}}{=} r_n(\varepsilon) = \exp(n^{1-\varepsilon}), \\ \nu_n &\stackrel{\text{def}}{=} \sqrt{c_{22}}\,/(\log\log r_n)^{3/2}, \\ \Xi_n &\stackrel{\text{def}}{=} \inf\Big\{ t \geq T(r_{n-1}) : \sup_{x\leq r_n-\nu_n\sqrt{t}} L_t^x < \sup_{x\geq r_n-\nu_n\sqrt{t}} L_t^x + (r_n - r_{n-1})\Big\}. \end{aligned}$$

Observe that Ξ_n is again a stopping time. Define, on $\{\Xi_n < \infty\}$,

$$\begin{aligned} E_n &= \Big\{ \sup_{x\leq r_n-\nu_n\sqrt{\Xi_n}} (L^x_{T(r_n)} - L^x_{\Xi_n}) < r_n - r_{n-1}\Big\}, \\ F_n &= \Big\{ \sup_{x\geq r_n-\nu_n\sqrt{T(r_n)}} L^x_{T(r_n)} > \sup_{x\leq r_n-\nu_n\sqrt{T(r_n)}} L^x_{T(r_n)} - 2(r_n - r_{n-1})\Big\}. \end{aligned}$$

Then $\{T(r_{n-1}) < \Xi_n \leq T(r_n)\} \cap E_n$ is included in F_n (cf. the argument leading to (3.10)).

If $T(r_{n-1}) < \Xi_n < \infty$, we have $W(\Xi_n) \geq r_n - \nu_n\sqrt{\Xi_n}$, which yields that, on $\{T(r_{n-1}) < \Xi_n \leq T(r_n)\}$,

$$E_n \supset \Big\{ \sup_{x\leq W(\Xi_n)} (L^x_{T(r_n)} - L^x_{\Xi_n}) < r_n - r_{n-1}\Big\}.$$

By considering the new Wiener process $\{W(t+\Xi_n) - W(\Xi_n);\, t\geq 0\}$, we arrive at:

$$\mathbb{P}(F_n) \geq \mathbb{E}\Big[\mathbb{1}_{\{T(r_{n-1})<\Xi_n\leq T(r_n)\}}\, \mathbb{P}\Big(\sup_{x\leq 0} L^x_{T(\nu_n\sqrt{a}\,)} < r_n - r_{n-1}\Big)_{a=\Xi_n}\Big].$$

By scaling and (3.8), we have

$$\begin{aligned} \mathbb{P}(F_n) &\geq \mathbb{E}\Big[\mathbb{1}_{\{T(r_{n-1})<\Xi_n\leq T(r_n)\}}\, c_7\, \frac{r_n - r_{n-1}}{\nu_n\sqrt{\Xi_n}}\, \mathbb{1}_{\{\sqrt{\Xi_n}>(r_n-r_{n-1})/\nu_n\}}\Big] \\ &\geq c_7\, \mathbb{E}\Big[\mathbb{1}_{\{T(r_{n-1})<\Xi_n\leq T(r_n)\}}\, \frac{r_n - r_{n-1}}{\nu_n\sqrt{\Xi_n}}\Big] \\ &\qquad - c_7\, \mathbb{E}\Big(\frac{r_n - r_{n-1}}{\nu_n\sqrt{T(r_{n-1})}}\, \mathbb{1}_{\{\sqrt{T(r_{n-1})}\leq (r_n-r_{n-1})/\nu_n\}}\Big) \\ &\stackrel{\text{def}}{=} \Lambda_6(n) - \Lambda_7(n), \end{aligned}$$

with obvious notation. Applying the Cauchy–Schwarz inequality and (4.4) leads to:

$$\begin{aligned}
\Lambda_7(n) &\le c_7\, \frac{r_n - r_{n-1}}{\nu_n} \left[\mathbb{E}\Big(\frac{1}{T(r_{n-1})}\Big)\, \mathbb{P}\Big(\sqrt{T(r_{n-1})} \le \frac{r_n - r_{n-1}}{\nu_n}\Big) \right]^{1/2} \\
&\le c_7\, \frac{r_n - r_{n-1}}{\nu_n} \left[\frac{1}{r_{n-1}^2}\, \exp\Big(-\frac{(\nu_n\, r_{n-1})^2}{2(r_n - r_{n-1})^2}\Big) \right]^{1/2} \\
&\le \exp(-n^{\varepsilon}). \qquad (4.13)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Lambda_6(n) &\ge c_7\, \frac{r_n - r_{n-1}}{\nu_n\, r_n\, n^{1+\varepsilon}}\, \mathbb{P}\Big(\, T(r_{n-1}) < \Xi_n \le T(r_n) \le r_n^2 n^{2+2\varepsilon} \Big) \\
&\ge n^{-(1+3\varepsilon)} \left[\mathbb{P}\big(T(r_{n-1}) < \Xi_n \le T(r_n)\big) - \mathbb{P}\big(T(r_n) > r_n^2 n^{2+2\varepsilon}\big) \right] \\
&\ge n^{-(1+3\varepsilon)} \left[\mathbb{P}\big(T(r_{n-1}) < \Xi_n \le T(r_n)\big) - n^{-(1+\varepsilon)} \right]. \qquad (4.14)
\end{aligned}$$

(We have used (4.5) in the last inequality). Since $\mathbb{P}(F_n) \ge \Lambda_6(n) - \Lambda_7(n)$, combining (4.13) and (4.14) implies

$$\mathbb{P}\Big(T(r_{n-1}) < \Xi_n \le T(r_n)\Big) \le n^{-(1+\varepsilon)} + n^{1+3\varepsilon}\Big(\exp(-n^{\varepsilon}) + \mathbb{P}(F_n)\Big).$$

Assume we could show

$$\sum_n \mathbb{P}\Big(\Xi_n = T(r_{n-1})\Big) < \infty, \qquad (4.15)$$

$$\sum_n n^{1+3\varepsilon}\, \mathbb{P}(F_n) < \infty. \qquad (4.16)$$

Then we would have $\sum_n \mathbb{P}(T(r_{n-1}) \le \Xi_n \le T(r_n)) < \infty$, which, according to the Borel–Cantelli lemma, would imply that almost surely for all large n and all $t \in [T(r_{n-1}), T(r_n)]$,

$$\sup_{x \le r_n - \nu_n \sqrt{t}} L_t^x \ge \sup_{x \ge r_n - \nu_n \sqrt{t}} L_t^x + (r_n - r_{n-1}).$$

Since for $t \in [T(r_{n-1}), T(r_n)]$,

$$r_n - r_{n-1} \ge \frac{r_n}{(\log r_n)^{2\varepsilon}} \ge \frac{\overline{W}(t)}{(\log \overline{W}(t))^{2\varepsilon}} \ge \frac{t^{1/2}}{(\log t)^{1+3\varepsilon}},$$

(using (3.6)), and since by (3.7),

$$r_n - \nu_n\sqrt{t} \le r_{n-1} - \frac{1}{2}\,\nu_n\sqrt{t} \le \overline{W}(t) - \frac{\sqrt{c_{22}}}{3(\log\log t)^{3/2}}\, t^{1/2},$$

this would yield Theorem 4.1.

It remains to check (4.15) and (4.16). By definition, $\mathbb{P}(\Xi_n = T(r_{n-1}))$ equals

$$\mathbb{P}\Big(\sup_{x \ge r_n - \nu_n \sqrt{T(r_{n-1})}} L^x_{T(r_{n-1})} \ge \sup_{x \le r_n - \nu_n \sqrt{T(r_{n-1})}} L^x_{T(r_{n-1})} - (r_n - r_{n-1}) \Big),$$

which, in light of the scaling property, is smaller than $\mathbb{P}(F_n)$. Therefore, we only have to prove (4.16).

Observe that by scaling,

$$\begin{aligned}\mathbb{P}(F_n) &= \mathbb{P}\Big(\sup_{0 \le t \le \nu_n \sqrt{T(1)}} L^{1-t}_{T(1)} \ge \sup_{t \ge \nu_n \sqrt{T(1)}} L^{1-t}_{T(1)} - \frac{2(r_n - r_{n-1})}{r_n} \Big) \\ &\le \mathbb{P}\Big(\sup_{0 \le t \le \nu_n \sqrt{T(1)}} L^{1-t}_{T(1)} > \sup_{t \ge \nu_n \sqrt{T(1)}} L^{1-t}_{T(1)} - \frac{2}{n^\varepsilon} \Big).\end{aligned}$$

Since $T(1) = \int_0^\infty L^{1-t}_{T(1)} \, \mathrm{d}t$, by distinguishing two possible situations $\nu_n \sqrt{T(1)} > 1$ and $\nu_n \sqrt{T(1)} \le 1$, the Ray–Knight theorem (cf. Fact 2.1) confirms that the probability term on the right hand side

$$\begin{aligned}&= \mathbb{P}\Big(X_n \le 1; \sup_{0 \le t \le X_n} Q(t) > \sup_{X_n \le t \le 1} Q(t) \vee Q(1) \sup_{t \ge 0} Z(t) - \frac{2}{n^\varepsilon} \Big) \\ &\qquad + \mathbb{P}\Big(X_n > 1; \sup_{0 \le t \le 1} Q(t) \vee Q(1) \sup_{0 \le t \le (X_n - 1)/Q(1)} Z(t) \\ &\qquad\qquad > Q(1) \sup_{t \ge (X_n - 1)/Q(1)} Z(t) - \frac{2}{n^\varepsilon} \Big) \\ &\stackrel{\text{def}}{=} \Lambda_8(n) + \Lambda_9(n),\end{aligned}$$

with the notation

$$X_n \stackrel{\text{def}}{=} \nu_n \Big(\int_0^1 Q(t) \, \mathrm{d}t + Q^2(1) \int_0^\infty Z(t) \, \mathrm{d}t \Big)^{1/2},$$

where, as before (cf. (2.2) and (2.4)), Q is a 2-dimensional squared Bessel process starting from 0, and Z is a squared Bessel process of dimension 0 starting from 1 (the processes Q and Z being independent). For our needs later, we insist that

$$X_n \stackrel{\text{law}}{=} \nu_n \sqrt{T(1)}. \tag{4.17}$$

The proof of (4.16) (hence of Theorem 4.1) will be complete once we prove the following lemma. □

Lemma 4.4. *We have,*

$$\sum_n n^{1+3\varepsilon}\,\Lambda_8(n) < \infty, \tag{4.18}$$

$$\sum_n n^{1+3\varepsilon}\,\Lambda_9(n) < \infty. \tag{4.19}$$

The proof of Lemma 4.4 is divided into two parts, namely, the two estimates (4.18) and (4.19) are established separately.

Proof of (4.18). Let

$$N = N(n) \stackrel{\text{def}}{=} 2^6\,(\log n)^2. \tag{4.20}$$

For brevity, we write, for $0 < s < t < \infty$ and $0 < x < y \le \infty$,

$$\overline{Q}(t) \stackrel{\text{def}}{=} \sup_{0\le u\le t} Q(u), \qquad \overline{Q}(s,t) \stackrel{\text{def}}{=} \sup_{s\le u\le t} Q(u), \tag{4.21}$$

$$\overline{Z}(y) \stackrel{\text{def}}{=} \sup_{0\le u\le y} Z(u), \qquad \overline{Z}(x,y) \stackrel{\text{def}}{=} \sup_{x\le u\le y} Z(u). \tag{4.22}$$

Observe that

$$\begin{aligned}\Lambda_8(n) &\le \sum_{k=1}^{N} \mathbb{P}\Big(\frac{k-1}{N} < X_n \le \frac{k}{N},\, \overline{Q}(\frac{k}{N}) > \overline{Q}(\frac{k}{N},1) \vee Q(1)\overline{Z}(\infty) - \frac{2}{n^{\varepsilon}}\Big)\\ &\stackrel{\text{def}}{=} \sum_{k=1}^{N} \Lambda_{10}(n,k). \end{aligned}\tag{4.23}$$

By the definition of X_n,

$$\begin{aligned}\Lambda_{10}(n,k) \le \mathbb{P}\Big(&\overline{Q}(1) + Q^2(1)\,\zeta_Z\,\overline{Z}(\infty) > \frac{(k-1)^2}{\nu_n^2\,N^2},\\ &\overline{Q}(\frac{k}{N}) > \overline{Q}(\frac{k}{N},1) \vee Q(1)\overline{Z}(\infty) - \frac{2}{n^{\varepsilon}}\Big),\end{aligned}$$

where, as before, ζ_Z denotes the life-time of Z. Applying Lemma 3.3 to $a = 1/N$ and $b = 2/n^{\varepsilon}$ gives (recalling that $j_0 > 2$):

$$\begin{aligned}\Lambda_{10}(n,1) &\le \mathbb{P}\Big(\overline{Q}(\frac{1}{N}) > \overline{Q}(\frac{1}{N},1) - \frac{2}{n^{\varepsilon}}\Big)\\ &\le c_8\,N\,\exp\Big(-j_0\sqrt{N-1}\Big)\\ &\le n^{-4}. \end{aligned}\tag{4.24}$$

On the other hand, for $2 \le k \le N$,

$$\begin{aligned}\Lambda_{10}(n,k) &\le \mathbb{P}\Big(\,\overline{Q}(1) \ge \frac{(k-1)^2}{2\nu_n^2\,N^2},\ \overline{Q}(\frac{k}{N}) > \overline{Q}(\frac{k}{N},1) - \frac{2}{n^{\varepsilon}}\Big)\\ &\qquad + \mathbb{P}\Big(\,Q^2(1)\,\zeta_Z\,\overline{Z}(\infty) \ge \frac{(k-1)^2}{2\nu_n^2\,N^2},\ Q(1)\overline{Z}(\infty) < \overline{Q}(\frac{k}{N}) + \frac{2}{n^{\varepsilon}}\Big)\\ &\overset{\text{def}}{=} \Lambda_{11}(n,k) + \Lambda_{12}(n,k).\end{aligned}$$

We have, for $2 \le k \le N$,

$$\begin{aligned}\Lambda_{11}(n,k) &\le \mathbb{P}\Big(\,\overline{Q}(\frac{k}{N}) \ge \frac{(k-1)^2}{2\nu_n^2\,N^2} - \frac{2}{n^{\varepsilon}}\Big)\\ &\le \mathbb{P}\Big(\,\overline{Q}(\frac{k}{N}) \ge \frac{(k-1)^2}{3\nu_n^2\,N^2}\Big)\\ &\le c_{16}\,\exp\Big(-\frac{(k-1)^2}{9k\nu_n^2 N}\Big),\end{aligned}$$

by virtue of (4.2). Noting $(k-1)^2/k \ge 1/2$ and in light of (4.11), we obtain:

$$\max_{2\le k\le N} \Lambda_{11}(n,k) \le n^{-4}. \tag{4.25}$$

To estimate $\Lambda_{12}(n,k)$, note that for all $2 \le k \le N$,

$$\begin{aligned}\Lambda_{12}(n,k) &\le \mathbb{P}\Big(\,\frac{1}{\zeta_Z}\overline{Z}(\infty) < \frac{2\nu_n^2\,N^2}{(k-1)^2}\Big(\overline{Q}(\frac{k}{N}) + \frac{2}{n^{\varepsilon}}\Big)^2\,\Big)\\ &\le \mathbb{P}\Big(\overline{Q}(\frac{k}{N}) > \frac{2^5\,(k-1)}{N}\log n - \frac{2}{n^{\varepsilon}}\Big)\\ &\qquad + \mathbb{P}\Big(\,\frac{1}{\zeta_Z}\sup_{t\ge 0} Z(t) < 2^{11}\,\nu_n^2\,(\log n)^2\Big)\\ &\overset{\text{def}}{=} \Lambda_{13}(n,k) + \Lambda_{14}(n,k).\end{aligned}$$

By (4.2), for $2 \le k \le N$,

$$\begin{aligned}\Lambda_{13}(n,k) &\le \mathbb{P}\Big(\overline{Q}(\frac{k}{N}) > \frac{31(k-1)}{N}\log n\Big)\\ &\le c_{16}\,\exp\Big(-\frac{31(k-1)}{3k}\log n\Big)\\ &\le c_{16}\,\exp\Big(-\frac{31}{6}\log n\Big)\\ &\le n^{-4},\end{aligned}$$

whereas by Lemma 4.2 and (4.12),

$$\Lambda_{14}(n,k) \le c_{18}\,\exp\Big(-\frac{c_{19}}{2^{11}\,\nu_n^2\,(\log n)^2}\Big) \le n^{-4}.$$

Therefore, for all sufficiently large n,

$$\max_{2\le k\le N} \Lambda_{12}(n,k) \le 2\,n^{-4}. \tag{4.26}$$

Since $\Lambda_{10}(n,k) \le \Lambda_{11}(n,k) + \Lambda_{12}(n,k)$ for all $2 \le k \le N$, assembling (4.24)–(4.26) yields

$$\sum_{k=1}^{N} \Lambda_{10}(n,k) \le 2N\,n^{-4} \le n^{-3},$$

which, in view of (4.23), completes the proof of (4.18). □

Proof of (4.19). Let N, $\overline{Q}$ and $\overline{Z}$ be as in (4.20)–(4.22). We have,

$$\begin{aligned}\Lambda_9(n) &\le \mathbb{P}\Big(X_n > n^3\nu_n\Big) + \sum_{k=1}^{[n^3\nu_n N]} \mathbb{P}\Big(\frac{k-1}{N}+1 < X_n \le \frac{k}{N}+1,\\ &\qquad \overline{Q}(1) \vee Q(1)\overline{Z}\big(\frac{k}{Q(1)\,N}\big) > Q(1)\overline{Z}\big(\frac{k}{Q(1)\,N},\infty\big) - \frac{2}{n^{\varepsilon}}\Big)\\ &\overset{\text{def}}{=} \mathbb{P}\Big(X_n > n^3\nu_n\Big) + \sum_{k=1}^{[n^3\nu_n N]} \Lambda_{15}(n,k).\end{aligned} \tag{4.27}$$

Clearly,

$$\begin{aligned}\Lambda_{15}(n,k) &\le \mathbb{P}\Big(\overline{Q}(1) + Q^2(1)\,\zeta_Z\,\overline{Z}(\infty) > \frac{(N+k-1)^2}{\nu_n^2\,N^2},\\ &\qquad \overline{Q}(1) \vee Q(1)\overline{Z}\big(\frac{k}{Q(1)\,N}\big) > Q(1)\overline{Z}\big(\frac{k}{Q(1)\,N},\infty\big) - \frac{2}{n^{\varepsilon}}\Big)\\ &\le \Lambda_{16}(n,k) + \Lambda_{17}(n,k) + \Lambda_{18}(n,k) + \Lambda_{19}(n,k),\end{aligned} \tag{4.28}$$

where

$$\Lambda_{16}(n,k) \stackrel{\mathrm{def}}{=} \mathbb{P}\Big(\,\overline{Q}(1) > \frac{(N+k-1)^2}{2\nu_n^2\,N^2}\Big),$$

$$\Lambda_{17}(n,k) \stackrel{\mathrm{def}}{=} \mathbb{P}\Big(\,\overline{Q}(1) > \frac{N+k-1}{\log n}\Big),$$

$$\Lambda_{18}(n,k) \stackrel{\mathrm{def}}{=} \mathbb{P}\Big(Q^2(1)\,\zeta_Z\,\overline{Z}(\infty) > \frac{(N+k-1)^2}{2\nu_n^2\,N^2}, \quad \overline{Z}\big(\frac{k}{Q(1)\,N},\infty\big) < \overline{Z}\big(\frac{k}{Q(1)\,N}\big) + \frac{2}{n^\varepsilon}\Big),$$

$$\Lambda_{19}(n,k) \stackrel{\mathrm{def}}{=} \mathbb{P}\Big(Q^2(1)\,\zeta_Z\,\overline{Z}(\infty) > \frac{(N+k-1)^2}{2\nu_n^2\,N^2}, \quad \frac{N+k-1}{\log n} \ge \overline{Q}(1) \ge Q(1)\overline{Z}\big(\frac{k}{Q(1)\,N}\big), \quad \overline{Q}(1) > Q(1)\overline{Z}\big(\frac{k}{Q(1)\,N},\infty\big) - \frac{2}{n^\varepsilon}\Big).$$

It is easy to estimate $\Lambda_{16}(n,k)$ and $\Lambda_{17}(n,k)$. Indeed, by (4.2),

$$\Lambda_{16}(n,k) \le c_{16}\,\exp\Big(-\frac{(N+k-1)^2}{6\nu_n^2\,N^2}\Big) \le c_{16}\,\exp\Big(-\frac{1}{6\nu_n^2}\Big),$$

$$\Lambda_{17}(n,k) \le c_{16}\,\exp\Big(-\frac{N+k-1}{3\log n}\Big) \le c_{16}\,\exp\Big(-\frac{N}{3\log n}\Big).$$

In view of (4.11), we have

$$\sum_{k=1}^{[n^3\nu_n N]} \Lambda_{16}(n,k) \le n^{-3}, \tag{4.29}$$

$$\sum_{k=1}^{[n^3\nu_n N]} \Lambda_{17}(n,k) \le n^{-3}. \tag{4.30}$$

To estimate $\Lambda_{18}(n,k)$, note that

$$\begin{aligned}\Lambda_{18}(n,k) &\le \mathbb{P}\Big(\,\overline{Z}\big(\frac{k}{Q(1)\,N}\big) > \frac{N+k-1}{Q(1)\log n} + 2\Big)\\ &\quad + \mathbb{P}\Big(Q^2(1)\,\zeta_Z\,\overline{Z}(\infty) > \frac{(N+k-1)^2}{2\nu_n^2\,N^2}, \quad \overline{Z}(\infty) < \frac{N+k-1}{Q(1)\log n} + 2 + \frac{2}{n^\varepsilon}\Big)\\ &\stackrel{\mathrm{def}}{=} \Lambda_{20}(n,k) + \Lambda_{21}(n,k).\end{aligned}$$

Conditioning on $Q(1)$ and using Lemma 4.3,

$$\Lambda_{20}(n,k) \le c_{21}\,\exp\Big(-\frac{N(N+k-1)}{9k\log n}\Big) \le c_{21}\,\exp\Big(-\frac{N}{9\log n}\Big) \le n^{-7},$$

whereas

$$\begin{aligned}\Lambda_{21}(n,k) &\le \mathbb{P}\Big(Q^2(1)\,\zeta_Z\,\overline{Z}(\infty) > \frac{(N+k-1)^2}{2\nu_n^2\,N^2},\ \overline{Z}(\infty) < \frac{2(N+k-1)}{Q(1)\log n}\Big)\\ &\qquad + \mathbb{P}\Big(\frac{N+k-1}{Q(1)\log n} < 3\Big)\\ &\le \mathbb{P}\Big(\frac{\overline{Z}(\infty)}{\zeta_Z} < \frac{8\nu_n^2\,N^2}{(\log n)^2}\Big) + \mathbb{P}\Big(Q(1) > \frac{N}{3\log n}\Big)\\ &\le c_{18}\,\exp\Big(-\frac{c_{19}\,(\log n)^2}{8\nu_n^2\,N^2}\Big) + c_{16}\,\exp\Big(-\frac{N}{9\log n}\Big),\end{aligned}$$

the last inequality following from (4.6) and (4.2). Together with (4.12), we obtain:

$$\sum_{k=1}^{[n^3\nu_n N]} \Lambda_{18}(n,k) \le n^{-3}. \tag{4.31}$$

Finally, to estimate $\Lambda_{19}(n,k)$, note that

$$\begin{aligned}\Lambda_{19}(n,k) &\le \mathbb{P}\Big(Q^2(1)\,\zeta_Z\,\overline{Z}(\infty) > \frac{(N+k-1)^2}{2\nu_n^2\,N^2},\\ &\qquad Q(1)\overline{Z}(\infty) \le \frac{N+k-1}{\log n} + \frac{2}{n^\varepsilon}\Big)\\ &\le \mathbb{P}\Big(\frac{\overline{Z}(\infty)}{\zeta_Z} < \frac{2\nu_n^2\,N^2}{(N+k-1)^2}\Big(\frac{N+k-1}{\log n} + \frac{2}{n^\varepsilon}\Big)^2\Big)\\ &\le \mathbb{P}\Big(\frac{\overline{Z}(\infty)}{\zeta_Z} < \frac{3\nu_n^2\,N^2}{(\log n)^2}\Big),\end{aligned}$$

which, according to Lemma 4.2, is $\le c_{18}\,\exp(-c_{19}\,(\log n)^2/3\nu_n^2\,N^2)$. Recalling (4.12), this leads to the following estimate:

$$\sum_{k=1}^{[n^3\nu_n N]} \Lambda_{19}(n,k) \le n^{-3}. \tag{4.32}$$

Assembling (4.27)–(4.32) gives

$$\Lambda_9(n) \le \mathbb{P}\Big(X_n > n^3\nu_n\Big) + 4\,n^{-3}.$$

By (4.17) and (4.5), we have $\mathbb{P}(X_n > n^3\nu_n) \le n^{-3}$, which completes the proof of (4.19). □

5. Large favourite sites of the Wiener process

The problem of favourite sites can be posed for the Wiener process W as well. Let L_t^x be the jointly continuous local time process of W, and we can define the set of the favourite sites of W:

$$\mathbb{V}(t) \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R} : \, L_t^x = \sup_{y \in \mathbb{R}} L_t^y \right\}.$$

It is known (cf. Leuridan [19], Eisenbaum [11]) that almost surely for all $t > 0$, $\mathbb{V}(t)$ is either a singleton or composed of two points. Let us choose

$$V(t) \stackrel{\text{def}}{=} \max_{x \in \mathbb{V}(t)} x,$$

the (largest) favourite site. Our Theorems 3.1 and 4.1 and a 0-1 argument (see Remark 5.1.1) imply the following analogues of Theorems 1.1 and 1.2 for the Wiener process. Recall that $\overline{W}(t) \stackrel{\text{def}}{=} \sup_{0 \le s \le t} W(s)$.

Theorem 5.1. *There exists a universal constant* $\overline{c}_0 \in (0, \infty)$ *such that*

$$\liminf_{t \to \infty} \frac{(\log \log t)^{3/2}}{t^{1/2}} \left(\overline{W}(t) - V(t) \right) = \overline{c}_0, \qquad \text{a.s.}$$

Remark 5.1.1. Similarly to the case of favourite site of the random walk, one can easily see that a 0–1 law applies also for $V(t)$. Indeed, Bass and Griffin [2] proved that $\lim_{t \to \infty} |V(t)| (\log t)^{12} t^{-1/2} = \infty$ a.s., so $V(t)$ depends on large values of the Wiener process W and hence the initial portion $\{W(s), \, 0 \le s \le \log t\}$ has no influence on $V(t)$. It follows that the lim inf in Theorem 5.1 should be a constant. We believe that $\overline{c}_0$ must be identical with c_0 of Theorem 1.1 but due to lack of strong invariance principle between U and V we can not prove it.

Theorem 5.2. *Almost surely,*

$$\liminf_{t \to \infty} (\log \log t)^2 \, \frac{\overline{W}(t) - V(t)}{\overline{W}(t)} = j_0^2,$$

where j_0 *is the smallest positive root of the Bessel function* $J_0(\cdot)$.

Acknowledgements

We are grateful to Omer Adelman who kindly communicated the problem to us, and to Nathalie Eisenbaum and Bálint Tóth for helpful discussions. Thanks are also due to an anonymous referee for insightful remarks. Cooperation between the authors was supported by the joint French–Hungarian Intergovernmental Grant 'Balaton' (grant no. F25/97).

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